

The Cosmological Vacuum from a Topological perspective

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Abstract: This article examines how the physical presence of field energy and particulate matter can be interpreted in terms of the topological properties of space time. The theory is developed in terms of vector and matrix equations of exterior differential forms. The theory starts from the sole postulate that field properties of the Cosmological Vacuum (a continuum) can be defined in terms of a vector space domain, of maximal rank, infinitesimal, neighborhoods, where exact differentials are mapped into exterior differential 1-forms, $|A\rangle$, by a Basis Frame of C2 functions, $[B]$, with non-zero determinant. The particle properties of the Cosmological Vacuum are defined in terms of topological defects (or compliments) of the field vector space, where the non-zero determinant condition fails. When the exterior differential 1-forms, $|A\rangle$, are not uniquely integrable, the fibers can be twisted, leading to possible Chiral matrix arrays of certain 3-forms of Topological Torsion and Topological Spin. In addition, there exist Chiral objects constructed from vector arrays of 1-forms that mimic the properties of the Einstein tensor.

Part I

The Cosmological Vacuum

1 Preface

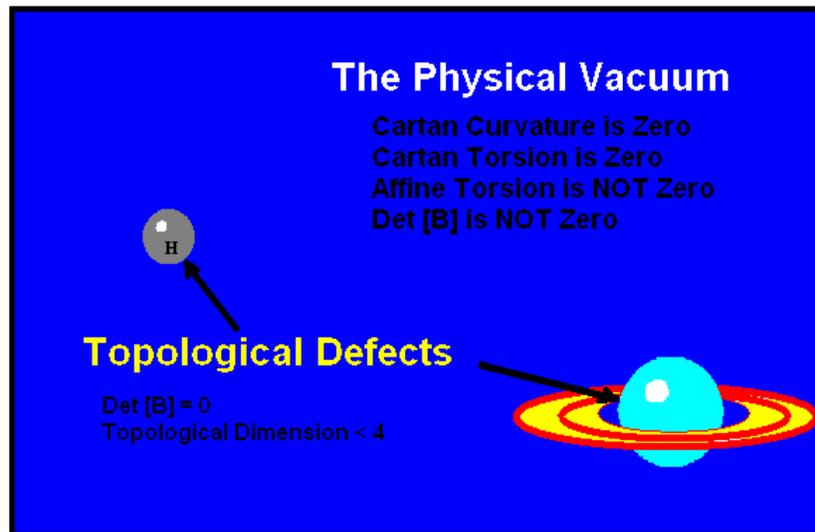
In 1993-1998, Gennady Shipov [29] presented his pioneering concept of a modified A_n space of "Absolute Parallelism". Shipov implied that a such geometric structures could exhibit "affine" torsion. Such non-Riemannian structures are different from those induced by Gauss curvature effects associated with gravitational fields, for the A_n space defines a domain of zero Gauss curvature.

This article, however, examines how the physical presence of field energy and particulate matter could influence the *topological* properties (not only the *geometrical* properties) of space time to form a "Cosmological Vacuum". The point of departure in this article consists of three parts:

I Shipov's vision of Absolute Parallelism, is extended to include a larger set of admissible systems. The larger set is based on the sole requirement that *infinitesimal* neighborhoods of a "Cosmological Vacuum" are elements of a vector space, defined by the mapping of a vector arrays of exact differentials into a vector arrays (almost always) of non-exact differential 1-forms¹. The matrix of of C2 functions that represents the vector space mapping is defined as a Basis Frame, $[\mathbb{B}(y^a)]$, and represents the vector space properties of infinitesimal neighborhoods. The basis vectors that make up the Basis Frame for *infinitesimal* neighborhoods exhibit topological, differential closure. That is, the differential of any column vector of the Basis Frame is a linear combination of all of the column vectors that make up the Basis Frame. The set of admissible Basis Frames for the vector space of infinitesimal neighborhoods is larger than the set of basis frames for global neighborhoods, for the infinitesimal maps need not be integrable, and would therefor represent non-trivial bundle concepts. In this article, the matrix format of exterior differential forms is maintained as the mathematical vehicle of choice, thus removing the "debauch des indices" associated with tensor analysis. Moreover, the "twisting" of the "fibers" producing chiral effects (missed by classical tensor methods) becomes evident. The Basis Frames are not necessarily Symmetric or Orthonormal, which are properties associated with specific gauge constraints imposed on the general system of infinitesimal mappings.

II In certain domains the Frame Matrix $[\mathbb{B}]$ is singular, and then one or more of its four eigenvalues is zero. These singular domains (or objects) may be viewed as topological defects of 3 (topological) dimensions or less embedded in the field domain of a 4 dimensional "Cosmological Vacuum". They can be thought of as condensates or particles or field discontinuities. The major theme of this article examines the field properties of the "Cosmological Vacuum", which is the domain free of singularities of the type $\det[\mathbb{B}] = 0$. The Basis Frame Matrix $[\mathbb{B}]$ will be assumed to consist of C2 functions, but only C1 differentiability is required for deriving a linear connection that defines infinitesimal differential closure. If the functions are not C2, singularities can occur in second order terms, such as curvatures (and accelerations).

¹Some authors have called this a "local mapping", but in order to preserve the topological implications, I prefer the words "infinitesimal mappings". The method permits the inclusion of non-trivial bundles when the 1-forms are not integrable.



The 4D Cosmological Vacuum with 3D topological defects

Although more complicated, the singular sets admit analysis, for example, in terms of propagating discontinuities and topologically quantized period integrals [18]. These topics will be considered in more detail in a subsequent article.

- III It is recognized that topological coherent structures (fields, and particles, along with fluctuations) in a "Cosmological Vacuum" can be put into correspondence with the concepts of topological thermodynamics based on continuous topological evolution [32] [22]. Perhaps surprising to many, topology can change continuously in terms of processes that are not diffeomorphic. For example, a blob of putty can be deformed continuously into a cylindrical rope, and then the ends can be "pasted" together to create a non-simply connected object from a simply connected object. Topological continuity requires only that the limit points of the initial state topology be included in the closure of the topology of the final state. Such continuous maps are not necessarily invertible; it is important to remember that topology need not be conserved by such continuous processes. Diffeomorphic processes require continuity of the map and its inverse and therefore are specializations of homeomorphisms which preserve topology. This observation demonstrates why tensor constraints cannot be applied to problems of irreversible evolution and topological change [16].

2 Topological Structure of a Cosmological Vacuum

2.1 The Fundamental Postulate:

2.1.1 The Cosmological Vacuum Field space

Assume the existence of a matrix array of 0-forms (C2 functions), $[\mathbb{B}] = [\mathbb{B}_{\text{col}}^{\text{row}}(y)] = [\mathbb{B}_a^k(y)]$, on a 4D variety of points $\{y^a\}$. The domain will be constrained such that there exists an inverse Frame, $[\mathbb{B}]^{-1}$, and will be used to study those cases such that

$$[\mathbb{B}] \circ |dx^k\rangle = |A^k\rangle. \quad (1)$$

The domain for which the determinant of $[\mathbb{B}]$ is not zero will define a Vector space. It is assumed that such regions will serve as the domain of the Cosmological Vacuum field.

The Basis Frame, $[\mathbb{B}]$, maps a vector array of exact differentials into a vector array of exterior differential forms, $|A^k\rangle$. Each component of $|A^k\rangle$ may or may not be exact. If the components of the 1-forms are integrable in the sense of Frobenius, $|(A \wedge dA)^k\rangle = 0$, then the differential mappings are said to define a "trivial" bundle of "tangent vectors". When the components, $|A^k\rangle$, are not uniquely integrable, the mappings are said to define a "non-trivial" bundle of tangent vectors.

The class of Basis Frames to be studied may have a determinant which is positive definite, or which consists of a negative domain, or a positive domain. In the latter case, it may be true that there exist non-unique (more than one) solutions which can describe the "twisting of the tangent vectors". It is this feature that leads to possible Chiral properties of the Cosmological Vacuum.

2.1.2 The Cosmological Vacuum Particle space.

The compliment of the Cosmological Vacuum field domain is defined as the singular domain, which is the realm of topologically coherent defect structures (such as particles and field discontinuities) which will be described later. From this sole topological assumption, that the Cosmological Vacuum field is a continuous collection of infinitesimal neighborhoods that form a vector space, the concepts displayed below are deducible results, based on the Cartan Calculus of exterior differential forms. The methods do not impose the constraints of a metric, or a linear connection, or a coframe of 1-forms, as a starting point, as is done in MAG and other theories [9] [10] Instead these concepts are deduced in terms of topological refinements of the single hypothesis:

Definition 1 *The Cosmological Vacuum is a maximal rank vector space on a 4D domain, which, as a differential ideal, supports topological differential closure.*

2.2 Constructive Results

Remark 2 *The idea is to use matrix representations of differential forms and the exterior matrix product to replicate and expand the ideas generated by Tensor analysis. It becomes apparent that Tensor methods miss the important concept of chirality. For every Basis Frame, $[\mathbb{B}]$, of a vector space, there exists TWO connections, a Right and a Left connection. In simpler cases, the two connections can be equivalent, but, in general, they are not the same. Each connection can be related to the other by a similarity transformation, which preserves their symmetric invariant properties, but does not exhibit the antisymmetry and chiral properties associated with torsion. The matrix methods utilized below correct this deficiency, and will demonstrate Chirality effects associated with both matrices of 3-forms and matrices of 1-forms.*

By applying algebraic and exterior differential processes developed by E. Cartan to an element, $[\mathbb{B}]$, of given equivalence class of Basis Frames of C2 functions, the following concepts are **derived, not postulated**, results. It is possible to encode these results in terms of existence theorems and constructive proofs. Starting from the assumption that the Basis Frame maps exact differentials on to 1-forms,

$$[\mathbb{B}] \circ |dx^k\rangle = |A^k\rangle, \quad (2)$$

$$[\mathbb{B}]^{-1} \circ [\mathbb{B}] = [\mathbb{I}], \quad (3)$$

the following constructions are possible:

1. A "flat" Right Cartan Connection as a matrix of 1-forms, $[\mathbb{C}]$, can be derived, leading to structural equations of curvature 2-forms and torsion 2-forms which are zero relative to this connection. However, the construction can support the concept of "Affine" Torsion. This derived Right Cartan Connection, $[\mathbb{C}]$, is algebraically compatible with the Basis Frame, and is differentially unique.

$$d[\mathbb{B}] = -[\mathbb{B}] \circ d[\mathbb{B}]^{-1} \circ [\mathbb{B}] = [\mathbb{B}] \circ [\mathbb{C}], \quad (4)$$

$$[\mathbb{C}] = -d[\mathbb{B}]^{-1} \circ [\mathbb{B}] = +[\mathbb{B}]^{-1} \circ d[\mathbb{B}]. \quad (5)$$

The Connection leads to the idea of differential closure.

It is also possible to construct a Left Cartan matrix of 2-forms, $[\Delta]$, also relative to the inverse Frame Matrix, $[\mathbb{B}]^{-1}$ such that:

$$d[\mathbb{B}] = [\Delta] \circ [\mathbb{B}] = [\mathbb{B}] \circ [\mathbb{C}] \quad (6)$$

$$[\Delta] = -[\mathbb{B}] \circ d[\mathbb{B}]^{-1} = +d[\mathbb{B}] \circ [\mathbb{B}]^{-1}. \quad (7)$$

A Right connection can be defined to express differential closure in terms of the Inverse matrix, and this turns out to be equal to $[-\Delta]$

$$d[\mathbb{B}]^{-1} = [\mathbb{B}]^{-1} \circ [-\Delta] = [-\mathbb{C}] \circ [\mathbb{B}]^{-1} \quad (8)$$

2. A symmetric quadratic congruence of functions, relative to a signature matrix, $[\eta]$, can be deduced by algebraic methods from Basis Frame, can play the role of a metric, $[g]$, compatible with the the Basis Frame:

$$[g] = [\mathbb{B}]^T \circ [\eta] \circ [\mathbb{B}]. \quad (9)$$

The quadratic form, $[g]$, can be used to generate another connection, $[\Gamma]$, not equal to the Cartan Connection, by using the Christoffel method. The Christoffel connection is free of "affine" torsion, but need not be flat. The Christoffel connection is not necessarily equal to the symmetric part of the Cartan Connection! The line element can be constructed from the formula,

$$\delta s^2 = \langle dx^j | \circ [g] \circ | dx^k \rangle = \langle dx^k | \circ [\mathbb{B}]^T \circ [\eta] \circ [\mathbb{B}] \circ | dx^k \rangle, \quad (10)$$

$$\delta s^2 = \langle A^m | \circ [\eta_{mn}] \circ | A^n \rangle. \quad (11)$$

The line element, δs^2 , may or may not be integrable.

3. The exterior matrix product of 1-forms leads to the expression for the vector of Affine Torsion 2-forms, $|G^m\rangle$ (not Cartan Torsion 2-forms):

$$\text{Vector of Affine Torsion 2-forms} = [\mathbb{C}] \wedge |dx^k\rangle = |G^m\rangle. \quad (12)$$

The exterior differential of $|G^m\rangle$ leads to a Vector of 3-form currents, which are divergence free:

$$\text{Vector of 3-form currents} = |J^k\rangle = d|G^m\rangle, \quad (13)$$

$$d|J^k\rangle = 0. \quad (14)$$

4. It is possible to construct the density of 3-forms that represent Topological Spin, S , and its divergence which yields the first Poincare function of Field Lagrangian density minus the Field Interaction energy .

$$\text{Topological Spin 3-form} = S = \langle A^m | \wedge |G^m\rangle, \quad (15)$$

$$dS = \langle dA^m | \wedge |G^m\rangle - \langle A^m | \wedge |dG^m\rangle, \quad (16)$$

$$\text{Poincare I 4-form} = \langle F^m | \wedge |G^m\rangle - \langle A^m | \wedge |J^m\rangle, \quad (17)$$

5. It is possible to construct a Scalar of 3-forms that represent Topological Torsion, H , and its divergence that which yields the second Poincare function as a 4-form, of Topological Parity, K , with a coefficient equal to what is best described as "Bulk viscosity".

$$\text{Topological Torsion 3-form} = H = \langle A^m | \wedge | F^m \rangle \quad (18)$$

$$dH = \langle dA^m | \wedge | F^m \rangle = \langle F^m | \wedge | F^m \rangle, \quad (19)$$

$$\text{Poincare II 4-form} = \langle F^m | \wedge | G^m \rangle - \langle A^m | \wedge | J^m \rangle, \quad (20)$$

6. For any Connection, $[\Gamma]$, the Matrix of Curvature 3-forms, $[\Phi]$, can be defined as:

$$[\Phi] = [\Gamma] \wedge [\Gamma] + d[\Gamma], \quad (21)$$

The exterior differential of the Curvature is, in general, not equal to zero. However, it can be decomposed into two component 3-forms which can have chiral properties, $[RH_3]$ and $[LH_3]$.

$$d[\Phi] = [\Phi] \wedge [\Gamma] - [\Gamma] \wedge [\Phi], \quad (22)$$

$$d[\Phi] = [RH_3] - [LH_3]. \quad (23)$$

In certain cases the two Chiral species are equivalent, for which the exterior differential of the Curvature vanishes. In other cases, the two Chiral species are NOT equal, and the exterior differential of the Curvature is not zero. It is remarkable that in all cases the exterior differential of each term, though possibly different, cancel each other. These 3-forms represent Vectors of 3-forms, which may or may not have Zero divergence, but in all non-zero cases the divergence of $[RH_3]$ is equal and opposite in sign to the other, $[LH_3]$.

