The Category Theory of Topological Thermodynamics

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Abstract: The Category theory of Topological Thermodynamics and Continuous Topological Evolution can be used as an abstract, but universal, foundation for understanding, among other things, scale invariance, chaos, emergence, self-similarity, dynamical systems, irreversible processes, non-linear propagating discontinuities, and even statistical phenomena of indistinguishable ingredients. The abstract Category theory, as an exterior differential system, has solutions which can be put into correspondence with numerous physical experiments in different disciplines.

1 Introduction

Mathematicians consider a Category Theory, C, to be a collection of abstract objects, ob(C), constructed in terms of ingredients, with morphsims that map an object X to another object Y. For example, the abstract objects might be the vertices of a lattice structure, LS, and the morphisms might be defined abstractly as directed arrows from one vertex to another. Representation theory is a highly specialized Category theory where the objects are vectors, and the morphisms are constrained to be generated by matrix groups.

Herein, the interest is focused on the abstract Category of objects and morphisms that represent Topological Thermodynamics and Continuous Topological Evolution. The objects are finite topological spaces of sets whose ingredients are exterior differential forms and/or differential form densities. The morphisms between sets are constrained to be homotopic classes of topologically continuous, but possibly non-invertible, maps. The coefficient functions of the exterior differential forms define abstract, and universal, thermodynamic *systems*. The homotopic, topologically continuous, morphisms define abstract thermodynamic *processes*. These morphisms include diffeomorphic processes that preserve geometry and metric scales, homeomorphic processes that preserve deformable topological features, which are scale invariant, and irreversible processes that permit topological change.

The finite topologies of 4 ingredients used in this article to define thermodynamic systems can be associated with three partitions of those 33 lattice structures that generate topologies.

1. The first partition is a discrete Hausdorff T2 topology, which is metrizable. The T2 separation axioms imply that singlet sets, or "particles", can be distinguished by *geometrical* processes.

2. The second partition is a disconnected Kolmogorov T0 topology (poset 3), which is not metrizable. The T0 separation axioms imply that "particles" can be distinguished by *topological* (not geometric) scale-invariant processes.

3. The third partition involves Not-T0 topologies which are not constrained by separation axioms. Some, if not all, of the singleton subsets are indistinguishable. The novel claim of this article is that Not-T0 partitions are applicable to the understanding of complex wave, diffusion, and statistical systems, which are based upon indistinguishable ingredients, herein defined as "complex wavelets, or Bosons, or Fermions", not "particles".

2 Particle Thermodynamic Systems

The topological T2 or T0 sets are based upon those 4 ingredients of exterior differential forms that form the Pfaff Sequence.

$$[A, dA, A\hat{}dA, dA\hat{}dA] = [A, F, H, K].$$

The Pfaff Sequence can be constructed from any exterior 1-form, $A = A_k(x^m)dx^k$, by use of the exterior differential, d, and the exterior product, $\hat{}$. The number of non-zero elements in the Pfaff Sequence determines the Pfaff Topological Dimension, (or irreducible class) of the exterior differential system.

Using the fact that the exterior differential, d, and the Identity, define a Kuratowski closure operator $Kcl = \mathbb{I} \cup d$, it becomes apparent that the exterior differential of a p-form, Σ, defines the limit set, dΣ, as a p+1 form. Hence the complete topological structure can be constructed from a 1-form of Action, A. The basis for the topology is composed of the elements in the Pfaff Sequence:

$$\{A, Kcl(A), A\hat{}dA, Kcl(A\hat{}dA)\} = \{A, A \cup F, H, H \cup K\}.$$

The Open sets of the Cartan - Kuratowski topology are generated from the basis by composing all unions of the basis sets. For 4 ingredients, the Cartan-Kuratowski topology becomes recognizable as the equivalent of the disconnected Kolmogorov T0 (poset 3) topology. In passing, it should be noted that the ubiquitous claim of homology theory that the boundary of a boundary is zero is false. The T0 poset 3 topological structure is one of many counter examples. The statement is true for the discrete, disconnected, metrizable Hausdorff T2 topology.

Following E. Cartan's method (called Cartan's magic formula by Marsden), the "laws of motion" that describe the evolution of the T2 or T0 "particle" topologies will be generated by applying the homotopic Lie differential (not derivative), $L_{(\mathbf{V}_4)}$, to the exterior differential 1-form of Action, A, that encodes the system:

$$L_{(\mathbf{V}_4)}A = i(\mathbf{V}_4)dA + d(i(\mathbf{V}_4)A) \Rightarrow Q.$$

With a change of notation,

$$L_{(\mathbf{V}_4)}A = \quad W \quad + \quad dU \quad = Q,$$

and Cartan's Magic formula becomes recognizable as a topological, universal, expression of the First Law of Thermodynamics.

The direction field, $\mathbf{V}_4$, representing a process can be an element of a differential semi-group, or an element of a complex spinor space. These processes are not thermodynamically reversible, if $Q\hat{\ }dQ \neq 0$.

These particle aspects have been well documented in numerous publications over the years, starting in 1962. The research is summarized with many examples in six monographs (see http://www22.pair.com/ebooks.htm) which will be available as free pdf downloads to conference attendees.

3 Statistical Thermodynamic Systems

The third partition involves Not-T0 topologies which are not constrained by separation axioms. Some, if not all, of the singleton subsets are indistinguishable. The claim is made herein that such Not-T0 partitions are applicable to the understanding of complex wave, diffusion, and, more generally, statistical systems, all of which are composed of indistinguishable ingredients. Equations that describe continuous topological evolution of Not-T0 topologies are constructed in terms of the homotopic operator, the Lie Differential, acting on the sets of exterior differential form densities. A number of examples will be described below.

Historically, for finite systems, the properties of the metrizable T2 Hausdorff, disconnected, discrete topology have been used to construct geometric models of thermodynamic systems of distinguishable particles in equilibrium. The subsets of the T2 topological structure imply that the boundary of a subset boundary, ∂S, vanishes, $\partial\partial S = 0$. However, it is easy to demonstrate that the theorem fails for non-equilibrium systems of distinguishable particles which utilize the non-metrizable, disconnected, T0 poset 3 topology.

