Non-Equilibrium Thermodynamics
and the Kolmogorov Topology

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Preface:

The ideas of continuous topological evolution and its application to topological Thermodynamics started about 1964, and were assembled into a set of four monographs (2004) after I retired in 1999 [1]. It is remarkable that in the period 1964-1969, (when I first became interested in the idea of a topological basis for physical systems) the first success was the prediction of the concept of Topological Spin Currents [68]. The idea was conceived using Cartan's methods of exterior differential forms. At that time, I recognized, intuitively, that the mathematics of exterior differential systems contained topological properties that went far beyond the geometric constraints of tensor analysis. The concept of a Spin Current was an exotic idea, and it was not appreciated how it could be used in a practical sense. Now, the concept of a Spin Current has entered the practical world of nanometer physics called Spintronics. It must be remembered however the idea is a topological concept; Topological Spin Current, does not depend upon scales or symmetries, and should have utilization in the macroscopic world.

Over the years, it became apparent that a thermodynamic system of synergetic parts could be encoded in terms of an exterior differential 1-form of Action, and that a physical process could be expressed in terms of vector, or macroscopic spinor, direction field, which represented the components of a exterior differential N-1 form density – a Current. The system and the process were dynamically connected by the Lie differential (with respect to a process) acting on the thermodynamic system(the Action 1-form of potentials per unit mole) to produce the non-exact 1-form of Heat. The result was a dynamical, cohomological, statement of the First Law of thermodynamics, which included topological fluctuations, and the ability to determine if a process acting on a thermodynamic system was irreversible or not.

The macroscopic "isotropic" Spinor direction fields [23] are those to be associated with the eigendirection fields of anti-symmetric matrices, or exterior differential 2-forms. Without macroscopic Spinors there would be no chaotic systems, or irreversible thermodynamic dissipation, or turbulence. Classic analysis has focused attention on symmetric matrices (metric, stress and strain) for which Spinors, as complex isotropic vectors (of zero quadratic form, or length), do not exist. Macroscopic Spinor direction fields are studied in much more detail in the fifth (2007) monograph [1]. One of the extrordinary results was the proof that C1 linear processes that approximated C2 smooth processes could be reversible, where the approximated C2 process were irreversible. Also it should be noted that not until early 2006 was it appreciated fully that the projective geometry concepts of correlations are related to the Jacobian matrix of the coefficients, A_k, of that 1-form of Action (per unit source) that encodes a thermodynamic system. On the other hand, projective concepts of collineations are related to the Jacobian matrix of the coefficients of a direction field, V^k, that encodes a thermodynamic process, and an (N-1)-form current. It became become apparent that the concept of Affine Torsion could be associated with the concept of field excitation 2-forms, $|G\rangle$. Such concepts of field excitation 2-forms, $|G\rangle$ have been long been part of EM theory and act as the source of charge-current density 3-forms, $J = dG$. Such charge current density 3-forms do not appear in classical hydrodynamics, where stress-strain relations have been constrained to those expositions in terms of symmetric matrices. When anti-symmetries are allowed, then Fluids also permit excitation 2-forms and current deinsity 3-forms can be constructed in terms of the Adjoint matrix of the Jacobian correlation matrix constructed in terms of the coeffiients of the 1-form, A, that defines the thermodynamic system. With this concept, the theories of electromagnetism and hydrodynamics become topologically equivalent. In addition, the concept of Topological Torsion (Currents) and Topological Spin (Currents) can be formated, and have found practical application in transport of collective spin states.

This essay consists of several chapters, and incorporates the idea that exterior differential forms are best defined on a "differential variety", $\{x, y, z, t, \; dx, dy, dz, dt\}$, rather than on the tensor domain of "coordinates" $\{x, y, z, t\}$. The differential varieties are elements of an equivalence class of C1 differential maps ϕ, and $d\phi$ between the differential varieties. These maps need not be diffeomorphisms, for the inverse of the neither the function nor is differential need exist. It can be demonstrated that any exterior differential 1-form, $A\{\xi^k, d\xi^k\}$, defined on a differential variety $\{\xi^k, d\xi^k\}$ can be used to generate a topological structure; this topological structure can be used to determine if processes are continuous or not. Without further constraints, the most primitive of the topologies that can be generated from a 1-form of Action will be the Kolmogorov-Cartan $\mathrm{T}_\mathbf{0}$ Spaces of Exterior Differential Forms.

It will be assumed that the differential variety of "measurement" is the ubiquitous 4D differential variety of three spatial and one temporal differentiable functions $\{x, y, z, t; \; dx, dy, dz, dt\}$ which are pre-geometric in the sense that the functions are considered *not* to be constrained by scales, metric, or shape. The differential variety $\{\xi^k, d\xi^k\}$ may be of Dimension N ≥ 4, and if so it will be presumed that there exist C1 differentiable maps that connect the $\{x^k, dx^k\}$ variety to the $\{\xi^k, d\xi^k\}$ variety. These mapping functions are not invertible if the dimension N is greater than 4. However, the magic of using differential forms is that any p-form on the differential variety $\{\xi^k, d\xi^k\}$ is well-defined on the variety $\{x^k, dx^k\}$ by means of functional substitution (called the Pullback, in the parlance of differential forms). The method transcends the diffeomorphic equivalences used in tensor analysis, which do not admit topological change. The utility of the higher dimensions is that they sometimes they permit easier computations. For example any curved Riemannian variety can be mapped into a variety of higher dimension which is flat.

For purposes of simplicity attention will be focused on exterior differential forms defined on the 4D differential variety of physical measurement. A 1-form of Action, $A\{x^k, dx^k\}$ and its Pfaff Sequence of differential forms, $\{A, F = dA, H = A\hat{}F, K = F\hat{}F\}$ can be used to generate a basis for a topological structure. The exterior differential, d, acting on a p-form, Σ, of the topological structure, is a generator of the Limit Sets of the p-form, and is expressed in terms of the elements of a p+1-form, $d\Sigma$. Continuous Topological Evolution can be described in terms of the Lie differential with respect to a process Vector (or Macroscopic Spinor) direction field, V, acting on the 1-form, A, which has been chosen to encode the thermodynamic system. The method develops a cohomological, universal, dynamical equivalent of the First Law of Thermodynamics:

$$L_{(V)}A = Q = i(V)dA + d(i(V)A) = W + dU. \tag{1}$$

Recall that all of the variables, including the Heat and Work 1-forms and their coefficients, are well defined functions on the differential variety. The process is a field, $\mathbf{V_4}(x, y, z, t)$, the Action is a 1-form, $A\{x^k, dx^k\}$, the incremental Heat is a 1-form $Q\{x^k, dx^k\}$, the incremental Work is a 1-form, $W\{x^k, dx^k\}$, the internal energy, U, is function of $\{x^k\}$. Every thing is well defined on the 4D differential variety of measurement, $\{x^k, dx^k\}$, without the geometric constraints of metric, connection, and gauge.

As such, the Kolmogorov-Cartan Topology gives credence to the idea that thermodynamics is one of the most fundamental physical laws, independent from the geometric constraints of metric, connection, scales and symmetry. It will be shown that C2 Macroscopic Spinor processes are the source of kinematic fluctuations, $d\xi^k - V^k dt \neq 0$, and thermodynamic irreversibility implies that $Q\hat{}dQ \neq 0$.