7. Perhaps the most intriguing thing about Einstein's formula² (which is used to define the concept of gravity in terms of the deformation of space-time) is that the Einstein tensor G_m^c is deduced from contractions of the Riemann curvature tensor. The formulatio of the Einstein tensor, G_m^c , is said to have "Zero divergence" for any choice of metric. The Riemann curvature tensor depends only upon the connection, but the formulation of the Einstein tensor also implicates the metric structure (to raise and lower the indices), as well.

As demonstrated below, the properties of the Einstein tensor can be replicated by a vector of 1-forms that has zero exterior derivative, and is constructed directly from the Connection without recourse to the metric. The formulas demonstrate by construction the concept of a vector of 1-forms, that always has a ZERO exterior differential, and yet can be composed of components, that may exhibit Chiral properties.

²The Einstein formula uses the contracted Ricci tensor and the Ricci scalar to construct

$$G_m^c = g^{cb} R_{bma}^a - \delta_m^c R/2$$

By exterior differentiation of the fundamental assumption, eq (1),

$$d\{[\mathbb{B}] \circ |dx^k\rangle\} = d[\mathbb{B}] \circ |dx^k\rangle = |dA^k\rangle, \quad (24)$$

$$\{[\mathbb{B}] \circ \mathbb{C} \wedge |dx^k\rangle\} = [\mathbb{B}] \wedge |G^k\rangle = |F^k\rangle \quad (25)$$

Recall that the vector of 2-forms, $|G^m\rangle = [\mathbb{C}] \wedge |dx^k\rangle$, defines the vector of Affine Torsion 2-forms relative to the Right Cartan connection, $[\mathbb{C}]$.

It is to be noted that the inverse Basis Frame, $[\mathbb{B}]^{-1}$, maps differential forms into exact differentials,

$$|dx^k\rangle = [\mathbb{B}]^{-1} \circ |A^k\rangle. \quad (26)$$

Exterior differentiation leads to the expression,:

$$d|dx^k\rangle = 0 = d\{[\mathbb{B}]^{-1} \circ |A^k\rangle\} \quad (27)$$

$$= [\mathbb{B}]^{-1} \circ \{[-\Delta] \circ |A^k\rangle + d|A^k\rangle\} \quad (28)$$

$$= [\mathbb{B}]^{-1} \circ \{[-\Delta] \circ |A^k\rangle + |F^k\rangle\} = 0. \quad (29)$$

The coefficient in brackets defines the vector of Cartan Torsion (not Affine Torsion) 2-forms (which are zero for the constraint given by the fundamental formula),

$$\text{The vector of Cartan Torsion 2-forms, } |\sigma\rangle = [-\Delta] \circ |A^k\rangle + |F^k\rangle. \quad (30)$$

As $[\mathbb{B}]^{-1}$ is not zero, it follows that

$$[-\Delta] \circ |A^k\rangle = [-\Delta] \circ [\mathbb{B}] \circ |dx^k\rangle = -|F^k\rangle, \quad (31)$$

to compare with

$$[\mathbb{B}] \circ [\mathbb{C}] \circ |dx^k\rangle = |F^k\rangle. \quad (32)$$

It follows that,

$$\{[\mathbb{B}] \circ [\mathbb{C}] + [\Delta] \circ [\mathbb{B}]\} \circ |dx^k\rangle = 0. \quad (33)$$

$$\{[\mathbb{B}] \circ [\mathbb{C}] - [\Delta] \circ [\mathbb{B}]\} \circ |dx^k\rangle = 2|F^k\rangle \quad (34)$$

The remarkable result is that the vector of 1-forms, $\{[\mathbb{B}] \circ [\mathbb{C}] - [\Delta] \circ [\mathbb{B}]\}$, is not ZERO, but is closed. The 1-form can be decomposed into two component vectors of 1-forms: $[RH_1]$ and $[LH_1]$

$$\{[\mathbb{B}] \circ [\mathbb{C}] - [\Delta] \circ [\mathbb{B}]\} \neq 0, \quad (35)$$

$$\{[RH_1] - [LH_1]\} \neq 0, \quad (36)$$

$$d[RH_1] \neq 0, \quad (37)$$

$$d[LH_1] \neq 0, \quad (38)$$

$$\text{but } d\{[\mathbb{B}] \circ [\mathbb{C}] - [\Delta] \circ [\mathbb{B}]\} = \quad (39)$$

$$\{d[RH_1] - d[LH_1]\} = 2 \langle dF^k \rangle = 2 \langle ddA^k \rangle = 0. \quad (40)$$

The exterior differential form, $\{[\mathbb{B}] \circ [\mathbb{C}] - [\Delta] \circ [\mathbb{B}]\}$ is the analog of the Einstein tensor in General Relativity theory. Again, note the Chirality features of the 1-forms.

A Maple worksheet for the Isotropic Schwarzschild case is attached as an example.

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is universal, and will appear at all levels from the microscopic to the galactic. gr-qc/0101098

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> restart: with (linalg):with(diffforms):
> deform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=0,Az=0,C=0,Phi=0,phi=0,theta=0,r=0,tau=0,nu=0,am=0,bm=0,f=0,g=0,S=0,K=const,c=const,aa=const,bb=const,cc=const,MM1=0,MM2=0,M=const,ss=0,cc=const,ee=const,Lx=0,Ly=0,Lz=0,vx=const,vy=const,vz=const,kk=const,Lambda=const,omega=const,mmm=const,A=0,lambda=0,nu=0);
setup(r,theta,phi,tau):wedgeset(1);

```

Warning, the protected names norm and trace have been redefined and unprotected

wedgeset(1)

>

The Cosmological Vacuum

Third Draft

The Isotropic Schwarzschild example

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INTRODUCTION

The theory is available in PDF format from
<http://www22.pair.com/csdc/download/cosmologicalvacuum.pdf>

The concept of the Cosmological Vacuum utilizes continuum field properties of a 4D Vector space. The vector space defined by a Basis Frame that maps exact differentials into exterior 1-forms. (This sometimes is referred to as the Basis Frame on the Tangent space.) The fundamental Formula defining the vector space of infinitesimal neighborhoods is:

$$[\mathbf{B}(y)] | dy \rangle = | \mathbf{A}(y,dy) \rangle$$

The vector of 1-forms $|A\rangle$ forms a set of "potentials".
 If uniquely integrable, the 1-forms define a trivial tangent bundle.
 If not uniquely integrable, the 1-forms can define a twisted tangent bundle.

The examples presented here will perturb the Basis Frame that represents the spherical to Cartesian Jacobian matrix, such that the resulting perturbed Basis Frame has a congruent quadratic symmetry equivalent to the metric of interest.

Relative to the Minkowski - Dirac signature (- - - +) relative to $\{r, \theta, \phi, \tau\}$ an interesting class of metrics is given by the Matrix

$$ggm := \begin{bmatrix} -am^2 & 0 & 0 & 0 \\ 0 & -am^2 & 0 & 0 \\ 0 & 0 & -am^2 & 0 \\ 0 & 0 & 0 & bm^2 \end{bmatrix}$$

Specific metric realizations metric can be chosen by fixing the functional form of the coefficient functions, a, f and b.

Several Examples include:

1. Schwarzschild metric with the choice $bm^2 = \exp(\nu(r,\tau))$, $am^2 = \exp(\lambda(r,\tau))$, $f(r,\tau)^2=1$,
2. Robertson-Walker-Friedmann-Lemaitre metric with the choice $bm = 1$, $am = S(t)/(1-K/4*r^2)$
3. The isotropic Schwarzschild metric with Minkowski-Dirac signature with

$$bm := \frac{2 - \frac{M}{r}}{2 + \frac{M}{r}}$$

T

[B] = Basis Frame of Functions (the Tetrad)

[CR] = the Right Cartan Connection as matrix of 1-forms, based on [B]

[DR] = Right Cartan Connection based on [Binverse]

[Christ] = the Christoffel Connection as a matrix of 1-forms based on a metric [g]

[T] = the Residue Connection such that [CR] = [Christ] + [T]

[Theta] = the matrix of curvature 2-forms based on [CR]

[Sigma] = the matrix of curvature 2-forms based on [T]

[Phi] = the matrix of curvature 2-forms based on [Christ]

[A> = the vector of differential 1-forms constructed from the fundamental formula

[B]|dx> = [A>

[G> = [CR]^|dx> = vector of Affine Torsion 2-forms.

[L1] = RH Chiral matrix of "angular momentum" 3-forms based on [Christ]

[L2] = LH Chiral matrix of "angular momentum" 3-forms based on [Christ]

[S1] = RH Chiral matrix of "spin" 3-forms based on [T]

[S2] = LH Chiral matrix of "spin" 3-forms based on [T]

[E1] = RH Chiral matrix of "Einstein" 1-forms based on [CR]

[E2] = LH Chiral matrix of "Einstein" 1-forms based on [DR]

Three metric signatures, Euclidean: eta {1,1,1,1}, Minkowski -Dirac: etam {-1,-1,-1,+1}, Minkowski -Majorana: etap {+1,+1,+1,-1}

```
> eta:=array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]]):etam:=array([[[-1,0,0,0],[0,-1,0,0],[0,0,-1,0],[0,0,0,1]])
:etap:=array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,-1]]):
```

The classic coordinate map from Spherical to Cartesian Coordinates

is given by the expressions:

>

```
> eta:=array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]]):etam:=evalm(array([[[-1,0,0,0],[0,-1,0,0],[0,0,-1,0],[0,0,0,1]])
):etap:=evwalm(array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,-1]])):
```

```
> x:=r*sin(theta)*cos(phi);y:=r*sin(theta)*sin(phi);z:=r*cos(theta);t:=tau;
```

```
> JJ:=jacobian([x,y,z,t],[r,theta,phi,tau]);
```

$$x := r \sin(\theta) \cos(\phi)$$

$$y := r \sin(\theta) \sin(\phi)$$

$$z := r \cos(\theta)$$

$$t := \tau$$

$$JJ := \begin{bmatrix} \sin(\theta) \cos(\phi) & r \cos(\theta) \cos(\phi) & -r \sin(\theta) \sin(\phi) & 0 \\ \sin(\theta) \sin(\phi) & r \cos(\theta) \sin(\phi) & r \sin(\theta) \cos(\phi) & 0 \\ \cos(\theta) & -r \sin(\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> MD:=array([[am,0,0,0],[0,am,0,0],[0,0,am,0],[0,0,0,bm]]):
> Metricformat:ggm:=innerprod(MD,etam,MD);
```

$$ggm := \begin{bmatrix} -am^2 & 0 & 0 & 0 \\ 0 & -am^2 & 0 & 0 \\ 0 & 0 & -am^2 & 0 \\ 0 & 0 & 0 & bm^2 \end{bmatrix}$$

Specialization of the functions a(r,tau) and b(r,tau) will yield specific cases. For example, the choice below yields the isotropic Schwarzschild metric.

```
> am:=((2+(M/r))/2)^2;bm:=(((2-M/r))/((2+M/r)));
>
```

$$am := \left(1 + \frac{\frac{1}{2}M}{r}\right)^2$$

$$bm := \frac{2 - \frac{M}{r}}{2 + \frac{M}{r}}$$

At first, the Minkowski-Dirac signature matrix, etam will be utilized to produce the metric ggm

```
> ggm:=innerprod(MD,etam,MD):`Minkowski - Dirac metric (possibility of Dirac spinors) ggm
:=evalm(ggm);DETggm:=factor(simplify(det(ggm),trig));gpp:=innerprod(MD,etam,MD)
: `Majorana metric (possibility of Majorana spinors)
:=evalm(gpp):DETgpp:=factor(simplify(det(gpp))):
```

Minkowski - Dirac metric (possibility of Dirac spinors) ggm :=

$$\begin{bmatrix} -\frac{1}{16} \frac{(2r+M)^4}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{1}{16} \frac{(2r+M)^4}{r^4} & 0 & 0 \\ 0 & 0 & -\frac{1}{16} \frac{(2r+M)^4}{r^4} & 0 \\ 0 & 0 & 0 & \frac{(2r-M)^2}{(2r+M)^2} \end{bmatrix}$$

$$DETggm := -\frac{1}{4096} \frac{(2r+M)^{10} (2r-M)^2}{r^{12}}$$

The Basis Frame of Choice is which produces a congruent quadratic form equal to the metric: ggm = [Btranspose][etam][B]

```
> B:=evalm(MD&^JJ);
```

$$B := \begin{bmatrix} \left(1 + \frac{1}{2} \frac{M}{r}\right)^2 \sin(\theta) \cos(\phi) & \left(1 + \frac{1}{2} \frac{M}{r}\right)^2 r \cos(\theta) \cos(\phi) & -\left(1 + \frac{1}{2} \frac{M}{r}\right)^2 r \sin(\theta) \sin(\phi) & 0 \\ \left(1 + \frac{1}{2} \frac{M}{r}\right)^2 \sin(\theta) \sin(\phi) & \left(1 + \frac{1}{2} \frac{M}{r}\right)^2 r \cos(\theta) \sin(\phi) & \left(1 + \frac{1}{2} \frac{M}{r}\right)^2 r \sin(\theta) \cos(\phi) & 0 \\ \left(1 + \frac{1}{2} \frac{M}{r}\right)^2 \cos(\theta) & -\left(1 + \frac{1}{2} \frac{M}{r}\right)^2 r \sin(\theta) & 0 & 0 \\ 0 & 0 & 0 & 2 - \frac{M}{r} \\ & & & 2 + \frac{M}{r} \end{bmatrix}$$

```
> DETB:=simplify(det(B));
> Bin:=evalm(simplify(inverse(B)));
>
>
```

$$DET B := \frac{1}{64} \frac{(2r - M) \sin(\theta) (2r + M)^5}{r^4}$$

$$Bin := \begin{bmatrix} 4 \frac{\cos(\phi) \sin(\theta) r^2}{(2r + M)^2} & 4 \frac{\sin(\phi) \sin(\theta) r^2}{(2r + M)^2} & 4 \frac{r^2 \cos(\theta)}{(2r + M)^2} & 0 \\ 4 \frac{\cos(\phi) r \cos(\theta)}{(2r + M)^2} & 4 \frac{\sin(\phi) r \cos(\theta)}{(2r + M)^2} & -4 \frac{r \sin(\theta)}{(2r + M)^2} & 0 \\ -4 \frac{\sin(\phi) r}{\sin(\theta) (2r + M)^2} & 4 \frac{\cos(\phi) r}{\sin(\theta) (2r + M)^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{2r + M}{2r - M} \end{bmatrix}$$

Note that the 4D vector space conditions fails at M=2r !