Extending a suggestion of Chern, the Top set, X, of classical set theory, will be replaced by the Top Pfaffian, μ, which is a differential 4-form density on a variety of 4D topological dimensions. There Lattice Structures of the Power set of X of 4 ingredients will produce 17 topologies that are not T0, and 16 topologies that are T0. All of the 17 Not-T0 topologies have indistinguishable subsets, some of which are dense, and others which are not dense. The boundary of a boundary theorem is true for those indistinguishable subsets that are dense, but not true for those subsets that are not dense.

For simplicity, attention will be focused on the simplest of the Not-T0 lattice structures, the indiscrete Topology of 4 ingredients which has only two subsets, { } and μ. The topology is the indiscrete, but connected, Not-T0 topology. All subsets of the power set of X are dense and indistinguishable. The boundary of the boundary theorem is true for the connected indiscrete topology, suggestively similar to the result obtained for the discrete, disconnected, T2 topology.

The exterior differential of the Top Pfaffian, μ, vanishes:

$$\mu = \rho(x^k)\Omega(dx^k) = \rho(x^k)\ \{dx\hat{}dy\hat{}dz\hat{}dt\},\quad d\mu = 0.$$

The homotopic Cartan-Lie differential applied to Top Pfaffian yields the equation of continuous topological evolution:

$$L_{(\mathbf{V}_4)}\mu = \kappa(x^k)\ \mu.$$

This coefficient expression, $\kappa(x^k)$, has been called the "bulk dissipation, or self-similarity, or fractal, or homogeneity, or Entropic" coefficient. If the function, $\kappa(x^k)$, vanishes, the density, μ, is a local evolutionary volume invariant. If $\kappa = +1$, the evolution is homogeneous of degree 1, a necessary requirement for positive definite additivity of indistinguishable Bosons and statistical ingredients. If $\kappa = -1$, it is conjectured that the system describes indistinguishable (mod Spin) Fermions. If $\kappa(x^k) > 0$, then the 4D universe is expanding.

Evaluating the formula of continuous topological evolution, above, yields the expression,

$$L_{(\mathbf{V}_4)}\mu = i(\mathbf{V}_4)d\mu + d[i(\mathbf{V}_4)\mu] = 0 + d[i(\mathbf{V}_4)\mu] \Rightarrow \kappa(x^k)\ \rho(x^k)\ \Omega(dx^k).$$

Factorization of this formula yields a necessary PDE equation that describes a 4D divergence relation with inhomogeneous source terms:

$$\mathbf{div}_4(\mathbf{V}_4) = [\kappa(x^k)\ \ - \{\Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k)\}].$$

Realizations of this ubiquitous partial differential system of homotopic topological evolution can be recognized as the inhomogeneous wave equation or diffusion equation, the fractal minimal surface equation, the Klein-Gordan equation, the Schroedinger equation, the Hopf conjugate Spinor equation, the Landau-Ginsburg equation of superconductivity, and the Petrov blackhole equations of relativity theory. The term in brackets, {}, represents a generalization of the non-linear Eikonal expression, whose zero sets represent propagating singularities and discontinuities in the Maxwell theory of electromagnetic waves, and the Raleigh-Taylor wavecrests so cherished by surfers, and the von Karmon mushroom cloud singularities seen in liquid and plasma fluids.

3.1 Examples of Homotopic evolution of N-form densities

The Top Pfaffian is defined in terms of a differential volume element, $\Omega(dx^k) = dx\hat{}dy\hat{}dz\hat{}dt\}$, and a density coefficient $\rho(x^k)$, that may or may not be a constant:

The exterior differential 4-form density, the Top Pfaffian, μ

$$\mu \ = \ \rho(x^k)\Omega(dx^k) = \rho(x^k)\ \Omega(dx^k) = \rho(x,y,z,t)\{dx\hat{}dy\hat{}dz\hat{}dt\} \tag{1}$$

Following the success and experience of homotopic evolution of the distinguishable topologies, it is presumed that the evolutionary "equations of motion" of the indiscrete topology of indistinguishable sets are deduced by applying the homotopic Lie differential operator $L_{(\ \mathbf{V}_4)}$ (relative to the process, $\mathbf{V}_4$) to the 4-form density, μ.

For a process, $\mathbf{V}_4(x^k)$, assumed to be of the form,

$$\textbf{Process}:\ \mathbf{V}_4(x^k) = [V^x(x^k), V^y(x^k), V^z(x^k), V^t(x^k)], \tag{2}$$

the application of the homotopic Lie differential operator to Top Pfaffian produces the equation,

$$L_{(\ \mathbf{V}_4)}\mu = \kappa(x^k)\ \mu. \tag{3}$$

As μ is the Top Pfaffian, the only possible homotopic values are function multiples $\kappa(x^k)$ of μ; hence,

$$L_{(\mathbf{V}_4)}\mu = \kappa(x^k)\ \mu = \kappa(x^k)\ \rho(x^k)\ \Omega(dx^k). \tag{4}$$

The function, $\kappa(x^k)$, is called the homogeneity coefficient [**?**], [**?**], which can take on values equal to zero, or greater or less than zero, $\kappa = (0, +1, -1)$. If $\kappa = 0$, then the evolution of the Top Pfaffian is an invariant of the process. If If $\kappa = +1$, the evolution is homogenous of degree 1, a requirement for positive definite additivity of indistinguishable Bosons. If $\kappa = -1$,it is conjectured that the system describes indistinguishable (Mod Spin) Fermions.

As demonstrated below special case of interest is when

$$\kappa(x^k) = ln(\rho + \delta), \tag{5}$$

and δ takes in values

$$\delta = \{0, +1, -1\}. \tag{6}$$

Remark 1 *Does the process, $L_{(\mathbf{V}_4)}$, cause the differential 4-form volume element, $\Omega(dx^k)$, to expand, or contract, or remain invariant? Can the expansion of the volume element compensate for a change in density?*

3.2 Homotopic evolution of the Not T0 indiscrete Topology

Using Cartan's magic formula to define the homotopic evolution operator, the evolution of the Top Pfaffian becomes:

$$\begin{aligned} L_{(\mathbf{V}_4)}\mu &= (L_{(\mathbf{V}_4)}\rho(x^k))\ \Omega(dx^k) + \rho(x^k)(L_{(\mathbf{V}_4)}\Omega(dx^k) && (7) \\ &= (V^k \circ \partial\rho/\partial x^k)\Omega(dx^k) + \rho(x^k)d(i(\ \mathbf{V}_4)\Omega(dx^k)) && (8) \\ &= \{\Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k) + \partial V^k/\partial x^k)\ \}\mu = \kappa\ \mu. && (9) \end{aligned}$$

