Turbulence and the Navier-Stokes Equations

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Abstract: The concept of Continuous Topological Evolution, based upon Cartan's methods of exterior differential systems, is used to develop a topological theory of non-equilibrium thermodynamics, within which there exist processes that exhibit continuous topological change and thermodynamic irreversibility. The technique furnishes a universal, topological foundation for the partial differential equations of hydrodynamics and electrodynamics; the topological technique does not depend upon a metric, connection or a variational principle. Certain topological classes of solutions to the Navier-Stokes equations are shown to be equivalent to thermodynamically irreversible processes. The method demonstrates, by example, how an irreversible dissipative process acting in an Open non-equilibrium system of Pfaff topological dimension 4 can decay, or create in finite time, topological defect structures, or Closed systems of Pfaff topological dimension 3. These Closed non-equilibrium systems admit a Hamiltonian process which can emulate the geometrical evolution of topological stationary states far from equilibrium. The theory of Continuous Topological Evolution gives formal credence, as well as analytic examples, to the Prigogine conjecture of self-organization in terms of disspative (thermo)dynamics.

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Prologue

THE POINT OF DEPARTURE

This presentation summarizes a portion of 40 years of research interests[1] in applied physics from a perspective of continuous topological evolution. The motivation for the

[1] This work is summarized in a series of reference monographs [42], [43], [44], [45], [46] which have been constructed and updated from numerous publications. These volumes contain many examples and proofs of the basic concepts.

past and present effort continues to be based on the recognition that topological evolution (not geometrical evolution) is required if non-equilibrium thermodynamic systems and irreversible turbulent processes are to be understood without the use of statistics. This essay is written (by an applied physicist) as an alternative response to the (more mathematical) challenge of the Clay Institute regarding the properties of the Navier-Stokes equations and their relationship to hydrodynamic turbulence. To replicate a statement made by the Clay Institute:

> "The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations."

The point of departure starts with a topological (not statistical) formulation of Thermodynamics, which furnishes a universal foundation for the Partial Differential Equations of classical hydrodynamics and electrodynamics [42]. The topology that is of significance is defined in terms of Cartan's topological structure [36], which can be constructed from an exterior differential 1-form, A, defined on a pre-geometric domain of base variables. The topological method extends the classical geometrical approach to the study of non-equilibrium thermodynamic systems.

Claim 1 *The topological method permits the conclusion that among the solutions to the Navier-Stokes equations there are C2 smooth, thermodynamically irreversible processes which permit description of topological change and the decay of turbulence.*

In addition, the method permits examples to be constructed showing the difference between certain piecewise-linear processes which are reversible, but which are different from certain smooth processes which are irreversible. Such concepts (of smoothness) seem to be of direct interest to the challenge of the Clay Institute, and are to be associated with the fact that there are differences between piecewise linear, smooth, and topological manifolds (see p. 106, [40]).

However, the methods of topological thermodynamics go well beyond these types of questions. In particular, the methods permit non-statistical engineering design criteria to be developed for non-equilibrium systems. The theory of Topological Thermodynamics, based upon Continuous Topological Evolution [37] of Cartan's topological structure, can explain why topologically coherent, compact structures, far from equilibrium, will emerge as long-lived artifacts of thermodynamically irreversible, turbulent, continuous processes. I want to present the idea that:

Theorem 2 *The Pfaff Topological Dimension (PTD) of a Thermodynamic System can change dynamically and continuously via irreversible dissipative processes from a non-equilibrium* *turbulent state of PTD = 4 to an excited "topologically stationary, but excited," state of PTD = 3, which is still far from equilibrium! The PTD=3 state admits an extremal Hamiltonian evolutionary process which, if dominant, produces a relatively long lifetime.*

There exist C2 smooth processes that can describe the topological evolution from an Open non-equilibrium turbulent domain of Pfaff Topological Dimension 4 to Closed, but non-equilibrium, domains of Pfaff Topological Dimension 3, and ultimately to isolated-equilibrium domains of Pfaff dimension 2 or less. The Topological domains of Pfaff Topological Dimension 3 emerge via thermodynamically irreversible, dissipative processes as topologically coherent, deformable defects, embedded in the turbulent environment of Pfaff Topological Dimension 4. The theory of Continuous Topological Evolution gives formal credence, as well as analytic examples, to the Prigogine conjecture of self-organization in terms of disspative dynamics.

Now I am well aware of the fact that Thermodynamics (much less Topological Thermodynamics) is a topic often treated with apprehension. In addition, I must confess, that as undergraduates at MIT we used to call the required physics course in Thermodynamics, The Hour of Mystery! Let me present a few quotations (taken from Uffink, [41]) that describe the apprehensive views of several very famous scientists:

> Any mathematician knows it is impossible to understand an elementary course in thermodynamics .... V. Arnold 1990.
>
> It is always emphasized that thermodynamics is concerned with reversible processes and equilibrium states, and that it can have nothing to do with irreversible processes or systems out of equilibrium ......Bridgman 1941
>
> No one knows what entropy really is, so in a debate (if you use the term entropy) you will always have an advantage ...... Von Neumann (1971)

On the other hand Uffink states:

> Einstein, ..., remained convinced throughout his life that thermodynamics is the only universal physical theory that will never be overthrown.

I wish to demonstrate that from the point of view of Continuous Topological Evolution (which is based upon Cartan's theory of exterior differential forms) many of the mysteries of non-equilibrium thermodynamics, irreversible processes, and turbulent flows, can be resolved.

In addition, the non-equilibrium methods can lead to many new processes and patentable devices and concepts.

There are many intuitive, yet disputed, definitions of what is meant by turbulence, but the one property of turbulence that everyone agrees upon is that turbulent evolution in a fluid is a thermodynamic irreversible process. Isolated or equilibrium thermodynamics can be defined on a 4D space-time variety in terms of a connected Cartan topology of Pfaff Topological Dimension 2 or less. Non-equilibrium thermodynamics can be constructed in terms of disconnected Cartan topology of Pfaff Topological Dimension of 3 or more. As irreversibility requires a change in topology, the point of departure for this article will be to use the thermodynamic theory of continuous topological evolution in 4D space-time. It will be demonstrated, by example, that the non-equilibrium component of the Cartan topology can support topological change, thermodynamic irreversible processes and turbulent solutions to the Navier-Stokes equations, while the equilibrium topological component cannot. In addition, it will be demonstrated that complex isotropic *macroscopic* Spinors are the source of topological fluctuations and irreversible processes in the topological dynamics of non-equilibrium systems. This, perhaps surprising, fact has been ignored by almost all researchers in classical hydrodynamics who use classic real vector analysis and symmetries to produce conservation laws, which do not require Spinor components. The flaw in such symmetrical based theories is that they describe evolutionary processes that are time reversible. Time irreversibility requires topological change.

EXTERIOR DIFFERENTIAL FORMS
overcomes the
LIMITATIONS of REAL VECTOR ANALYSIS

During the period 1965-1992 it became apparent that new theoretical foundations were needed to describe non-equilibrium systems and continuous irreversible processes - which require topological (not geometrical) evolution. I selected Cartan's methods of exterior differential topology to encode Continuous Topological Evolution. The reason for this choice is that many years of teaching experience indicated that such methods were rapidly learned by both research scientists and engineers. In short:

1. Vector and Tensor analysis is not adequate to study the evolution of topology. The tensor constraint of diffeomorphic equivalences implies that the topology of the initial state must be equal to the topology of the final state. Turbulence is a thermodynamic, irreversible process which can not be described by tensor fields alone.

2. However, Cartan's methods of exterior differential systems and the topological perspective of Continuous Topological Evolution (not geometrical evolution) CAN be

used to construct a theory of non-equilibrium thermodynamic systems and irreversible processes.

3. Bottom Line: Exterior differential forms carry topological information and can be used to describe topological change induced by processes. Real "Vector" directionfields alone cannot describe processes that cause topological change; but Spinor directionfields can.

A cornerstone of classic Vector (tensor) analysis is the constraint of functional equivalence with respect to diffeomorphisms. However, diffeomorphisms are a subset of a homeomorphisms, and homeomorphisms preserve topology. Hence to study topological change, Vector (tensor) analysis is not adequate. In topological thermodynamics, processes are defined in terms of directionfields which may or may not be tensors. The ubiquitous concepts of 1-1 diffeomorphic equivalence, and non-zero congruences, for the eigen directionfields of symmetric matrices do not apply to the eigen directionfields of antisymmetric matrices. The eigen direction fields of antisymmetric matrices (which are equivalent to Cartan's isotropic Spinors) may be used to define components of a thermodynamic process, but such Spinors have a null congruence (zero valued quadratic form), admit chirality, and are not 1-1. Where classic geometric evolution is described in terms of symmetries and conservation laws, topological evolution is described in terms of antisymmetries.

Cartan's theory of exterior differential forms is built over completely antisymmetric structures, and therefore is the method of choice for studying topological evolution. The exterior differential defines limit sets; the Lie differential defines continuous topological evolution. The concept of Spinors arise naturally in theories using Cartan's methods of exterior differential forms; i.e., Spinors are not added to the theory ad hoc. The Cartan theory of extended differential forms can be used to study topological change. The word *extended* is used to emphasize the fact that differential forms are functionally well defined with respect a larger class of transformations than those used to define tensors. Extended differential forms behave as scalars with respect to C1 maps which do not have an inverse, much less an inverse Jacobian. Both the inverse map and the inverse Jacobian are required by a diffeomorphism. The exterior differential form on the final state of such C1 non-invertible maps permits the functional form of the differential form on the initial state to be functionally well defined in a retrodictive, pullback sense - not just at a point, but over a neighborhood.

Theorem 3 *Tensor fields can be neither retrodicted nor predicted in functional form by maps that are not diffeomorphisms [14].*

CONTINUOUS TOPOLOGICAL EVOLUTION

Objectives of CTE The objectives of the theory of Continuous Topological Evolution are to:

1. Establish the long sought for connection between irreversible thermodynamic processes and dynamical systems – without statistics!

2. Demonstrate the connection between thermodynamic irreversibility and Pfaff Topological Dimension equal to 4. The result suggests that "2-D Turbulence is a myth" for it is a thermodynamic system of Pfaff Topological Dimension equal to 3 [23].

3. Demonstrate that topological thermodynamics leads to universal topological equivalences between Electromagnetism, Hydrodynamics, Cosmology, and Topological Quantum Mechanics.

4. Demonstrate that Cartan's methods of exterior differential forms permits important topological concepts to be displayed in a useful, engineering format.

New Concepts deduced from CTE The theory of Continuous Topological Evolution introduces several new important concepts that are not apparent in a geometric equilibrium analysis.

1. Continuous Topological Evolution is the dynamical equivalent of the FIRST LAW OF THERMODYNAMICS.

2. The Pfaff Topological Dimension, PTD, is a topological property associated with any Cartan exterior differential 1-form, A. The PTD can change via topologically continuous processes.

3. Topological Torsion is a 3-form (on any 4D geometrical domain) that can be used to describe irreversible processes. Topological Torsion is a unique 4D non-equilibrium directionfield that is completely determined by the coefficient functions that encode the thermodynamic system. Other process directionfields are determined by the system topology based upon the 1-form of Action, A, and the refinements based on the topology of the 1-form of work, W.

4. Closed thermodynamic topological defects of Pfaff Topological Dimension 3 can emerge from Open thermodynamic systems of Pfaff Topological Dimension 4 by means of irreversible dissipative processes that represent topological evolution and change. When

the topologically coherent defect structures emerge, their evolution can be dominated by a Hamiltonian component (modulo topological fluctuations), which maintains the topological deformation invariance, and yields hydrodynamic wakes [22] and other Soliton structures. These objects are of Pfaff Topological Dimension 3 and are far from equilibrium. They behave as if they were "stationary excited" states above the equilibrium ground state. Falaco Solitons are an easily reproduced hydrodynamic example that came to my attention in 1986 [43] [35]. A movie is available online [17].

PRESENTATION OUTLINE

The essay is constructed in several sections:

Section 1. Topological Thermodynamics In Section 1, the concepts of topological thermodynamics in a space-time variety are reviewed (briefly) in terms of Cartan's method of exterior differential forms. A thermodynamic system is encoded in terms of a 1-form of Action[2], A. Thermodynamic processes are encoded in terms of the Lie differential with respect to a directionfield, V, acting on the 1-form, A, to produce a 1-form, Q. The process directionfield can have Vector and Spinor components. The definition of the Lie differential is a statement of cohomology and defines Q as the composite of a 1-form, W, and a perfect differential, dU. The formula abstractly represents a dynamical version of the First Law of Thermodynamics. It is a statement about cohomology theory, where the difference between the inexact 1-form of Heat, Q, and the inexact 1-form of Work, W, is a perfect differential, $dU = Q - W$.

The existence of a 1-form on a 4D space-time variety generates a Cartan topology. If the Pfaff Topological (not geometrical) Dimension of the 1-form of Action, A, is 2 or less, then the thermodynamic system is an isolated or equilibrium system on the 4D variety. If the Pfaff Topological Dimension of A is greater than 3, then the system is a non-equilibrium system on the 4D variety. Examples of systems of Pfaff Topological Dimension 4 which admit processes which are thermodynamically irreversible are given in the reference monographs (see footnote page 1).

Section 2. Applications In Section 2, the abstract formalism will be given a specific realization appropriate for plasmas and fluids in general. First, an electromagnetic format will be described because my teaching experience has demonstrated that the concepts of non-equilibrium phenomena are more readily recognized in in terms of an electromagnetic format.

[2] The units of the Action 1-form for themodynamic systems are defined in terms of angular momentum per mole number. In EM theory, the electric charge is used as the mole-number, such that the units of A are h/e.

Then it will be demonstrated how the PDE's representing the Hamiltonian version of the hydrodynamic Lagrange-Euler equations arise from the constraint that the work 1-form, W, should vanish (Pfaff Topological Dimension of $W = 0$).

The Bernoulli flow will be obtained by constraining the thermodynamic Work 1-form to be exact, $W = d\Theta$ (Pfaff Topological Dimension 1), and the Helmholtz flow will follow from the constraint that the thermodynamic Work 1-form be closed, but not necessarily exact, $dW = 0$. Such reversible dynamical processes belong to the connected component of the Work 1-form; irreversible processes belong to the disconnected component of the Work 1-form.

Irreversible processes belong to the disconnected topological component of the Work 1-form. An important example is the process defined in terms of the Topological Torsion direction field on a Symplectic manifold. Such processes are self-similar relative to the 1-form of Action, A, and are thermodynamically irreversible.