Chapter 1
THERMODYNAMICS FROM A TOPOLOGICAL PERSPECTIVE

1.1 The Hour of Mystery

Now I am well aware of the fact that Thermodynamics (much less Topological Thermodynamics) is a topic often treated with apprehension. In addition, I must confess, that as undergraduates at MIT (1949) we used to call the required physics course in Thermodynamics, "The Hour of Mystery!"

Let me present a few quotations (taken from Uffink, [108]) that describe the apprehensive views of several very famous scientists:

> A*ny mathematician knows it is impossible to understand an elementary course in thermodynamics* .... V. Arnold 1990.
>
> *It is always emphasized that thermodynamics is concerned with reversible processes and equilibrium states, and that it can have nothing to do with irreversible processes or systems out of equilibrium* ......Bridgman 1941
>
> *No one knows what entropy really is, so in a debate (if you use the term entropy) you will always have an advantage* ...... Von Neumann (1971)

On the other hand Uffink states:

> *Einstein, ..., remained convinced throughout his life that thermodynamics is the only universal physical theory that will never be overthrown.*

The original classical development of thermodynamics was phenomenological, but it became motivated - and then dominated -by the concept of microscopic "molecules" after the start of the 20th century. However, as Sommerfeld has written (without explicit reference to topology, but inferring that "microscopic molecules" are not of thermodynamic importance):

> *"The atomistic, microscopic point of view is alien to thermodynamics. Consequently, as suggested by Ostwald, it is better to use moles rather than molecules."* Arnold Sommerfeld p. 11 [6].

1.1.1 Intensities and Excitations

The classical theory of Thermodynamics is often presented as a number of phenomenological "Laws" to be written in stone and taken on faith. Indeed, contrived experiments are conducted to demonstrate a measure of credence in the "Laws", but the universality of the "Laws" is always left a bit clouded and mirky. Part of the thermodynamic mystery is due to the fact that the thermodynamic variables are of two types,

1. "Intensities" such as Pressure, Temperature ....,
2. and additive quantities, or "Excitations"such as Entropy, Internal Energy....

Remark 1 *Following Arnold Sommerfeld, think of the first category in terms of fields of "Excitation" and the second category in terms of fields of "Intensities". The first category is related to "Sources" leading to additive, extensive particle properties which are homogeneous of degree 1. The second category is related to potentials and "Forces" leading to field wave properties, and intensities, which are not additive, and are homogeneous of degree 0.*

For systems constrained by diffeomorphic (tensor analysis) equivalences, the intensities behave like the components of a covariant vector, while the processes behave like the components of a contravariant vector. However, in terms of a topological perspective, such is not the case. The Jacobian matrix of the Intensities form a correlation matrix in the sense of projective geometry. The Jacobian matrix of the Processes form a collineation matrix in the sense of projective geometry. The correlation matrix need not be a "polarity", which would require that the matrix is symmetric. In fact, it is the anti-symmetric parts of the correlation matrix that are of most importance to the topological theory of thermodynamics.

A major part of the "Mystery" of thermodynamics can be related to the fact that:

> ..."there are thermodynamic variables which are uniquely specified by the equilibrium state (independent from the past history of the system) and which are not conclusions deduced logically from some philosophical first principles. They are conclusions ineluctably drawn from more that two centuries of experiments" ... P. M. Morse p.8 in [58].

In addition, the "thermodynamic coordinates" are not well defined as functions on the usual 4D space time differential variety of measurement, $\{x, y, z, t, dx, dy, dz, dt\}$. In fact, most of classical thermodynamic theories define equilibrium when the Intensities are constants with zero differentials across a finite domain.

This monograph rectifies this geometric problem, by formulatiing the first principles in terms of topological ideas.

1.1.2 The "Laws"

Permit me to inscribe the Stone tablet with the following tongue-in-cheek "Laws" of thermodynamics:

1. Thou shall not destroy Energy.
2. Entropy must always flow uphill from lower entropy to higher entropy.
3. Heat must always flow downhill from high-temperature to low-temperature.
4. Thou shall not destroy Entropy.
5. Thou must not admit Negative Pressure.
6. Thou must not admit Negative Temperature.
7. The Laws of Nature must predict unique final data from unique initial data.
8. The Laws of Nature must be based upon geometry and symmetries.
9. The Laws of Nature must be explained in terms of microscopic quantum mechanics.
10. The Laws of Nature must have a probabilistic, statistical, foundation and interpretation.

My objective is to present a universal theory of Thermodynamics based upon Continuous Topological Evolution and Topological sets of Exterior Differential forms. As a matter of faith, it will be presumed that measurement processes are defined by functions that must be evaluated, ultimately, in terms of a 4 dimensional space-time differential variety, $\{x, y, z, t\ ; dx, dy, dz, dt\} = \{x^k, dx^k\}$. This (pre-geometric) differential variety will not be constrained by metric or scales.

From this universal foundation, the meaning of the "Laws" are to be rationally deduced. Many of the mysteries of thermodynamics, especially those currently found in non-equilibrium thermodynamics, will be removed and be made transparent. Indeed, new practical applications can be devised as the result of a logical dynamical description of emergence. The evolutionary emergence of topological defects and other singularities is produced by processes acting on thermodynamic systems in the universal non-equilibrium topological environment. The topological environment is neither a vacuum, nor an empty set, but should be considered as a non-equilibrium thermodynamic system of topological dimension 4.

1.1.3 The Arrow of Time

Topology is the study of the number of disconnected parts, the number of connected parts, and the number of obstructions, topological defects, or "holes" in the connected parts. Topological evolution focuses on processes where these numbers change. Geometric evolution focuses on processes when these numbers stay the same. Processes of topological change can be identified with *topologically continuous* processes of "pasting" together disconnected parts, or with the deformation and pasting together certain parts of a boundary of a connected part to form a hole, or with the collapse and pasting together of a cyclic portion of boundary to destroy a hole All such continuous processes can create topological defects ot "holes". Disconnected parts can continuously evolve into connected parts thereby causing topological changes (condensation of a vapor), but connected parts cannot evolve continuously into disconnected parts (evaporation of a liquid). Both processes involve topological change; but they can not be geometrical processes for which the topology is an continuous evolutionary invariant. Topological change is a necessary, but not sufficient, property of an irreversible process.

Paraphrasing Eddington:

Remark 2 *Aging and the arrow of time have slipped through the net of geometric analysis.*

Yet it is the interplay of topological evolution and continuity (of disconnected sets into connected sets which can be continuous) that establishes the arrow of time. The reciprocal process of topological evolution (of connected sets into disconnected sets) can NOT be continuous. As a geometrical process preserves topology, a geometrical analysis cannot describe an irreversible process, which requires topological change.