Note that the evolution of the differential 4-form volume element, $\Omega(dx^k)$, can be described in terms of the limit points, dC, of a current 3-form, C:

$$\begin{aligned} C &= i(\ \mathbf{V}_4)\Omega(dx^k) && (10) \\ &= [V^x\hat{}dy\hat{}dz\hat{}dt - V^y\hat{}dx\hat{}dz\hat{}dt && (11) \\ &\quad +V^z\hat{}dx\hat{}dy\hat{}dt - V^t dx\hat{}dy\hat{}dz], && (12) \\ L_{(\ \mathbf{V}_4)}\Omega(dx^k) &= d(C) = \mathbf{div}_4(\mathbf{V}_4)\Omega(dx^k) && (13) \\ \mathbf{div}_4(\mathbf{V}_4) &= \mathrm{Trace}(\mathbb{J}) && (14) \\ \mathbb{J} &= [\partial V^m/\partial x^n] = \text{Jacobian matrix} && (15) \end{aligned}$$

It follows that the Homotopic Evolution of the Top Pfaffian density 4-form, μ, is governed by the equation:

$$L_{(\mathbf{V}_4)}\mu = \{\Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k) + \mathbf{div}_4(\mathbf{V}_4))\ \}\mu = \kappa\ \mu. \qquad (16)$$

Note that if the density is not zero, $\mu \neq 0$, then the universal evolutionary equation above has a common factor, μ, which - when removed - leads to a partial differential system, that must vanish universally. By re-arranging terms to emphasize the divergence component, the 2nd Fundamental Partial Differential system becomes:

2nd Fundamental Partial Differential system:

$$\mathbf{div}_4(\mathbf{V}_4) = \kappa - \{\Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k)\}, \qquad (17)$$

It is this differential system, **PDE II**, that defines the Second Fundamental Equation of indiscrete Homotopic Evolution.

3.3 The Wave Equation

For example assume that

$$\begin{aligned} \left(x^k\right) &= \{x, y, z, s = ict\} && (18) \\ \rho &= \Psi\left(x^k\right), \quad \varepsilon\mu c^2 = 1, && (19) \\ grad_4(\rho) &= [\partial\Psi/\partial x, \partial\Psi/\partial y, \partial\Psi/\partial z, (\sqrt{\varepsilon\mu}/i)\partial\Psi/\partial t] = \mathbf{V}_4(x^k) && (20) \end{aligned}$$

then the Fundamental Partial Differential system becomes:

$$\begin{aligned} \mathbf{div}_4(\mathbf{V}_4) &= \partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 + \partial^2\Psi/\partial z^2 - \varepsilon\mu\partial^2\Psi/\partial t^2 && (21) \\ \mathbf{div}_4(\mathbf{V}_4) &= \{\kappa\ - (1/\Psi)\{(\partial\Psi/\partial x)^2 + (\partial\Psi/\partial y)^2 + (\partial\Psi/\partial z)^2 - \varepsilon\mu(\partial\Psi/\partial t)^2\} && (22) \\ \mathbf{div}_4(\mathbf{V}_4) &= k - (1/\Psi)\{Eikonal\ PDE\}. && (23) \end{aligned}$$

This set of equations is to be recognized as the inhomogeneous wave equations (with the non-linear quadratic PDE known as the Eikonal constraint.). If the RHS is zero, then the result is the linear, second order, Wave equation. The wave speed, c, is determined from the formula, $\varepsilon\mu c^2 = 1$.

If the RHS = MΨ then the Partial Differential System generates the Klein-Gordan equation, with the constant M related to "mass" in the Klein =-Gordan theory.

3.4 The Diffusion Equation

Consider a possible solution to the homotopic evolution of the indiscrete topology generated by the Top Pfaffian, μ, in terms the process, $\mathbf{V}_4$:

$$\mathbf{V}_4 = [V^k] = [\partial\Psi/\partial x, \partial\Psi/\partial y, \partial\Psi/\partial z, -D\Psi], \tag{24}$$

$$\mathbf{div}_4(\mathbf{V}_4) = \partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 + \partial^2\Psi/\partial z^2 - D\partial\Psi/\partial t \tag{25}$$

Substitution of the example expression for $\mathbf{div}_4(\mathbf{V}_4)$ leads to the equation,

$$\begin{aligned} &\partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 + \partial^2\Psi/\partial z^2 - D\partial\Psi/\partial t \\ = &\{\kappa \ - \Sigma_k(V^k \circ \partial(ln\rho)/\partial x^k)\} \end{aligned}$$

This equation is to be recognized as the inhomogeneous Diffusion equation, where the terms on the RHS can be interpreted as interactions, or source terms. The solutions to the diffusion equation consists of indistinguishable distributions. /if the diffusion coefficient is imaginary, $D \to \hbar/i$, then the equation of homotopic evolution has the format of the Schroedinger equation.

3.5 The Minimal Surface equation

Consider a possible solution to the homotopic evolution of the indiscrete topology generated by the Top Pfaffian, μ, in terms the process direction field, $\mathbf{V}_4/\lambda_H$, where λ_H is a Holder norm, with arbitrary constant coefficients, $\{a, b, c, e\}$ and constant exponents, (p, h):

$$\lambda_H = [a(V^x)^p + b(V^y)^p + c(V^z)^p + \varepsilon(V^s)^p]^{h/p} \tag{26}$$

$$\mathbb{J} = [(\partial \mathbf{V}_4/\lambda_H)/\partial x^n] = \text{Jacobian matrix} \tag{27}$$

$$\mathbf{div}_4(\mathbf{V}_4) = Trace[\mathbb{J}] \tag{28}$$

The Holder norm can be used to adjust the homogeneity properties of the "rescaled" process direction field. The Jacobian matrix always supports a Hamilton-Jacobi characteristic polynomial, $Ch[Jac(\mathbf{V}_4/\lambda_H)] = \Theta$, of the form

$$\Theta = \eta^4 - M\eta^3 + G\eta^2 - A\eta + K = 0, \tag{29}$$

where the coefficients $\{M, G, A, K\}$ are the similarity coefficients, invariant with respect to similarity transformations of the Jacobian Matrix, $\mathbb{J}$.