Section 3. The Navier-Stokes system In Section 3, the topological constraints of isolated equilibrium systems will be relaxed to produce more general PDE's defining the topological evolution of the system relative to an applied process. These relaxed topological constraints will include the partial differential equations known as the Navier-Stokes equations. The method used will be to augment the topology induced by the 1-form of Action, A, by studying the topological refinements induced by the 1-form of Work, W. It will be demonstrated that when the Pfaff Topological Dimension of A and W and Q are 4, there exist C2 solutions (processes) to the Navier-Stokes equations which are thermodynamically irreversible (the most significant property of turbulent flow). An interesting result is the set of conditions on solutions of the Navier-Stokes equations that produce an adiabatic *irreversible* flow.

Those topological refinements of the Work 1-form, required to include the Navier-Stokes equations, can be related directly to the concept of macroscopic Spinors. Macroscopic, complex Spinor solutions occur naturally in terms of the eigendirection fields of (real) anti-symmetric matrices with non-zero eigenvalues, whenever the thermodynamic Work 1-form is not zero. Spinors can also be associated with topological fluctuations of position and velocity about kinematic perfection generated by 1-parameter groups. These topological fluctuations are presumed to be representations of pressure and temperature.

Section 4. Closed States of Topological Coherence embedded as deformable defects in Turbulent Domains One of the key interests of the Clay problem has to do with the smoothness of the solutions to the Navier-Stokes equations. In Section 4, the problem will be attacked from the point of view of thermodynamics. First, the properties of the different species of topological defects will be discussed. These defects are non-equilibrium closed domains (of $PTD = 3$) which can emerge by C2 smooth irreversible process in open domains (of $PTD = 4$), as excited states far from equilibrium, yet with long relative lifetimes. Falaco Solitons are an easily reproduced *experimental* example of such topological defects, and are discussed in detail in [43].

The analytic properties of two different species of $PTD = 3$ defect domains will be given in detail. In addition, an analytic solution of a thermodynamically irreversible process that causes the defect domain to emerge will be demonstrated. An example of a process that creates the topological defect in finite time will be given.

Finally, an example will be given where by combinations of Spinor solutions produce piecewise linear processes. These piecewise linear processes are thermodynamically reversible, while the Spinor solutions of which they are composed are not.

Section 5. Topological Fluctuations and Spinors In Section 5, a few concluding remarks will be made about the ongoing research concerning topological fluctuations, as generated by Spinors. Such topological fluctuations can be associated with fluid pressure and temperature. The methods of fiber bundle theory can be used extend the 4D thermodynamic domain.

1 Topological Thermodynamics

1.1 The Axioms of Topological Thermodynamics

The topological methods used herein are based upon Cartan's theory of exterior differential forms. The thermodynamic view assumes that the physical systems to be studied can be encoded in terms of a 1-form of Action Potentials (per unit source, or, per mole), A, on a four-dimensional variety of ordered independent variables, $\{\xi^1, \xi^2, \xi^3, \xi^4\}$. The variety supports a differential volume element $\Omega_4 = d\xi^1\hat{}d\xi^2\hat{}d\xi^3\hat{}d\xi^4$. This statement implies that the differentials of the $\mu = 4$ base variables are functionally independent. No metric, no connection, no constraint of gauge symmetry is imposed upon the four-dimensional pre-geometric variety. Topological constraints can be expressed in terms of exterior differential systems placed upon this set of base variables [1].

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables will be changed to the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety, at this, thermodynamic, level. For instance, it is NOT assumed that the variety is spatially Euclidean.

With this notation, the Axioms of Topological Thermodynamics can be summarized as:

Axiom 1. *Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_\mu(x, y, z, t...)$, on a four-dimensional abstract variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports differential volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$.*

Axiom 2. *Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x, y, z, t...)$, in terms of a contravariant Vector and/or complex Spinor directionfields, symbolized as $V_4(x, y, z, t)$.*

Axiom 3. *Continuous Topological Evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [10]). The Lie differential with respect to the process, ρV_4, when applied to an exterior differential 1-form of Action, $A = A_\mu dx^\mu$, is equivalent, abstractly, to the first law of thermodynamics.*

$$\text{Cartan's Magic Formula } L_{(\rho \mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A), \tag{1}$$

$$\text{First Law} : W + dU = Q, \tag{2}$$

$$\text{Inexact Heat 1-form } Q = W + dU = L_{(\rho\mathbf{V}_4)}A, \tag{3}$$

$$\text{Inexact Work 1-form } W = i(\rho\mathbf{V}_4)dA, \tag{4}$$

$$\text{Internal Energy } U = i(\rho\mathbf{V}_4)A. \tag{5}$$

Axiom 4. *Equivalence classes of systems and continuous processes can be defined in terms of the Pfaff Topological Dimension and topological structure generated by the 1-forms of Action,* A, Work, W, and Heat, Q.

Axiom 5. If $Q\hat{}dQ \neq 0$, *then the thermodynamic process is irreversible.*

1.2 Cartan's Magic Formula ≈ First Law of Thermodynamics

The Lie differential (not Lie derivative) is the fundamental generator of Continuous Topological Evolution. When acting on an exterior differential 1-form of Action, $A = A_\mu dx^\mu$, Cartan's magic (algebraic) formula is equivalent *abstractly* to the first law of thermodynamics:

$$L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A), \tag{6}$$

$$= W + dU = Q. \tag{7}$$

Cartan's magic formula leads to a topological basis of thermodynamics, where the thermodynamic Work, W, thermodynamic Heat, Q, and the thermodynamic internal energy, U, are defined *dynamically* in terms of Continuous Topological Evolution. In effect, the First Law is a statement of Continuous Topological Evolution in terms of deRham cohomology theory; the difference between two non-exact differential forms is equal to an exact differential, $Q - W = dU$.

My recognition (some 30 years ago) of this correspondence between the Lie *differential* and the First Law of thermodynamics has been the corner stone of my research efforts in applied topology.

It is important to realize that the Cartan formula is to be interpreted algebraically. Many textbook presentations of the Cartan-Lie differential formula presume a dynamic constraint, such that the vector field $\mathbf{V}_4(x, y, z, t)$ be the generator of a single parameter group. If true, then the topological constraint of Kinematic Perfection can be established as an exterior differential system of the format:

$$\textbf{Kinematic Perfection :} \quad dx^k - \mathbf{V}^k dt \Rightarrow 0. \tag{8}$$

The topological constraint of Kinematic Perfection, in effect, defines (or presumes) a limit process. This constraint leads to the concept of the Lie *derivative*[3] of the 1-form A. The evolution then is represented by the infinitesimal propagation of the 1-form, A, down the

[3] Professor Zbigniew Oziewicz told me that Slebodzinsky was the first to formulate the idea of the Lie derivative in his thesis (in Polish).

flow lines generated by the 1-parameter group. Cartan called this set of flow lines "the tube of trajectories".

However, such a topological, kinematic constraint is *not* imposed in the presentation found in this essay; the directionfield, $\mathbf{V}_4$, may have multiple parameters. This observation leads to the important concept of topological fluctuations (about Kinematic Perfection), such as given by the expressions:

$$\begin{aligned} \textbf{Topological} &: \textbf{Fluctuations} \\ (dx^k - \mathbf{V}^k dt) &= \Delta\mathbf{x}^k \neq 0, \qquad (\sim \text{Pressure}) \quad (9) \\ (dV^k - \mathbf{A}^k dt) &= (\Delta\mathbf{V}^k) \neq 0, \quad (\sim \text{Temperature}) \quad (10) \\ d(\Delta\mathbf{x}^k) &= -(d\mathbf{V}^k - \mathbf{A}^k dt)\hat{}dt = -(\Delta\mathbf{V}^k)\hat{}dt, \quad (11) \end{aligned}$$

In this context it is interesting to note that in Felix Klein's discussions [4] of the development of calculus, he says

> "The primary thing for him (Leibniz) was not the differential quotient (the derivative) thought of as a limit. The differential, dx, of the variable x had for him (Leibniz) actual existence..."

The Leibniz concept is followed throughout this presentation. It is important for the reader to remember that the concept of a differential form is different from the concept of a derivative, where a (topological) limit has been defined, thereby constraining the topological evolution.

The topological methods to be described below go beyond the notion of processes which are confined to equilibrium systems of kinematic perfection. Non-equilibrium systems and processes which are thermodynamically irreversible, as well as many other classical thermodynamic ideas, can be formulated in precise mathematical terms using the topological structure and refinements generated by the three thermodynamic 1-forms, A, W, and Q.

1.3 The Pfaff Sequence and the Pfaff Topological Dimension

1.3.1 The Pfaff Topological Dimension of the System 1-form, A

It is important to realize that the Pfaff Topological Dimension of the system 1-form of Action, A, determines whether the thermodynamic system is Open, Closed, Isolated or in Equilibrium. Also, it is important to realize that the Pfaff Topological Dimension of the thermodynamic Work 1-form, W, determines a specific category of reversible and/or irreversible processes. It is therefore of some importance to understand the meaning of the

Pfaff Topological Dimension of a 1-form. Given the functional format of a general 1-form, A, on a 4D variety it is an easy step to compute the Pfaff Sequence, using one exterior differential operation, and several algebraic exterior products. For a differential 1-form, A, defined on a geometric domain of 4 base variables, the Pfaff Sequence is defined as:

$$\textbf{Pfaff Sequence} \quad \{A, dA, A\hat{}dA, dA\hat{}dA...\} \tag{12}$$

It is possible that over some domains, as the elements of the sequence are computed, one of the elements (and subsequent elements) of the Pfaff Sequence will vanish. The number of non-zero elements in the Pfaff Sequence (PS) defines the Pfaff Topological Dimension (PTD) of the specified 1-form[4]. Modulo singularities, the Pfaff Topological Dimension determines the minimum number M of N functions of base variables ($N \geq M$) required to define the topological properties of the connected component of the 1-form A.

The Pfaff Topological Dimension of the 1-form of Action, A, can be put into correspondence with the four classic topological structures of thermodynamics: Equilibrium, Isolated, Closed, and Open systems. The classic thermodynamic interpretation is that the first two structures do not exchange mass (mole numbers) or radiation with their environment. The Closed structure can exchange radiation with its environment but not mass (mole numbers). The Open structure can exchange both mass and radiation with its environment. The following table summarizes these properties. For reference purposes, I have given the various elements of the Pfaff sequence specific names:

Topological p-form name	**PS** element	**Nulls**	**PTD**	Thermodynamic system
Action	A	$dA = 0$	1	**Equilibrium**
Vorticity	dA	$A\hat{}dA = 0$	2	**Isolated**
Torsion	$A\hat{}dA$	$dA\hat{}dA = 0$	3	**Closed**
Parity	$dA\hat{}dA$	$-$	4	**Open**

Table 1 Applications of the Pfaff Topological Dimension.

The four thermodynamic systems can be placed into two disconnected topological categories. If the Pfaff Topological Dimension of A is 2 or less, the first category is determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$. This Cartan topology is a connected topology. In the case that the Pfaff Topological Dimension is greater than 2, the Cartan topology is based on the union of two closures, $\{A \cup dA \ \cup \ A\hat{}dA \cup dA\hat{}dA\}$, and is a disconnected topology.

[4] The Pfaff Topological dimension has been called the "class" of a 1-form in the old literature. I prefer the more suggestive name.

It is a topological fact that there exists a (topologically) continuous C2 process from a disconnected topology to a connected topology, but there does not exist a C2 continuous process from a connected topology to a disconnected topology. This fact implies that topological change can occur continuously by a "pasting" processes representing the *decay* of turbulence by "condensations" from non-equilibrium to equilibrium systems. On the other hand, the *creation* of Turbulence involves a discontinuous (non C2) process of "cutting" into parts. This warning was given long ago [21] to prove that computer analyses that smoothly match value and slope will not replicate the *creation* of turbulence, but can faithfully replicate the *decay* of turbulence.

1.3.2 The Pfaff Topological Dimension of the Thermodynamic Work 1-form, W

The topological structure of the thermodynamic Work 1-form, W, can be used to refine the topology of the physical system; recall that the physical system is encoded by the Action 1-form, A.

Claim 4 *The PDE's that represent the system dynamics are determined by the Pfaff Topological Dimension of the 1-form of Work, W, and the 1-form of Action, A, that encodes the physical system.*

The Pfaff Topological Dimension of the thermodynamic Work 1-form depends upon both the physical system, A, and the process, $\mathbf{V}_4$. In particular if the Pfaff Dimension of the thermodynamic Work 1-form is zero, $(W = 0)$, then system dynamics is generated by an extremal vector field which admits a Hamiltonian realization. However, such extremal direction fields can occur only when the Pfaff Topological Dimension of the system encoded by A is odd, and equal or less than the geometric dimension of the base variables.

For example, if the geometric dimension is 3, and the Pfaff Topological Dimension of A is 3, then there exists a unique extremal field on the Contact manifold defined by dA. This unique directionfield is the unique eigen directionfield of the 3x3 antisymmetric matrix (created by the 2-form $F = dA$) with eigenvalue equal to zero.

If the geometric dimension is 4, and the Pfaff Topological Dimension of A is 3, then there exists a two extremal fields on the geometric manifold. These directionfields are those generated as the eigen directionfields of the 4x4 antisymmetric matrix (created by the 2-form $F = dA$) with eigenvalue equal to zero.

If the geometric dimension is 4, and the Pfaff Topological Dimension of A is 4, then there do not exist extremal fields on the Symplectic manifold defined by dA. All of the eigen directionfields of the 4x4 antisymmetric matrix (created by the 2-form $F = dA$) are complex isotropic spinors with pure imaginary eigenvalues not equal to zero.

1.4 Topological Torsion and other Continuous Processes.

1.4.1 Reversible Processes

Physical Processes[5] are determined by directionfields, with the symbol, $\mathbf{V}_4$, to within an arbitrary function, ρ. A direction field is defined by the components of a vector (or spinor) field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

There are several classes of direction fields that are defined as follows [5]:

$$\textbf{Associated Class} \quad : \quad i(\rho\mathbf{V}_4)A = 0, \tag{13}$$

$$\textbf{Extremal Class} \quad : \quad i(\rho\mathbf{V}_4)dA = 0, \tag{14}$$

$$\textbf{Characteristic Class} \quad : \quad i(\rho\mathbf{V}_4)A = 0, \tag{15}$$

$$\text{and} \quad : \quad i(\rho\mathbf{V}_4)dA = 0, \tag{16}$$

$$\textbf{Helmholtz Class} \quad : \quad d(i(\rho\mathbf{V}_4)dA) = 0, \tag{17}$$

Extremal Vectors (relative to the 1-form of Action, A) produce zero thermodynamic work, $W = i(\rho\mathbf{V}_4)dA = 0$, and admit a Hamiltonian representation. Associated Vectors (relative to the 1-form of Action, A) can be adiabatic if the process remains orthogonal to the 1-form, A. Helmholtz processes (which include Hamiltonian processes, Bernoulli processes and Stokes flow) conserve the 2-form of Topological vorticity, dA. All such processes are thermodynamically reversible. Many examples of these systems are detailed in the reference monographs (see footnote on page 1).