From this Basis Frame it is possible to derive the Right Cartan Matrix of Connection 1-forms.

```
> dB:=d(B):CR:=simpform(simplify(Bin*dB)):
> ` Right Cartan Connection CR `:= evalm(CR);
```

$$\text{Right Cartan Connection } CR := \begin{bmatrix} -2 \frac{M d(r)}{r(2r + M)} & -d(\theta) r & d(\phi) r (-1 + \cos(\theta)^2) & 0 \\ \frac{d(\theta)}{r} & \frac{d(r) (2r - M)}{r(2r + M)} & -d(\phi) \cos(\theta) \sin(\theta) & 0 \\ \frac{d(\phi)}{r} & \frac{d(\phi) \cos(\theta)}{\sin(\theta)} & \frac{\cos(\theta) d(\theta)}{\sin(\theta)} + \frac{d(r) (2r - M)}{r(2r + M)} & 0 \\ 0 & 0 & 0 & 4 \frac{M d(r)}{4r^2 - M^2} \end{bmatrix}$$

It is also possible to compute the Left Cartan Connection matrix, [CL], based on [B] and the Right Cartan Connection matrix [DR] based on [Binverse], where [DR] = -[CL]

where [CL] is the Left Cartan Connection matrix based on [B]

```
> dB:=d(B):CL:=simpform(simplify(evalm(dB*Bin))):
> ` Left Cartan Connection CL = -DR `:= simpform(simplify(evalm(CL))):
```

Left Cartan Connection CL = -DR :=

$$\begin{aligned}
& \left[-\frac{(-1 + \cos(\phi)^2) \cos(\theta) d(\theta)}{\sin(\theta)} \right. \\
& + \frac{(-M \cos(\phi)^2 + \cos(\phi)^2 \cos(\theta)^2 M + 2 \cos(\phi)^2 \cos(\theta)^2 r + 2 r - 2 \cos(\phi)^2 r - M) d(r)}{r(2r + M)}, \\
& - \frac{\cos(\phi) \sin(\phi) \cos(\theta) d(\theta)}{\sin(\theta)} - d(\phi) + \frac{\cos(\phi) \sin(\phi) (-1 + \cos(\theta)^2) d(r)}{r}, \\
& \left. - \frac{\cos(\theta) \sin(\theta) d(r) \cos(\phi)}{r} + d(\theta) \cos(\phi), 0 \right] \\
& \left[-\frac{\cos(\phi) \sin(\phi) \cos(\theta) d(\theta)}{\sin(\theta)} + d(\phi) + \frac{\cos(\phi) \sin(\phi) (-1 + \cos(\theta)^2) d(r)}{r}, \frac{\cos(\theta) \cos(\phi)^2 d(\theta)}{\sin(\theta)} \right. \\
& - \frac{(2M - \cos(\theta)^2 M - M \cos(\phi)^2 + \cos(\phi)^2 \cos(\theta)^2 M - 2r \cos(\theta)^2 + 2 \cos(\phi)^2 \cos(\theta)^2 r - 2 \cos(\phi)^2 r) d(r)}{r(2r + M)}, \\
& \left. - \frac{\cos(\theta) \sin(\theta) d(r) \sin(\phi)}{r} + d(\theta) \sin(\phi), 0 \right] \\
& \left[-d(\theta) \cos(\phi) - \frac{\cos(\theta) \sin(\theta) d(r) \cos(\phi)}{r}, -d(\theta) \sin(\phi) - \frac{\cos(\theta) \sin(\theta) d(r) \sin(\phi)}{r}, \right. \\
& \left. - \frac{(\cos(\theta)^2 M - 2r + 2r \cos(\theta)^2 + M) d(r)}{r(2r + M)}, 0 \right] \\
& \left[0, 0, 0, 4 \frac{M d(r)}{4r^2 - M^2} \right]
\end{aligned}$$

> DR:=evalm(-CL);

DR :=

$$\begin{aligned}
& \left[\frac{(-1 + \cos(\phi)^2) \cos(\theta) d(\theta)}{\sin(\theta)} \right. \\
& - \frac{(-M \cos(\phi)^2 + \cos(\phi)^2 \cos(\theta)^2 M + 2 \cos(\phi)^2 \cos(\theta)^2 r + 2 r - 2 \cos(\phi)^2 r - M) d(r)}{r(2r + M)}, \\
& \frac{\cos(\phi) \sin(\phi) \cos(\theta) d(\theta)}{\sin(\theta)} + d(\phi) - \frac{\cos(\phi) \sin(\phi) (-1 + \cos(\theta)^2) d(r)}{r}, \frac{\cos(\theta) \sin(\theta) d(r) \cos(\phi)}{r} - d(\theta) \cos(\phi) \\
& \left. , 0 \right] \\
& \left[\frac{\cos(\phi) \sin(\phi) \cos(\theta) d(\theta)}{\sin(\theta)} - d(\phi) - \frac{\cos(\phi) \sin(\phi) (-1 + \cos(\theta)^2) d(r)}{r}, -\frac{\cos(\theta) \cos(\phi)^2 d(\theta)}{\sin(\theta)} \right. \\
& + \frac{(2M - \cos(\theta)^2 M - M \cos(\phi)^2 + \cos(\phi)^2 \cos(\theta)^2 M - 2r \cos(\theta)^2 + 2 \cos(\phi)^2 \cos(\theta)^2 r - 2 \cos(\phi)^2 r) d(r)}{r(2r + M)}, \\
& \left. \frac{\cos(\theta) \sin(\theta) d(r) \sin(\phi)}{r} - d(\theta) \sin(\phi), 0 \right] \\
& \left[d(\theta) \cos(\phi) + \frac{\cos(\theta) \sin(\theta) d(r) \cos(\phi)}{r}, d(\theta) \sin(\phi) + \frac{\cos(\theta) \sin(\theta) d(r) \sin(\phi)}{r}, \right. \\
& \left. \frac{(\cos(\theta)^2 M - 2r + 2r \cos(\theta)^2 + M) d(r)}{r(2r + M)}, 0 \right] \\
& \left[0, 0, 0, -4 \frac{M d(r)}{4r^2 - M^2} \right]
\end{aligned}$$

>

[It is also possible to come to the same conclusions using tensor methods.

[**The right Cartan Connection is based on the perturbed Frame: [C] = [Bin] [dF]= - [dBin][F]**

[First compute the differentials of the inverse matrix [Bin]

```
> dim:=4;coord:=[r,theta,phi,tau];
                                     dim := 4
                                     coord := [r, θ, φ, τ]
```

Compute the elements of the matrix product of - d[Bin][B]= Bin[dB]

[First compute the differentials of the inverse matrix [Bin]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1Bin[i,j,k] := (diff(Bin[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[Bin][B]= Bin[dB]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for
  m from 1 to dim do ss := ss+(d1Bin[a,m,k]*B[m,b]*d(coord[k]));
  C2C[a,b,k]:=simplify(-ss) od od od od ;
```

Right Cartan connection coefficients by tensor methods are displayed below:

```
>
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if
  C2C[a,b,k]=0 then else print(`CRight`(a,b,k)=factor(C2C[a,b,k])) fi od od od ;
```

$$\begin{aligned} \text{CRight}(1, 1, 1) &= -2 \frac{M d(r)}{r(2r+M)} \\ \text{CRight}(2, 1, 2) &= \frac{d(\theta)}{r} \\ \text{CRight}(3, 1, 3) &= \frac{d(\phi)}{r} \\ \text{CRight}(1, 2, 2) &= -d(\theta) r \\ \text{CRight}(2, 2, 1) &= \frac{d(r)(2r-M)}{r(2r+M)} \\ \text{CRight}(3, 2, 3) &= \frac{d(\phi) \cos(\theta)}{\sin(\theta)} \\ \text{CRight}(1, 3, 3) &= d(\phi) r (\cos(\theta) - 1) (\cos(\theta) + 1) \\ \text{CRight}(2, 3, 3) &= -d(\phi) \cos(\theta) \sin(\theta) \\ \text{CRight}(3, 3, 1) &= \frac{d(r)(2r-M)}{r(2r+M)} \\ \text{CRight}(3, 3, 2) &= \frac{\cos(\theta) d(\theta)}{\sin(\theta)} \\ \text{CRight}(4, 4, 1) &= 4 \frac{M d(r)}{(2r+M)(2r-M)} \end{aligned}$$

```
> CGamma11:=C2C[1,1,1]+C2C[1,1,2]+C2C[1,1,3]+C2C[1,1,4]:
> CGamma12:=C2C[1,2,1]+C2C[1,2,2]+C2C[1,2,3]+C2C[1,2,4]:
> CGamma13:=C2C[1,3,1]+C2C[1,3,2]+C2C[1,3,3]+C2C[1,3,4]:
> CGamma14:=C2C[1,4,1]+C2C[1,4,2]+C2C[1,4,3]+C2C[1,4,4]:
> CGamma21:=C2C[2,1,1]+C2C[2,1,2]+C2C[2,1,3]+C2C[2,1,4]:
> CGamma22:=C2C[2,2,1]+C2C[2,2,2]+C2C[2,2,3]+C2C[2,2,4]:
```

```

> CGamma23:=C2C[2,3,1]+C2C[2,3,2]+C2C[2,3,3]+C2C[2,3,4]:
> CGamma24:=C2C[2,4,1]+C2C[2,4,2]+C2C[2,4,3]+C2C[2,4,4]:
> CGamma31:=C2C[3,1,1]+C2C[3,1,2]+C2C[3,1,3]+C2C[3,1,4]:
> CGamma32:=C2C[3,2,1]+C2C[3,2,2]+C2C[3,2,3]+C2C[3,2,4]:
> CGamma33:=C2C[3,3,1]+C2C[3,3,2]+C2C[3,3,3]+C2C[3,3,4]:
> CGamma34:=C2C[3,4,1]+C2C[3,4,2]+C2C[3,4,3]+C2C[3,4,4]:
> CGamma41:=C2C[4,1,1]+C2C[4,1,2]+C2C[4,1,3]+C2C[4,1,4]:
> CGamma42:=C2C[4,2,1]+C2C[4,2,2]+C2C[4,2,3]+C2C[4,2,4]:
> CGamma43:=C2C[4,3,1]+C2C[4,3,2]+C2C[4,3,3]+C2C[4,3,4]:
> CGamma44:=C2C[4,4,1]+C2C[4,4,2]+C2C[4,4,3]+C2C[4,4,4]:
> CartanC:=array([[CGamma11,CGamma12,CGamma13,CGamma14],[CGamma21,CGamma22,CGamma23,CGamma24],[CGamma31,CGamma32,CGamma33,CGamma34],[CGamma41,CGamma42,CGamma43,CGamma44]]):print(Cartan_Connection_by_tensor_methods);evalm(CartanC);print(Cartan_Connection_by_matrix_methods);simpform(factor(CR));
> CHECKSUM:=simpform(simplify(evalm(CartanC-CR)));
>

```

Cartan_Connection_by_tensor_methods

$$\begin{bmatrix} -2 \frac{M d(r)}{r(2r+M)} & -d(\theta) r & d(\phi) r (-1 + \cos(\theta)^2) & 0 \\ \frac{d(\theta)}{r} & \frac{d(r)(2r-M)}{r(2r+M)} & -d(\phi) \cos(\theta) \sin(\theta) & 0 \\ \frac{d(\phi)}{r} & \frac{d(\phi) \cos(\theta)}{\sin(\theta)} & \frac{\cos(\theta) d(\theta)}{\sin(\theta)} + \frac{d(r)(2r-M)}{r(2r+M)} & 0 \\ 0 & 0 & 0 & 4 \frac{M d(r)}{4r^2 - M^2} \end{bmatrix}$$

Cartan_Connection_by_matrix_methods

$$\begin{bmatrix} -2 \frac{M d(r)}{r(2r+M)} & -d(\theta) r & d(\phi) r (-1 + \cos(\theta)^2) & 0 \\ \frac{d(\theta)}{r} & \frac{d(r)(2r-M)}{r(2r+M)} & -d(\phi) \cos(\theta) \sin(\theta) & 0 \\ \frac{d(\phi)}{r} & \frac{d(\phi) \cos(\theta)}{\sin(\theta)} & \frac{\cos(\theta) d(\theta)}{\sin(\theta)} + \frac{d(r)(2r-M)}{r(2r+M)} & 0 \\ 0 & 0 & 0 & 4 \frac{M d(r)}{4r^2 - M^2} \end{bmatrix}$$

$$CHECKSUM := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next check for Affine Torsion using the tensor methods:

```

> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do ss :=
(C2C[i,j,k]/d(coord[k])-C2C[i,k,j]/d(coord[j]))/2; CCTTS[i,j,k]:=ss od od od ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
CCTTS[i,j,k]=0 then else print(`G>_AffineTorsion`(i,k,j)=factor(CCTTS[i,k,j]))
fi od od od ;

```

$$[G>_AffineTorsion(2, 2, 1) = -\frac{M}{r(2r+M)}$$

$$[G>_AffineTorsion(2, 1, 2) = \frac{M}{r(2r+M)}$$

$$[G>_AffineTorsion(3, 3, 1) = -\frac{M}{r(2r+M)}$$

$$[G>_AffineTorsion(3, 1, 3) = \frac{M}{r(2r+M)}$$

$$[G>_AffineTorsion(4, 4, 1) = 2\frac{M}{(2r-M)(2r+M)}$$

$$[G>_AffineTorsion(4, 1, 4) = -2\frac{M}{(2r-M)(2r+M)}$$

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

Now use matrix methods to compute the vector of affine torsion 2-forms, |G> .