The zero set of the characteristic polynomial, $\Theta = 0$, defines a hypersurface family (with parameter, η) in the 4 dimensional space whose coordinates are $\{M, G, A, K\}$. It is easily demonstrated (with Maple) that the similarity coefficients depend upon a choice for the homogeneity index h in the Holder Norm, λ_H.

$$\text{Mean Curvature: } M = (4-h)/\lambda_H = Trace\ [Jac(\mathbf{V}_4/\lambda_H)], \tag{30}$$

$$\text{Gauss Curvature: } G = (6-3h)/\lambda_H^2, \text{ gravitational energy} \tag{31}$$

$$\text{Cubic Curvature: } A = (4-3h)/\lambda_H^3 = Adoint[Jac(\mathbf{V}_4/\lambda_H)]\text{Interaction energy} \tag{32}$$

$$\text{Quartic Curvature: } K = (1-h)/\lambda_H^4 = det[Jac(\mathbf{V}_4/\lambda_H)], \text{ Irreversible Dissipation.} \tag{33}$$

The Hamilton-Jacobi characteristic polynomial, $\Theta = 0$, can be interpreted as a universal thermodynamic equation of state that describes the Homotopic Evolution of the indiscrete topology, $Not - T0$.

$$\Theta = (\eta\lambda_H - 1)^3\,(\eta\lambda_H - 1 + h) = 0. \tag{34}$$

When the mean curvature vanishes, $M \Rightarrow 0$, the Hypersurface has a normal field with zero divergence, and a homogeneity index, $h = 4$.

When there is no irreversible Dissipation, $K \Rightarrow 0$, and the homogeneity index is finite, $h = 1$. In this situation, the universal equation of state has three non-zero eigen values, and one zero eigenvalue, that mimics the equation of state for a van der Walls gas.

3.6 Dissipation interaction

Assume that the volume element on space-time is proportion to the volume element created by the differentials of the components of the process:

$$\begin{aligned}
\Omega(dV^k) &\Rightarrow \rho\,\Omega(dx^k) = det\ [\mathbb{J}]\ dx\hat{}dy\hat{}dz\hat{}dt && (35)\\
[\mathbb{J}] &= [\mathbb{J}^m_n] = [\partial V^m/\partial x^n], && (36)\\
\rho &= det\ [\mathbb{J}]. && (37)\\
\text{and define } A_k &= \mathbf{grad4}\ \rho = \mathbf{grad4}\ (det\ [\mathbb{J}]) && (38)
\end{aligned}$$

Then the Fundamental Differential system becomes

$$\begin{aligned}
\mathbf{div}_4(\mathbf{V}_4) &= \{\kappa\ - (V^k \circ A_k)/det\ [\mathbb{J}]\} && (39)\\
\text{where } \mathbb{J} &= [\partial V^k/\partial x^n] = \text{Jacobian matrix} && (40)\\
\mathbf{div}_4(\mathbf{V}_4) &= \text{Trace}(\mathbb{J}). && (41)
\end{aligned}$$

The term $(V^k \circ A_k)$ plays the role of an interaction dissipation coefficient, which vanishes if the innerproduct is zero, $(V^k \circ A_k) = 0$. The interaction energy vanishes if either the gradient of the density distribution (determinant of the process Jacobian) is zero, or the gradient of the density distribution (determinant of the process Jacobian) is orthogonal to the process. If $(V^k \circ A_k)$ is not zero, then a thermodynamic argument indicates that such processes are irreversible.

3.7 The Characteristic Polynomial of Homogeneous Processes

The process direction fields can include homogeneous structures, where the process directions are renormalized by divisors that are functions of the coefficients of the process. A particularly useful divisor is given by the Holder norm, λ_H:

$$\begin{aligned} \mathbf{V}_4 &= [V^k/\lambda_H], & (42)\\ \lambda_H &= [a(V^x)^p + b(V^y)^p + c(V^z)^p + \varepsilon(V^s)^p]^{h/p} & (43)\\ C &= i(V^k/\lambda_H)dx\hat{}dy\hat{}dz\hat{}dt = i(V^k/\lambda_H)\Omega(dx^k). & (44) \end{aligned}$$

The coefficients $\{a, b, c, \varepsilon, p, h)$ are presumed to be constant. The coefficient, h, is defined to be the homogeneity index In 4D, the characteristic polynomial of $[Jac(\mathbf{V}_4/\lambda_H)]$ is of 4th degree, where for (possibly complex) eigen values, γ, the polynomial (by the Cayley Hamilton theorem) generates the hypersurface,

$$\begin{aligned} Ch[Jac(\mathbf{V}_4/\lambda_H)] &= \gamma^4 - M\gamma^3 + G\gamma^2 - A\gamma + K = 0, & (45)\\ &= \gamma^4 - (4-h)\gamma^3/\lambda_H + (6-3h)\gamma^2/\lambda_H^2 - (4-3h)\gamma/\lambda_H^3 + (1-h)/\lambda_H^4. & (46)\\ &= (\gamma\lambda_H - 1)^3(\gamma\lambda_H - 1 + h) = 0. & (47) \end{aligned}$$

For different values of the homogeneity index, h, the Holder norm, λ_H, creates different homogeneity criteria. It is easily demonstrated (with Maple) that

$$\begin{aligned} \text{Mean Curvature: } M &= (4-h)/\lambda_H = Trace\ [Jac(\mathbf{V}_4/\lambda_H)], & (48)\\ \text{Gauss Curvature: } G &= (6-3h)/\lambda_H^2, & (49)\\ \text{Cubic Curvature: } A &= (4-3h)/\lambda_H^3, & (50)\\ \text{Quartic Curvature: } K &= (1-h)/\lambda_H^4, & (51) \end{aligned}$$

So, as mentioned above, for h = 4, the trace of $Ch[Jac(\mathbf{V}_4/\lambda_H)]$ vanishes and the Mean Curvature of the hypersurface is zero. Hence the homogeneous process is a minimal surface. The 4-divergence of the process $\mathbf{V}_4/\lambda_H$ is then zero, and the volume element is invariant for such all such homogeneous processes (h=4). If the Mean curvature is not zero, then divergence of the homogeneous process depends upon the Holder norm, λ_H.

$$L_{(\mathbf{V}_4)}\rho\Omega = \rho\ d(i(V^k/\lambda_H)\Omega\} + (L_{(\mathbf{V}_4)}\rho)\ \Omega = \tag{52}$$

$$\text{where } \mathbb{J}{=}[\partial(V^m/\lambda_H)/\partial x^n] = \text{Jacobian matrix of } \mathbf{V}_4 \tag{53}$$

$$\{div_4(\mathbf{V}_4)\} = \Psi = (4-h)/\lambda_H \neq 0. \tag{54}$$

is well defined for any h, and for all forms of the Holder norm. When h $>$ 4 the 4D volume is contracting; when h $<$ 4, the 4D volume is expanding, due to the homogeneous process $\mathbf{V}_4/\lambda_H$.