1.4.2 Irreversible Processes

There is one directionfield that is uniquely defined by the coefficient functions of the 1-form, A, that encodes the thermodynamic system on a 4D geometric variety. This vector exists only in non-equilibrium systems, for which the Pfaff Topological Dimension of A is 3 or 4. This 4 vector is defined herein as the Topological Torsion vector. In the Frenet theory of spaces curves, Frenet torsion is an indicator that the space curve resides in 3 (geometrical) dimensions. . In the Frenet theory of spaces curves, Frenet torsion is an indicator that the geometrical dimension of the space curve is irreducibly three. The reason for defining $\mathbf{T}_4$ as the "Topological Torsion" vector is that the existence $\mathbf{T}_4$ indicates the topological dimension is at least three.

[5] Processes include both vector and spinor processes.

To within a factor, this directionfield has the four coefficients of the 3-form $A\hat{}dA$, with the following properties:

$$\begin{aligned}
\textbf{Properties of} &\ : \ \textbf{Topological Torsion } \mathbf{T}_4 \text{ on } \Omega_4 && (18)\\
i(\mathbf{T}_4)\Omega_4 &= i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA, && (19)\\
W &= i(\mathbf{T}_4)dA = \sigma\ A, && (20)\\
dW &= d\sigma\hat{}A + \sigma dA = dQ && (21)\\
U &= i(\mathbf{T}_4)A = 0, \quad \mathbf{T}_4 \text{ is associative} && (22)\\
i(\mathbf{T}_4)dU &= 0 && (23)\\
i(\mathbf{T}_4)Q &= 0 \qquad \mathbf{T}_4 \text{ is adiabatic} && (24)\\
L_{(\mathbf{T}_4)}A &= \sigma\ A, \quad \mathbf{T}_4 \text{ is homogeneous} && (25)\\
L_{(\mathbf{T}_4)}dA &= d\sigma\hat{}A + \sigma dA = dQ, && (26)\\
Q\hat{}dQ &= L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0, && (27)\\
dA\hat{}dA &= d(A\hat{}dA) = d\{(i(\mathbf{T}_4)\Omega_4\} = (div_4\mathbf{T}_4)\Omega_4, && (28)\\
L_{(\mathbf{T}_4)}\Omega_4 &= d\{(i(\mathbf{T}_4)\Omega_4\} = (2\sigma)\Omega_4, && (29)
\end{aligned}$$

If the Pfaff Topological Dimension of A is 4 (an Open thermodynamic system), then $\mathbf{T}_4$ has a non-zero 4 divergence, (2σ), representing an expansion or a contraction of the 4D volume element Ω_4. The Heat 1-form, Q, generated by the process, $\mathbf{T}_4$, is NOT integrable. Q is of Pfaff Topological Dimension greater that 2, when the self-similarity function, σ is not zero. Furthermore the $\mathbf{T}_4$ process is locally adiabatic as the change of internal energy in the direction of the process path is zero. Therefore, in the Pfaff Topological Dimension 4 case, where $dA\hat{}dA \neq 0$, the $\mathbf{T}_4$ direction field represents an *irreversible, adiabatic process.*

When σ is zero *and* $d\sigma = 0$, but $A\hat{}dA \neq 0$, the Pfaff Topological Dimension of the system is 3 (a Closed thermodynamic system). In this case, the $\mathbf{T}_4$ direction field on a 4D geometric set of base variables becomes a characteristic vector field which is both extremal and associative, and induces a Hamilton-Jacobi representation (the ground state of the system for which $dQ = 0$).

For any process and any system, equation (27) can be used as a test for irreversibility.

It seems a pity, that the concept of the Topological Torsion vector and its association with non-equilibrium systems, where it can be used to establish design criteria to minimize energy dissipation, has been ignored by the engineering community.

1.4.3 The Spinor class

It is rather remarkable (and only fully appreciated by me in February, 2005) that there is a large class of direction fields useful to the topological dynamics of thermodynamic systems (given herein the symbol $\rho\mathbf{S}_4$) that do not behave as vectors (with respect to rotations). They are isotropic complex vectors of zero length, defined as Spinors by E. Cartan [2], but which are most easily recognized as the eigen directionfields relative to the antisymmetric matrix, $[F]$, generated by the component of the 2-form $F = dA$:

$$\textbf{The Spinor Class} \quad [F] \circ |\rho\mathbf{S}_4\rangle = \lambda\, |\rho\mathbf{S}_4\rangle \neq 0, \tag{30}$$

$$\langle\rho\mathbf{S}_4| \circ |\rho\mathbf{S}_4\rangle = 0, \quad \lambda \neq 0 \tag{31}$$

In the language of exterior differential forms, if the Work 1-form is not zero, the process must contain Spinor components:

$$W = i(\rho\mathbf{S}_4)dA \neq 0 \tag{32}$$

As mentioned above, Spinors have metric properties, behave as vectors with respect to transitive maps, but do not behave as vectors with respect to rotations (see p. 3, [2]). Spinors generate harmonic forms and are related to conjugate pairs of minimal surfaces. The notation that a Spinor is a complex isotropic *directionfield* is preferred over the names "complex isotropic *vector*", or "null *vector*" that appear in the literature. As shown below, the familiar formats of Hamiltonian mechanical systems exclude the concept of Spinor process directionfields, for the processes permitted are restricted to be represented by direction fields of the extremal class, which have zero eigenvalues.

Remark 5 *Spinors are normal consequences of antisymmetric matrices, and, as topological artifacts, they are not restricted to physical microscopic or quantum constraints. According to the topological thermodynamic arguments, Spinors are implicitly involved in all processes for which the 1-form of thermodynamic Work is not zero. Spinors play a role in topological fluctuations and irreversible processes.*

The thermodynamic Work 1-form, W, is generated by a completely antisymmetric 2-form, F, and therefore, if not zero, must have Spinor components. In the odd dimensional Contact manifold case there is one eigen Vector, with eigenvalue zero, which generates the extremal processes that can be associated with a Hamiltonian representation. The other two eigendirection fields are Spinors. In the even dimensional Symplectic manifold case, any non-zero component of work requires that the evolutionary directionfields must contain Spinor components. All eigen directionfields on symplectic spaces are Spinors.

The fundamental problem of Spinor components is that there can be more than one Spinor direction field that generates the same geometric path. For example, there can be Spinors of left or right handed polarizations and Spinors of expansion or contraction that produce the same optical (null congruence) path. This result does not fit with the classic arguments of mechanics, which require unique initial data to yield unique paths. Furthermore, the concept of Spinor processes can annihilate the concept of time reversal symmetry, inherent in classical hydrodynamics. The requirement of uniqueness is not a requirement of non-equilibrium thermodynamics, where Spinor "entanglement" has to be taken into account.

1.5 Emergent Topological Defects

Suppose an evolutionary process starts in a domain of Pfaff Topological Dimension 4, for which a process in the direction of the Topological Torsion vector, $\mathbf{T}_4$, is known to represent an irreversible process. Examples can demonstrate that the irreversible process can proceed to a domain of the geometric variety for which the dissipation coefficient, σ, becomes zero. Physical examples [43] such as the skidding bowling ball proceed with irreversible dissipation ($PTD = 6$) until the "no-slip" condition is reached ($PTD = 5$). In fluid systems the topological defects can emerge as long lived states far from equilibrium. The process is most simply visualized as a "condensation" from a turbulent gas, such as the creation of a star in the model which presumes the universe is a very dilute, turbulent van der Waals gas near its critical point. The red spot of Jupiter, a hurricane, the ionized plasma ring in a nuclear explosion, Falaco Solitons, the wake behind an aircraft are all exhibitions of the emergence process to long lived topological structures far from equilibrium. It is most remarkable that the emergence of these experimental defect structures occurs in finite time.

The idea is that a subdomain of the original system of Pfaff Topological Dimension 4 can evolve continuously with a change of topology to a region of Pfaff Topological Dimension 3. The emergent subdomain of Pfaff Topological Dimension 3 is a topological defect, with topological coherence, and often with an extended lifetime (as a soliton structure with a dominant Hamiltonian evolutionary path), embedded in the Pfaff dimension 4 turbulent background.

The Topological Torsion vector in a region of Pfaff Topological Dimension 3 is an extremal vector direction field in systems of Pfaff Topological Dimension 3; it then has a zero 4D divergence, and leaves the volume element invariant. Moreover the existence of an extremal direction field implies that the 1-form of Action can be given a Hamiltonian representation, $P_k dq^k + H(P, q, t)dt$. In the domain of Pfaff dimension 3 for the Action, A, the subsequent continuous evolution of the system, A, relative to the process $\mathbf{T}_4$, can proceed in an energy conserving, Hamiltonian manner, representing a "stationary" or "excited" state far from equilibrium (the ground state). This argument is based on the assumption that the Hamiltonian component of the direction field is dominant, and any Spinor components in the $PTD = 3$ domain, representing topological fluctuations, can be ignored. These excited states, far from equilibrium, can be interpreted as the evolutionary topological defects that emerge and self-organize due to irreversible processes in the turbulent dissipative system of Pfaff dimension 4.

The descriptive words of self-organized states far from equilibrium have been abstracted from the intuition and conjectures of I. Prigogine [9]. The methods of Continuous Topological Evolution correct the Prigogine conjecture that "dissipative structures" can be caused by dissipative processes and fluctuations. The long-lived excited state structures created by irreversible processes are non-equilibrium, deformable topological defects; they are almost void of irreversible dissipation. The topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non-equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Using Cartan's methods of exterior differential systems, thermodynamic irreversibility and the arrow of time can be well defined in a topological sense, a technique that goes beyond (and without) statistical analysis [25]. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

2 Applications

2.1 An Electromagnetic format

The thermodynamic identification of the terms in Cartan's magic formula are not whimsical. To establish an initial level of credence in the terminology, consider the 1-form of Action, A, where the component functions are the symbols representing the familiar vector and scalar potentials in electromagnetic theory. The coefficient functions have arguments over the four independent variables $\{x, y, z, t\}$,

$$A = A_\mu(x, y, z, t)dx^\mu = \mathbf{A} \circ d\mathbf{r} - \phi\ dt. \tag{33}$$

Construct the 2-form of field intensities as the exterior differential of the 1-form of Action[6],

$$\begin{aligned} F &= dA = (\partial A_k/\partial x^j - \partial A_j/\partial x^k)dx^j\hat{}dx^k & (34) \\ &= F_{jk}dx^j\hat{}dx^k = +\mathbf{B}_z dx\hat{}dy... + \mathbf{E}_x dx\hat{}dt... \quad . & (35) \end{aligned}$$

The engineering variables are defined as electric and magnetic field intensities:

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\ \phi, \qquad \mathbf{B} = curl\ \mathbf{A}. \tag{36}$$

Relative to the ordered set of base variables, $\{x, y, z, t\}$, define a process directionfield, $\rho\mathbf{V}_4$, as a 4-vector with components, $[\mathbf{V}, 1]$, with a scaling factor, ρ.

$$\rho[\mathbf{V}_4] = \rho[\mathbf{V}, 1]. \tag{37}$$

Note that this direction field can be used to construct a useful 3-form of (matter) current, C, in terms of the 4-volume element, $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$:

$$C = i(\rho\mathbf{V}_4)dx\hat{}dy\hat{}dz\hat{}dt = i(\mathbf{C}_4)\Omega_4. \tag{38}$$

The process 3-form, C, is not necessarily the same as electromagnetic charge current density 3-form of electromagnetic theory, J. The 4-divergence of C, need not be zero: $dC \neq 0$.

Using the above expressions, the evaluation of the thermodynamic work 1-form in terms

[6]Recall that the physical dimension of the 1-form of Action , A, (applicable to electromagnetic systems) is h/e.

of 3-vector engineering components becomes:

$$\begin{aligned}
\textbf{The thermodynamic Work 1-form:} \quad & W = i(\rho\mathbf{V}_4)dA = i(\rho\mathbf{V}_4)F, && (39)\\
\Rightarrow \quad & -\rho\{\mathbf{E}+\mathbf{V}\times\mathbf{B}\}\circ d\mathbf{r} + \rho\{\mathbf{V}\circ\mathbf{E}\}dt && (40)\\
= \quad & -\rho\{\mathbf{f}_{Lorentz}\}\circ d\mathbf{r} + \rho\{\mathbf{V}\circ\mathbf{E}\}dt. && (41)\\
\textbf{The Lorentz force} \quad = \quad & -\{\mathbf{f}_{Lorentz}\}\circ d\mathbf{r} \text{ (spatial component)} && (42)\\
\textbf{The dissipative power} \quad = \quad & +\{\mathbf{V}\circ\mathbf{E}\}dt \text{ (time component).} && (43)
\end{aligned}$$

For those with experience in electromagnetism, note that the construction yields the format, automatically and naturally, for the "Lorentz force" as a *derivation consequence*, without further ad hoc assumptions. The dot product of a 3 component force, $\mathbf{f}_{Lorentz}$, and a differential spatial displacement, $d\mathbf{r}$, defines the elementary classic concept of "differential work". The 4-component thermodynamic Work 1-form, W, includes the spatial component and a differential time component, Pdt, with a coefficient which is recognized to be the "dissipative power", $P = \{\mathbf{V}\circ\mathbf{E}\}$. The thermodynamic Work 1-form, W, is not necessarily a perfect differential, and therefore can be path dependent. Closed cycles of Work need not be zero.

Next compute the Internal Energy term, relative to the process defined as $\rho\mathbf{V}_4$:

$$\textbf{Internal Energy:}\ U = i(\rho\mathbf{V}_4)A = \rho(\mathbf{V}\circ\mathbf{A}-\phi). \qquad (44)$$

The result is to be recognized as the "interaction" energy density in electromagnetic plasma systems. It is apparent that the internal energy, U, corresponds to the interaction energy of the physical system; that is, U is the internal stress energy of system deformation. Therefore, the electromagnetic terminology can be used to demonstrate the premise that Cartan's magic formula is not just another way to state that the first law of thermodynamics makes practical sense. The topological methods permit the long sought for integration of mechanical and thermodynamic concepts, without the constraints of equilibrium systems, and/or statistical analysis.

It is remarkable that although the symbols are different, the same basic constructions and conclusions apply to many classical physical systems. The correspondence so established between the Cartan magic formula acting on a 1-form of Action, and the first law of thermodynamics is taken both literally and seriously in this essay. The methods yield explicit constructions for testing when a process acting on a physical system is irreversible. The methods permit irreversible adiabatic processes to be distinguished from reversible adiabatic processes, analytically. Adiabatic processes need not be "slow" or quasi-static.