1.1.4 Extensive and Intensive Functions from a Topological Perspective

Topological thermodynamics has two distinct categories of exterior differential forms. The first category describes a *process* in terms of a vector (or spinor) direction field, $\mathbf{V}^k$, of 4 ordered functions that form the components of a differential N-1=3 form density, C, or current. The 3-form currents, C, are the topological analogues of the thermodynamic extensive properties. The second category describes a thermodynamic *system* in terms of an ordered array of of 4 functions, $\mathbf{A}_k$, that form the components of an exterior differential 1-form, A. The 1-forms are the topological analogues of the intensive variables in classical thermodynamics.

Ultimately, the exterior differential forms are presumed to be defined in open vector space domains of Pffaf topological dimension 4; these Open domains are defined as the thermodynamic physical environment. The "particulate" matter is defined in terms of the topological closed defect structures of Pfaff topological dimension 3 that emerge from, and interact with, the thermodynamic physical environment. The process current that induces the emergence is dissipative and irreversible.

For example, the number of components is a topological concept that can change continuously by "pasting" together (or condensing) various components. The number of components can change discretely by the "pasting" process, but the process is topologically continuous in a formal sense.

Theorem 3 *A map is topologically continuous iff the limit points of every subset in the domain permute into the closure*

It will be demonstrated in Chapter 3 that the closure of a diffential form, Σ, (in the KCT_0 topology) is equivalent to the the differential ideal, $\{\Sigma \cup d\Sigma\}$, and limit points are equivalent to $d\Sigma$. An extended discussion of topological continuity is to be found in Chapter 4where it demonstrated that there is a common topological thread that links the sciences of Thermodynamics, Hydrodynamics, and Electrodynamics [1]. Both Hydrodynamics and Electrodynamics (as well as almost any other of the physical specializations) have a topologicalfoundation in terms of Thermodynamics. As will be shown below, topological thermodynamics can be built upon:

i: a 1-form of Action, $A(x, y, z, t)$, that encodes a specific Thermodynamic System, and

ii: a set of vector-spinor direction fields, $V(x, y, z, t)$, that define the Dynamic Processes acting on the specific Thermodynamic System.

iii: A Kolmogorov-Cartan T_0 topology with subsets in terms of exterior differential forms.

The methods lead to precise, non-statistical, methods for determining when a process, $V(x, y, z, t)$, applied to a specific thermodynamic system, $A(x, y, z, t)$, is

1. Thermodynamically irreversible or not,
2. Adiabatic or not.
3. Adiabatically irreversible or reversible.

1.2 Historical Developments

1.2.1 Cartan's Exterior Differential forms on Differentiable Varieties

A topological perspective can be used to extract those properties of physical systems and their evolution that are independent from the geometrical constraints of connections and/or metrics. It is subsumed that the presence of a physical system establishes a *topological structure** on a differentiable variety of independent variables. This concept is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a differentiable variety of independent variables, and the dynamics is established in terms of a connection. These are assumptions of constraint on the differentiable variety, and are avoided when a topological perspective, not a geometric perspective, is assumed. Note that a given differentiable variety may support many different topological structures simultaneously; hence a given differentiable variety may support many different co-existent physical systems. A major success of theory is that continuous non-homeomorphic processes of topological evolution establish a logical basis for thermodynamic irreversibility and the arrow of time [91] without the use of statistics.

In the period from 1899 to 1926, Eli Cartan developed his theory of exterior differential systems [19], [20], which included the ideas of spinor systems [23] and the differential geometry of projective spaces and spaces with torsion [21]. The theory was appreciated by only a few contemporary researchers, and made little impact on the main stream of the physical sciences until about the 1960's. Even specialists in differential geometry (with a few notable exceptions [25]) made little use of Cartan's methods until the 1950's. Even today, many physical scientists and engineers have the impression that Cartan's theory of exterior differential forms is just another formalism of fancy. That conclusion is wrong. The Cartan methods transcend the geometrical constraints in current vogue.

Cartan's theory of exterior differential systems has several advantages over the methods of tensor analysis that were developed during the same period of time. The principle fact is that differential forms are well behaved with respect to functional substitution of C1 differentiable maps. Such maps need not be invertible even locally, yet differential forms are always deterministic in a retrodictive sense [75], by means of functional substitution. Such determinism is not to be associated with contravariant tensor fields, if the map is not a diffeomorphism. Cartan's theory of exterior differential systems contains topological information, and admits non-diffeomorphic maps which can describe topological evolution.

Although the word "topology" had not become popular when Cartan developed his ideas (topological ideas were described as part of the theory of "analysis situs"), there is no doubt that Cartan's intuition was directed towards a topological development. For example, Cartan did not define what were the open sets of his topology, nor did he use, in his early works, the words "limit points or accumulation

*The concept of the Cartan Topological Structure was developed 1984-1991. The main ideas were presented as a talk given in August, 1991, at the Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara, UCSB. Part of the Cartan topological truth table was due to Phil Baldwin. The recognition that the Cartan topology was a disconnected topology is due to the author. In 2009 it was determined that the Topology of Thermodynamics was a Kolmogorov T_0 topology.

points" explicitly, but he did describe the union of a differential form and its exterior differential as the "closure" of the form. With this concept, Cartan effectively used the idea that the closure of a subset is the union of the subset with its topological limit points. What was never stated (until 1990) is the idea that the exterior differential is indeed a limit point generator relative to a Cartan topology. The union of the identity operator and the exterior differential satisfy the axioms of a Kuratowski closure operator [52], which can be used to define a topology. The other operator of the Cartan calculus, the exterior product, also has topological connotations when it is interpreted as an intersection operator.

In a perhaps over simplistic comparison, it might be said that ubiquitous tensor methods are restricted to geometric applications, while Cartan's methods can be applied directly to both topological concepts and geometrical concepts. Cartan's theory of exterior differential systems is a topological theory not necessarily limited by geometrical constraints suvh as the class of diffeomorphic transformations that serve as the foundations of tensor calculus. It is possible to show how limit points, intersections, closed sets, continuity, connectedness and other elementary concepts of modern topology are inherent in Cartan's theory of exterior differential systems. These topological ideas do not depend upon the geometrical ideas of size and shape. Hence Cartan's theory, as are all topological theories, is renormalizable (perhaps a better choice of words is that the topological components of the theory are independent from scale). The idea of nearby of far away is to be replaced by the idea that a point b is reachable from a point a (connected set) or not reachable (a and b in different disconnected sets). In fact the most useful of Cartan's ideas do not depend explicitly upon the geometric ideas of a metric, distance, nor upon the choice of a differential connection between basis frames, as in fiber bundle theories. The topological theory of thermodynamics explores the physical usefulness of those topological features of Cartan's methods which are independent from the constraints and refinements imposed by a connection and/or a metric.

Continuous Topological Evolution (developed in the period 1974-1984) is encoded in terms of the Lie differential with respect to the process direction field, V, acting on the 1-form of Action, A, that encodes the thermodynamic system. For C2 functions it can be demonstrated that this formulation not only represents continuous topological evolution,

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q \tag{1.1}$$

$$Q_{heat} = W + dU, \tag{1.2}$$

$$W_{work} = i(V)dA, \quad U_{internal\_energy} = i(V)A, \tag{1.3}$$

but also represents the abstract, Cohomological, dynamical, equivalent to the First Law of Thermodynamics (1984-2009). The dynamics are topological dynamics, not geometrical dynamics. The dynamics of the Thermodynamic system can be refined by the Topological structures associated with the 1-forms of Q_{heat} and the 1-forms of W_{work}†.