Now return to the "diffusion" format,

$$\mathbf{V}_4 = [\partial\psi/\partial x, \partial\psi/\partial y, \partial\psi/\partial z, D\psi], \qquad \rho\mathbf{V}_4 = \mathbf{V}_4/\lambda_H\ \ , \tag{55}$$

$$\lambda_H = [a(\partial\psi/\partial x)^p + b(\partial\psi/\partial y)^p + c(\partial\psi/\partial z)^p + \varepsilon(D\psi)^p]^{h/p}. \tag{56}$$

Then the divergence of $\mathbf{V}_4/\lambda_H$ then yields a modified diffusion equation:

$$div_4(\mathbf{V}_4/\lambda_H) = (div_4\mathbf{V}_4)/\lambda_H - \mathbf{V}_4 \circ grad(\lambda_H)/(\lambda_H)^2 = (4-N)/\lambda_H, \tag{57}$$

$$= (div_4\mathbf{V}_4) - \mathbf{V}_4 \circ grad_4(\lambda_H)/(\lambda_H) = (4-h)/\lambda_H, \tag{58}$$

$$-(\varepsilon D)\partial\psi/\partial t = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 - (4-h) - \mathbf{V}_4 \circ grad_4(\ln\rho). \tag{59}$$

If the Characteristic Polynomial defines a minimal surface, $M = 0$, for $h = 4$, then the Gauss curvature is negative for any signature, indicating the surface is unstable. The volume element is invariant, but the Minimal Surface is unstable. Suppose that $N \leq 2$; then the Gauss curvature is positive, indicating that the Hypersurface generated by the Characteristic polynomial is stable, but it is not a Minimal surface. However, the Volume element is expanding.

Conjecture 2 *Is the Expansion of the universe required to stabilize the Hypersurface generated by the Characteristic polynomial?*

3.8 Ginsburg Landau Currents

With a change of notation ($\eta \Rightarrow \Psi$), the Universal Phase function (eq. 29) can be solved for the determinant of the Jacobian matrix, which is equal to the similarity invariant T_K,

$$T_K = -\{\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi\}. \tag{60}$$

The similarity invariant, T_K, represents the determinant of the Jacobian matrix. All determinants are in effect N-forms on the domain of independent variables. All N-forms can be related to the exterior differential of some (N-1)-form or current, J. Hence

$$dJ = T_K\Omega_4 = (div\mathbf{J} + \partial\rho/\partial t)\Omega_4 = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi)\Omega_4. \tag{61}$$

For currents of the form,

$$[\mathbf{J}, \rho] = [grad\ \Psi, -D\Psi], \tag{62}$$

the Universal Phase function generates the format of the universal Ginsburg Landau equations of superconductivity:

$$\nabla^2\Psi - D\partial\Psi/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi). \tag{63}$$

It is also possible to construct conjugate minimal surfaces (see page187 in [**?**]) using the holomorphic function, Φ,

$$\Phi = A\ ln\rho + B + C\rho + D\rho^2, \tag{64}$$

which couples indistinguishable sets to Gibbs entropy, $A\ ln\rho + C\rho$, and Mandelbrot term $B \pm D\rho^2$.

4 Significant Results

As stated above, the Category of Topological Thermodynamics and Continuous Topological Evolution can be used as an abstract, but universal, foundation for understanding, among other things, scale invariance, chaos, emergence, self-similarity, dynamical systems, non-linear propagating discontinuities, and even statistical phenomena of indistinguishable ingredients. The abstract categorical foundation, as an exterior differential system, admits solutions which can be put into correspondence with numerous physical experiments, in different physical disciplines.

- The method Pfaff Topological Dimension explains topological differences between equilibrium PTD<3 (connected) and non-equilibrium PTD>2 (disconnected) thermodynamic systems of distinguishable particles.
- The method explains topological distinctions between thermodynamically reversible, integrable processes $(Q\hat{}dQ = 0)$ and irreversible non-integrable $(Q\hat{}dQ \neq 0)$ processes.
- The method explains topological distinctions between thermodynamic systems based on distinguishable sets, and thermodynamic systems based upon statistical distributions of indistinguishable sets (Not-T0 topologies).
- The method explains that Entropy can consist of two simultaneous components: one component is based on distinguishable sets (massive particles) and the other component is based upon indistinguishable sets (radiation).
- The method demonstrates that the First Law of Thermodynamics is a statement of cohomology, where the difference of the two not necessarily exact differentials (PTD >1) of Heat Q and Work W are equal to an exact differential of internal energy, dU.
- The method demonstrates that the decay of turbulence can be described in terms of a topologically continuous process, but the creation of turbulence can not.
- The method indicates that the top down nested sequence of Open, Closed, Isolated, and Equilibrium thermodynamic domains is in 1-1 correspondence with the Pfaff Topological Dimension [4,3,2,1] of the 1-form, A, used to create the Cartan topological structure.
- The method suggests that the environment of the universe is not a vacuum (a void empty set PTD(A) = 0), but is best described as the disconnected Open system of PTD(A) = 4, in which domains of PTD < 4 are distinguishable as topological defects in the environment.
- Non-Equilibrium open systems of Pfaff Topological Dimension 4 (PTD=4) can decay (in finite time) to Closed systems of Pfaff Topological Dimension 3 (PTD=3), thereby justifying Prigogine's conjecture of emergence by means of dissipative, but topologically continuous, processes. These metastable PTD=3 Closed states are topological defects, far from equilibrium, in the Open systems of PTD=4.
- Topological change is a necessary condition for thermodynamic irreversibility.
- The constraints of topological continuity and non-homeomorphic processes establish a logical basis for thermodynamic irreversibility and the arrow of time.