Given any 1-form, A, W, and/or Q, the concept of Pfaff Topological Dimension (for each of the three 1-forms, A, W, Q) permits separation of processes and systems into equivalence classes. For example, dynamical processes can be classified in terms of the topological Pfaff dimension of the thermodynamic Work 1-form, W. All extremal Hamiltonian systems have a thermodynamic Work 1-form, W, of topological Pfaff dimension of 1, ($dW = 0$). Hamiltonian systems can describe reversible processes in non-equilibrium systems for which the topological Pfaff dimension is 3. Such systems are topological defects whose topology is preserved by the Hamiltonian dynamics, but all processes which preserve topology are reversible. In non-equilibrium systems, topological fluctuations can be associated with Spinors of the 2-form, $F = dA$. Even if the dominant component of the process is Hamiltonian, Spinor fluctuations can cause the system (ultimately) to decay.

2.1.1 Topological 3-forms and 4-forms in EM format

Construct the elements of the Pfaff Sequence for the EM notation,

$$\{A, F = dA, A\hat{}F, F\hat{}F\}, \tag{45}$$

and note that the algebraic expressions of Topological Torsion, $A\hat{}F$, can be evaluated in terms of 4-component engineering variables $\mathbf{T}_4$ as:

$$\textbf{Topological Torsion vector} \tag{46}$$

$$A\hat{}F = i(\mathbf{T}_4)\Omega_4 = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt \tag{47}$$

$$\mathbf{T}_4 = [\mathbf{T}, h] = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}]. \tag{48}$$

The exterior 3-form, $A\hat{}F$, with physical units of $(\hbar/e)^2 = \hbar Z_{Hall}$, is not found (usually) in classical discussions of electromagnetism.

If $\mathbf{T}_4$ is used as to define the direction field of a process, then

$$L_{(\mathbf{T}_4)}A = \sigma A, \qquad i(\mathbf{T}_4)A = 0. \tag{49}$$

$$\text{where } 2\sigma = \{div_4(\mathbf{T}_4)\} = 2(\mathbf{E} \circ \mathbf{B}). \tag{50}$$

The important (universal) result is that if the acceleration associated with the direction field, **E**, is parallel to the vorticity associated with the direction field, **B,** then according to the equations starting with eq. (18) et. seq. the process is dissipatively irreversible. This result establishes the design criteria for engineering applications to minimize dissipation from turbulent processes.

The Topological Torsion vector has had almost no utilization in applications of classical electromagnetic theory.

2.1.2 Topological Torsion quanta

The 4-form of Topological Parity, $F\hat{}F$, can be evaluated in terms of 4-component engineering variables as:

Topological Parity

$$d(A\hat{}F) \;=\; F\hat{}F = \{div_4(\mathbf{T}_4)\}\Omega_4 = \{2\mathbf{E}\circ\mathbf{B}\}\ \Omega_4. \tag{51}$$

This 4-form is also known as the second Poincare Invariant of Electromagnetic Theory.

The fact that $F\hat{}F$ need not be zero implies that the Pfaff Topological Dimension of the 1-form of Action, A, must be 4, and therefore A represents a non-equilibrium Open thermodynamic system. Similarly, if $F\hat{}F = 0$, but $A\hat{}F \neq 0$, then the Pfaff Topological Dimension of the 1-form of Action, A, must be 3, and the physical system is a non-equilibrium Closed thermodynamic system. When $F\hat{}F = 0$, the corresponding three-dimensional integral of the closed 3-form, $A\hat{}F$, when integrated over a closed 3D-cycle, becomes a deRham period integral, defined as the Torsion quantum. In other words, the closed integral of the (closed) 3-form of Topological Torsion becomes a deformation (Hopf) invariant with integral values proportional to the integers.

$$\textbf{Torsion quantum} = \iiint_{3D\_cycle} A\hat{}F. \tag{52}$$

On the other hand, topological evolution and transitions between "quantized" states of Torsion require that the respective Parity 4-form is not zero. As,

$$L_{(\mathbf{T}_4)}\Omega_4 = d\{i(\mathbf{T}_4)\Omega_4\} = \ (2\sigma)\ \Omega_4 = 2(\mathbf{E}\circ\mathbf{B})\ \Omega_4 \neq 0, \tag{53}$$

it is apparent that the evolution of the differential volume element, Ω_4, depends upon the existence and colinearity of both the electric field, $\mathbf{E}$, and the magnetic field, $\mathbf{B}$. It is here that contact is made with the phenomenological concept of "4D bulk" viscosity $= 2\sigma$. It is tempting to identify σ^2 with the concept of entropy production. Note that the Topological Torsion directionfield appears only in non-equilibrium systems of topological dimension 3 or more. These results are universal and can be used in hydrodynamic systems discussed in that which follows.

2.2 A Hydrodynamic format

2.2.1 The Topological Continuum vs. the Geometrical Continuum

In many treatments of fluid mechanics the (geometrical) continuum hypothesis is invoked from the start. The idea is "matter" occupies all points of the space of interest, and that properties of the fluid can be represented by piecewise continuous functions of space and time, as long as length and time scales are not too small. The problem is that at very small scales, one has been led to believe the molecular or atomic structure of particles will become evident, and the "macroscopic" theory will breakdown. However, these problems of scale are geometric issues, important to many applications, but not pertinent to a topological perspective, where shape and size are unimportant. Suppose that the dynamics can be formulated in terms of topological concepts which are independent from sizes and shapes. Then such a theory of a Topological Continuum would be valid at all scales. Such is the goal of this monograph.

Remark 6 *However, one instance where "scale" many have topological importance is associated with the example of a surface with a "teeny" hole. If the hole, no matter what its size, has a twisted ear (Moebius band) then the whole surface is non-orientable, no matter how "small" the hole. Could it be that the world of the quantum is, in effect, that of non-orientable defects embedded in an otherwise orientable manifold that originally had no such defects. Note further the strong correspondence with Fermions with non-oriented (half-integer) multiplet ribbons, and Bosons with oriented (integer) multiplet ribbons of both right and left twists [46].*

As will be developed below, the fundamental equations of exterior differential systems can lead to field equations in terms of systems of Partial Differential Equations (PDE's). The format of the fundamental theory will be in terms of objects (exterior differential forms) which, although composed of algebraic constructions of tensorial[7] things, are in a sense scalars (or pseudo scalars) that are homogeneous with respect to concepts of scale. The theory then developed is applicable to hydrodynamics at all scales, from the microworld to the cosmological arena. The "breakdown" of the continuum model is not relevant. The topological system may consist of many disconnected parts when the system is not in thermodynamic equilibrium or isolation, and the parts can have topological obstructions or defects, some of which can be used to construct period integrals that are topologically "quantized". Hence the "quantization" of the micro-scaled geometric systems can have it genesis in the non-equilibrium theory of thermodynamics. However, from the topological perspective, the rational topological quantum values can also occur at all scales.

2.2.2 Topological Hydrodynamics

The axioms of Topological Thermodynamics are summarized in Section 1.1. For hydrodynamics (or electrodynamics) the axioms are essentially the same. Just exchange the

[7] Relative to diffeomorphisms.

word, hydrodynamics (or electrodynamics), for the word, thermodynamics, in the formats of Section 1.1

By 1969 it had become evident to me that electromagnetism (without geometric constraints), when written in terms of differential forms, was a topological theory, and that the concept of dissipation and irreversible processes required more than that offered by Hamiltonian mechanics. At that time I was interested in possible interactions of the gravitational field and the polarizations of an electromagnetic signal. One of the first ideas discovered about topological electrodynamics was that there existed an intrinsic transport theorem [11] that introduced the concept of what is now called Topological Spin[8], $A\hat{}G$, into electromagnetic theory [45]. As a transport theorem not recognized by classical electromagnetism, the first publication was as a letter to Physics of Fluids. That started my interest in a topological formulation of fluids.

It was not until 1974 that the Lie differential acting on exterior differential forms was established as the key to the problem of intrinsically describing dissipation and the production of topological defects in physical systems; but methods of visualization of such topological defects in classical electrodynamics were not known [12]. It was hoped that something in the more visible fluid mechanics arena would lend credence to the concepts of topological defects. The first formulations of the PDE's of fluid dynamics in terms of differential forms and Cartan's Magic formula followed quickly [13].

In 1976 it was argued that topological evolution was at the cause of turbulence in fluid dynamics, and the notion of what is now called Topological Torsion, $A\hat{}F$, became recognized as an important concept. It was apparent that streamline flow imposed the constraint that $A\hat{}F = 0$ on the equations of hydrodynamics. Turbulent flow, being the antithesis of streamline flow, must admit $A\hat{}F \neq 0$. In 1977 it was recognized that topological defect structures could become "quantized" in terms of deRham period integrals [15], forming a possible link between topology and both macroscopic and microscopic quantum physics. The research effort then turned back to a study of topological electrodynamics in terms of the dual polarized ring laser, where it was experimentally determined that the speed of an electromagnetic signal outbound could be different from the speed of an electromagnetic signal inbound [20]: a topological result not within the realm of classical theory.

Then in 1986 the long sought for creation and visualization of topological defects in fluids [16] became evident. The creation of Falaco Solitons in a swimming pool was the experiment that established credence in the ideas of what had, by that time, become a theory of continuous topological evolution. It was at the Cambridge conference in 1989 [18] that the notions of topological evolution, hydrodynamics and thermodynamics were put

[8] Topological Spin has physical units of h, Planck's constant. $A\hat{}G$ does not depend explicitly on charge, e

together in a rudimentary form, but it was a year later at the Permb conference in 1990 [19] that the ideas were well established. The Permb presentation also suggested that the ambiguous (at that time) notion of coherent structures in fluids could be made precise in terms of topological coherence. A number of conference presentations followed in which the ideas of continuous thermodynamic irreversible topological evolution in hydrodynamics were described [21], but the idea that the topological methods of thermodynamics could be used to distinguish non-equilibrium processes and non-equilibrium systems and irreversible processes with out the use of statistics slowly came into being in the period 1985-2005 [24]. These efforts have been summarized in [42], and a collection of the old publications appears in [47].

2.3 Classical Hydrodynamic Theory

There are two classical techniques for describing the evolutionary motion of a fluid: the Lagrangian method and the Eulerian method. Both methods treat the fluid relative to a Euclidean 3D manifold, with time as a parameter. The first (Lagrangian) technique treats a fluid as a collection of "particles, or parcels" and the flow is computed in terms of "initial" data $\{a, b, c, \tau\}$ imposed upon solutions to Newtonian equations of motion for "particles, or parcels". The solution functions describe a map from an initial state $\{a, b, c; \tau\}$ to a final state $\{x, y, z, t\}$. This method is related to solutions of kinematic equations, and is *contravariant* in the sense of an immersion to velocities (the tangent space). The kinematic basis for the Lagrangian motion draws heavily from the Frenet-Serret analysis of a point moving along a space curve.

The second technique treats a fluid as "field", and is representative of a "wave" point of view of a Hamiltonian system. The functions that define the field (the covariant momenta) depend on "final data" $\{x, y, z, t\}$, and are covariant in the sense of a submersion. Each method has its preimage in the form of an exterior differential 1-form of Action. The primitive classical Lagrangian Action concept is written in the form

$$A_L = \mathsf{L}(x^k, V^m, t)dt, \tag{54}$$

and the primitive Eulerian Action is written as

$$A_E = p_k dx^k + \mathsf{H}dt. \tag{55}$$

As both Actions supposedly describe the same fluid, are they equivalent? That is,

$$\text{does } A_E \Leftrightarrow A_L \text{ ?} \tag{56}$$

Note that A_L is composed from only two functions (L and t) such that at most A_L is of Pfaff Topological Dimension 2. On the other hand, the Pfaff Topological Dimension of A_E (as written) could be as high as 8, if all functions and differentials are presumed to be independent. So the two formulations are NOT equivalent, unless constraints reduce the topological dimension of A_E to 2, or if additions are made to the 1-form A_L to increase its Pfaff dimension.

2.3.1 The Lagrange-Hilbert Action

The classic addition to A_L is of the form, $p_k(dx^k - V^k dt)$, where the p_k are presumed to be Lagrange multipliers of the fluctuations in kinematic velocity. The result is defined as the Cartan - Hilbert 1-form of Action:

$$A_{CH} = \mathsf{L}(x^k, V^k, t)dt + p_k(dx^k - V^k dt). \tag{57}$$

Note that at first glance it appears that there are 10=3N+1 independent geometric variables $\{x^k, V^k, p_k, t\}$ in the formula for A_{CH}, but if the Pfaff Sequence is constructed, the Pfaff Topological Dimension turns out to be 8. So with this addition of Lagrange multipliers to A_L, the topological dimensions of the two actions are the same. However, note that by rearranging variables,

$$A_{CH} = p_k dx^k + (\mathsf{L}(x^k, V^m, t) - p_k V^k)dt, \tag{58}$$

$$= p_k dx^k + \mathsf{H}dt = A_E. \tag{59}$$

For a fluid the Eulerian "momenta" per unit parcel of mass is usually defined as

$$p_k/m = \mathbf{v}_k, \tag{60}$$

such that the Eulerian Action per unit mass becomes

$$A_E \Rightarrow \mathbf{v}_k dx^k + \mathsf{H}dt. \tag{61}$$

The bottom line is that the Lagrangian and Eulerian point of view can be made compatible if fluctuations in Kinematic Perfection are allowed.

Recall that the development of elasticity theory (and its emphasis on symmetrical tensors) also spawned the development of hydrodynamics. Much of the theory of classical hydrodynamics was phrased geometrically in the language of vector analysis. The development followed the phenomenological concepts of an extended Newtonian theory of elasticity. The classical theory was developed from "balance" equations for a bounded sample, or parcel, of matter (mass), which express assumptions (defined as the conservation of mass, momenta and energy) in terms of integrals over the bounded sample, or parcel, of matter. The classical integrals are performed usually over three-dimensional volumes (and not over 4D space-time). The classical Cauchy result (for the momentum equations) is

$$\rho\{\partial\mathbf{v}/\partial t + \mathbf{v}\circ\boldsymbol{\nabla}\mathbf{v}\} \;=\; \boldsymbol{\nabla}\circ\mathbb{T} + \rho\mathbf{f}, \tag{62}$$

$$\rho\{\partial\mathbf{v}/\partial t + grad(\mathbf{v}\circ\mathbf{v}/\mathbf{2}) - \mathbf{v}\times curl\mathbf{v}\} \;=\; \boldsymbol{\nabla}\circ\mathbb{T} + \rho\mathbf{f}, \tag{63}$$

Constitutive assumptions are then made for the 3D stress tensor $\mathbb{T}$, such that (in matrix format)

$$[\mathbb{T}] = (-P + \lambda(\boldsymbol{\nabla}\circ\mathbf{v})\,[\mathbb{I}] + \nu\{[\boldsymbol{\nabla}\mathbf{v}] + [\boldsymbol{\nabla}\mathbf{v}]^T\}, \tag{64}$$

where P is the Pressure, ν the affine "shear" viscosity and λ the "expansion" viscosity. The antisymmetric components $\{[\boldsymbol{\nabla}\mathbf{v}] - [\boldsymbol{\nabla}\mathbf{v}]^T\}$ have been ignored.