```
> AR:=array([[d(r)], [d(theta)], [d(phi)], [d(tau)]]):
> G:=simpform(evalm(CR)&^evalm(AR)):
  G11:=G[1,1]-1/2*&^^(diff(lambda(r,tau),r)*d(r),d(r)):G21:=factor(G[2,1]):G31:=f
actor(G[3,1]):G41:=G[4,1]:`the Field excitations [G> (affine torsion 2-forms)
`:= array([[G11],[G21],[G31],[G41]]);
>
>
```

$$\text{the Field excitations } [G> \text{ (affine torsion 2-forms)} := \begin{bmatrix} 0 \\ 2\frac{(d(\theta) \&^ d(r)) M}{r(2r+M)} \\ 2\frac{(d(\phi) \&^ d(r)) M}{r(2r+M)} \\ 4\frac{M(d(r) \&^ d(\tau))}{4r^2 - M^2} \end{bmatrix}$$

A column Vector array of 2-forms, |G>, above

For this example, the affine torsion is an artifact of the choice of Basis Frames.

Note that for the isotropic Schwarzschild solution, the Affine torsion is Zero if the mass M = 0.

Next, display the Vector of field potentials |A> .

```
> A:=simpform(simplify(evalm((B&^AR)))):AT:=transpose(A):`the Potentials, |A>`:=
evalm(A);subs(M=0,evalm(A));
```

the Potentials, |A> :=

$$\left[\begin{array}{c} \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(r)}{r^2} + \frac{\frac{1}{4} (2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r} - \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r} \\ \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(r)}{r^2} + \frac{\frac{1}{4} (2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r} + \frac{\frac{1}{4} (2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r} \\ \frac{\frac{1}{4} (2r+M)^2 \cos(\theta) d(r)}{r^2} - \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) d(\theta)}{r} \\ \frac{(2r-M) d(\tau)}{2r+M} \end{array} \right]$$

$$\left[\begin{array}{c} \sin(\theta) \cos(\phi) d(r) + r \cos(\theta) \cos(\phi) d(\theta) - r \sin(\theta) \sin(\phi) d(\phi) \\ \sin(\theta) \sin(\phi) d(r) + r \cos(\theta) \sin(\phi) d(\theta) + r \sin(\theta) \cos(\phi) d(\phi) \\ \cos(\theta) d(r) - r \sin(\theta) d(\theta) \\ d(\tau) \end{array} \right]$$

A column Vector array of 1-forms above representing the field potentials, A.
(Rovelli calls this vector of 1-forms the "gravity field". I am "anxious" about this set of words)

Next, display the Vector array of Field Intensity 2-forms, |F> = |dA>.

```
> F:=simpform(simplify(evalm((d(A))))):`the Field intensities, |F>`
:=simplify(evalm(F)):
> ` the Field Intensity 2-forms |F> `:=evalm(F);
>
```

the Field Intensity 2-forms |F> :=

$$\left[\begin{array}{c} \frac{1}{2} \frac{\cos(\phi) \cos(\theta) (2r+M) M (d(\theta) \wedge d(r))}{r^2} - \frac{1}{2} \frac{\sin(\theta) \sin(\phi) (2r+M) M (d(\phi) \wedge d(r))}{r^2} \\ \frac{1}{2} \frac{\cos(\theta) \sin(\phi) (2r+M) M (d(\theta) \wedge d(r))}{r^2} + \frac{1}{2} \frac{\sin(\theta) \cos(\phi) (2r+M) M (d(\phi) \wedge d(r))}{r^2} \\ - \frac{1}{2} \frac{\sin(\theta) (2r+M) M (d(\theta) \wedge d(r))}{r^2} \\ \frac{1}{4} \frac{M (d(r) \wedge d(\tau))}{(2r+M)^2} \end{array} \right]$$

A column Vector array of Intensity 2 forms, |F>, above.
Note that for the isotropic Schwarzschild solution, the 2-forms |F> vanish if the mass M = 0.

Next, display the vector array of individual 3-forms of topological torsion, |A^F> relative to the 1-forms of potentials

```
> ` 3-forms of topological torsion |A^F>
:=array([[simpform(simplify(A[1,1]&^F[1,1]))],[simpform(simplify(A[2,1]&^F[2,1]
))],[simpform(simplify(A[3,1]&^F[3,1]))],[simpform(simplify(A[4,1]&^F[4,1]))]]);
```

$$3\text{-forms of topological torsion } |A^F> := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A column Vector of 3-forms, |A^F>, above.

Note that as the 3-forms vanish, and there exist integrating factors, such that the 2-forms of field intensities vanish, as the new 1-form of potentials are exact. Each 1-form component of $|A\rangle$ is integrable. A new Basis Frame can be constructed in terms of the product of a diagonal matrix of integrating factors, $[DIF]$ and the old Basis Frame, $[B_{new}] = [DIF][B_{old}]$, which will not have any Affine Torsion ($F = 0$ and $G=0$). In the new (integrable) basis, there is no intrinsic Affine Torsion, no intrinsic Field intensities, no intrinsic charge current, no intrinsic topological torsion, no intrinsic topological spin. HOWEVER, the new metric is given by $[g_{new}] = [B_{transpose}][DIF][\eta][DIF][B]$ replacing $[g_{old}] = [B_{transpose}][\eta][B]$. The Cartan Connection of the new Basis Frame is equal to the Christoffel Connection for the new metric, $[T_{new}] = 0$.

Next display the scalar 3-form of Topological Torsion, $\langle A||F\rangle$, and the second Poincare invariant, $d\langle A||F\rangle$.

```
> `Scalar Topological Torsion = <A^F>`:=simplify(AT&^F); `Scalar PoincareII =
d(<A^F>) `:=d(evalm((AT&^F)));
Scalar Topological Torsion = <A^F> := [ 0]
Scalar PoincareII = d(<A^F>) := [ 0]
```

It is remarkable that the Topological Torsion 3-forms of each of the individual 4 components is not zero, but the scalar sum vanishes.

This would imply that the Affine Torsion 2-forms can be transformed away by a projective (not diffeomorphic) transformation.

Otherwise the Field intensities of 2-forms and the associated 3-forms are NOT reducible.

The Affine Torsion 2-forms are irreducible in the non-integrable case.

Next, display the vector array of individual 4-forms of topological parity, $|F^F\rangle$ relative to the 1-forms of potentials

```
> K11:=(F[1,1]&^F[1,1]):
> K21:=(F[2,1]&^F[2,1]):
> K31:=(F[3,1]&^F[3,1]):
> K41:=(F[4,1]&^F[4,1]): ` |F^F> ` :=array([[K11],[K21],[K31],[K41]]);
|F^F> := [
0
0
0
0]
```

A column Vector Array of Topological Parity 4-forms, $|F^F\rangle$.

They are ALL zero, so that the 4 individual thermodynamic systems, based on the 1-forms of Potentials, are closed in a thermodynamic sense. Each system is of Pfaff topological dimension 2, as all of the 3-forms of topological torsion are zero.

Next display the scalar of total topological Parity scalar

```
>
>
> ` the TOTAL Topological Parity scalar K `:= simplify(transpose(F)&^F);
the TOTAL Topological Parity scalar K := [ 0]
```

Next, compute the Vector of induced charge current density 3-forms, $|J\rangle = |dG\rangle$

```
> J:=simpform(simplify(d(G))): ` "charge current density" 3-form, |J> = |dG> `:=
```

```
evalm(J);
```

$$\text{"charge current density" 3-form, } |J\rangle = |dG\rangle := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

>
A column vector array of 3-forms |J>, above, which vanishes identically for the Schwarzschild example

Next compute the individual components of the Topological Spin 3-forms, |A^G>

```
> AG:=array([[simpform(simplify(A[1,1]&^G[1,1])],[simpform(simplify(A[2,1]&^G[2,1]
))]],[simpform(simplify(A[3,1]&^G[3,1])],[simpform(simplify(A[4,1]&^G[4,1])]]])
:
```

```
> ` Individual Topological Spin 3-form densities |A^G> `:=evalm(AG);
```

$$\text{Individual Topological Spin 3-form densities } |A^G\rangle := \begin{bmatrix} 0 \\ \frac{1}{2} \frac{M(2r+M)\sin(\theta)\cos(\phi) \&^{\wedge}(d(\phi), d(\theta), d(r))}{r^2} \\ -\frac{1}{2} \frac{M(2r+M)\sin(\theta) \&^{\wedge}(d(\theta), d(\phi), d(r))}{r^2} \\ 0 \end{bmatrix}$$

Next, compute the Total Topological Spin density

```
> ` Total Topological Spin 3-form density <A|^G>
`:=evalm(simpform(simplify(AT&^G)));
```

$$\text{Total Topological Spin 3-form density } \langle A|^G\rangle := \begin{bmatrix} \frac{1}{2} \frac{\sin(\theta)(2r+M)M(\cos(\phi)+1) \&^{\wedge}(d(\phi), d(\theta), d(r))}{r^2} \end{bmatrix}$$

However, this construction is an "artifact" of the choice of the BASIS FRAME, and can be Eliminated, as there exist integrating factors for the potentials, and then |F> = 0, so that |G> = 0 relative to the new basis.

Note that for the isotropic Schwarzschild solution, the 3-form Spin densities are Zero if the mass M = 0.

Next compute the individual 4-forms of the first Poincare Invariants

```
> ` PoincareI 4-form densities = d|A^G> `:= d(AG);
```

$$\text{PoincareI 4-form densities = } d|A^G\rangle := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A column vector of 4-form densities above

As the PoincareI components are Zero, the Topological Spin 3-form is closed, and could yield topologically quantized Spin in terms of 3D period integral.

Next the Cartan matrix of curvature 2-forms, [Theta], based upon d[C] + [C]^2

first compute the d[C] + [C]^2 terms using matrix methods

```

[ >
> C2Cx2C:=simplify(simplify(evalm(CR)&^evalm(CR))):` the term CR^CR
  `:=evalm(C2Cx2C);
> dC2C:=simplify(d(evalm(CR))): ` the term d(CR) `:=evalm(dC2C);
>
> Theta:=simplify(simplify(evalm(evalm(dC2C)+evalm(C2Cx2C)))):`Cartan Curvature
  matrix of 2-forms form d[CR]+[CR]^[CR]} = [Theta]`:=simplify(evalm(Theta));
the term CR^CR :=

$$\begin{bmatrix} 0 & d(r) \wedge d(\theta) & \sin(\theta)^2 (d(r) \wedge d(\phi)) + 2 r \cos(\theta) \sin(\theta) (d(\theta) \wedge d(\phi)) & 0 \\ -\frac{d(\theta) \wedge d(r)}{r^2} & 0 & (-1 + 2 \cos(\theta)^2) (d(\theta) \wedge d(\phi)) & 0 \\ -\frac{d(\phi) \wedge d(r)}{r^2} & \frac{d(\phi) \wedge d(\theta)}{-1 + \cos(\theta)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the term d(CR) :=

$$\begin{bmatrix} 0 & -(d(r) \wedge d(\theta)) & (-1 + \cos(\theta)^2) (d(r) \wedge d(\phi)) - 2 r \cos(\theta) \sin(\theta) (d(\theta) \wedge d(\phi)) & 0 \\ -\frac{d(r) \wedge d(\theta)}{r^2} & 0 & (1 - 2 \cos(\theta)^2) (d(\theta) \wedge d(\phi)) & 0 \\ -\frac{d(r) \wedge d(\phi)}{r^2} & \frac{d(\theta) \wedge d(\phi)}{-1 + \cos(\theta)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Cartan Curvature matrix of 2-forms form d[CR]+[CR]^[CR]} = [Theta] :=

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$


```

The matrix of curvature 2-forms (based upon d[C]+[C]^[C]) is the zero matrix

as it should be for the right Cartan connection (also it has a zero trace)!

Note that the exterior derivative of the Cartan Curvature Matrix of Curvature 2-forms , d[Theta], is zero.