- Any synergetic system of parts defines a topology such that the Category of Topological Thermodynamics is applicable to economic systems, political systems, as well as to biological systems. Such systems admit long-lived states far from equilibrium (adulthood). However, the theory can be used to design subtle perturbations to such systems such that their metastable lifetimes can be extended or destroyed.

- C2 Continuous Topological Evolution permits irreversible processes, for which, $Q\hat{}dQ \neq 0$. Segmented C1 processes approximating smooth C2 processes can be reversible, $Q\hat{}dQ = 0$.

- On odd-dimensional spaces, sequential C1 (translational) processes can be thermodynamically reversible, while intransitive C2 processes (rotation and expansion with a fixed point) can be thermodynamically irreversible.

- The relativistic Twin Paradox is resolved if the evolutionary paths of each twin are defined by processes that cause (unequal) topological change. If there is no topological change, or if each twin suffers the same topological change, there is no disparate biological aging.

- Adiabatic processes are transverse to the heat 1-form, $(i(\rho\mathbf{V}_4)Q) = 0$. Adiabatic processes need not be quasi-static, and can be reversible or irreversible. Topological Transitions can occur in finite (often short) time.

- The work 1-form, W, is always *transverse* to the 4D process direction field, $\rho\mathbf{V}_4$, but the heat 1-form, Q, may or may not be transverse, The heat 1-form, Q can have longitudinal components in the direction of the process, corresponding to irreversible dissipation.

- Engineers should be guided by the universal concept of minimizing longitudinal heat in order to improve efficiency. For fluids, this idea can be translated to reducing vorticity components in the direction of fluid accelerations (by using wing tip tabs on commercial jets).

- For non-equilibrium systems, the 3-form of Topological Torsion (an N-1 form current) is not zero: $A\hat{}dA = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt \neq 0$. The Topological Torsion vector, $\mathbf{T}_4$, is deduced intrinsically from the 1-form that encodes the thermodynamic system. It can be used as a direction field for a process current density, $\rho\mathbf{T}_4$.

- For PTD=3 "closed" thermodynamic systems, the process current has zero divergence, and the 4D volume element is a conformal invariant (any ρ). This result is the space-time extension of the Liouville theorem that preserves the phase-space volume element in classical theory.

- For a PTD=4 "open" thermodynamic systems, the Topological Torsion vector does not have zero divergence, and so the process current $\rho\mathbf{T}_4$ may not be closed for arbitrary ρ (that is, the divergence of the process current is not zero).

- The combination of continuity and non-equilibrium, PTD(Q)>2, requires a causal direction to the arrow or time.

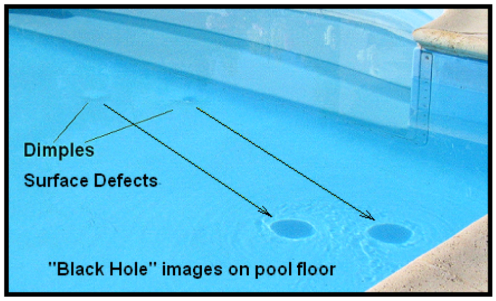
- PTD(A)=3 domains can have a local basis in terms of one complex Spinor pair with complex conjugate eigenvalues, and one real vector with eigenvalue zero. The eigenvalue 0 state can represent metastable (long-lived) configurations far from equilibrium. Such domains are locally "contact" manifolds.

- A key artifact of non-equilibrium is the existence of Topological Torsion current 3-forms, Topological Spin current 3-forms and Topological Adjoint current 3-forms. All of these current 3-forms are similar to the Amperian charge current 3-form of electromagnetic theory, but are related to different species of dissipative phenomena, which occur only in non-equilibrium systems. The dissipation coefficients are related to the non-zero divergences of the vector coefficients of the various 3-forms. For example, in electromagnetic systems, the dissipation coefficient is proportional to $\mathbf{E} \circ \mathbf{B}$ (see Vol 4) ; in hydrodynamics, the dissipation coefficient is called "Bulk Viscosity" (see Vol 3).

- PTD(A)=4 domains can have a local basis in terms of two complex Spinor pairs. Locally, such domains are symplectic manifolds. In such symplectic domains there exist local density distributions (subdomains), ρ, such that the divergence of any process current is zero in that subdomain. Such subdomains are metastable contact manifolds. In other words, the symplectic domain can contain defect structures in the form of contact subdomains. It can be demonstrated in terms of continuous topological evolution that such local density distributions, which define a "stationary" state, can emerge as a topological defect in a PTD=4 system, by means of a dissipative processes.

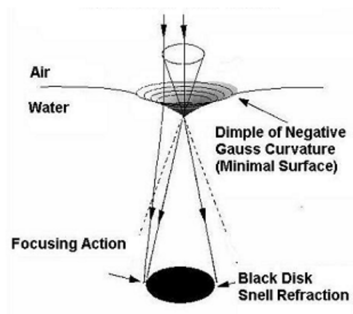
Emergence of Topological Defects

FALACO SOLITONS Movie by D. Radabaugh



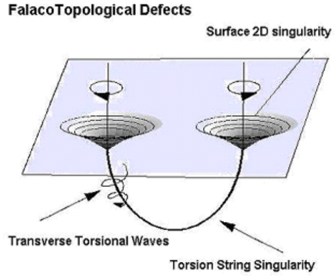
Solar elevation about 30 degrees. See movie at
http://www22.pair.com/csdc/download/spotsmovie.avi

Emergence of Topological Defects



Snell refraction of Falaco Soliton Spin Pairs

Emergence of Topological Defects



The first measurableTorsion String coupling between branes

This real world effect has been ignored by string theorists !!!

- The Adjoint Current, $J_{adjoint}$, like the Torsion Current, can be constructed entirely from the topological features of the thermodynamic system as determined by the functional coefficients of the 1-form of Action, A. The Adjoint Current also admits an infinity of integrating factors. However, when the 1-form coefficients, A_k are divided by a Holder Norm, λ, of homogeneity index one (known as the Gauss map), then the components of the Adjoint Current are defined as $|\mathbf{J}_{adjoint}\rangle = [Jacobian(A_k/\lambda)]^{adjoint} \circ |A_k/\lambda\rangle$. The 3-form of the adjoint current is defined as $J_{adjoint} = i(\mathbf{J}_{adjoint})dx\hat{}dy\hat{}dz\hat{}dt$,and has zero divergence globally, and a pre-image 2-form that is not closed, $G_{adjoint}$, such that $dG_{adjoint} = J_{adjoint}$. It can be demonstrated that the adjoint current is related to the cubic curvature of the shape matrix, leading to the idea that the source of electromagnetic charge is related to cubic curvatures, similar to the idea that mass is related to quadratic curvatures.