It will be demonstrated how the Axioms of Topological Hydrodynamics yield topological information about the classic Cauchy development. The topological theory goes beyond the symmetrical geometrical formulations by recognizing that antisymmetries can introduce complex spinor contributions to the dynamics.

2.4 Euler flows and Hamiltonian fluids

Consider the Action 1-form per unit source (in thermodynamics, the unit source is mole number, or sometimes mass), constructed from a covariant 3D velocity field, $\mathbf{v} = \mathbf{v}_k(x, y, z, t)$, and a scalar potential function, ϕ:

$$A = \mathbf{v}\circ\mathbf{dr} - \phi dt = \mathbf{v}_k(x, y, z, t)dx^k - \phi dt. \tag{65}$$

Compute the exterior differential dA and define the following (3D vector) functions as,

$$\boldsymbol{\omega} = curl\ \mathbf{v} \quad and \quad \mathbf{a} = +\{\partial\mathbf{v}/\partial t + grad(\phi)\}, \tag{66}$$

such that,

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k \tag{67}$$

$$= \boldsymbol{\omega}_z dx\hat{}dy + \boldsymbol{\omega}_x dy\hat{}dz + \boldsymbol{\omega}_y dz\hat{}dx - \mathbf{a}_x dx\hat{}dt - \mathbf{a}_y dy\hat{}dt - \mathbf{a}_z dz\hat{}dt.$$

These vector fields always satisfy the Poincare-Faraday induction equations, $dF = ddA = 0$, or,

$$curl\ (-\mathbf{a}) + \partial\boldsymbol{\omega}/\partial t = 0, \qquad div\ \boldsymbol{\omega} = 0. \tag{68}$$

The Eulerian Fluid Consider a process created by the contravariant vector directionfield, $\mathbf{V}_4 = [\mathbf{V}^x, \mathbf{V}^y, \mathbf{V}^z, 1]$ and use Cartan's magic formula,

$$L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) = W + dU = Q, \tag{69}$$

to compute the thermodynamic Work 1-form, W. The expressions for Work, W, and internal energy, U, become:

$$\begin{aligned} W &= i(\rho\mathbf{V}_4)dA = -\rho\{-\partial\mathbf{v}/\partial t - grad(\phi) + \mathbf{V}\times\boldsymbol{\omega}\}\circ d\mathbf{r} \\ &\quad -\rho\mathbf{V}\circ\{\partial\mathbf{v}/\partial t + grad(\phi)\}dt, && (70) \\ &= \rho\{\mathbf{a} - \mathbf{V}\times\boldsymbol{\omega}\}\circ d\mathbf{r} - \rho\mathbf{V}\circ\{\mathbf{a})\}dt && (71) \\ U &= i(\mathbf{V}_4)A = \rho(\mathbf{V}\cdot\mathbf{v} - \phi). && (72) \end{aligned}$$

At first, topologically constrain the thermodynamic Work 1-form to be of the Bernoulli class in terms of the exterior differential system:

$$\begin{aligned} W &= -dP && (73) \\ \rho\{+\mathbf{a} - \mathbf{V}\times\boldsymbol{\omega}\}\circ d\mathbf{r} - \rho\mathbf{V}\circ\{\mathbf{a})\}dt &= -grad\ P\circ d\mathbf{r} - \partial P/\partial t\ dt && (74) \end{aligned}$$

Assume that formally $\mathbf{V} = \mathbf{v}$, and the potential is equal to $\phi = \mathbf{v}\cdot\mathbf{v}/2$. Compare the coefficients of $d\mathbf{r}$ to deduce the classic equations of motion for the Eulerian fluid.

$$\{\partial\mathbf{v}/\partial t + grad(\mathbf{v}\cdot\mathbf{v}/2) - \mathbf{v}\times\boldsymbol{\omega}\} = -grad(P)/\rho. \tag{75}$$

This formula should be compared to the derivation of the Lorentz force term for Work in electromagnetic systems. The functional format of the hydrodynamic 1-form of Action, A, is the same as that specified above for the electromagnetic system. All that is changed is the notation. In essence, the two topological theories are equivalent to the extent that there is a correspondence between functions:

$$\begin{aligned} \mathbf{A} &\Leftrightarrow \mathbf{v}, & \phi &\Leftrightarrow \mathbf{v}\cdot\mathbf{v}/2, && (76) \\ \mathbf{E} &\Leftrightarrow -\mathbf{a}, & \mathbf{B} &\Leftrightarrow \boldsymbol{\omega}. && (77) \end{aligned}$$

All the results of the preceding section using an electromagnetic format can be translated into the hydrodynamic format.

Note that the Bernoulli "pressure", P, is an evolutionary invariant along a trajectory,

$$L_{(\rho\mathbf{V}_4)}P = i(\mathbf{V}_4)dP = i(\mathbf{V}_4)i(\mathbf{V}_4)dA = 0. \tag{78}$$

If the Pressure is barotropic, then the Bernoulli function becomes, $d\Theta = dP/\rho$. The function Θ can be amalgamated with the potential, ϕ, such that the thermodynamic Work 1-form becomes equal to zero. The system then admits an extremal Hamiltonian direction field such that the thermodynamic Work 1-form is zero.

$$W = i(V_H)dA = 0. \tag{79}$$

For a process defined in terms of an extremal directionfield, the First Law indicates that the 1-form of Heat, Q, is exact, $dQ = 0$, and equal to the change in internal energy, $Q = dU$. Any Hamiltonian process is reversible, as $Q\hat{}dQ = 0$.

The time-like component of the exterior differential system $W + dP = 0$ leads to the equation,

$$\partial P/\partial t = -\rho \mathbf{v} \circ \{\partial \mathbf{v}/\partial t + grad(\mathbf{v} \cdot \mathbf{v}/2) = \rho(\mathbf{v} \circ \mathbf{a}). \tag{80}$$

It is apparent that if the velocity, $\mathbf{v}$, and the acceleration, $\mathbf{a}$, are orthogonal, then the time rate of change of the Bernoulli pressure is zero.

It also follows that the "Master" equation is valid, with the only difference being that $curl\ \mathbf{v}$ is defined as $\boldsymbol{\omega}$, the vorticity of the hydrodynamic flow. The master equation becomes,

$$curl(\mathbf{v} \times \boldsymbol{\omega}) = \partial \boldsymbol{\omega}/\partial t, \tag{81}$$

and this equation is to be recognized as the equivalent of Helmholtz' equation for the conservation of vorticity.

In the hydrodynamic sense, conservation of vorticity implies uniform continuity. In other words, the Eulerian flow is not only Hamiltonian, it is also uniformly continuous, and satisfies both the master equation and the conservation of vorticity constraints. In addition, it may be demonstrated that such systems are at most of Pfaff dimension 3, and admit a relative integral invariant which generalizes the hydrodynamic concept of invariant helicity. In the electromagnetic topology, the Hamiltonian constraint is equivalent to the statement that the Lorentz force vanishes, a condition that has been used to define the "ideal" plasma or "force-free" plasma state [48].

3 The Navier-Stokes fluid

3.1 The classic Navier-Stokes equations

It can be demonstrated that the "ideal fluid" has a Hamiltonian representation, for which the dynamics preserves a "Hamiltonian" energy. This result is in disagreement with experiment in that it is observed that motions of "non-ideal" fluids exhibit decay to a stationary state. The Lagrange Euler equations must be modified to accommodate dissipation of kinetic energy and angular momentum. In fact, the ideal fluid constraint of zero affine shear stresses should be replaced by dissipative terms related to both affine shears and a new phenomena of rotational and expansion shears which have a fixed point. The classical phenomenological outcome is the Navier-Stokes PDE's,

$$\begin{aligned} \partial \mathbf{v}/\partial t + grad(\mathbf{v} \cdot \mathbf{v}/2) - \mathbf{v}\times\ curl\ \mathbf{v} &= -grad\varphi - (1/\rho)\ gradP \\ &+\nu\Delta\mathbf{v} \\ &-(\mu_B + \nu)\ grad\ div\ \mathbf{v}, \end{aligned} \tag{82}$$

where μ_B is the "bulk" viscosity coefficient and ν is the "shear" viscosity coefficient. If the fluid is "incompressible" then the last term, which includes corrections due to bulk viscosity, vanishes; the incompressible constraint requires that $div\ \mathbf{v} \Rightarrow 0$.

In that which follows, the basic momentum equation (82) will be *deduced* from the perspective of Continuous Topological Evolution. Different topological equivalence classes of thermodynamic processes depend upon the Pfaff Topological Dimension of the Work 1-form. The different classes of thermodynamic processes are related to the velocity field in a Hydrodynamic system. The phenomenological (geometrical) derivation of the equations of hydrodynamics will be replaced by determining the format of the PDE's that agree with the constraint required to satisfy the various PTD equivalence classes the Work 1-form. The 1-form of Work (for barotropic flows as in eq. (75)) will be of Pfaff Topological Dimension 1. The Pfaff Topological Dimension of the 1-form of Work for the Navier-Stokes fluid can be as high as 4, and is required to be 4 if the flow is fully turbulent. Various intermediate classes of the work 1-form are of interest, as well. In particular, the Pfaff Topological Dimension of the work 1-form must be 3 for a baroclinic system, a result that admits frontal systems with propagating tangential discontinuities as found in weather systems.

3.2 The Navier-Stokes equations embedded in a non-equilibrium thermodynamic system

In this subsection, the topological refinement due to the Pfaff Topological Dimension of the Work 1-form will be employed to demonstrate that processes in a non-equilibrium thermodynamic system can be put into correspondence with solutions of the Navier-Stokes equations.

In order to go beyond extremal (Hamiltonian) processes, it is necessary that the Pfaff Topological Dimension of the Work 1-form must be greater than 1. Recall that for any process, the Work done is transverse to the process trajectory,

$$(i(\rho \mathbf{V}_4)W = (i(\rho \mathbf{V}_4)(i(\rho \mathbf{V}_4)dA = 0. \tag{83}$$

Hence, if the PTD of the Work 1-form, W, is to be greater than 1, it must have the format,

$$W = i(\rho \mathbf{V}_4)dA = -dP + \varpi_j(d\mathbf{x}^j - \mathbf{v}^j dt) = -dP + \varpi_j \Delta \mathbf{x}^j, \tag{84}$$

where the "Bernoulli function", P, if it exists, must be a first integral (a process invariant),

$$L(\rho \mathbf{V}_4)P = (i(\rho \mathbf{V}_4)dP = 0. \tag{85}$$

It is also important to remember that such non-zero contributions to the work 1-form are due to the complex, isotropic Cartan Spinors, which are the eigen directionfields of the 2-form, F.

The coefficients, ϖ_j, of the topological fluctuations, $\Delta \mathbf{x}^j$, act in the manner of Lagrange multipliers, and mimic the concept of system forces. If ϖ_j/ρ is defined (arbitrarily[9]) as $\upsilon\, curl\, curl\, \mathbf{v}$ then the spatial components of the thermodynamic Work 1-form, W, are constrained to yield the partial differential equations for a constant density Navier-Stokes fluid:

$$\{\partial \mathbf{v}/\partial t + grad(\mathbf{v} \cdot \mathbf{v}/2) - \mathbf{v} \times \boldsymbol{\omega}\} = -grad(P)/\rho + \upsilon\, curl\, curl\, \mathbf{v}. \tag{86}$$

Density variations can be included by adding a term $\lambda div(\mathbf{V})$ to the potential $\{\mathbf{v} \cdot \mathbf{v}/2\}$ to yield:

$$\partial \mathbf{v}/\partial t + grad\{\mathbf{v}.\mathbf{v}/2\} - \mathbf{v} \times curl\ \mathbf{v} = -gradP/\rho \tag{87}$$

$$+\lambda\{grad(div\ \mathbf{v})\} \tag{88}$$

$$+\upsilon\{curl\ curl\ \mathbf{v}\}. \tag{89}$$

Classically, υ can be identified with the geometric kinematic shear viscosity, and $\lambda = \mu_B - \upsilon$. The coefficient μ_B can be identified with the topological (space-time) bulk viscosity.

[9] This is one of many formal choices, but the choice demonstrates that the Navier-Stokes equations reside within the domain of non-equilibrium thermodynamics. QED

3.3 The Topological Torsion process for the Navier-Stokes fluid

The Navier-Stokes constraint implies that the thermodynamic Work 1-form need not be closed. Then there are thermodynamic processes represented by solutions to the Navier-Stokes equations that are thermodynamically irreversible. In this subsection, the Topological Torsion vector will be expressed in terms of the solutions of the Navier-Stokes equations.

From the work in section 2, the 1-form of Action will generate a 3-form of Topological Torsion, $A\hat{}dA = i(\mathbf{T}_4)\Omega_4$, and leads to the 4 Vector of Topological Torsion (written in hydrodynamic notation):

$$\mathbf{T}_4 = [-\mathbf{a}\times\mathbf{v} + \{\mathbf{v}.\mathbf{v}/2\}\ curl\ \mathbf{v}, (\mathbf{v}\circ curl\,\mathbf{v})], \tag{90}$$

$$= [-\mathbf{a}\times\mathbf{v} + \{\mathbf{v}.\mathbf{v}/2\}\ \boldsymbol{\omega}, (\mathbf{v}\circ\boldsymbol{\omega})] = [\mathbf{T}, h]. \tag{91}$$

Use the Navier-Stokes equations (82) to solve for $\mathbf{a}$,

$$\mathbf{a} = [grad\{\mathbf{v}.\mathbf{v}/2\} + \partial\mathbf{v}/\partial t] \tag{92}$$

$$= \mathbf{v}\times curl\ \mathbf{v} - gradP/\rho + \lambda\{grad(div\ \mathbf{v})\} + \upsilon\{curl\ curl\ \mathbf{v}\}, \tag{93}$$

and then substitute this result into the expression for $\mathbf{T}_4$, to yield:

$$\mathbf{T} = [h\mathbf{v} - (\mathbf{v}\circ\mathbf{v}/2)curl\ \mathbf{v} - \mathbf{v}\times(gradPl\rho) + \lambda\{\mathbf{v}\times grad(div\ \mathbf{v})\} - \upsilon\{\mathbf{v}\times(curl\ curl\ \mathbf{v})\}], \tag{94}$$

$$h = \mathbf{v}\cdot curl\,\mathbf{v}, \tag{95}$$

Note that $\mathbf{T}_4$ exists even for Euler flows, where $\upsilon = 0$, if the flow is baroclinic. The measurement of the components of the Torsion vector, $\mathbf{T}_4$, have been completely ignored by experimentalists in hydrodynamics.