In 1969, it was recognized that Van Dantzig's concept [109] of a topological basis for electromagnetism required the additional imposition of a tensor density 2-form, G, to accompany the thermodynamic 1-form, A, and the process, V. The 2-form density, G, was necessary to account for discrete Charge and the Amperian charge current density found in experimental electrodynamic-thermodynamic systems. This suggested that the 3-form density, $A\hat{}G$, of a thermodynamic - electromagnetic theory would have useful physical application [68]. In fact, if the 3-form density was closed over certain subsets of the variety, then in that region, the closed integral of $A\hat{}G$ obeyed a conservation law (a transport theorem) equivalent to the First Poincare Invariant of electromagnetism. Based upon dimensional analysis, the 3-form density $A\hat{}G$ had the (suggestive) physical dimensions of Planck's constant (Spin angular momentum). The 2-form density, G, if without limit points, and integrated over closed 2-chains (which are not boundaries), represents discrete Charge (via deRham's theorems [67]). The 3-form density, $A\hat{}G$, if without limit points and integrated over closed 3-chains, represents discrete Spins. All of this follows from deRham cohomology theory. In other words, the discrete features of quantum mechanics are contained in the topological approach.

It was only several years later that I appreciated [74] that the thermodynamic system also supported a 3-form density, $A\hat{}F$, which I called Topological Torsion (not equivalent to affine torsion in geometric systems). Topological Torsion $= 0$ defines integrable equilibrium thermodynamic systems and Topological Torsion $\neq 0$ defines non-equilibrium non-integrable systems. Equilibrium systems permit unique integrability; non-equilibrium systems lead to non-unique solutions, such as envelopes and edges of regression. Remarkably, Topological Torsion is defined entirely by the 1-form, A, that encodes the thermodynamic system, and its limit points, $dA = F$:

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA. \tag{1.4}$$

Topological Torsion does not depend upon any particular external process, V; it is intrinsic to the thermodynamic system.

It is extraordinary, but the struggle to understand fully the extraordinary properties of Cartan topological structure only came about recently (Feb-2009). After some 45 years of study, I became aware that the Cartan topology of a 1-form had the topological structure of a Kolmogorov T_0 topology, where the topological subsets are exterior differential forms. I now call this topology, the Kolmogorov-Cartan Topology, KCT. An important feature of the KCT toplogy is that its subsets of exterior differential forms support the Kuratowski closure axioms, expressed in terms of the Identity operator and the exterior differential, d:

†The increment of Heat is represented by the symbol Q , a differential 1-form, that need not be closed, much less exact. The older literature has used the symbol $đQ$. Similar remarks apply to the increment of Work, W.

Theorem 4 *The closure of an exterior differential p-form, Σ, is the union of the p-form and its exterior differential,*

$$\text{Relative to the KCT topology,} \qquad Kcl(\Sigma) = \Sigma \cup d\Sigma \tag{1.5}$$

The exterior differential, d, in effect is a Limit Point generator. The 2-form $F = dA$ represents the limit points of the 1-form of Action, A. It will be demonstrated that in terms of EM notation, the field intensities, $F(\mathbf{E}, \mathbf{B})$ are to be interpreted as the "Limit Points" of the vector and scalar potentials. A similar statement can be made for hydrodynamimc systems. A first result is both Electromagnetic and Hydrodynamic systems obey a Faraday induction law.

My mantra over the years had been to go beyond (that is, avoid, or ignore) techniques, or constraints, that utilized statistics and probability theory, metric, tensor analysis with affine connections, diffeomorphic and group (symmetry) gauge constraints. Although each of these various constraint disicplines have interesting features, I was convinced early on that such constraints impeded the development of physical understanding of thermodynamic irreversibility and biological non-equilibrium evolution to the ultimate state of equilibrium, and death.

Now the 45 years of effort seems to have born fruit, for the Kolmogorov-Cartan T_0 topological structure means that the topology is NOT a metric topology, NOT a Hausdorff topology, does NOT satisfy the separation axioms to be a T_1 topology, is not necessarily constrained by symmetries, and is NOT the space of any topological group! I find it more remarkable and pleasing that such a topology apparently can find broad application to the science of thermodynamics, and therefor to other allied physical systems, To repeat, this T_0 topology of thermodynamics is far afield from the metric based, diffeomorphically constrained, gauge theoretic, group theoretic symmetries found in many of the current physical theories, which, due to the constraints, inherent time reversibility, in disagreement with experiment. Topological change is a neccesary condition for thermodynamic irreversibility.

The "course" topological features of Thermodynamics, whose sets are based upon Cartan's theory of exterior differential forms, originally [68], [76], [1] were motivated by the recognition that the Pfaff Sequence for any 1-form A,

$$\text{Pfaff Sequence:} \quad \{A, dA, A\hat{}dA, dA\hat{}dA\}, \tag{1.6}$$

contains certain topological information, now described as the Pfaff Topological Dimension (or class, see p. 290- [36]) of any 1-form, A. It soon became obvious that thermodynamic irreversibility was associated with topological change, for if the topology was an evolutionary invariant, it could be described by a homeomorphism, which has an inverse. That is, homeomorphic evolution is reversible.

Then it was noted that topological evolution of discrete collections could occur *continuously* if the limit points of any subset relative to the topology of the initial state was to be found within the closure of the subset relative to final state

[52], [80],. For example, the number of components is a topological concept that can change continuously by "pasting" together (or condensing) various components. The number of components can change discretely by the "pasting" process, but the process is topologically continuous in a formal sense.

Theorem 5 *A map is topologically continuous iff the limit points of every subset in the domain permute into the closure of the subsets in the range.*

1.2.2 The Kolmogorov-Cartan T_0 topology

For any given 1-form, A, it is possible to construct the Pfaff Sequence $\{A, dA, A\hat{}dA, dA\}$. The Pfaff Topological Dimension (PTD) is defined as the minimum number of terms in the Pfaff Sequence. The non-zero elements in the Pfaff Sequence can be used to define a topological basis of exterior differential forms. The details of such a construction are found in chapter 2. The outcome is that the topology of Cartan's exterior differential forms is a Kolmogorov T_0 topology.

The Kolmogorov-Cartan T_0 (KCT_0) topology can be constructed explicitly for an arbitrary exterior differential 1-form. All elements of the KCT_0 topology can be evaluated quickly. The limit points, the boundary sets and the closure of every subset can be computed abstractly. These constructions will be explained in chapter 2. Earlier intuitive results, which utilized the notion that Cartan's concept of the exterior product may be used as an intersection operator, and his concept of the exterior differential may be used as a limit point operator acting on differential forms, can be given formal substance. A major result is the fact that the Kolmogorov-Cartan T_0 topology is a disconnected topology for non-equilibrium systems (PTD=4,PTD=3) and is a connected topology for equilibrium systgems (PTD=2,PTD=1). A key artifact of non-equilibrium is the existence of Topological Torsion current 3-forms, Topological Spin current 3-forms, and Topological Adjoint interaction current 3-forms, all similar to the charge-current 3-form densities of electromagnetic theory, but related to different species of dissipative phenomena, which only occur in non-equilibrium systems.