```

> CRR3form:=evalm(Theta)&^evalm(CR);CRL3form:=evalm(CR)&^evalm(Theta);;

```

$$CRR3form := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$CRL3form := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The right and left 3-forms - whose difference compose the exterior differential of the Cartan curvature are individually zero.

NEXT, obtain the Christoffel Connection coefficients from the induced perturbed pullback [metric g] using tensor formulas. The non zero Christoffel Connection coefficients 2nd kind

for the perturbed frame and perturbed metric, are:

Gamma2 (i,j,k) index (upper,lower,lower)

```
>
>
>
> metric:=evalm(ggm);
> metricinverse:=inverse(metric):dim:=4:coord:=[r,theta,phi,tau]:
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k]
  := 0 od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to
  dim do C1S[i,j,k] := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k] od od
  od;
> for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0;
  for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then else print(`Christoffel_connection of ggm of the second kind
  `(i,j,k)=simplify((C2S[i,j,k])) fi od od od:
```

The non zero Christoffel Connection coefficients 2nd kind for the perturbed frame and perturbed metric, on the initial space (domain) are:

Gamma2 (i,j,k) index (i up,j down, k down)

```
>
Now compute the matrix elements of the matrix of connection 1-forms based upon the Christoffel connection
> Gamma11:=C2S[1,1,1]*d(r)+C2S[1,1,2]*d(theta)+C2S[1,1,3]*d(phi)+C2S[1,1,4]*d(tau)
:
> Gamma12:=C2S[1,2,1]*d(r)+C2S[1,2,2]*d(theta)+C2S[1,2,3]*d(phi)+C2S[1,2,4]*d(tau)
:
> Gamma13:=C2S[1,3,1]*d(r)+C2S[1,3,2]*d(theta)+C2S[1,3,3]*d(phi)+C2S[1,3,4]*d(tau)
:
> Gamma14:=C2S[1,4,1]*d(r)+C2S[1,4,2]*d(theta)+C2S[1,4,3]*d(phi)+C2S[1,4,4]*d(tau)
:
> Gamma21:=C2S[2,1,1]*d(r)+C2S[2,1,2]*d(theta)+C2S[2,1,3]*d(phi)+C2S[2,1,4]*d(tau)
:
> Gamma22:=C2S[2,2,1]*d(r)+C2S[2,2,2]*d(theta)+C2S[2,2,3]*d(phi)+C2S[2,2,4]*d(tau)
:
> Gamma23:=C2S[2,3,1]*d(r)+C2S[2,3,2]*d(theta)+C2S[2,3,3]*d(phi)+C2S[2,3,4]*d(tau)
:
> Gamma24:=C2S[2,4,1]*d(r)+C2S[2,4,2]*d(theta)+C2S[2,4,3]*d(phi)+C2S[2,4,4]*d(tau)
:
> Gamma31:=C2S[3,1,1]*d(r)+C2S[3,1,2]*d(theta)+C2S[3,1,3]*d(phi)+C2S[3,1,4]*d(tau)
```

```

:
> Gamma32:=C2S[3,2,1]*d(r)+C2S[3,2,2]*d(theta)+C2S[3,2,3]*d(phi)+C2S[3,2,4]*d(tau)
:
> Gamma33:=C2S[3,3,1]*d(r)+C2S[3,3,2]*d(theta)+C2S[3,3,3]*d(phi)+C2S[3,3,4]*d(tau)
:
> Gamma34:=C2S[3,4,1]*d(r)+C2S[3,4,2]*d(theta)+C2S[3,4,3]*d(phi)+C2S[3,4,4]*d(tau)
:
>
> Gamma41:=C2S[4,1,1]*d(r)+C2S[4,1,2]*d(theta)+C2S[4,1,3]*d(phi)+C2S[4,1,4]*d(tau)
:
> Gamma42:=C2S[4,2,1]*d(r)+C2S[4,2,2]*d(theta)+C2S[4,2,3]*d(phi)+C2S[4,2,4]*d(tau)
:
> Gamma43:=C2S[4,3,1]*d(r)+C2S[4,3,2]*d(theta)+C2S[4,3,3]*d(phi)+C2S[4,3,4]*d(tau)
:
> Gamma44:=C2S[4,4,1]*d(r)+C2S[4,4,2]*d(theta)+C2S[4,4,3]*d(phi)+C2S[4,4,4]*d(tau)
:
> Christ:=array([[Gamma11,Gamma12,Gamma13,Gamma14],[Gamma21,Gamma22,Gamma23,Gamma24],
[Gamma31,Gamma32,Gamma33,Gamma34],[Gamma41,Gamma42,Gamma43,Gamma44]]):`
Christoffel Connection Matrix [Christ] of 1-forms based on the metric
`:=eval(Christ);
> `Compare to Cartan Connection Matrix [CR] of
1-forms`:=evalm(CR);T:=simpform(simplify(evalm(evalm(CR)-evalm(Christ)))):
>
> `Compare to Residue Connection Matrix [T] = [CR]-[Christ] of 1-forms`:=evalm(T);
>

```

$$metric := \begin{bmatrix} -\frac{1}{16} \frac{(2r+M)^4}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{1}{16} \frac{(2r+M)^4}{r^4} & 0 & 0 \\ 0 & 0 & -\frac{1}{16} \frac{(2r+M)^4}{r^4} & 0 \\ 0 & 0 & 0 & \frac{(2r-M)^2}{(2r+M)^2} \end{bmatrix}$$

$$\text{Christoffel_connection of ggm of the second kind } (1, 1, 1) = -2 \frac{M}{r(2r+M)}$$

$$\text{Christoffel_connection of ggm of the second kind } (1, 2, 2) = 2 \frac{M}{r(2r+M)}$$

$$\text{Christoffel_connection of ggm of the second kind } (1, 3, 3) = 2 \frac{M}{r(2r+M)}$$

$$\text{Christoffel_connection of ggm of the second kind } (1, 4, 4) = 64 \frac{r^4 (2r-M) M}{(2r+M)^7}$$

$$\text{Christoffel_connection of ggm of the second kind } (2, 1, 2) = -2 \frac{M}{r(2r+M)}$$

$$\text{Christoffel_connection of ggm of the second kind } (2, 2, 1) = -2 \frac{M}{r(2r+M)}$$

$$\text{Christoffel_connection of ggm of the second kind } (3, 1, 3) = -2 \frac{M}{r(2r+M)}$$

$$\text{Christoffel_connection of ggm of the second kind } (3, 3, 1) = -2 \frac{M}{r(2r+M)}$$

$$\text{Christoffel_connection of ggm of the second kind } (4, 1, 4) = 4 \frac{M}{4r^2 - M^2}$$

$$\text{Christoffel_connection of ggm of the second kind } (4, 4, 1) = 4 \frac{M}{4r^2 - M^2}$$

Christoffel Connection Matrix [Christ] of 1-forms based on the metric :=

$$\begin{bmatrix} -2 \frac{M d(r)}{r(2r+M)} & 2 \frac{M d(\theta)}{r(2r+M)} & 2 \frac{M d(\phi)}{r(2r+M)} & 64 \frac{r^4 (2r-M) M d(\tau)}{(2r+M)^7} \\ -2 \frac{M d(\theta)}{r(2r+M)} & -2 \frac{M d(r)}{r(2r+M)} & 0 & 0 \\ -2 \frac{M d(\phi)}{r(2r+M)} & 0 & -2 \frac{M d(r)}{r(2r+M)} & 0 \\ 4 \frac{M d(\tau)}{(2r-M)(2r+M)} & 0 & 0 & 4 \frac{M d(r)}{(2r+M)(2r-M)} \end{bmatrix}$$

Compare to Cartan Connection Matrix [CR] of 1-forms :=

$$\begin{bmatrix} -2 \frac{M d(r)}{r(2r+M)} & -d(\theta) r & d(\phi) r (-1 + \cos(\theta)^2) & 0 \\ \frac{d(\theta)}{r} & \frac{d(r)(2r-M)}{r(2r+M)} & -d(\phi) \cos(\theta) \sin(\theta) & 0 \\ \frac{d(\phi)}{r} & \frac{d(\phi) \cos(\theta)}{\sin(\theta)} & \frac{\cos(\theta) d(\theta)}{\sin(\theta)} + \frac{d(r)(2r-M)}{r(2r+M)} & 0 \\ 0 & 0 & 0 & 4 \frac{M d(r)}{4r^2 - M^2} \end{bmatrix}$$

Compare to Residue Connection Matrix [T] = [CR]-[Christ] of 1-forms :=

$$\begin{bmatrix} 0, -\frac{d(\theta)(2r^3 + r^2 M + 2M)}{r(2r+M)}, \frac{d(\phi)(-2r^3 - r^2 M + 2r^3 \cos(\theta)^2 + r^2 \cos(\theta)^2 M - 2M)}{r(2r+M)}, -64 \frac{r^4 (2r-M) M d(\tau)}{(2r+M)^7} \\ \left[\frac{d(\theta)(2r+3M)}{r(2r+M)}, \frac{d(r)}{r}, -d(\phi) \cos(\theta) \sin(\theta), 0 \right] \\ \left[\frac{d(\phi)(2r+3M)}{r(2r+M)}, \frac{d(\phi) \cos(\theta)}{\sin(\theta)}, \frac{\cos(\theta) d(\theta)}{\sin(\theta)} + \frac{d(r)}{r}, 0 \right] \\ \left[-4 \frac{M d(\tau)}{4r^2 - M^2}, 0, 0, 0 \right] \end{bmatrix}$$

Compute Curvature 2-forms for the different connections

First, Curvature 2-forms based on the Cartan connection

```
> ` Curvature 2-forms based on Cartan connection [CR] ` := Theta;
Theta := simpform(simplify(evalm(d(CR)+CR^2))); ` Closed 3-forms based on the
Cartan Connection [CR] =d(Theta) ` := d(evalm(Theta));
```

Curvature 2-forms based on Cartan connection [CR] := Θ

$$\Theta := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Closed 3-forms based on the Cartan Connection [CR]} = d(\text{Theta}) := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, Curvature 2-forms based on the Christoffel metric connection

>

```
> ` Curvature 2-forms based on Christoffel metric connection `:=
Phi;Phi:=simpform(simplify(evalm(d(Christ)+Christ^Christ)));` Closed 3-forms
based on the Christoffel metric Connection [Christ]computes from d(Phi)
`:=d(evalm(Phi));
```

Curvature 2-forms based on Christoffel metric connection := Φ

Φ :=

$$\begin{bmatrix} 0, -2 \frac{M(4r+M)(d(r) \wedge d(\theta))}{(2r+M)^2 r^2}, -2 \frac{M(4r+M)(d(r) \wedge d(\phi))}{r^2(2r+M)^2}, \\ -128 \frac{r^3 M(4r^2 + M^2 - 4Mr)(d(r) \wedge d(\tau))}{(2r+M)^8} \end{bmatrix}$$

$$\begin{bmatrix} 2 \frac{M(4r+M)(d(r) \wedge d(\theta))}{(2r+M)^2 r^2}, 0, -4 \frac{M^2(d(\theta) \wedge d(\phi))}{r^2(2r+M)^2}, -128 \frac{M^2 r^3(2r-M)(d(\theta) \wedge d(\tau))}{(2r+M)^8} \end{bmatrix}$$

$$\begin{bmatrix} 2 \frac{M(4r+M)(d(r) \wedge d(\phi))}{r^2(2r+M)^2}, -4 \frac{M^2(d(\phi) \wedge d(\theta))}{r^2(2r+M)^2}, 0, -128 \frac{M^2 r^3(2r-M)(d(\phi) \wedge d(\tau))}{(2r+M)^8} \end{bmatrix}$$

$$\begin{bmatrix} -8 \frac{M(d(r) \wedge d(\tau))}{r(2r+M)^2}, 8 \frac{M^2(d(\tau) \wedge d(\theta))}{(2r-M)(2r+M)^2 r}, 8 \frac{M^2(d(\tau) \wedge d(\phi))}{(2r-M)(2r+M)^2 r}, 0 \end{bmatrix}$$

Closed 3-forms based on the Christoffel metric Connection [Christ]computes from d(Phi) :=

$$\begin{bmatrix} 0, & & 0, & & 0, & & 0 \\ 0, & 0, & 8 \frac{M^2(4r+M) \wedge (d(r), d(\theta), d(\phi))}{r^3(2r+M)^3}, & & 128 \frac{M^2 r^2(16r^2 + 3M^2 - 18Mr) \wedge (d(r), d(\theta), d(\tau))}{(2r+M)^9} \\ 0, & 8 \frac{M^2(4r+M) \wedge (d(r), d(\phi), d(\theta))}{r^3(2r+M)^3}, & & 0, & 128 \frac{M^2 r^2(16r^2 + 3M^2 - 18Mr) \wedge (d(r), d(\phi), d(\tau))}{(2r+M)^9} \\ 0, & -8 \frac{M^2(16r^2 - 2Mr - M^2) \wedge (d(r), d(\tau), d(\theta))}{(2r-M)^2(2r+M)^3 r^2}, & & -8 \frac{M^2(16r^2 - 2Mr - M^2) \wedge (d(r), d(\tau), d(\phi))}{(2r-M)^2(2r+M)^3 r^2}, & & 0 \end{bmatrix}$$

Define the Christoffel chiral3forms: L1 = [Phi]^Christ and L2 = [Christ]^Phi