By a similar substitution, the topological parity 4-form, $F\hat{}F$, becomes expressible in terms of engineering quantities as,

$$K = \{2(-\mathbf{a}\circ\boldsymbol{\omega})\}\Omega_4 = \{2(\sigma)\} \Leftrightarrow 2(\mathbf{E}\circ\mathbf{B})$$

$$\sigma = \{gradP/\rho\circ curl\ \mathbf{v} \tag{96}$$

$$-\lambda\{\ grad(div\ \mathbf{v})\circ curl\ \mathbf{v}\} - \upsilon\{curl\ \mathbf{v}\circ(curl\ curl\ \mathbf{v})\}\}\Omega_4. \tag{97}$$

The coefficient σ is a measure of the space-time bulk dissipation coefficient (not λ), and it is the square of this number which must not be zero if the process is irreversible (see eq (27)).

Recall that a turbulent dissipative irreversible flow is defined when the Pfaff dimension of the Action 1-form is equal to 4, which implies that $K \neq 0$.

From the expression for σ, it is apparent that if the 3D vector of vorticity is of Pfaff dimension 2, such that $\boldsymbol{\omega} \circ curl\ \boldsymbol{\omega} = 0$, then the last term vanishes, and there is no irreversible dissipation due to shear viscosity, υ (a result useful in the theory of wakes).

Other useful situations and design criteria for dissipation, or the lack thereof, can be gleaned from the formula. If the vector field is harmonic, then an irreversible process requires that,

$$\sigma = (-\mathbf{a} \circ \boldsymbol{\omega}) = \{(gradP/\rho - \mu_B grad(div\ \mathbf{v})) \circ curl\ \mathbf{v}\} \neq 0. \tag{98}$$

(Recall that harmonic vector fields are generators of minimal surfaces.) For fluids where $(\mu_B) \Rightarrow 0$, if the pressure gradient is orthogonal to the vorticity and the flow field is harmonic, then there is no irreversible dissipation as $\sigma = 0$, and the flow is not turbulent. Note that for many fluids the bulk viscosity is much greater than the shear viscosity. When $\sigma = 0$, no topological torsion defects are created; the acceleration, $\mathbf{a}$, and the vorticity, $\boldsymbol{\omega}$, of the Navier-Stokes fluid are orthogonal.

Theorem 7 *It is thereby demonstrated that solutions to the Navier-Stokes equations correspond to processes of a non-equilibrium thermodynamic system of $PTD(A) = 4$, and Work 1-forms of $PTD(W) > 2$. Such processes include Spinor direction fields generated by the Topological Torsion vector. The Topological Torsion vector generates processes, and hence solutions to the Navier-Stokes equations, that are thermodynamically irreversible..*

These results should be compared to those generated by Lamb and Eckart [3] for the fluid dissipation function, which is defined by the requirement that the dissipative flow has a (geometric) entropy production rate greater than or equal to zero. More examples can be found in, "Wakes, Coherent Structures, and Turbulence" [44].

4 Closed States of Topological Coherence embedded as deformable defects in Turbulent Domains

In this section 4, the problem of C2 smoothness will be attacked from the point of view of topological thermodynamics. First, two distinct examples will be given demonstrating two different emergent $PTD = 3$ states, that emerge from different 4D rotations (see p. 108, [40]). Then, an example demonstrating the decay of a $PTD = 4$ state into a $PTD = 3$ state will be given in detail. The electromagnetic notation will be used, but the results can be converted into hydrodynamic format using the techniques found in section 3.

4.1 Examples of PTD = 3 domains and their Emergence

In section 3, it was demonstrated that there are solutions (thermodynamic processes) to the Navier Stokes equations in non-equilibrium thermodynamic domains. The properties of those $PTD = 3$ domains which emerge by C2 irreversible solutions from domains of $PTD = 4$ are of particular interest. From section 2, it is apparent that the key feature of $PTD = 3$ domains is that the electric $\mathbf{E}$ field (acceleration field $\mathbf{a}$ in hydrodynamics) must be orthogonal to the magnetic $\mathbf{B}$ field (vorticity field $\boldsymbol{\omega}$ in hydrodynamics). There are 8 cases to consider (including chirality),

Pfaff Topological Dimension 3

$$\mathbf{E} = \mathbf{0}, \qquad \pm\mathbf{B} \neq \mathbf{0}, \tag{99}$$

$$\mathbf{B} = \mathbf{0}, \qquad \pm\mathbf{E} \neq \mathbf{0}, \tag{100}$$

$$\mathbf{E} \circ \mathbf{B} = \mathbf{0}, \qquad \text{with chirality choices, } \pm\mathbf{E} = \pm\mathbf{B} \neq \mathbf{0}, \tag{101}$$

of which two will be discussed in detail.

The Finite Helicity case (both E and B finite) PTD = 3 Start with the 4D thermodynamic domain, and first consider the 1-form of Action, A, with the format[10]:

$$A = A_x(z)dx + A_y(z)dy - \phi(z)dt, \tag{102}$$

and its induced 2-form, $F = dA$,

$$\begin{aligned} F &= dA = (\partial A_x(z)/\partial z)dz\hat{}dx + (\partial A_y(z)/\partial z)dz\hat{}dy - (\partial \phi(z)/\partial z)dz\hat{}dt, & (103)\\ &= B_x(z)dz\hat{}dx - B_y(z)dz\hat{}dy + E_z(z)dz\hat{}dt. & (104)\end{aligned}$$

[10] The +E, +B chirality has been selected.

The 3-form of Topological Torsion 3-form becomes

$$\begin{aligned} i(\mathbf{T}_4)\Omega_4 &= A\hat{}F \text{ where} && (105)\\ \mathbf{T}_4(z) &= [E_z A_y + \phi B_x,\ -E_z A_x + \phi B_y,\ 0,\ A_x B_x + A_y B_y] && (106)\\ \text{with } div_4(\mathbf{T}_4(z)) &= 2(\mathbf{E}\circ\mathbf{B}) = 0, \qquad \mathbf{A}\circ\mathbf{B} \neq 0. && (107)\end{aligned}$$

The Zero Helicity case (both E and B finite) PTD = 3 Start with the 4D thermodynamic domain, and consider the 1-form of Action, A, with the format:

$$A = A_x(x,y)dx + A_y(x,y)dy - \phi(x,y)dt, \tag{108}$$

and its induced 2-form, $F = dA$,

$$\begin{aligned} F &= dA = \{(\partial A_y(x,y)/\partial x) - (\partial A_x(x,y)/\partial x)dx\hat{}dy\} \\ &\quad -(\partial\phi(x,y)/\partial x)dx\hat{}dt - (\partial\phi(x,y)/\partial y)dy\hat{}dt, && (109)\\ &= B_z(x,y)dx\hat{}dy + E_x(x,y)dx\hat{}dt + E_y(x,y)dy\hat{}dt. && (110)\end{aligned}$$

The 3-form of Topological Torsion 3-form becomes,

$$\begin{aligned} i(\mathbf{T}_4)\Omega_4 &= A\hat{}F \qquad \text{where} && (111)\\ \mathbf{T}_4(x,y) &= [0, 0, (E_x A_y - E_y A_x) + \phi B_z, 0] && (112)\\ \text{with } div_4(\mathbf{T}_4(x,y)) &= 2(\mathbf{E}\circ\mathbf{B}) = 0, \qquad \mathbf{A}\circ\mathbf{B} = 0. && (113)\end{aligned}$$

This case of zero helicity ($\mathbf{A}\circ\mathbf{B} = \mathbf{0}$), has the Topological Torsion vector, $\mathbf{T}_4(x,y)$, colinear with the $\mathbf{B}$ field.

Zero Helicity case: PTD = 4 decays to PTD = 3 The two distinct cases, modulo chirality, are suggestive of the idea (see p. 108 [40]) that the rotation group of a 4D domain is not simple. The example, immediately above, is particularly useful because the algebra of the decay from Pfaff dimension 4 to 3 is transparent.

Start with the 4D thermodynamic domain, and consider the 1-form of Action, A, with the format:

$$A = A_x(x,y)dx + A_y(x,y)dy - \phi(x,y,z,t)dt, \tag{114}$$

and its induced 2-form, $F = dA$,

$$\begin{aligned} F &= dA = \{(\partial A_y(x,y)/\partial x) - (\partial A_x(x,y)/\partial x)dx\hat{}dy\} \\ &\quad -(\partial\phi(x,y,z)/\partial x)dx\hat{}dt - (\partial\phi(x,y,z)/\partial y)dy\hat{}dt - (\partial\phi(x,y,z)/\partial z)dz\hat{}dt, && (115)\\ &= B_z(x,y)dx\hat{}dy && (116)\\ &\quad +E_x(x,y,z,t)dx\hat{}dt + E_y(x,y,z,t)dy\hat{}dt + E_z(x,y,z,t)dz\hat{}dt. && (117)\end{aligned}$$

The 3-form of Topological Torsion 3-form becomes,

$$i(\mathbf{T}_4)\Omega_4 = A\hat{}F \text{ with PTD}(A) = 4 \tag{118}$$

$$\mathbf{T}_4(x,y,z,t) = [-E_z A_y, +E_z A_x, (E_x A_y - E_y A_x) + \phi B_z, 0] \tag{119}$$

$$\text{with } div_4(\mathbf{T}_4(x,y,z,t)) = 2\{E_z(x,y,z,t)B_z(x,y)\} \neq 0. \tag{120}$$

In this case, the helicity ($\mathbf{A} \circ \mathbf{B} = \mathbf{0}$) is still zero, but now the Topological Torsion vector, $\mathbf{T}_4(x,y,z,t)$, has three spatial components. Moreover, the Process generated by $\mathbf{T}_4(x,y,z)$ is thermodynamically irreversible, as $(\mathbf{E} \circ \mathbf{B}) \neq 0$. The example 1-form is of PTD = 4.

Zero Helicity case: PTD = 4 decays exponentially to PTD = 3 To demonstrate the emergence of the $PTD = 3$ state, suppose the potential function in this example has the format,

$$\phi = \psi(x,y) + \varphi(z)e^{-\alpha t} \tag{121}$$

$$E_z(z,t) = -(\partial\varphi(z)/\partial z)e^{-\alpha t} = E_z(z)e^{-\alpha t}. \tag{122}$$

Then the irreversible dissipation function decays as

$$\sigma = \{E_z(z)B_z\}e^{-\alpha t}.$$

By addition of Spinor fluctuation terms to represent the very small components of irreversible dissipation at late times, the $PTD = 3$ solution,

$$\mathbf{T}_4(x,y) = [0, 0, (E_x A_y - E_y A_x) + \phi B_z, 0] \tag{123}$$

becomes dominant, and represents a long lived "stationary" state far from equilibrium, modulo the small Spinor decay terms[11].

Zero Helicity case: PTD = 4 decays in finite time to PTD = 3 To demonstrate the emergence of the $PTD = 3$ state in finite time, suppose the potential function in this example has the format,

$$\phi = \psi(x,y) \pm \varphi(z)\sqrt{(-t-t_c)^3} \tag{124}$$

$$E_z(z,t) = \mp(\partial\varphi(z)/\partial z)\sqrt{(-t-t_c)^3} = \pm E_z(z)\sqrt{(-t-t_c)^3}. \tag{125}$$

[11] The experimental fact that the defect structures emerge in finite time is still an open *topological* problem, although some *geomtric* success has been achieved through Ricci flows.

Then the irreversible dissipation function decays in a cuspoidal way (typical of the approach to an edge of regression of an envelope function) according to the formula,

$$\sigma = \{E_z(z)B_z\}\sqrt{(-t - t_c)^3}.$$

The PTD of the system is 4 for $t < t_c$, and becomes equal to 3 for $t = t_c$. The PTD =3 state admits a Hamiltonian field that joins smoothly to the cuspoidal singularity.

4.2 Piecewise Linear Vector Processes vs. C2 Spinor processes

It will be demonstrated on thermodynamic spaces of Pfaff Topological Dimension 3, that there exist piecewise continuous processes (solutions to the Navier-Stokes equations) which are thermodynamically reversible. These Vector processes can be fabricated by combinations of Spinor processes, each of which is irreversible. This topological result demonstrates, by example, the difference between piecewise linear 3-manifolds and smooth complex manifolds. It appears that the key feature of the irreversible processes is that they have a fixed point of "rotation or expansion".

Consider those abstract physical systems that are represented by 1-forms, A, of Pfaff Topological Dimension 3. The concept implies that the topological features can be described in terms of 3 functions (of perhaps many geometrical coordinates and parameters) and their differentials. For example, if one presumes the fundamental independent base variables are the set $\{x, y, z\}$, with an exterior differential oriented volume element consisting of a product[12] of exact 1-forms $\Omega_3 = +dx\hat{}dy\hat{}dz$, (then a local) Darboux representation for a physical system could have the appearance,

$$A = xdy + dz. \tag{126}$$

The objective is to use the features of Cartan's magic formula to compute the possible evolutionary features of such a system. The evolutionary dynamics is essentially the first law of thermodynamics:

$$L_{\rho\mathbf{V}}A = i(\rho\mathbf{V})dA + di(\rho\mathbf{V})A) = W + dU = Q. \tag{127}$$

The elements of the Pfaff sequence for this Action become,

$$\begin{aligned} A &= xdy + dz., & (128)\\ dA &= dx\hat{}dy, & (129)\\ A\hat{}dA &= dx\hat{}dy\hat{}dz, & (130)\\ dA\hat{}dA &= 0. & (131) \end{aligned}$$

[12] More abstract systems could be constructed from differential forms which are not exact.

Note that for this example the coefficient of the 3-form of Topological Torsion is not zero. In fluid notation it becomes evident that the coefficient of the 3-form depends upon the Enstrophy (square of the Vorticity) of the fluid flow.