1.2.3 Continuous Topological Evolution

The stated objective of this monograph is to formulate a theoretical basis for non-equilibrium thermodynamics and irreversible processes, based upon topological arguments, without the constraints of metric, connections, statistics, or gauge groups. The effort is a work in progress, with a number of significant results that demonstrate that a topological perspective is worthy of further research in the engineering sciences. As I am an applied engineering physicist, this article uses only the those simplest of topological methods, especially those that appear to have practical value and can be used to explain away some of the mysteries of non-equilibrium thermodynamics. Previous topological approaches to thermodynamics [13], [12] missed the point that the fundamental topological structure of thermodynamics is based upon the disconnected Kolmogorov T_0 topology. When written in terms of Cartan's theory of exterior differential forms, the First Law of thermodynamics becomes a dynamical statement of Cohomology, $Q = W + dU$: the difference between two non-exact 1-forms $(Q - W)$ is equal to an exact differential dU.

$$\text{The First Law} \quad W + dU = Q \tag{1.7}$$

The fundamental idea is that a topological analysis of a thermodynamic system can be based upon the Kolmogorov-Cartan T_0 topological structure. This KCT_0 topology will have subsets defined in terms of exterior differential forms. The KCT_0 topological structure can be composed from a (any) 1-form of Action,

$$A(x^k, dx^k) = A_x(x^m)dx^k, \tag{1.8}$$

defined herein on a differential variety, of 4 components, say $\{x, y, z, t\}$. The topological structure can be used to determine if an evolutionary mapping is continuous or not. Features of the KCT_0 topology will be detailed below.

The Pfaff Topological dimension

The coefficients of the 1-form, A, will be assumed to be C2 differentiable, which permits the the construction of the Pfaff sequence, using only 1 differential process and numerous exterior algebraic products:

$$\textbf{Pfaff Sequence} \quad \{A, dA, A\hat{}dA, dA\hat{}dA\} \tag{1.9}$$

. The Pfaff Topological Dimension of a specific 1-form, A, is equal to the (easily computed) number of non-zero elements in the Pfaff Sequence. Open, Closed, Isolated and Equilibrium thermodynamic systems will be associated with 1-forms of PTD=4, PTD=3, PTD=2, and PTD=1. The KCT_0 topology is a *disconnected* topology if the PTD > 2.

It is also possible to define C2 differentiable arrays (vector or spinor) of ordered functions, $J = \rho V_4(x^k)$, that encode thermodynamic processes acting on the various thermodynamic systems. The KCT_0 topological structure permits the definition of (topologically) continuous processes to be evaluated in terms of the Lie exterior differential "propagator" acting on an exterior differential p-form, Σ,

$$\text{"Cartan's Magic Formula"}^{\ddagger} \quad L_{(\rho \mathbf{V}_4)}\Sigma = i(\rho \mathbf{V}_4)d\Sigma + d(i(\rho \mathbf{V}_4)\Sigma). \tag{1.10}$$

Using the Lie differential, the continuous topological evolution of a p-form, Σ, will yield a p-form, Ξ.

$$L_{(\rho \mathbf{V}_4)}\Sigma = i(\rho \mathbf{V}_4)d\Sigma + d(i(\rho \mathbf{V}_4)\Sigma) = \Xi. \tag{1.11}$$

When applied to a 1-form of Action that defines a thermodynamic system,

$$L_{(\rho \mathbf{V}_4)}A = i(\rho \mathbf{V}_4)dA + d(i(\rho \mathbf{V}_4)A) = Q. \tag{1.12}$$

Using the notation, Work, $W = i(\rho\mathbf{V}_4)dA$ and Internal energy, $U = i(\rho\mathbf{V}_4)A$, it becomes apparent that continuous topological evolution is an abstract dynamical (cohomological) equivalent of the First Law of Thermodynamics. The 1-form of Action, A, "evolves" into the 1-form of Heat, Q, due to the process, $\rho\mathbf{V}_4$:

$$\text{The First Law}: \ L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A) = W + dU = Q. \tag{1.13}$$

This topological perspective separates the concept of a thermodynamic system (defined in terms of a 1-form of Action, A) and the concept of a 4D process (defined in terms of an N-1 form current, J with $\mathbf{J}_4 = \rho\mathbf{V}_4$) acting on the thermodynamic system. For particular processes, $\rho\mathbf{V}_4$, the 1-form of Heat, Q, need not have the same Pfaff Topological Dimension as the 1-form of Action, A. Therefor it is apparent that Cartan's magic formula can encode topological change[§]. Perhaps surprisingly, and not intuitively, such topological changes can appear to be discrete geometrically, but are topologically continuous.

[§]Note that the covariant differential based upon the constraint of diffeomorphic processes cannot describe topological change, as diffeomorphisms preserve topology.

Chapter 2
TOPOLOGICAL PROPERTIES OF DIFFERENTIAL FORMS

2.1 Closure,Cohomology and Homology

The topological structure of interest is the Kolmogorov-Cartan topology constructed in terms of sets of differential forms. As shown below, in this topology the exterior differential acting on an exterior differential p-form Σ generates the limit sets $d\Sigma$ (of exterior differential p+1forms) for the form, Σ. The (Kuratowski) closure of the p-form is defined as the Union of Σ and $d\Sigma$:

$$Kcl(\Sigma) = \Sigma \cup d\Sigma. \tag{2.1}$$

Cartan never defined his topological structure explicitly, but he did refer to the closure of a p-form, Σ as the union of Σ and its exterior differential, $d\Sigma$. It is apparent that the use of exterior differential forms as the sets of a topological space emphasizes the concept of Cohomology, rather than concept of Homology. For an arbitrary topological space:

$$\begin{aligned}
\textbf{Cohomology} \quad &: \quad \text{Closure of Set } \; Cl(\Sigma) = \Sigma \cup d\Sigma, & (2.2)\\
& \quad \{\text{Set}\} \cup \{\text{Limit points of the Set}\} & (2.3)\\
\textbf{Homology} \quad &: \quad \text{Closure of Set } \; Cl(\Sigma) = \Sigma^0 \cup \partial\Sigma, & (2.4)\\
&= \{\text{interior of the Set}\} \cup \{\text{boundary of the Set}\} & (2.5)
\end{aligned}$$

The use of exterior differential forms as the topological sets of the KCT topology, demonstrates that the exterior differential is indeed a limit point generator.

2.2 Ordered Arrays and Differential Form Densities

The exterior differential forms are presumed to be defined on a variety of 4 independent functions and their differentials, say $\{x, y, z, t; \quad dx, dy, dz, dt\}$. It is presumed that there exists a maximal system with a differential volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$, upon which can be described a differential p-form density, $\rho_4(x^k)\Omega_4 = \rho_4(x^k)dx\hat{}dy\hat{}dz\hat{}dt$. The p-form density can be sensitive to a permutation of the ordering (~orientation). The ordering is important, for $dx\hat{}dy\hat{}dz\hat{}dt = -(dy\hat{}dx\hat{}dz\hat{}dt$. Depending upon special properties of the density coefficient, $\rho_4(x^k)$, the differential p-form densities may, or may not, be sensitive to ordering (orientation). If the p-form density is sensitive to orientation it is described as an "impair-p-form density"; If the p-form is not sensitive to orientation it is described as an "pair-p-form density". For example, the 2-form density G associated with discrete charge in electromagnetic theory is impair, and exhibits both plus values and minus values of charge. The 2-form density associated with mole number (baryons) in fluid dynamics is pair, as "mass" is positive definite.