```
> L1:=evalm(d(Christ)^Christ);dL1:=d(evalm(L1));L2:=simpform(simplify(evalm((Ch
rist^d(Christ)))));dL2:=d(evalm(L2));NETCH:=simpform(simplify(evalm(evalm(L1)-e
valm(L2))));dNETCH:=d(evalm(NETCH));` Proof that d(Phi)- NETCH = 0
`:=simpform(simplify(evalm(d(Phi)-NETCH)));
```

>

$$L1 := \begin{bmatrix} 0, & & 0, & & 0, & & 0 \\ 0, & 0, & 4 \frac{M^2(4r+M) \wedge (d(r), d(\theta), d(\phi))}{r^3(2r+M)^3}, & & 128 \frac{M^2(4r+M) r^2(2r-M) \wedge (d(r), d(\theta), d(\tau))}{(2r+M)^9} \\ 0, & 4 \frac{M^2(4r+M) \wedge (d(r), d(\phi), d(\theta))}{r^3(2r+M)^3}, & & 0, & 128 \frac{M^2(4r+M) r^2(2r-M) \wedge (d(r), d(\phi), d(\tau))}{(2r+M)^9} \\ 0, & & -64 \frac{M^2 \wedge (d(r), d(\tau), d(\theta))}{(2r-M)^2(2r+M)^3}, & & -64 \frac{M^2 \wedge (d(r), d(\tau), d(\phi))}{(2r-M)^2(2r+M)^3}, & & 0 \end{bmatrix}$$

$$dL1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L2 := \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, 0, 4 \frac{M^2 (4r+M) \wedge (d(\theta), d(r), d(\phi))}{r^3 (2r+M)^3}, 512 \frac{M^2 r^2 (2r^2 + M^2 - 4Mr) \wedge (d(\theta), d(r), d(\tau))}{(2r+M)^9} \\ 0, 4 \frac{M^2 (4r+M) \wedge (d(\phi), d(r), d(\theta))}{r^3 (2r+M)^3}, 0, 512 \frac{M^2 r^2 (2r^2 + M^2 - 4Mr) \wedge (d(\phi), d(r), d(\tau))}{(2r+M)^9} \\ 0, -8 \frac{M^2 (4r+M) \wedge (d(\tau), d(r), d(\theta))}{(2r-M)(2r+M)^3 r^2}, -8 \frac{M^2 (4r+M) \wedge (d(\tau), d(r), d(\phi))}{(2r-M)(2r+M)^3 r^2}, 0 \end{bmatrix}$$

$$dL2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

NETCH :=

$$\begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, 0, 8 \frac{M^2 (4r+M) \wedge (d(r), d(\theta), d(\phi))}{r^3 (2r+M)^3}, 128 \frac{M^2 r^2 (16r^2 + 3M^2 - 18Mr) \wedge (d(r), d(\theta), d(\tau))}{(2r+M)^9} \\ 0, 8 \frac{M^2 (4r+M) \wedge (d(r), d(\phi), d(\theta))}{r^3 (2r+M)^3}, 0, 128 \frac{M^2 r^2 (16r^2 + 3M^2 - 18Mr) \wedge (d(r), d(\phi), d(\tau))}{(2r+M)^9} \\ 0, -8 \frac{M^2 (16r^2 - 2Mr - M^2) \wedge (d(r), d(\tau), d(\theta))}{(2r-M)^2 (2r+M)^3 r^2}, -8 \frac{M^2 (16r^2 - 2Mr - M^2) \wedge (d(r), d(\tau), d(\phi))}{(2r-M)^2 (2r+M)^3 r^2}, 0 \end{bmatrix}$$

$$dNETCH := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Proof that } d(\text{Phi})\text{-NETCH} = 0 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Relative to the Schwarzschild isotropic metric:

The symmetric metric based connection (affine torsion vanishes) produces a non-zero curvature [Phi], both the Right and the Left Chirality 3-forms have Zero divergence, individually.

These results should be compared with the results obtained for the Cartan Connection.

The Chirality 3-forms are Zero for the Cartain Connection (Curvature 2-forms vanish, Affine Torsion 2-forms need not vanish)

The Chirality 3-forms, L1 and L2 are not Zero for the metric Christoffel (symmetric) connection (Curvature 2-forms do not vanish, Affine Torsion 2-forms vanish)

Both Chirality 3-forms have Zero divergence for the metric Christoffel connection: d(L1)=0, d(L2) = 0

L1 and L2 vanish for the Schwarzschild example as M => 0.

(It is tempting to associate L1 and L2 with orbital angular momentum properties)

> T:=simpform(simplify(evalm(evalm(CR) - evalm(Christ))));

T:=

$$\begin{bmatrix} 0, -\frac{d(\theta) (2r^3 + r^2 M + 2M)}{r(2r+M)}, \frac{d(\phi) (-2r^3 - r^2 M + 2r^3 \cos(\theta)^2 + r^2 \cos(\theta)^2 M - 2M)}{r(2r+M)}, -64 \frac{r^4 (2r-M) M d(\tau)}{(2r+M)^7} \end{bmatrix}$$

$$\left[\begin{array}{l} \frac{d(\theta)(2r+3M)}{r(2r+M)}, \frac{d(r)}{r}, -d(\phi)\cos(\theta)\sin(\theta), 0 \\ \frac{d(\phi)(2r+3M)}{r(2r+M)}, \frac{d(\phi)\cos(\theta)}{\sin(\theta)}, \frac{\cos(\theta)d(\theta)}{\sin(\theta)} + \frac{d(r)}{r}, 0 \\ -4\frac{Md(\tau)}{4r^2-M^2}, 0, 0, 0 \end{array} \right]$$

> Sigma:=simpform(simplify(evalm(evalm(d(T))+evalm(T&^T)))); ` Curvature 2-forms based on the T connection `:= evalm(Sigma);
dSigma:=simpform(simplify(d(evalm(Sigma)))); ` closure 3-forms generated by dSigma `:=simpform(simplify(evalm(dSigma)));

Σ :=

$$\left[\begin{array}{l} 0, 4\frac{M(3r+M)(d(r)\&^d(\theta))}{r^2(2r+M)^2}, 4\frac{M(3r+M)(d(r)\&^d(\phi))}{r^2(2r+M)^2} - \frac{2(-2+\cos(\theta)^2)\cos(\theta)M(d(\theta)\&^d(\phi))}{(2r+M)r\sin(\theta)}, \\ 256\frac{r^3M(2r^2+M^2-4Mr)(d(r)\&^d(\tau))}{(2r+M)^8} \\ -4\frac{M(d(r)\&^d(\theta))}{r(2r+M)^2}, 0, 2\frac{M(-2r^3-r^2M+r^2\cos(\theta)^2M-2r-3M+2r^3\cos(\theta)^2)(d(\theta)\&^d(\phi))}{r^2(2r+M)^2}, \\ -64\frac{(2r+3M)r^3(2r-M)M(d(\theta)\&^d(\tau))}{(2r+M)^8} \\ -4\frac{M(d(r)\&^d(\phi))}{r(2r+M)^2}, 2\frac{M(2r^3+r^2M+2r+3M)(d(\theta)\&^d(\phi))}{r^2(2r+M)^2}, 0, \\ -64\frac{(2r+3M)r^3(2r-M)M(d(\phi)\&^d(\tau))}{(2r+M)^8} \\ 32\frac{Mr(d(r)\&^d(\tau))}{(4r^2-M^2)^2}, 4\frac{M(2r^3+r^2M+2M)(d(\tau)\&^d(\theta))}{(4r^2-M^2)r(2r+M)}, \\ -4\frac{M(-2r^3-r^2M+2r^3\cos(\theta)^2+r^2\cos(\theta)^2M-2M)(d(\tau)\&^d(\phi))}{(4r^2-M^2)r(2r+M)}, 0 \end{array} \right]$$

Curvature 2-forms based on the T connection :=

$$\left[\begin{array}{l} 0, 4\frac{M(3r+M)(d(r)\&^d(\theta))}{r^2(2r+M)^2}, 4\frac{M(3r+M)(d(r)\&^d(\phi))}{r^2(2r+M)^2} - \frac{2(-2+\cos(\theta)^2)\cos(\theta)M(d(\theta)\&^d(\phi))}{(2r+M)r\sin(\theta)}, \\ 256\frac{r^3M(2r^2+M^2-4Mr)(d(r)\&^d(\tau))}{(2r+M)^8} \\ -4\frac{M(d(r)\&^d(\theta))}{r(2r+M)^2}, 0, 2\frac{M(-2r^3-r^2M+r^2\cos(\theta)^2M-2r-3M+2r^3\cos(\theta)^2)(d(\theta)\&^d(\phi))}{r^2(2r+M)^2}, \\ -64\frac{(2r+3M)r^3(2r-M)M(d(\theta)\&^d(\tau))}{(2r+M)^8} \\ -4\frac{M(d(r)\&^d(\phi))}{r(2r+M)^2}, 2\frac{M(2r^3+r^2M+2r+3M)(d(\theta)\&^d(\phi))}{r^2(2r+M)^2}, 0, \\ -64\frac{(2r+3M)r^3(2r-M)M(d(\phi)\&^d(\tau))}{(2r+M)^8} \end{array} \right]$$

$$\left[\begin{array}{l} 32 \frac{M r (d(r) \wedge d(\tau))}{(4 r^2 - M^2)^2}, 4 \frac{M (2 r^3 + r^2 M + 2 M) (d(\tau) \wedge d(\theta))}{(4 r^2 - M^2) r (2 r + M)}, \\ -4 \frac{M (-2 r^3 - r^2 M + 2 r^3 \cos(\theta)^2 + r^2 \cos(\theta)^2 M - 2 M) (d(\tau) \wedge d(\phi))}{(4 r^2 - M^2) r (2 r + M)}, 0 \end{array} \right]$$

closure 3-forms generated by $dSigma :=$

$$\left[\begin{array}{l} 0, 0, 2 \frac{(-2 + \cos(\theta)^2) \cos(\theta) M (4 r + M) \wedge (d(r), d(\theta), d(\phi))}{\sin(\theta) r^2 (2 r + M)^2}, 0 \\ 0, 0, -4 \frac{M (-2 r^4 - r^3 M + r^3 \cos(\theta)^2 M - 6 r^2 - 13 M r + 2 r^4 \cos(\theta)^2 - 3 M^2) \wedge (d(r), d(\theta), d(\phi))}{r^3 (2 r + M)^3}, \\ 64 \frac{r^2 M (24 r^3 - 46 r M^2 + 12 r^2 M + 9 M^3) \wedge (d(r), d(\theta), d(\tau))}{(2 r + M)^9} \\ 0, -4 \frac{M (2 r^4 + r^3 M + 6 r^2 + 13 M r + 3 M^2) \wedge (d(r), d(\theta), d(\phi))}{r^3 (2 r + M)^3}, 0, \\ 64 \frac{r^2 M (24 r^3 - 46 r M^2 + 12 r^2 M + 9 M^3) \wedge (d(r), d(\phi), d(\tau))}{(2 r + M)^9} \\ 0, -4 \frac{(8 r^5 + 4 r^4 M + 2 r^3 M^2 + r^2 M^3 + 32 r^2 M - 4 r M^2 - 2 M^3) M \wedge (d(r), d(\tau), d(\theta))}{r^2 (2 r + M)^2 (2 r - M) (4 r^2 - M^2)}, 4 (-8 r^5 + 8 r^5 \cos(\theta)^2 \\ + 4 r^4 \cos(\theta)^2 M - 4 r^4 M - 2 r^3 M^2 + 2 r^3 \cos(\theta)^2 M^2 - r^2 M^3 - 32 r^2 M + r^2 \cos(\theta)^2 M^3 + 4 r M^2 + 2 M^3) M \\ \wedge (d(r), d(\tau), d(\phi)) / (r^2 (2 r + M)^2 (2 r - M) (4 r^2 - M^2)) + \frac{8 M r \cos(\theta) \sin(\theta) \wedge (d(\theta), d(\tau), d(\phi))}{4 r^2 - M^2}, 0 \end{array} \right]$$

Define T connection chiral3forms: $S1 = [Sigma] \wedge [T]$ and $S2 = [T] \wedge [Sigma]$

```
> S1:=simpform(simplify(evalm(d(T)&^T))):S1Mzero:=subs(M=0,evalm(S1));dS1:=simpform(simplify(d(evalm(S1)))):S2:=simpform(simplify(evalm((T&^d(T))))):S2Mzero:=subs(M=0,evalm(S2));dS2:=simpform(simplify(d(evalm(S2)))):NETCHT:=simpform(simplify(evalm(evalm(dS1) - evalm(dS2))));
```

$S1Mzero :=$

$$\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{8} \frac{(-16 r^5 + 24 r^5 \cos(\theta)^2) \wedge (d(r), d(\theta), d(\phi))}{r^6} & 0 \\ 0 & \frac{1}{8} \frac{(-16 r^5 + 8 r^5 \cos(\theta)^2) \wedge (d(r), d(\phi), d(\theta))}{r^6 (-1 + \cos(\theta)^2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$dS1 := \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -64 \frac{\wedge (d(\theta), d(r), d(\tau), d(\phi)) r^2 \cos(\theta) M \sin(\theta)}{(4 r^2 - M^2)^2} & 0 \end{array} \right]$$

$S2Mzero :=$

$$\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} \frac{(-16 r^5 + 24 r^5 \cos(\theta)^2) \wedge (d(\theta), d(r), d(\phi))}{r^6} & 0 \\ 0 & -\frac{1}{8} \frac{(-16 r^5 + 8 r^5 \cos(\theta)^2) \wedge (d(r), d(\theta), d(\phi))}{r^6 (-1 + \cos(\theta)^2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$dS2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 64 \frac{r^2 \cos(\theta) M \sin(\theta) \wedge (d(\theta), d(\tau), d(r), d(\phi))}{(4 r^2 - M^2)^2} & 0 \end{bmatrix}$$

$$NETCHT := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that for the isotropic Schwarzschild solution, the chiral components, S1 and S1 are NOT zero when the mass, M, goes to zero. !!!!

However, the divergence of each term separately is zero when the mass goes to zero.

(IT is tempting to associate S1 and S2 with intrinsic angular momentum, which can have a Spin-Orbit interaction when M is not zero -- as exhibited by the non-zero divergence of S1 and S2 when M is not zero.

Note the divergence that occurs at $M=2r$!!

A SURPRISE DUE TO AFFINE TORSION is generated by T, which is NOT generated by the symmetric connection, [Christ]. For the symmetric connection, both left and right components have zero divergence. For the asymmetric connection (with affine torsion) the two 3-forms do not have zero divergence separately, but the difference of the two three forms is exact and has zero divergence.

THIS IS AN EXAMPLE OF topological CHIRALITY !!!

Relative to the Schwarzschild isotropic metric:

The asymmetric connection, T, (affine torsion does not vanish) which produces a non-zero curvature [Phi], both the Right and the Left Chirality 3-forms DO NOT have Zero divergence, individually.

These results should be compared with the results obtained for the Cartan Connection and the Christoffel Connection.

The Chirality 3-forms are Zero for the Cartan Connection (Curvature 2-forms vanish, Affine Torsion 2-forms need not vanish)

The Chirality 3-forms are not Zero for the Christoffel (symmetric) connection (Curvature 2-forms do not vanish, Affine Torsion 2-forms vanish)

Both Chirality 3-forms have Zero divergence for the metric Christoffel connection

The Chirality 3-forms DO NOT have Zero divergence in the presence of Torsion.

(However, the difference between the Right and Left Chirality 3-forms is not Zero, and the composite has Zero divergence.

Other 3-forms of interest:

```
> AwedgeSigma:=simplform(simplify(evalm(evalm(transpose(A)&^evalm(Sigma)))));
dAwedgeSigma:=simplform(simplify(d(AwedgeSigma)));AwedgePhi:=simplform(simplify(evalm(evalm(transpose(A))&^evalm(Phi))));
```

>

AwedgeSigma :=

$$\left[-\frac{M \sin(\theta) (\cos(\phi) + 1) \wedge (d(\phi), d(r), d(\theta))}{r^2}, \frac{1}{2} M \right]$$

$$(-6 \sin(\theta) \sin(\phi) r^2 - 2 M \sin(\theta) \sin(\phi) r + 2 \cos(\theta) r^3 + M \cos(\theta) r^2 + 2 r \cos(\theta) + 3 M \cos(\theta))$$

$$\begin{aligned} & \wedge^4(d(\phi), d(r), d(\theta)) / r^4, \frac{1}{2} M \wedge^4(d(r), d(\theta), d(\phi)) (-2 \cos(\phi) \cos(\theta) r^2 - 2 \cos(\phi) \cos(\theta)^3 r^2 \\ & - \cos(\phi) \cos(\theta)^3 r M - 2 \sin(\theta) \sin(\phi) r^3 - \sin(\theta) \sin(\phi) r^2 M + \sin(\theta) \sin(\phi) r^2 \cos(\theta)^2 M - 2 r \sin(\theta) \sin(\phi) \\ & - 3 \sin(\theta) \sin(\phi) M + 2 \sin(\theta) \sin(\phi) r^3 \cos(\theta)^2) / r^4, 16 M r (8 r^3 \cos(\phi) \cos(\theta) + 4 M^2 r \cos(\phi) \cos(\theta) \\ & - 16 M r^2 \cos(\phi) \cos(\theta) + 4 \sin(\theta) \sin(\phi) r^2 + 4 M \sin(\theta) \sin(\phi) r - 3 M^2 \sin(\theta) \sin(\phi)) \wedge^4(d(\theta), d(r), d(\tau)) / \\ & (2 r + M)^6 - 16 M r \\ & (8 \sin(\theta) \sin(\phi) r^3 + 4 M^2 \sin(\theta) \sin(\phi) r - 16 \sin(\theta) \sin(\phi) r^2 M - 4 \cos(\theta) r^2 - 4 M \cos(\theta) r + 3 M^2 \cos(\theta)) \\ & \wedge^4(d(\phi), d(r), d(\tau)) / (2 r + M)^6 \\ & - \frac{16 \sin(\theta) r^2 M (4 r^2 \cos(\phi) + 4 M r \cos(\phi) - 3 M^2 \cos(\phi) + 4 r^2 + 4 M r - 3 M^2) \wedge^4(d(\phi), d(\theta), d(\tau))}{(2 r + M)^6} \end{aligned}$$

dA wedge Sigma :=

$$\begin{aligned} & [0, 0, 0, 16 M r \sin(\theta) \\ & (24 r^3 \cos(\phi) + 20 r^2 \cos(\phi) M - 38 M^2 r \cos(\phi) + 3 M^3 \cos(\phi) + 24 r^3 + 20 r^2 M - 38 r M^2 + 3 M^3) \\ & \wedge^4(d(\phi), d(\theta), d(r), d(\tau)) / (2 r + M)^7] \end{aligned}$$

A wedge Phi :=

$$\begin{aligned} & \left[\frac{1}{2} \frac{M (4 r + M) \sin(\theta) (\cos(\phi) + 1) \wedge^4(d(\phi), d(r), d(\theta))}{r^3}, \right. \\ & \frac{1}{2} \frac{M (4 \sin(\theta) \sin(\phi) r^2 + M \sin(\theta) \sin(\phi) r + 2 M \cos(\theta)) \wedge^4(d(\phi), d(r), d(\theta))}{r^4}, \\ & \left. - \frac{1}{2} \frac{M (4 \cos(\phi) \cos(\theta) r^2 + M r \cos(\phi) \cos(\theta) - 2 \sin(\theta) \sin(\phi) M) \wedge^4(d(\theta), d(r), d(\phi))}{r^4}, -32 M r \right. \\ & (4 r^3 \cos(\phi) \cos(\theta) + M^2 r \cos(\phi) \cos(\theta) - 4 M r^2 \cos(\phi) \cos(\theta) - 2 M \sin(\theta) \sin(\phi) r + M^2 \sin(\theta) \sin(\phi)) \\ & \wedge^4(d(\theta), d(r), d(\tau)) / (2 r + M)^6 + 32 M r \\ & (4 \sin(\theta) \sin(\phi) r^3 + M^2 \sin(\theta) \sin(\phi) r - 4 \sin(\theta) \sin(\phi) r^2 M + 2 M \cos(\theta) r - M^2 \cos(\theta)) \wedge^4(d(\phi), d(r), d(\tau)) \\ & \left. / (2 r + M)^6 - \frac{32 M^2 r^2 \sin(\theta) (2 r \cos(\phi) - M \cos(\phi) + 2 r - M) \wedge^4(d(\phi), d(\theta), d(\tau))}{(2 r + M)^6} \right] \end{aligned}$$

>

Lets Check the effect of the different connections of the Affine Torsion 2-forms.

> evalm(DR); evalm(CR);

$$\begin{aligned} & \left[\frac{(-1 + \cos(\phi)^2) \cos(\theta) d(\theta)}{\sin(\theta)} \right. \\ & - \frac{(-M \cos(\phi)^2 + \cos(\phi)^2 \cos(\theta)^2 M + 2 \cos(\phi)^2 \cos(\theta)^2 r + 2 r - 2 \cos(\phi)^2 r - M) d(r)}{r (2 r + M)}, \\ & \frac{\cos(\phi) \sin(\phi) \cos(\theta) d(\theta)}{\sin(\theta)} + d(\phi) - \frac{\cos(\phi) \sin(\phi) (-1 + \cos(\theta)^2) d(r)}{r}, \frac{\cos(\theta) \sin(\theta) d(r) \cos(\phi)}{r} - d(\theta) \cos(\phi) \\ & \left. , 0 \right] \\ & \left[\frac{\cos(\phi) \sin(\phi) \cos(\theta) d(\theta)}{\sin(\theta)} - d(\phi) - \frac{\cos(\phi) \sin(\phi) (-1 + \cos(\theta)^2) d(r)}{r}, - \frac{\cos(\theta) \cos(\phi)^2 d(\theta)}{\sin(\theta)} \right. \\ & \left. + \frac{(2 M - \cos(\theta)^2 M - M \cos(\phi)^2 + \cos(\phi)^2 \cos(\theta)^2 M - 2 r \cos(\theta)^2 + 2 \cos(\phi)^2 \cos(\theta)^2 r - 2 \cos(\phi)^2 r) d(r)}{r (2 r + M)}, \right. \end{aligned}$$

$$\begin{bmatrix} \frac{\cos(\theta) \sin(\theta) d(r) \sin(\phi)}{r} - d(\theta) \sin(\phi), 0 \\ \left[d(\theta) \cos(\phi) + \frac{\cos(\theta) \sin(\theta) d(r) \cos(\phi)}{r}, d(\theta) \sin(\phi) + \frac{\cos(\theta) \sin(\theta) d(r) \sin(\phi)}{r}, \right. \\ \left. \frac{(\cos(\theta)^2 M - 2 r + 2 r \cos(\theta)^2 + M) d(r)}{r(2 r + M)}, 0 \right] \\ \left[0, 0, 0, -4 \frac{M d(r)}{4 r^2 - M^2} \right] \\ \begin{bmatrix} -2 \frac{M d(r)}{r(2 r + M)} & -d(\theta) r & d(\phi) r (-1 + \cos(\theta)^2) & 0 \\ \frac{d(\theta)}{r} & \frac{d(r)(2 r - M)}{r(2 r + M)} & -d(\phi) \cos(\theta) \sin(\theta) & 0 \\ \frac{d(\phi)}{r} & \frac{d(\phi) \cos(\theta)}{\sin(\theta)} & \frac{\cos(\theta) d(\theta)}{\sin(\theta)} + \frac{d(r)(2 r - M)}{r(2 r + M)} & 0 \\ 0 & 0 & 0 & 4 \frac{M d(r)}{4 r^2 - M^2} \end{bmatrix} \end{bmatrix}$$

```
>
> AffineTorsionCR:=simpform(simplify(CR^(AR)));AffineTorsionChrist:=simpform(simplify(Christ^(AR)));AffineTorsionC:=simpform(simplify(T^(AR)));
```

$$AffineTorsionCR := \begin{bmatrix} 0 \\ 2 \frac{(d(\theta) \wedge d(r)) M}{r(2 r + M)} \\ 2 \frac{(d(\phi) \wedge d(r)) M}{r(2 r + M)} \\ 4 \frac{M(d(r) \wedge d(\tau))}{4 r^2 - M^2} \end{bmatrix}$$

$$AffineTorsionChrist := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AffineTorsionC := \begin{bmatrix} 0 \\ 2 \frac{(d(\theta) \wedge d(r)) M}{r(2 r + M)} \\ 2 \frac{(d(\phi) \wedge d(r)) M}{r(2 r + M)} \\ -4 \frac{M(d(\tau) \wedge d(r))}{4 r^2 - M^2} \end{bmatrix}$$

As expected the Christoffel Connection matrix does not contribute to the affine connection.

The entire affine torsion 2-forms are included in the [T] Connection matrix.

>

Now check the fundamental formula: [Theta] = [Phi] + [Sigma] + [IT]

First get the IT interaction terms

```
GxT:=evalm(simpform(evalm(evalm(simpform(T&^Christ))+evalm(simpform(Christ&^T))))):`Interaction 2-forms IT =
[Gamma]^[T]+[T]^[Gamma]^:=simpform(simplify(evalm(GxT)));IT:=evalm(GxT):
```

Interaction 2-forms $IT = [\Gamma]^{[T]} + [T]^{[\Gamma]} :=$

$$\left[0, 2 \frac{M(d(\theta) \wedge d(r))}{r^2(2r+M)}, 2 \frac{(-2 + \cos(\theta)^2) \cos(\theta) M(d(\theta) \wedge d(\phi))}{(2r+M)r \sin(\theta)} + \frac{2M(d(\phi) \wedge d(r))}{r^2(2r+M)}, \right. \\ \left. -128 \frac{r^3 M^2(4r-M)(d(\tau) \wedge d(r))}{(2r+M)^8} \right] \\ \left[2 \frac{M(d(\theta) \wedge d(r))}{r^2(2r+M)}, 0, -2 \frac{M(-2r-5M-2r^3-r^2M+2r^3 \cos(\theta)^2+r^2 \cos(\theta)^2M)(d(\theta) \wedge d(\phi))}{r^2(2r+M)^2}, \right. \\ \left. 64 \frac{r^3(2r-M)M(2r+5M)(d(\theta) \wedge d(\tau))}{(2r+M)^8} \right] \\ \left[2 \frac{M(d(\phi) \wedge d(r))}{r^2(2r+M)}, 2 \frac{M(2r+5M+2r^3+r^2M)(d(\phi) \wedge d(\theta))}{r^2(2r+M)^2}, 0, \right. \\ \left. 64 \frac{r^3(2r-M)M(2r+5M)(d(\phi) \wedge d(\tau))}{(2r+M)^8} \right] \\ \left[8 \frac{M^2(4r-M)(d(\tau) \wedge d(r))}{(4r^2-M^2)^2 r}, -4 \frac{(2r^3+r^2M+4M)M(d(\tau) \wedge d(\theta))}{(2r-M)(2r+M)^2 r}, \right. \\ \left. 4 \frac{(-2r^3+2r^3 \cos(\theta)^2-r^2M+r^2 \cos(\theta)^2M-4M)M(d(\tau) \wedge d(\phi))}{(2r-M)(2r+M)^2 r}, 0 \right]$$

Now check to see that the Compatibility formal is correct:

$$[\Theta] = [\Phi] + [\Sigma] + [\text{Christ}]^{[T]} + [T]^{[\text{Christ}]} = :0$$

The Computation is difficult, and is done by matrix element by matrix element (all sums should be zero).

>