4.3 The Vector Processes

Relative to the position vector $\mathbf{R} = [x, y, z]$ of ordered topological coordinates $\{x, y, z\}$, consider the 3 abstract, linearly independent, orthogonal (supposedly) vector direction fields:

$$\mathbf{V}_x = \left| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\rangle, \tag{132}$$

$$\mathbf{V}_y = \left| \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right\rangle, \tag{133}$$

$$\mathbf{E} = \left| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\rangle. \tag{134}$$

These direction fields can be used to define a class of (real) Vector processes, but these real vectors do not exhibit the complex Spinor class of eigendirection fields for the 2-form, dA. The Spinor eigendirection fields are missing from this basis frame. The important fact is that thermodynamic processes defined in terms of a real basis frame (and its connection) are incomplete, as such processes ignore the complex spinor direction fields.

For each of the real direction fields, deform the (assumed) process by an arbitrary function, ρ. Then construct the terms that make up the First Law of topological thermodynamics. First construct the contractions to form the internal energy for each process,

$$U_{\mathbf{V}_x} = i(\rho\mathbf{V}_x)A = 0, \quad dU_{\mathbf{V}_x} = 0, \tag{135}$$

$$U_{\mathbf{V}_y} = i(\rho\mathbf{V}_y)A = \rho x, \quad dU_{\mathbf{V}_y} = d(\rho x), \tag{136}$$

$$U_{\mathbf{E}} = i(\rho\mathbf{E})A = \rho, \quad dU_{\mathbf{E}} = d\rho. \tag{137}$$

The *extremal* vector $\mathbf{E}$ is the unique eigenvector with eigenvalue zero relative to the maximal rank antisymmetric matrix generated by the 2-form, dA. The *associated* vector $\mathbf{V}_x$ (relative to the 1-form of Action, A, is orthogonal to the y, z plane. Recall that any associated vector represents a local adiabatic process, as the Heat flow is transverse to the process. The linearly independent thermodynamic Work 1-forms for evolution in the direction of the 3 basis vectors are determined to be,

$$W_{\mathbf{V}_x} = i(\rho\mathbf{V}_x)dA = +\rho dy, \tag{138}$$
$$W_{\mathbf{V}_y} = i(\rho\mathbf{V}_y)dA = -\rho dx, \tag{139}$$
$$W_{\mathbf{E}} = i(\rho\mathbf{E})dA = 0. \tag{140}$$

From Cartan's Magic Formula representing the First Law as a description of topological evolution,

$$L_{(\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V})A) \equiv Q, \tag{141}$$

it becomes apparent that,

$$Q_{\mathbf{V}_x} = +\rho dy, \qquad dQ_{\mathbf{V}_x} = +d\rho\hat{}dy, \tag{142}$$
$$Q_{\mathbf{V}_y} = +xd\rho, \qquad dQ_{\mathbf{V}_y} = -d\rho\hat{}dx, \tag{143}$$
$$Q_{\mathbf{E}} = d\rho \qquad dQ_{\mathbf{E}} = 0, \tag{144}$$

All processes in the extremal direction satisfy the conditions that $Q_{\mathbf{E}}\hat{}dQ_{\mathbf{E}} = 0$. Hence, all extremal processes are reversible. It is also true that evolutionary processes in the direction of the other basis vectors, separately, are reversible, as the 3-form $Q\hat{}dQ$ vanishes for $\mathbf{V}_x$, $\mathbf{V}_y$, or $\mathbf{E}$. Hence all such *piecewise* continuous, *transitive*, processes are thermodynamically reversible.

Note further that the "rotation" induced by the antisymmetric matrix $[dA]$ acting on $\mathbf{V}_x$ yields $\mathbf{V}_y$ and the 4th power of the matrix yields the identity rotation,

$$[dA] \circ |\mathbf{V}_x\rangle = |\mathbf{V}_y\rangle, \tag{145}$$
$$[dA]^2 \circ |\mathbf{V}_x\rangle = -|\mathbf{V}_x\rangle, \tag{146}$$
$$[dA]^4 \circ |\mathbf{V}_x\rangle = +|\mathbf{V}_x\rangle. \tag{147}$$

This concept is a signature of Spinor phenomena.

4.4 The Spinor Processes

Now consider processes defined in terms of the Spinors. The eigendirection fields of the antisymmetric matrix representation of $F = dA$,

$$[F] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{148}$$

are given by the equations:

$$\text{EigenSpinor1} \quad |Sp1\rangle = \left| \begin{matrix} 1 \\ \sqrt{\text{-}1} \\ 0 \end{matrix} \right\rangle \quad \text{Eigenvalue} = +\sqrt{\text{-}1}, \tag{149}$$

$$\text{EigenSpinor2} \quad |Sp2\rangle = \left| \begin{matrix} 1 \\ -\sqrt{\text{-}1} \\ 0 \end{matrix} \right\rangle \quad \text{Eigenvalue} = \text{-}\sqrt{\text{-}1} \tag{150}$$

$$\text{EigenVector1} \, |E\rangle = \left| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\rangle \quad \text{Eigenvalue} = 0 \tag{151}$$

Now consider the processes defined by ρ times the Spinor eigendirection fields. Compute the change in internal energy, dU, the Work, W and the Heat, Q, for each Spinor eigendirection field:

$$U_{\rho\mathbf{Sp_1}} = i(\rho\mathbf{Sp_1})A = \sqrt{-1}\rho x \qquad d(U_{\rho\mathbf{Sp_1}}) = \sqrt{-1}d(\rho x), \tag{152}$$

$$U_{\rho\mathbf{Sp_2}} = i(\rho\mathbf{Sp_2})A = -\sqrt{-1}\rho x \qquad d(U_{\rho\mathbf{Sp_2}}) = -\sqrt{-1}d(\rho x), \tag{153}$$

$$U_{\rho\mathbf{E}} = i(\rho\mathbf{E})A = \rho, \qquad d(U_{\rho\mathbf{E}}) = d\rho,. \tag{154}$$

$$W_{\rho\mathbf{Sp_1}} = i(\rho\mathbf{Sp_1})dA = \rho(dy - \sqrt{\text{-}1}dx), \tag{155}$$

$$W_{\rho\mathbf{Sp_2}} = i(\rho\mathbf{Sp_2})dA = +\rho(dy + \sqrt{\text{-}1}dx) \tag{156}$$

$$W_{\rho\mathbf{E}} = i(\rho\mathbf{V_1})dA = 0,. \tag{157}$$

$$Q_{\rho\mathbf{Sp_1}} = L_{i(\rho\mathbf{Sp_1})}A = \rho(dy - \sqrt{\text{-}1}dx) + \sqrt{\text{-}1}d(\rho x), \tag{158}$$

$$Q_{\rho\mathbf{Sp_2}} = L_{i(\rho\mathbf{Sp_2})}A = \rho(dy + \sqrt{\text{-}1}dx) - \sqrt{\text{-}1}d(\rho x), \tag{159}$$

$$Q_{\rho\mathbf{E}} = L_{i(\rho\mathbf{E})}A = d\rho. \tag{160}$$

4.5 Irreversible Spinor processes

Next compute the 3-forms of $Q\hat{}dQ$ for each direction field, including the spinors:

$$Q_{\rho\mathbf{E}}\hat{}dQ_{\rho\mathbf{E}} = 0, \tag{161}$$

$$Q_{\rho\mathbf{Sp_1}}\hat{}dQ_{\rho\mathbf{Sp_1}} = -\sqrt{\text{-}1}\rho d\rho\hat{}dx\hat{}dy, \tag{162}$$

$$Q_{\rho\mathbf{Sp_2}}\hat{}dQ_{\rho\mathbf{Sp_2}} = +\sqrt{\text{-}1}\rho d\rho\hat{}dx\hat{}dy. \tag{163}$$

It is apparent that evolution in the direction of the Spinor fields can be irreversible in a thermodynamic sense, if $d\rho\hat{}dx\hat{}dy$ is not zero. This is not true for the "piecewise linear" combinations of the complex Spinors that produce the real vectors, $\mathbf{V}_x$ and $\mathbf{V}_y$.

Evolution in the direction of "smooth" combinations of the base vectors may not satisfy the reversibility conditions, $Q\hat{}dQ = 0$, when the combination involves a fixed point in the x, y plane. For example, it is possible to consider smooth rotations (polarization chirality) in the x, y plane:

$$V_{\text{rotation right}} = \mathbf{V}_x + \sqrt{\text{-}1}\mathbf{V}_y = Sp1, \tag{164}$$
$$Q\hat{}dQ = -\sqrt{\text{-}1}\rho d\rho\hat{}dx\hat{}dy. \tag{165}$$

$$V_{\text{rotation left}} = \mathbf{V}_x - \sqrt{\text{-}1}\mathbf{V}_y = Sp2, \tag{166}$$
$$Q\hat{}dQ = +\sqrt{\text{-}1}\rho d\rho\hat{}dx\hat{}dy. \tag{167}$$

The non-zero value of $Q\hat{}dQ$ for the continuous rotations are related to the non-zero Godbillon-Vey class [7]. A key feature of the rotations is that they have a fixed point in the plane; the motions are not transitive. If the physical system admits an equation of state of the form, $\theta = \theta(x, y, \rho) = 0$, then the rotation or expansion processes are not irreversible.

Note that the (supposedly) Vector processes of the preceding subsection are special combinations of the Spinor processes,

$$\mathbf{V}_x = (a \cdot Sp1 + b \cdot Sp2)/2 \tag{168}$$
$$\mathbf{V}_y = -\sqrt{\text{-}1}(a \cdot Sp1 - b \cdot Sp2)/2. \tag{169}$$

Almost always, a process defined in terms a linear combinations of the Spinor direction fields will generate a Heat 1-form, Q, that does not satisfy the Frobenius integrability theorem, and therefore all such processes are thermodynamically irreversible: $Q\hat{}dQ \neq 0$. However, with the requirement that a^2 is precisely the same as b^2, then either piecewise linear process is reversible, for $Q\hat{}dQ = 0$.

If the coefficients, and therefore the Spinor contributions, have slight fluctuations, the cancellation of the complex terms is not precise. Then either of the (now approximately) piecewise continuous process will NOT be reversible due to Spinor fluctuations.

Remark 8 *The facts that piecewise (sequential) C1 transitive evolution along a set of direction fields in odd (3) dimensions can be thermodynamically reversible, $Q\hat{}dQ = 0$, while (smooth) C2 evolution processes composed from complex Spinors can be thermodynamically irreversible, $Q\hat{}dQ \neq 0$, is a remarkable result which appears to have a relationship to Nash's theorem on C1 embedding. Physically, the results are related to tangential discontinuities such as hydrodynamic wakes.*

For systems of Pfaff dimension 4, all of the eigendirection fields are Spinors. The Spinors occur as two conjugate pairs. If the conjugate variables are taken to be x,y and z,t then the z,t spinor pair can be interpreted in terms of a chirality of expansion or contraction, where the x,y pair can be interpreted as a chirality of polarization. In this sense it may be said that thermodynamic time irreversibility is an artifact of dimension 4.

It is remarkable that a rotation and an expansion can be combined (eliminating the fixed point) to produce a thermodynamically *reversible* process.

Ian Stewart points out that there are three types of manifold structure: piecewise linear, smooth, topological. Theorems on piecewise-linear manifolds may not be true on smooth manifolds. The work above seems to describe such an effect. Piecewise continuous processes are reversible, where smooth continuous processes are not (see page 106, [40])!

5 Epilogue: Topological Fluctuations and Spinors

This Section 5 goes beyond the original objective of demonstrating that the Navier-Stokes equations, based upon continuous topological evolution, can describe the irreversible decay of turbulence, but not its creation. However, the key features of process irreversibility and turbulence are entwined with the concept of Topological Torsion and Spinors. Hence this epilogue calls attention to the fact that the Cartan topological methods permit the analysis of Spinor entanglement, as well as the analysis of fluctuations about kinematic perfection. This research area is in its infancy, and extends the thermodynamic approach to the realm of fiber bundles. A few of the introductory ideas are presented below.

Remark 9 *These concepts go beyond the major scope of this essay which has the objective of presenting the important topological ideas in a manner palatable (if not recognizable) to the engineering community of hydrodynamics and plasmas.*

5.1 The Cartan-Hilbert Action 1-form

To start, consider those physical systems that can be described by a function $\mathsf{L}(\mathbf{q},\mathbf{v},t)$ and a 1-form of Action given by Cartan-Hilbert format,

$$A = \mathsf{L}(\mathbf{q}^k,\mathbf{v}^k,t)dt + \mathbf{p}_k\cdot(d\mathbf{q}^k - \mathbf{v}^k dt). \tag{170}$$

The classic Lagrange function, $\mathsf{L}(\mathbf{q}^k,\mathbf{v}^k,t)dt$, is extended to include fluctuations in the kinematic variables, $(d\mathbf{q}^k - \mathbf{v}^k dt) \neq 0$. It is no longer assumed that the equation of Kinematic Perfection is satisfied. Fluctuations of the topological constraint of Kinematic Perfection are permitted;

$$\textbf{Topological Fluctuations in position:}\ \ \Delta\mathbf{q} = (d\mathbf{q}^k - \mathbf{v}^k dt) \neq 0. \tag{171}$$

As the fluctuations are 1-forms, it is some interest to compute their Pfaff Topological Dimension. The first step in the construction of the Pfaff Sequence is to compute the exterior differential of the fluctuation 1-form:

$$\textbf{Fluctuation 2-form}:\quad d(\Delta\mathbf{q}) = -(d\mathbf{v}^k - \mathbf{a}^k dt)\hat{}dt \tag{172}$$

$$= -\Delta\mathbf{v}\hat{}dt, \tag{173}$$

$$\textbf{Topological Fluctuations in velocity}\ :\ \Delta\mathbf{v} = (d\mathbf{v}^k - \mathbf{a}^k dt) \neq 0. \tag{174}$$

It is apparent that the Pfaff Topological Dimension of the fluctuations is at most 3, as $\Delta\mathbf{q}\hat{}\Delta\mathbf{v}\hat{}dt \neq 0$, and has a Heisenberg component, When dealing with fluctuations in this prologue, the geometric dimension of independent base variables will not be constrained to the 4 independent base variables of the Thermodynamic model. At first glance it appears that the domain of definition is a (3n+1)-dimensional variety of independent base variables, $\{\mathbf{p}_k, \mathbf{q}^k, \mathbf{v}^k, t\}$. Do not make the assumption that the $\mathbf{p}_k$ are constrained to be canonically defined. Instead, consider $\mathbf{p}_k$ to be a (set of) Lagrange multiplier(s) to be determined later. Also, do not assume at this stage that $\mathbf{v}$ is a kinematic velocity function, such that $(d\mathbf{q}^k - \mathbf{v}^k dt) \Rightarrow 0$. The classical idea is to assert that topological fluctuations in position are related to pressure, and topological fluctuations in velocity are related to temperature.