2.3 The Pfaff Sequence

As mentioned above, for any 1-form, A, on a differentiable variety, it is possible to construct a Pfaff Sequence:

$$\textbf{Pfaff sequence} \quad : \quad \{A, dA, A\hat{}dA, dA\hat{}dA\}, \tag{2.6}$$

$$= \quad \{A, F = dA, \mathrm{H} = A\hat{}F, \mathrm{K} = F\hat{}F\}. \tag{2.7}$$

Surprisingly, for a 1-form written in terms of the 4D base variables,

$$A = \sum_{k=1-4} A_k(x, y, z, t) dx^k, \tag{2.8}$$

the functional format of the coefficients, $A_k(x, y, z, t)$, will determine how many non-zero entries appear in the Pfaff sequence. This fact will be used to define the Pfaff Topological Dimension..

What became apparent to me (and Phil Baldwin, a post doc at the University of Houston) about 1990 was that it was possible to construct a topological structure in terms of the properties of the exterior differential form elements in the Pfaff Sequence (see Chapter 6, vol1 [1]). The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology was constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures,

$$\text{the Cartan topological basis} : \{A, Cl(A), A\hat{}dA, Cl(A\hat{}dA)\}. \tag{2.9}$$

Cartan referred to the union of Σ and $d\Sigma$ as the "closure" of Σ, which is agreement with the Kuratowski closure axioms. As mentioned above, the exterior differential can be considered to be a limit point generator:

$$\text{Closure} = Cl(\Sigma) = (I \cup d) \circ \Sigma = \Sigma + d\Sigma = \text{subset} + \text{limit points}. \tag{2.10}$$

A most important feature of the Cartan topological structure (detailed below), is that it turned out to represent a "Disconnected" topology – A surprise that startled me*. This fact allowed the concept of irreversibility to be well defined with respect

*The concept of the Cartan Topological Structure was developed in the period 1984-1991. The main ideas were presente as a talk given in August, 1991, at the Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB. Part of the Cartan truth table was due to an assignment sugggest toPhil Baldwin. The recognition that the Cartan topology was a disconnected topology is due to the author.

to C2 continuous processes. The arrow of time is to be associated with the fact that Continuous Evolutionary Mappings of a disconnected topology to a connected topology are possible, but Continuous Evolutionary Mappings of a connected topology to a disconnected topology are **impossible**! Continuity and Topological Evolution are unidirectional.

For emphasis, I repeat a previous paragraph. Only recently, (Feb 2009), did I appreciate that the Cartan Topology that Baldwin and I had constructed - a topology whose sets were exterior differential forms - was in fact a Kolmogorov T_0 space! This means that the topology is NOT a metric topology, NOT a Hausdorff topology, does NOT satisfy the separation axioms to be a T_1 topology, is not necessarily constrained by symmetries, and is NOT the space of any topological group! I find it more remarkable that such a primitive topology could find broad application to the science of thermodynamics. This T_0 topology of thermodynamics is far afield from the metric based, diffeomorphically constrained, gauge theoretic, group theoretic symmetries found in many of the current physical theories.

2.4 The Pfaff Topological Dimension of a 1-form, PTD(A)

Perhaps one of the most important topological tools to be used within the theory of the Kolmogorov-Cartan T_0 spaces is the concept of Pfaff Topological Dimension. The PTD(A) is equal to the number of non-zero entries in the Pfaff Sequence. The maximum Pfaff Topological Dimension (or class of the form) is 4 on the 4D base variety of "coordinate functions".

For a given 1-form of Action,

$$A = \sum_{k=1-4} A_k(x,y,z,t)dx^k, \tag{2.11}$$

defined on the 4D base differentiable variety of $\{x,y,z,t;dx,dy,dz,dt\}$, it is possible to ask what is the irreducible minimum number of independent functions, $\theta^i(x,y,z,t)$, required to describe the topological features that can be generated by the specified 1-form, A. This irreducible number of functions gives topological importance to the PTD(A). It is remarkable that the irreducible Pfaff Topological Dimension for any given 1-form A is readily computed by constructing the Pfaff Sequence of forms,

$$\textbf{Pfaff sequence for A} \quad = \{A, dA, A\hat{}dA, dA\hat{}dA\}, \tag{2.12}$$

and determining the number of non-zero entries in the sequence.

2.4.1 Example 1: PTD(A) = 1

For example, if only one C2 differentiable function, $\theta(x,y,z,t)$, is required to describe the Action:

$$\begin{aligned} A &= A_k dx^k \Rightarrow d\theta(x,y,z,t)_{irreducible}, & (2.13)\\ \text{such that } A_k &= \partial\theta(x,y,z,t)/\partial x^k, & (2.14)\\ A &= d\theta(x,y,z,t),\ dA=0,\quad A\hat{}dA=0,\ dA\hat{}dA=0, & (2.15)\end{aligned}$$

then the Pfaff Sequence becomes

$$\text{Pfaff sequence} \quad = \{A, 0, 0, 0\}. \tag{2.16}$$

and has only 1 non-zero term. The PTD(A) = 1, even though the number of independent "coordinate" functions, $\{x, y, z, t\}$ is 4. Note that no metric or other geometric constraint is attached the the basis variety.

2.4.2 Example 2: PTD(A) = 2

For example, if the Pfaff sequence for a given 1-form A is $\{A, dA, 0, 0\}$ in a region $U \subset \{x, y, z, t\}$, then the Pfaff Topological Dimension of A is 2 in the region, U. The 1-form A, in the region U, then admits a topologically faithful description in terms of only 2, but not less than 2, independent variables, say $\{u^1, u^2\}$. For a differentiable map φ from $\{x, y, z, t\} \Rightarrow \{u^1, u^2\}$, the exterior differential 1-form defined on the target variety U of 2 pre-geometry dimensions as

$$A(u^1, u^2) = A_1(u^1, u^2)du^1 + A_2(u^1, u^2)du^2, \tag{2.17}$$

has a functionally well defined pre-image $A(x, y, z, t)$ on the base variety $\{x, y, z, t\}$ of 4 pre-geometric dimensions. This functionally well defined pre-image is obtained by functional substitution of u^1, u^2, du^1, du^2 in terms of $\{x, y, z, t\}$ as defined by the mapping φ. The process of functional substitution is called the pullback,

$$A(x, y, z, t) = A_k(x)dx^k \Leftarrow \varphi^*(A(u^1, u^2)) \Leftarrow \varphi^*(A_\sigma du^\sigma). \tag{2.18}$$

It may be true that the functional form of A yields a Pfaff Topological Dimension equal to 2 globally over the domain $\{x, y, z, t\}$, except for sub regions where the Pfaff dimension of A is 3 or 4. These sub regions represent topological defects in the almost global domain of Pfaff dimension 2. Conversely, the Pfaff dimension of A could be 4 globally over the domain, except for sub regions where the Pfaff dimension of A is 3, or less. These sub regions represent topological defects in the almost global domain of Pfaff dimension 4. Applications of both viewpoints will be described below. The important concept of Pfaff Topological Dimension also can be used to define equivalence classes of physical systems and processes.