```
> CG:=simplify(simplify(evalm(Phi))):CT:=simplify(simplify(evalm(Sigma))):Cx:=simplify(simplify(evalm(GxT))):
> SUM11:=simplify(simplify(CG[1,1]+CT[1,1]+Cx[1,1]));SUM12:=simplify(simplify(CG[1,2]+CT[1,2]+Cx[1,2]));SUM13:=simplify(simplify(CG[1,3]+CT[1,3]+Cx[1,3]));SUM14:=simplify(simplify(CG[1,4]+CT[1,4]+Cx[1,4]));
> SUM21:=simplify(simplify(CG[2,1]+CT[2,1]+Cx[2,1]));SUM22:=simplify(simplify(CG[2,2]+CT[2,2]+Cx[2,2]));SUM23:=simplify(simplify(CG[2,3]+CT[2,3]+Cx[2,3]));SUM24:=simplify(simplify(CG[2,4]+CT[2,4]+Cx[2,4]));
> SUM31:=simplify(simplify(CG[3,1]+CT[3,1]+Cx[3,1]));SUM32:=simplify(simplify(CG[3,2]+CT[3,2]+Cx[3,2]));SUM33:=simplify(simplify(CG[3,3]+CT[3,3]+Cx[3,3]));SUM34:=simplify(simplify(CG[3,4]+CT[3,4]+Cx[3,4]));
> SUM41:=simplify(simplify(CG[4,1]+CT[4,1]+Cx[4,1]));SUM42:=simplify(simplify(CG[4,2]+CT[4,2]+Cx[4,2]));SUM43:=simplify(simplify(CG[4,3]+CT[4,3]+Cx[4,3]));SUM44:=simplify(simplify(CG[4,4]+CT[4,4]+Cx[4,4]));
```

$$SUM11 := 0$$

$$SUM12 := 0$$

$$SUM13 := 0$$

$$SUM14 := 0$$

$$SUM21 := 0$$

$$SUM22 := 0$$

$$SUM23 := 0$$

$$SUM24 := 0$$

$$SUM31 := 0$$

$$SUM32 := 0$$

SUM33 := 0
 SUM34 := 0
 SUM41 := 0
 SUM42 := 0
 SUM43 := 0
 SUM44 := 0

 These 3-forms balance as they should

SOME SPECULATIONS

THE NEXT STEP IS TO FORMULATE THE EQUIVALENT OF THE EINSTEIN TENSOR in terms of DIFFERENTIAL FORMS

To this end note that the the vector of 1-forms [B][CR] and the vector of 1-forms [D][B] are not the same. [CR] is the Right Cartan matrix of the connection [B] and [Delta] is the Right Cartan matrix of [Binverse]

The sum of the two 1-forms is zero

$$[B][CR] + [DR][B] = 0.$$

The difference is not zero, but is equal to an exact differential

$$[Einstein] = [B][CR] - [DR][B] = 2[F > = 2] dA >.$$

Hence, the Einstein tensor, represented by the matrix of 1-forms consists of 2 - parts

The Right handed part is defined as [B][CR],

The Left handed part is defined as [DR][B]

>
>
>
>

> SUM:=simpform(simplify(evalm(B&^CR+DR&^B)));DIFF:=simpform(simplify(evalm(B&^CR-DR&^B)));
>
>

$$SUM := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

DIFF :=

$$\begin{aligned} & \left[\frac{1}{2} \frac{(2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r^2} - \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r^2} - \frac{\sin(\theta) \cos(\phi) (2r+M) M d(r)}{r^3}, \right. \\ & \frac{1}{2} \frac{(2r+M) \cos(\phi) (\cos(\theta)^2 M + 2r \cos(\theta)^2 - 2r - M) d(\theta)}{r \sin(\theta)} - \frac{1}{2} \frac{(2r+M)^2 \cos(\theta) \sin(\phi) d(\phi)}{r} \\ & \left. + \frac{\frac{1}{2} \cos(\phi) \cos(\theta) (2r+M) (2r-M) d(r)}{r^2}, \right] \end{aligned}$$

$$\begin{aligned}
& \left[\begin{aligned}
& -\frac{1}{2} \frac{(2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r} - \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r} - \frac{1}{2} \frac{\sin(\theta) \sin(\phi) (2r+M) (2r-M) d(r)}{r^2}, \\
& 0 \end{aligned} \right] \\
& \left[\begin{aligned}
& \frac{1}{2} \frac{(2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r^2} + \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r^2} - \frac{\sin(\theta) \sin(\phi) (2r+M) M d(r)}{r^3}, \\
& -\frac{1}{2} \frac{(2r+M)^2 \sin(\phi) \sin(\theta) d(\theta)}{r} + \frac{1}{2} \frac{(2r+M)^2 \cos(\theta) \cos(\phi) d(\phi)}{r} + \frac{1}{2} \frac{\cos(\theta) \sin(\phi) (2r+M) (2r-M) d(r)}{r^2}, \\
& \frac{1}{2} \frac{(2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r} - \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r} + \frac{1}{2} \frac{\sin(\theta) \cos(\phi) (2r+M) (2r-M) d(r)}{r^2}, 0 \end{aligned} \right] \\
& \left[\begin{aligned}
& -\frac{1}{2} \frac{(2r+M)^2 \sin(\theta) d(\theta)}{r^2} - \frac{\cos(\theta) (2r+M) M d(r)}{r^3}, \\
& -\frac{1}{2} \frac{(2r+M)^2 \cos(\theta) d(\theta)}{r} - \frac{1}{2} \frac{\sin(\theta) (2r+M) (2r-M) d(r)}{r^2}, 0, 0 \end{aligned} \right] \\
& \left[0, 0, 0, 8 \frac{M d(r)}{(2r+M)^2} \right]
\end{aligned}$$

The Einstein matrix of 1-forms is defined as the DIFF. The exterior differential of the Einstein matrix of 1-forms is always zero!

```
> Einstein:=simpform(simplify(evalm(B&^CR-DR&^B)));dEinstein:=simpform(simplify(evalm(d(Einstein))));
```

Einstein :=

$$\begin{aligned}
& \left[\begin{aligned}
& \frac{1}{2} \frac{(2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r^2} - \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r^2} - \frac{\sin(\theta) \cos(\phi) (2r+M) M d(r)}{r^3}, \\
& \frac{1}{2} \frac{(2r+M) \cos(\phi) (\cos(\theta)^2 M + 2r \cos(\theta)^2 - 2r - M) d(\theta)}{r \sin(\theta)} - \frac{1}{2} \frac{(2r+M)^2 \cos(\theta) \sin(\phi) d(\phi)}{r} \\
& + \frac{1}{2} \frac{\cos(\phi) \cos(\theta) (2r+M) (2r-M) d(r)}{r^2}, \\
& -\frac{1}{2} \frac{(2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r} - \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r} - \frac{1}{2} \frac{\sin(\theta) \sin(\phi) (2r+M) (2r-M) d(r)}{r^2}, \\
& 0 \end{aligned} \right]
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{1}{2} \frac{(2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r^2} + \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r^2} - \frac{\sin(\theta) \sin(\phi) (2r+M) M d(r)}{r^3}, \right. \\
& - \frac{1}{2} \frac{(2r+M)^2 \sin(\phi) \sin(\theta) d(\theta)}{r} + \frac{1}{2} \frac{(2r+M)^2 \cos(\theta) \cos(\phi) d(\phi)}{r} + \frac{1}{2} \frac{\cos(\theta) \sin(\phi) (2r+M) (2r-M) d(r)}{r^2}, \\
& \left. \frac{1}{2} \frac{(2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r} - \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r} + \frac{1}{2} \frac{\sin(\theta) \cos(\phi) (2r+M) (2r-M) d(r)}{r^2}, 0 \right] \\
& \left[- \frac{1}{2} \frac{(2r+M)^2 \sin(\theta) d(\theta)}{r^2} - \frac{\cos(\theta) (2r+M) M d(r)}{r^3}, \right. \\
& \left. - \frac{1}{2} \frac{(2r+M)^2 \cos(\theta) d(\theta)}{r} - \frac{1}{2} \frac{\sin(\theta) (2r+M) (2r-M) d(r)}{r^2}, 0, 0 \right] \\
& \left[0, 0, 0, 8 \frac{M d(r)}{(2r+M)^2} \right]
\end{aligned}$$

$$dEinstein := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

> ERM:=simpform(simplify(evalm(B&^CR)));ELM:=simpform(simplify(evalm(DR&^B)));

>

ERM:=

$$\begin{aligned}
& \left[\frac{1}{4} \frac{(2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r^2} - \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r^2} - \frac{1}{2} \frac{\sin(\theta) \cos(\phi) (2r+M) M d(r)}{r^3}, \right. \\
& - \frac{1}{4} \frac{(2r+M)^2 \cos(\phi) \sin(\theta) d(\theta)}{r} - \frac{1}{4} \frac{(2r+M)^2 \cos(\theta) \sin(\phi) d(\phi)}{r} + \frac{1}{4} \frac{\cos(\phi) \cos(\theta) (2r+M) (2r-M) d(r)}{r^2}, \\
& - \frac{1}{4} \frac{(2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r} - \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r} - \frac{1}{4} \frac{\sin(\theta) \sin(\phi) (2r+M) (2r-M) d(r)}{r^2}, \\
& \left. 0 \right] \\
& \left[\frac{1}{4} \frac{(2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r^2} + \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r^2} - \frac{1}{2} \frac{\sin(\theta) \sin(\phi) (2r+M) M d(r)}{r^3}, \right. \\
& - \frac{1}{4} \frac{(2r+M)^2 \sin(\phi) \sin(\theta) d(\theta)}{r} + \frac{1}{4} \frac{(2r+M)^2 \cos(\theta) \cos(\phi) d(\phi)}{r} + \frac{1}{4} \frac{\cos(\theta) \sin(\phi) (2r+M) (2r-M) d(r)}{r^2}, \\
& \left. 0, 0, 0 \right]
\end{aligned}$$

$$\left[\frac{1}{4} \frac{(2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r} - \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r} + \frac{1}{4} \frac{\sin(\theta) \cos(\phi) (2r+M) (2r-M) d(r)}{r^2}, 0 \right]$$

$$\left[-\frac{1}{4} \frac{(2r+M)^2 \sin(\theta) d(\theta)}{r^2} - \frac{1}{2} \frac{\cos(\theta) (2r+M) M d(r)}{r^3}, \right.$$

$$\left. -\frac{1}{4} \frac{(2r+M)^2 \cos(\theta) d(\theta)}{r} - \frac{1}{4} \frac{\sin(\theta) (2r+M) (2r-M) d(r)}{r^2}, 0, 0 \right]$$

$$\left[0, 0, 0, 4 \frac{M d(r)}{(2r+M)^2} \right]$$

ELM :=

$$\left[-\frac{1}{4} \frac{(2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r^2} + \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r^2} + \frac{1}{2} \frac{\sin(\theta) \cos(\phi) (2r+M) M d(r)}{r^3}, \right.$$

$$\left. -\frac{1}{4} \frac{(2r+M) \cos(\phi) (\cos(\theta)^2 M + 2r \cos(\theta)^2 - 2r - M) d(\theta)}{r \sin(\theta)} + \frac{1}{4} \frac{(2r+M)^2 \cos(\theta) \sin(\phi) d(\phi)}{r} \right.$$

$$\left. -\frac{1}{4} \frac{\cos(\phi) \cos(\theta) (2r+M) (2r-M) d(r)}{r^2}, \right.$$

$$\left. \frac{1}{4} \frac{(2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r} + \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r} + \frac{1}{4} \frac{\sin(\theta) \sin(\phi) (2r+M) (2r-M) d(r)}{r^2}, 0 \right]$$

$$\left[-\frac{1}{4} \frac{(2r+M)^2 \sin(\phi) \cos(\theta) d(\theta)}{r^2} - \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \cos(\phi) d(\phi)}{r^2} + \frac{1}{2} \frac{\sin(\theta) \sin(\phi) (2r+M) M d(r)}{r^3}, \right.$$

$$\left. \frac{1}{4} \frac{(2r+M)^2 \sin(\phi) \sin(\theta) d(\theta)}{r} - \frac{1}{4} \frac{(2r+M)^2 \cos(\theta) \cos(\phi) d(\phi)}{r} - \frac{1}{4} \frac{\cos(\theta) \sin(\phi) (2r+M) (2r-M) d(r)}{r^2}, \right.$$

$$\left. -\frac{1}{4} \frac{(2r+M)^2 \cos(\phi) \cos(\theta) d(\theta)}{r} + \frac{1}{4} \frac{(2r+M)^2 \sin(\theta) \sin(\phi) d(\phi)}{r} - \frac{1}{4} \frac{\sin(\theta) \cos(\phi) (2r+M) (2r-M) d(r)}{r^2}, \right.$$

$$\left. 0 \right]$$

$$\left[\frac{1}{4} \frac{(2r+M)^2 \sin(\theta) d(\theta)}{r^2} + \frac{1}{2} \frac{\cos(\theta) (2r+M) M d(r)}{r^3}, \right.$$

$$\left[\frac{1}{4} \frac{(2r+M)^2 \cos(\theta) d(\theta)}{r} + \frac{\frac{1}{4} \sin(\theta) (2r+M) (2r-M) d(r)}{r^2}, 0, 0 \right]$$

$$\left[0, 0, 0, -4 \frac{M d(r)}{(2r+M)^2} \right]$$

> `simplify(simplify(evalm(ERM+ELM)))`;

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

> `d(RightEinstein matrix of 1-forms) := simplify(simplify(evalm(d(ERM))))`;
`d(LeftEinstein matrix of 1-forms) := simplify(simplify(evalm(d(ELM))))`;

$$d(\text{RightEinstein matrix of 1-forms}) := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$d(\text{LeftEinstein matrix of 1-forms}) := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

>

>

>

> `E1:=simplify(simplify(RightEinstein&^AR))`: `Vector of Right Einstein 2-forms`
`:=evalm(E1)`;

Vector of Right Einstein 2-forms :=

$$\left[\frac{1}{2} \frac{\cos(\phi) \cos(\theta) (2r+M) M (d(\theta) \&^{\wedge} d(r))}{r^2} - \frac{1}{2} \frac{\sin(\theta) \sin(\phi) (2r+M) M (d(\phi) \&^{\wedge} d(r))}{r^2} \right]$$

$$\left[\frac{1}{2} \frac{\cos(\theta) \sin(\phi) (2r+M) M (d(\theta) \&^{\wedge} d(r))}{r^2} + \frac{1}{2} \frac{\sin(\theta) \cos(\phi) (2r+M) M (d(\phi) \&^{\wedge} d(r))}{r^2} \right]$$

$$\left[-\frac{1}{2} \frac{\sin(\theta) (2r+M) M (d(\theta) \&^{\wedge} d(r))}{r^2} \right]$$

$$\left[\frac{M (d(r) \&^{\wedge} d(\tau))}{4 (2r+M)^2} \right]$$

> `E2:=simplify(simplify(LeftEinstein&^AR))`: `Vector of Left Einstein 2-forms`
`:=evalm(E2)`;

Vector of Left Einstein 2-forms :=

$$\left[-\frac{1}{2} \frac{\cos(\phi) \cos(\theta) (2r+M) M (d(\theta) \&^{\wedge} d(r))}{r^2} + \frac{1}{2} \frac{\sin(\theta) \sin(\phi) (2r+M) M (d(\phi) \&^{\wedge} d(r))}{r^2} \right]$$

$$\left[-\frac{1}{2} \frac{\cos(\theta) \sin(\phi) (2r+M) M (d(\theta) \&^{\wedge} d(r))}{r^2} - \frac{1}{2} \frac{\sin(\theta) \cos(\phi) (2r+M) M (d(\phi) \&^{\wedge} d(r))}{r^2} \right]$$

$$\left[\frac{1}{2} \frac{\sin(\theta) (2r+M) M (d(\theta) \&^{\wedge} d(r))}{r^2} \right]$$

$$\left[-4 \frac{M (d(r) \&^{\wedge} d(\tau))}{(2r+M)^2} \right]$$

[Note that the vectors of chiral 2-forms, E1 and E2, are not the same as the vector of Affine Torsion 2-forms

> evalm(G);

$$\begin{bmatrix} 0 \\ \frac{(d(\theta) \wedge d(r)) M}{2 r (2 r + M)} \\ \frac{(d(\phi) \wedge d(r)) M}{2 r (2 r + M)} \\ \frac{M (d(r) \wedge d(\tau))}{4 r^2 - M^2} \end{bmatrix}$$

[For this example note that each vector of chiral 2-forms, E1 and E2 have zero divergence

> `dE1`:=d(E1); `dE2`:=d(E2);

>

>

$$dE1 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$dE2 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

[>