For the given Action, construct the Pfaff Sequence (12) in order to determine the Pfaff dimension or class [8] of the Cartan-Hilbert 1-form of Action. The top Pfaffian is defined as the non-zero p-form of largest degree p in the sequence. The top Pfaffian for the Cartan-Hilbert Action is given by the formula,

Top Pfaffian is 2n+2

$$(dA)^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial \mathsf{L}/\partial v^k - p_k)dv^k\}\hat{}\,\Omega_{2n+1}, \tag{175}$$

$$\Omega_{2n+1} = dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{176}$$

The formula is a bit surprising in that it indicates that the Pfaff Topological Dimension of the Cartan-Hilbert 1-form is 2n+2, and not the geometrical dimension $3n+1$. For n = 3 "degrees of freedom", the top Pfaffian indicates that the Pfaff Topological Dimension of the 2-form, dA is $2n+2=8$. The value $3n+1=10$ might be expected as the 1-form was defined initially on a space of $3n+1$ "independent" base variables. The implication is that there exists an irreducible number of independent variables equal to $2n+2=8$ which completely characterize the differential topology of the first order system described by the Cartan-Hilbert Action. It follows that the exact 2-form, dA, satisfies the equations

$$(dA)^{n+1} \neq 0, \; but \quad A\hat{}(dA)^{n+1} = 0. \tag{177}$$

Remark 10 *The idea that the 2-form, dA, is a symplectic generator of even maximal rank, 2n+2, implies that ALL eigendirection fields of the 2-form, F = dA, are complex isotropic Spinors, and all processes on such domains have Spinor components.*

The format of the top Pfaffian requires that the bracketed factor in the expression above, $\{\Sigma_{k=1}^{n}(\partial \mathsf{L}/\partial v^k - p_k)dv^k\}$, can be represented (to within a factor) by a perfect differential, dS.

$$dS = (n+1)!\{\Sigma_{k=1}^{n}(\partial \mathsf{L}/\partial v^k - p_k)dv^k\}. \tag{178}$$

The result is also true for any closed addition γ added to A; e.g., the result is "gauge invariant". Addition of a closed 1-form does not change the Pfaff dimension from even to odd. On the other hand the result is not renormalizable, for multiplication of the Action 1-form by a function can change the algebraic Pfaff dimension from even to odd.

On the 2n+2 domain, the components of (2n+1)-form $T = A\hat{}(dA)^n$ generate what has been defined herein as the Topological Torsion vector, to within a factor equal to the Torsion Current. The coefficients of the (2n+1)-form are components of a contravariant vector density $\mathbf{T}^m$ defined as the Topological Torsion vector, the same concept as defined previously on a 4D thermodynamic domain, but now extended to (2n+2)-dimensions. This vector is orthogonal (transversal) to the 2n+2 components of the covector, $\mathbf{A}_m$. In other words,

$$A\hat{}T = A\hat{}(A\hat{}(dA)^n) = 0 \Rightarrow i(\mathbf{T})(A) = \sum \mathbf{T}^m \mathbf{A}_m = 0. \tag{179}$$

This result demonstrates that the extended Topological Torsion vector represents an adiabatic process. This topological result does not depend upon geometric ideas such as metric. It was demonstrated above that, on a space of 4 independent variables, evolution in the direction of the Topological Torsion vector is irreversible in a thermodynamic sense, subject to the symplectic condition of non-zero divergence, $d(A\hat{}dA) \neq 0$. The same concept holds on dimension 2n+2.

The 2n+2 symplectic domain so constructed can not be compact without boundary for it has a volume element which is exact. By Stokes theorem, if the boundary is empty, then the surface integral is zero, which would require that the volume element vanishes; but that is in contradiction to the assumption that the volume element is finite. For the 2n+2 domain to be symplectic, the top Pfaffian can never vanish. The domain is therefore orientable, but has two components, of opposite orientation. Examination of the constraint that the symplectic space be of dimension 2n+2 implies that the Lagrange multipliers, $\mathbf{p}_k$, cannot be used to define momenta in the classical "conjugate or canonical" manner.

Define the non-canonical components of the momentum, $\hbar k_j$, as,

$$\textbf{non-canonical momentum: } \hbar k_j = (p_j - \partial \mathsf{L}/\partial v^j), \tag{180}$$

such that the top Pfaffian can be written as,

$$(dA)^{n+1} = (n+1)!\{\Sigma_{j=1}^n \hbar k_j dv^j\}\hat{}\Omega_{2n+1}, \tag{181}$$

$$\Omega_{2n+1} = dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{182}$$

For the Cartan-Hilbert Action to be of Pfaff Topological Dimension 2n+2, the factor $\{\Sigma_{j=1}^n \hbar k_j dv^j\} \neq 0$. It is important to note, however, that as $(dA)^{n+1}$ is a volume element of geometric dimension 2n+2, the 1-form $\Sigma_{j=1}^n \hbar k_j dv^j$ is exact (to within a factor, say $T(\mathbf{q}^k, t,\ \mathbf{p}_k, S_{\mathbf{v}})$); hence,

$$\Sigma_{j=1}^n \hbar k_j dv^j = T dS_{\mathbf{v}}. \tag{183}$$

Tentatively, this 1-form, $dS_{\mathbf{v}}$, will be defined as the Topological Entropy production relative to topological fluctuations of momentum, kinematic differential position and velocity. If $\hbar k_j$ is defined as the deviation about the canonical definition of momentum, $\hbar k_j = \Delta p_j$, and noting the the expression for the top Pfaffian can be written as $(n+1)!\{\Sigma_{j=1}^{n}\hbar k_j \Delta v^j\}\hat{}\,\Omega_{2n+1}$, leads to an expression for the entropy production rate in the suggestive "Heisenberg" format:

$$TdS_{\mathbf{v}} = \Delta p_j \Delta v^j. \tag{184}$$

References

[1] Bryant, R.L., Chern, S.S., Gardner, R.B., Goldschmidt, H.L., and Griffiths, P. A. (1991), "Exterior Differential Systems", Springer, Verlag.

[2] Cartan, E. (1966) "The Theory of Spinors", Dover, NY, 39.

[3] Eckart, C., (1940), "The thermodynamics of irreversible processes I: The simple fluid, II: Fluid mixtures, III: Relativistic theory of the simple fluid", Phys. Rev., 58.

[4] Klein, Felix. (1925), "Elementary Mathematics from an advanced standpoint – Arithmetic, Algebra, Analysis", Dover, p. 214.

[5] Klein, J. (1962), Ann. Inst. AIFUB Fourier, Grenoble 12, 1.

[6] Landau, L. D. and Liftshitz, E. M. (1959) "Fluid Mechanics", Pergamon Press, London, 94.

[7] Pittie, H.V. (1976) "Characteristic classes of Foliations", Res Notes in Math., Ditman, London

[8] Libermann P. and Marle, C-M. (1987), "Symplectic Geometry and Analytical Mechanics" Riedel, Dordrecht/Boston, 65.

[9] Kondepudi, D. and Prigogine, I. (1998) "Modern Thermodynamics", Wiley, NY.

[10] Marsden, J.E. and Riatu, T. S. (1994) "Introduction to Mechanics and Symmetry", Springer-Verlag, 122.

[11] Kiehn, R.M. and Pierce, J. F. (1969), An Intrinsic Transport Theorem, Phys. Fluids, 12, #9, 1971. (http://www22.pair.com/csdc/pdf/inttrans.pdf)

[12] Kiehn, R.M. (1974), Extensions of Hamilton's Principle to Include Dissipative Systems, J. Math Phys. 15, 9.

[13] Kiehn, R.M. (1975), Intrinsic hydrodynamics with applications to space-time fluids, Int. J. of Eng. Sci. 13, 941. (http://www22.pair.com/csdc/pdf/inthydro.pdf) arXiv/math-ph/0101032

[14] Kiehn, R.M. (1976), Retrodictive Determinism, Int. J. of Eng. Sci. 14, 749. (http://www22.pair.com/csdc/pdf/retrodic.pdf)

[15] Kiehn, R.M. (1977), Periods on manifolds, quantization and gauge, J. Math. Phy. 18, 614. (http://www22.pair.com/csdc/pdf/periods.pdf)

[16] Kiehn, R. M., (1987), The Falaco Effect as a topological defect was first noticed by the present author in the swimming pool of an old MIT friend, during a visit in Rio de Janeiro, at the time of Halley's comet, March 1986. The concept was presented at the Austin Meeting of Dynamic Days in Austin, January 1987, and caused some interest among the resident topologists. The easily reproduced experiment added to the credence of topological defects in fluids. It is now perceived that this topological phenomena is universal, and will appear at all levels from the microscopic to the galactic. (http://www22.pair.com/csdc/pdf/falaco85o.pdf), arXiv.org/gr-qc/0101098 (http://www22.pair.com/csdc/pdf/falaco97.pdf and (http://www22.pair.com/csdc/pdf/topturb.pdf)

[17] http://www22.pair.com/csdc/download/blackspots.avi

[18] Kiehn, R.M. (1990), Topological Torsion, Pfaff Dimension and Coherent Structures, in: "Topological Fluid Mechanics", H. K. Moffatt and T. S. Tsinober eds, Cambridge University Press, 449-458. (http://www22.pair.com/pdf/csdc/camb89.pdf)

[19] Kiehn, R. M. (1991), Compact Dissipative Flow Structures with Topological Coherence Embedded in Eulerian Environments, in: "Non-linear Dynamics of Structures", edited by R.Z. Sagdeev, U. Frisch, F. Hussain, S. S. Moiseev and N. S. Erokhin, 139-164, World Scientific Press, Singapore.

[20] Kiehn, R. M., Kiehn, G. P., and Roberds, B. (1991) Parity and time-reversal symmetry breaking, singular solutions and Fresnel surfaces, Phys. Rev **A 43**, 5165-5671. (http://www22.pair.com/csdc/pdf/timerev.pdf)

[21] Kiehn, R. M. (1992), Topological Defects, Coherent Structures and Turbulence in Terms of Cartan's Theory of Differential Topology, in "Developments in Theoretical and Applied Mathematics, Proceedings of the SECTAM XVI conference", B. N. Antar, R. Engels, A.A. Prinaris and T. H. Moulden, Editors, The University of Tennessee Space Institute, Tullahoma, TN 37388 USA.

[22] Kiehn, R. M. (1993), Instability Patterns, Wakes and Topological Limit Sets, in "Eddy Structure Identification in Free Turbulent Shear Flows", J.P.Bonnet and M.N. Glauser, (eds), Kluwer Academic Publishers, 363.

[23] Kiehn, R. M. (2000), 2D turbulence is a Myth, (invited speaker EGS XXIV General Assembly IUTAM, the Hague, 1999 (http://www22.pair.com/csdc/pdf/hague6.pdf)

[24] Kiehn, R.M. (2002), The Photon Spin and other Topological Features of Classical Electromagnetism, in "Gravitation and Cosmology, From the Hubble Radius to the Planck Scale", Amoroso, R., et al., eds., Kluwer, Dordrecht, Netherlands, 197-206. Vigier 3 conference in 2000. (http://www22.pair.com/csdc/pdf/vig2000.pdf)

[25] Kiehn, R. M. (2003), Thermodynamic Irreversibility and the Arrow of Time, in "The Nature of Time: Geometry, Physics and Perception", R. Bucher et al. (eds.), Kluwer, Dordrecht, Netherlands, 243-250. (http://www22.pair.com/csdc/pdf/arwfinal.pdf)

[26] Kiehn, R. M. (2005), Propagating topological singularities: the photon, in "The Nature of Light: What Is a Photon?"; Chandrasekhar Roychoudhuri, Katherine Creath; Eds, Proc. SPIE 5866, 192-206.

[27] Kiehn, R. M., "Thermodynamics and quantum cosmology – Continuous topological evolution of topologically coherent defects", arXiv:gr-qc/0603072

[28] Kiehn, R. M.,"A topological theory of the Physical Vacuum", arXiv:gr-qc/0602118

[29] Kiehn, R. M., "Instability patterns, wakes and topological limit sets",arXiv:physics/0102005

[30] Kiehn, R. M., "Topology and Turbulence", arXiv:physics/0102003

[31] Kiehn, R. M., "Some closed form solutions to the Navier Stokes equations", rmkarXiv:physics/0102002

[32] Kiehn, R. M., "Topological-Torsion and Topological-Spin as coherent structures in plasmas:, rmkarXiv:physics/0102001

[33] Kiehn, R. M., "Curvature and torsion of implicit hypersurfaces and the origin of charge", rmkarXiv:gr-qc/0101109

[34] Kiehn, R. M., "Chirality and helicity in terms of topological spin and topological torsion", rmkarXiv:physics/0101101

[35] Kiehn, R. M., "Falaco Solitons, Cosmic Strings in a Swimming Pool", rmkarXiv:gr-qc/0101098

[36] Kiehn, R. M. and Baldwin, P. "Cartan's topological structure", rmkarXiv:math-ph/0101033

[37] Kiehn, R. M., "Continuous topological evolution", rmkarXiv:math-ph/0101032

[38] Kiehn, R. M., "Electromagnetic Waves in the Vacuum with Torsion and Spin", rmkarXiv:physics/9802033

[39] W. Slebodzinsky, (1970), "Exterior Forms and their Applications", PWN, Warsaw.

[40] Stewart, I. (1988), The Problems of Mathematics, Oxford, NY..

[41] Uffink, J. arXiv.org/cond-mat/0005327

[42] Kiehn, R. M. (2007), Non-Equilibrium Thermodynamics, 2nd Edition, "Non-Equilibrium Systems and Irreversible Processes Vol 1", see (http://www.lulu.com/kiehn).

[43] Kiehn, R. M. (2007), Cosmology, Falaco Solitons and the Arrow of Time, 2nd Edition, "Non-Equilibrium Systems and Irreversible Processes Vol 2", see (http://www.lulu.com/kiehn).

[44] Kiehn, R. M. (2007), Wakes, Coherent Structures and Turbulence, 2nd Edition, "Non-Equilibrium Systems and Irreversible Processes Vol 3", see (http://www.lulu.com/kiehn).

[45] Kiehn, R. M. (2007), Plasmas and non-equilibrium Electrodynamics, 2nd Edition, "Non-Equilibrium Systems and Irreversible Processes Vol 4", see (http://www.lulu.com/kiehn).

[46] Kiehn, R. M. (2007), Exterior Differential Forms and Differential Topology, 1st Edition "Topological Torsion and Macroscopic Spinors Vol 5", see (http://www.lulu.com/kiehn).

[47] Kiehn, R. M. (2004), Selected Publications "Non-Equilibrium Systems and Irreversible Processes Vol 7", see (http://www.lulu.com/kiehn).

[48] Wesson, P. S. (2000) "Space Time Matter - Modern Kaluza Klein theory", World Scientific, Singapore.