2.4.3 Example 3: PTD(A) =3, or 4

When the 3-form $A\hat{}dA$ is not zero, the thermodynamic system is not uniquely integrable. If the limit points, $d(A\hat{}dA) = 0$, then the PTD(A) =3. If solutions exist, they are not uniquely determined form a unique set of initial data. The non-unique solutions take the form of envelopes and edges of regression. These solutions have been called singular solutions, but it is apparent in the light of more moder physics that these non-unique soluions are "Collective" solutions of many states of initial conditions. I have defined the three form, $A\hat{}F = A\hat{}dA$, Topological Torsion. If the Limit points of the Topological Torsion 3-form are not zero, $d(A\hat{}dA) = F\hat{}F$, then the 4D volume element, $F\hat{}F = 2(\mathbf{E} \circ \mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt \neq 0$ exhibits irreversible dissipation of expansion or contraction, and the PTD(A)=4 In hydrodynamics notation, the dissipation coefficient becomes the "bulk viscosity coefficient" [31].

2.4.4 Example 4: PTD(W) =3,

It is also possible to examine the Pfaff Topological dimension for any 1-form. For example, the Work 1-form, W_{work}, and the Heat 1-form, Q_{heat}, are of interest. The thermodynamic system encoded by the Action 1-form, A, does not depend (explicitly) upon an evolutionary process, V. The 1-forms of Work and Heat depend upon the concept of a process, V, as well as the thermodynamic system, A. These 1-forms depend dynamically upon the process direction field, V. The topological properties of the 1-forms, W and Q, will "refine" the topology of the thermodynamic system.

Consider a special case where the Work 1-form, W_γ, is closed, but not exact, and constructed in terms of two independent functions, $\Phi(x,y,z,t)$ and $\Psi(x,y,z,t)$.

$$W_\gamma = i(V)dA = (\Phi d\Psi - \Psi d\Phi)/(a\Phi^2 + b\Psi^2), \tag{2.19}$$
$$dW_\gamma = 0 \quad mod \quad \text{zeros of } (a\Phi^2 + b\Psi^2), \tag{2.20}$$
$$W_\gamma\hat{}dW_\gamma = 0, \quad dW_\gamma\hat{}dW_\gamma = 0. \tag{2.21}$$

This representation for the 1-form W_γ is closed, but not exact, which requires that the (example) divisor, $(a\Phi^2 + b\Psi^2)$, to be homogenous of degree 2 in the independent functions, $\Phi(x,y,z,t)$ and $\Psi(x,y,z,t)$. The divisor, which makes the 1-form homogeneous of degree zero, has an infinite number of realizations, not just $h^2 = (a\Phi^2 + b\Psi^2)$. For example, $h^2 = (a\Phi^p + b\Psi^p)^{2/p}$ (which is a Holder Norm) will generate similar results of closure, $dW_\gamma = 0$, for any choice of constants, a, b, p.

Note that for a closed but not exact 1-form, only the first term in the Pfaff sequence is non zero; hence the Pfaff Topological dimension of W_γ is $\text{PTD}(W_\gamma) = 1$.

Next, for example purposes, construct another representation for the Work 1-form , W, using W_γ plus other independent functions. There are three non-zero terms in the Pfaff sequence for this construction. Note that the 3-form $W\hat{}dW$ is not necessarily zero.

$$\widehat{W} = \Gamma(x,y,z,t)W_\gamma + d\Theta, \tag{2.22}$$
$$d\widehat{W} = d\Gamma\hat{}W_\gamma \tag{2.23}$$
$$\widehat{W}\hat{}d\widehat{W} = d\Theta\hat{}d\Gamma\hat{}W_\gamma, \tag{2.24}$$
$$d\widehat{W}\hat{}d\widehat{W} = 0. \tag{2.25}$$

Hence the $\text{PTD}(\widehat{W})$ for the modified example above is 3. Topological structures which are Pfaff Topological Dimension 3 are associated with Contact Manifolds. (Topological structures of Pfaff Topological Dimension 4 are associated with Symplectic Manifolds.)

Note that for thermodynamic systems (to be detailed below) the representation for the Work 1-form is dictated by the First Law of Thermodynamics. The work 1-form, W, (as well as the Heat 1-form, Q) depends upon **both** the 1-form of Action, A per unit source, and the process direction field, V:

$$\text{Thermodynamic Work 1-form,} \quad W = i(V)dA. \tag{2.26}$$

The physical thermodynamic dimensions of A are Action (angular momentum) per "particle". The physical thermodynamic dimensions of W are energy per "particle". To generalize a comment of A. Sommerfeld, "particle" is best expressed as mole number (which can be coagulates of particles, spin, and charge, representing nucleii, or molecules or even galaxies). The mole number is the number of coherent topological structures in the topology.

The 3-form of Work, $W\hat{}dW$, will be defined as the Topological Torsion 3-form of Energy, analogous to the definition of the 3-form, $A\hat{}dA$, as the Topological Torsion 3-form of Action. For the given example above, it is apparent that the induced 3-form of Topological Torsion for Energy consists of the exterior product of two exact 1-forms, and one closed, but not exact, 1-form. As such, the topological parity, $d\widehat{W}\hat{}d\widehat{W}$, of this energy-dynamic system is zero. The closed, but not exact, 3-form, $W\hat{}dW$, therefor represents a closed current density, but is not a 3-form monomial (volume element).

$$\text{Topological Torsion of energy} \quad : \quad \widehat{W}\hat{}d\widehat{W} = (\Phi d\Psi - \Psi d\Phi)\hat{}d\Theta\hat{}d\Gamma/(a\Phi^2 + b\Psi^2) \tag{2.27}$$
$$d\widehat{W}\hat{}d\widehat{W} = 0 \tag{2.28}$$

Limit cycles

Now consider the special case where $\Gamma = f(h)$ is some function of the (specific) divisor, $a\Phi^2+b\Psi^2$. Then the topological torsion for energy 3-form for energy, $\widehat{W}\hat{}d\widehat{W}$, becomes

$$\widehat{W}\hat{}d\widehat{W} = (\partial f/\partial h)d\Theta\hat{}dh\hat{}W_\gamma, \tag{2.29}$$
$$d\widehat{W}\hat{}d\widehat{W} = 0. \tag{2.30}$$

Although the example 3-form is not zero almost everywhere, the parity 4-form is zero globally. For special choices of the function, h, the 3-form of topological torsion for energy also vanishes. If, for example,

$$h^2 = a\Phi^2 + b\Psi^2 \tag{2.31}$$
$$f(h) = (b + h - h^3/3), \tag{2.32}$$

then the zeros of $\partial f/\partial h$ generate an elliptical orbit in the two dimensional plane defined by Φ and Ψ. For $a = b = 1$, this orbit is a limit cycle with a circular orbit, of radius 1.

$$\partial f/\partial h = 1 - h^2 \Rightarrow 0, \tag{2.33}$$
$$a\Phi^2 + b\Psi^2 = 1 \tag{2.34}$$

The Topological Torsion of energy vanishes on the limit cycle, which defines a subset of PTD(W) $= 2$.