- Examples of thermodynamic systems can be given to demonstrate that the conjectured format of the London Current of superconductivity, where $\mathbf{J} = \chi\mathbf{A}$, can be deduced as an emergent consequence of the Topological Theory of Thermodynamics (see Vol 5).

- Examples can generate a Spin Current 3-form, $\mathbf{S} = \mathbf{A}\hat{}\mathbf{G}$,where, formally, the Spin current is proportional to the Lorentz force (the space-time components of the Work 1-form, W). This is a new interpretation of an old result, $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{VxB})$, which is Ohm's law. The new part is due to the idea that the dissipation is due to Spin Currents and the transport of collective spins, $A\hat{}G$.

- The topological structure of domains of Pfaff dimension 2 or less creates a connected, but not necessarily simply connected topology. Evolutionary predictive solution uniqueness is possible.

- The topological structure of domains of Pfaff dimension 3, or more, creates a disconnected topology of multiple components. If solutions to a particular evolutionary problem exist, then the solutions are not unique. Envelope solutions, such as Huygen wavelets and propagating tangential discontinuities (called signals, or wakes) are classic examples of solution non-uniqueness. Topological Torsion is an artifact of non-uniqueness, and must be non-zero if a hydrodynamic system exhibits turbulence.

- All Hamiltonian, Symplectic-Bernoulli and Helmholtz processes are thermodynamically reversible. In particular, the work 1-form, W, created by Hamiltonian processes is of Pfaff Topological Dimension 1 or less. In all reversible cases the Work 1-form is closed, $dW = 0$. The evolutionary equations in such cases are time reversal invariant.

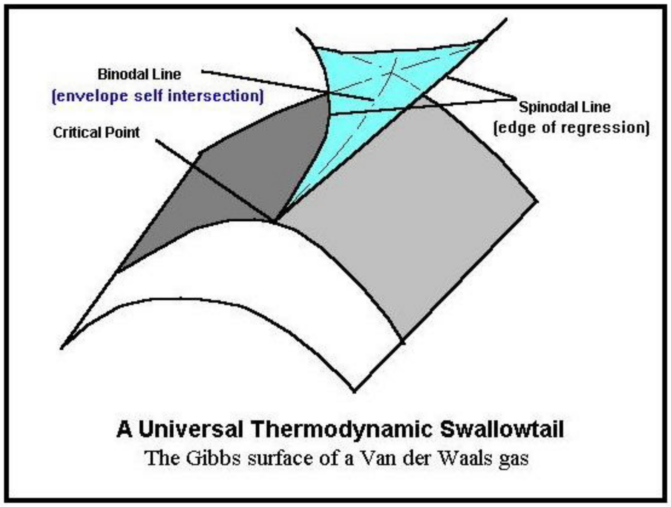
- The assumption of uniqueness of evolutionary solutions (which implies the Pfaff Topological Dimension of the thermodynamic system be equal to 2 or less), or the assumption of homeomorphic evolution, have imposed constraints upon classical mechanics that eliminate any time asymmetry, and the existence of isotropic spinors.

- The Lie differential acting on differential forms is not necessarily the same as a linear affine covariant differential acting on differential forms. The covariant differential always defines an adiabatic process, where the Lie differential does not.

- The particle view is based upon the 1-form of action whose coefficients, $[\mathrm{A}_m]$, admit a *correlation* Jacobian matrix $[\partial \mathrm{A}_m/\partial \mathrm{x}^n]$.

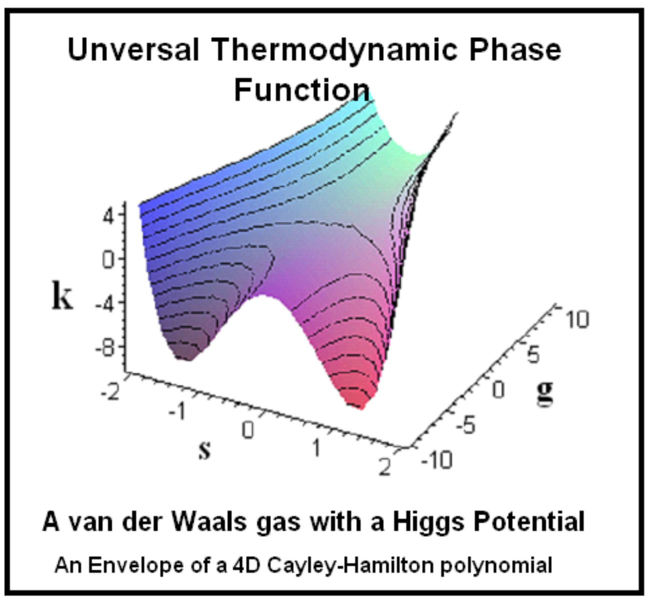
- The statistical point of view is based upon a current N-1 form whose coefficients, $[\mathrm{C}^m]$, admit a *collineation* Jacobian matrix, $[\partial \mathrm{C}^m/\partial \mathrm{x}^n]$, with a trace equal to a divergence of the Current.

- On spaces of PTD=4, the correlation Jacobian matrix has a characteristic Cayley-Hamilton polynomial that defines an quartic equation of state in terms of similarity invariants. The characteristic polynomial produces an implicit hypersurface function that can have envelopes and edges of regression in the format of the Gibbs function for a (universal and deformable) van der Waals gas.



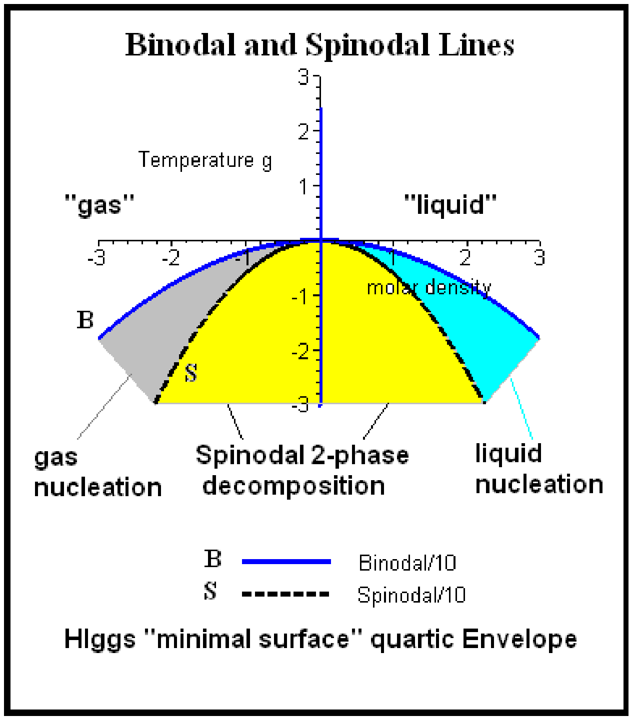
A Universal Thermodynamic Swallowtail
The Gibbs surface of a Van der Waals gas

- The correlation can be mapped into a reduced characteristic polynomial representing a quartic Universal Thermodynamic Phase function, with an envelope, which, below the critical point, has features of a Higgs potential.



A van der Waals gas with a Higgs Potential
An Envelope of a 4D Cayley-Hamilton polynomial

- The reduced polynomial yields universal analytic expressions for the critical point, and the binodal and spinodal lines, in terms of the similarity invariants. The same technique can be applied to dynamical systems.



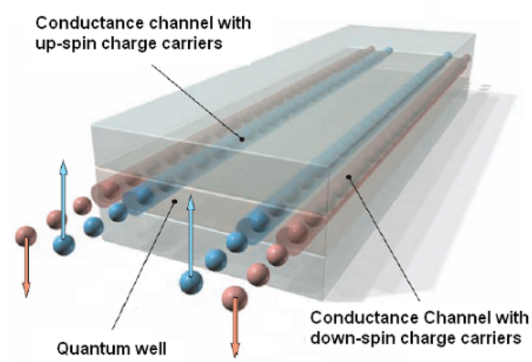
- The collineation can have complex eigenvalues and eigenvectors, even though the maximal rank matrix elements are real. These complex eigenvectors have a zero norm, and are generators of conjugate minimal surfaces, that have both statistical and fractal components.

- The Jacobian matrix can have complex eigenvalues and eigenvectors, even though the matrix elements are real. These complex eigenvectors have a zero norm, and are generators of conjugate minimal surfaces. Cartan defined such sets as Isotropic Spinors (Majorana Spinors, not Dirac Spinors). Pairs of non-colinear Spinors define an area, but the "norm" of each Spinor is Zero!

- The hypersurface minimal surface can be generated by a holomorphic function that includes both the Gibbs entropy and a Mandelbrot fractal germ, $\Theta = (z \ln z - z) + (a - bz^2)$. The third partial derivative of Θ leads to conjugate pairs of minimal surfaces. and the Mandelbrot germ vanishes. All functional iterates remain holomorphic and hence generate minimal surfaces with fractal boundaries.

- Topological fluctuations can be induced by processes that have components in terms of thermodynamic macroscopic Spinors. Thermodynamic Macroscopic Spinors are non-zero complex eigenvectors with complex eigenvalues (and zero quadratic form) of the antisymmetric 2-form (or matrix) representing the "Limit Points", dA, of the 1-form of Action, A. Such Macroscopic Spinors are capable of representing minimal surface conjugate pairs.

- Topological Insulators correspond to Impedances defined in terms of quantized Topological Spin, and Spin pairs coupled via Falaco Solitons. If the divergence of the Spin 4-vector vanishes, the Spin Current is time reversal invariant.



Spin Currents

Courtesy of C. Bickel and Science Magazine

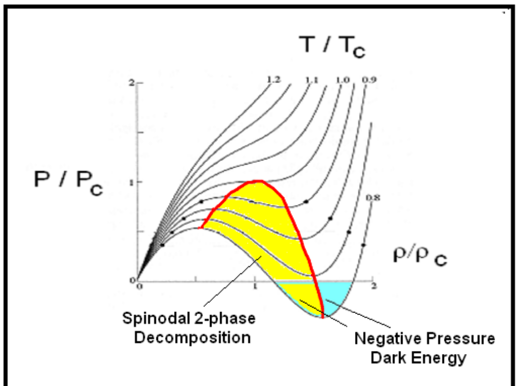
- The thermodynamic processes that lead to self-similarity of a Current 3-form $L_{(J)}\mathbf{C}=\sigma\mathbf{C}$ can generate fractals and holographic effects where small neighborhoods replicate the whole, approximately. The homogeneity coefficient is the trace of the Jacobian Collineation: $\sigma=$Trace $[\partial \mathrm{C}_m/\partial \mathrm{x}^n]$, or the divergence of the Process vector field.

- Following Landau, it is conjectured that a turbulent non-equilibrium thermodynamic cosmology can be constructed in terms of a dilute van der Waals gas near its critical point. The conjecture yields an explanation for:

1. The granularity of the night sky as exhibited by stars and galaxies is due to density fluctuations near the critical point, and the Newtonian law of gravitational attraction proportional to $1/r^2$ is due to a correlation between fluctuations (due to Landau).

2. The conformal expansion of the universe as an irreversible phenomenon-associated with Quartic similarity invariants in the thermodynamic phase function, and conformally related to dissipative effects.

3. The possibility of domains of negative pressure (explaining what has recently been called "*dark energy*") could be due to a *classical* "Higgs" mechanism for aggregates below the critical temperature.



Non Equilibrium van der Waals gas

4, The possibility of domains of negative temperature (explaining what has recently been called "*dark matter*") could be due to macroscopic collective states of ordered spins. The conjecture is that Positive temperature radiates, Negative temperature does not. The conjecture is that black holes could be negative temperature states of collective spins.

5. The possibility of domains where gravitational effects (quadratic similarity invariants, or 2nd order Gauss curvature effects) appear to be related to entropy and temperature properties of the thermodynamic system, and where cubic curvature effects could impede gravitational collapse.

6. Black Holes (generated by Petrov Type D solutions in gravitational theory) are related to Minimal Surface solutions to the Universal thermodynamic 4th order Phase function.