Note that the limit cycle can be an attractive orbit or a repelling orbit depending upon the function $f(h)$. The fundamental idea is that the limit cycle is the evolutionary limit of a topological process that evolves with a topological change of PTD $= 3$ to PTD $= 2$.

If the function $f(h)$ has no zeros, the Space of Pfaff Topological dimension 3 is said to be a "tight" Contact structure; When limit cycles exist, the Space of Pfaff Topological dimension 3 is said to be an "overtwisted" Contact structure.

Conclusion 6 *The production of a thermodynamic limit cycle corresponds to the topological evolution of a system of Pfaff Topological Dimension 3 to a system of Pfaff Topological dimension 2.*

I will come back to this analysis and apply it to thermodynamic processes where the Heat 1-form Q is of Pfaff topological Dimension 2, but the Work 1-form, W, is of Pfaff topological dimension 3. In such systems, process paths which are C1 differentiable can exhibit behavior that is different from the behavior of C2 process paths. In fact the process paths that are C1 appear to be reversible, and the process paths that are smooth C2 appear to be irreversible.

More Remarks about Topological Torsion, and Topological Parity, of the Action 1-form, A

The concept defined herein as the "Pfaff Topological Dimension" was developed more than 110 years ago (see page 290 of Forsyth [36]), and has been called the "class" of a differential 1-form in the mathematical literature. The term "Pfaff Topological Dimension" (instead of class) was introduced by me in order to emphasize the topological foundations of the concept. More mathematical developments can be found in Van der Kulk [99]. The method and its properties have been little utilized in the applied world of physics and engineering, where most classical analysis is only in "equilibrium regions" or uniquely integrable regions of PTD(A) <3.

Of key importance is the fact that the non-zero existence of the 3-form $A\hat{}dA$, or,

$$\textbf{Topological Torsion for } A, \quad \mathrm{H} = A\hat{}F, \tag{2.35}$$

implies that the Pfaff Topological Dimension of the region is 3 or more, and the non-zero existence of the 4-form of *Topological Parity*, $dA\hat{}dA = F\hat{}F$ implies that the Pfaff Topological Dimension of the region is 4. Either value is an indicator that the physical system (in the sub region) is NOT in thermodynamic equilibrium. It is also important to recall that non-zero values of Topological Torsion imply that the Frobenius unique integrability Theorem for the Pfaffian equation, $A = 0$, fails. The concept of *topological parity*, $F\hat{}F$, has its foundations in the theory of Pfaff's problem, with a recognizable four-dimensional formulation appearing in Forsyth [36] page 100. On a variety of 4 variables, the coefficient of the 4-form $F\hat{}F$ will be defined as the topological parity (or orientation) function, K, such that

$$\textbf{Topological Parity for } A, \quad \mathrm{K} = F\hat{}F = \sigma_4 dx\hat{}dy\hat{}dz\hat{}dt = \sigma_4\Omega_4. \tag{2.36}$$

It is possible to ascribe the idea of entropy production (due to bulk viscosity) to the coefficient $\sigma_4$of the Parity 4-form.

The idea of *Topological Torsion*, $A\hat{}F$, has been associated with the idea of magnetic helicity density, a concept that apparently had its electromagnetic genesis with the study of plasmas in WWII. However, the concept of helicity density is but one component of the four-dimensional *Topological Torsion 4-vector*.

Recall that a space curve with non-zero Frenet-Serret torsion does not reside in a two-dimensional plane. Non-zero Frenet-Serret torsion of a space curve is an indicator that the *geometrical* dimension of the space curve is at least 3. The fact that the Pfaff Topological Dimension of the 1-form, A, is at least 3, when $A\hat{}F$ is non-zero, is the basis of why the 3-form, $A\hat{}F$, was called "Topological Torsion". The idea of a non-zero 3-form $A\hat{}F$ also appears in the theory of the Hopf Invariant [14].

The concept of $A\hat{}F$ has also appeared in the differential geometry of connections, where a matrix valued 3-form is known as the Chern-Simons 3-form. However, on varieties without connection or metric, the Chern-Simons concept is not well defined, but the Topological Torsion concept exists and is acceptable, for it does not depend upon the geometric features of metric and/or connection. The concepts can be extended to "pre-geometrical", and therefore topological, domains of dimension greater than 4. Pre-geometry implies that constraints of metric or connection have not been (necessarily) imposed on the base variety.

It is possible to define a "curvature" dimension (at a point) in terms of the number of non-null eigenvectors of the Jacobian matrix built from the partial derivatives of the C1 functional components that define the 1-form of Action. The "Curvature" dimension is always less than the dimension of the base variety. The implication is that the determinant of the shape matrix is zero. It is possible that the Pfaff Topological Dimension can exceed the "curvature" dimension.

The idea of the Pfaff Topological Dimension is analogous to the idea of the number of "essential parameters" in the theory of continuous groups [33].

2.5 The Exterior Differential and Limit Points

Cartan referred to the Closure of an exterior differential system as the union of the exterior differential form Σ and its exterior differential $d\Sigma$. This was somewhat before mathematicians had determined that the topological closure of a subset of points consisted of the subset and its limit points. It would appear that the exterior differential is connected to the concept of limit point. It will be demonstrated below that in terms of the Kolomogorov-Cartan T_0 topological structure on sets which are exterior differential forms, the exterior differential is indeed a limit point generator. $I \cup d$ plays the role of a Kuratowski closure operator.

2.6 Disconnected Spaces and Disconnected Subsets.

An essential feature of the Kolmogorov-Cartan topology on subsets of exterior differential forms is that is a disconnected topology.

2.6.1 Disconnected Topology

Two subsets are **disjoint** if $A \cap B = \varnothing$. A disconnected topology (space) has disjoint Open Subsets G and H such that

$$X = G \cup H, \text{ and } \varnothing = G \cap H, \; G \cap H = \varnothing. \tag{2.37}$$

If the only sets that are Open and Closed are X and $\varnothing$, then the Topology (and X) is connected. The Kolmorgorov-Cartan topology on sets of exterior differential forms is a **disconnected** topology (space)

2.6.2 Disconnected Subsets

A subset $A \subset X$ is said to be disconnected if there are open subsets, G and H, such that $(A \cap G)$ and $(A \cap H)$ are not empty, and

$$(A \cap G) \cap (A \cap H) = \varnothing, \tag{2.38}$$
$$(A \cap G) \cup (A \cap H) = A. \tag{2.39}$$

Note that G and H are not necessarily disjoint. If the subset is NOT disconnected, it is connected. In the Kolmorgorov-Cartan topology, the pairs of subsets that are not both Open and Closed, are disconnected.

2.6.3 Separated Subsets

Two (distinct) subsets A and B, are said to be separated if they are disjoint and satisfy the rules,

$$A \cap B = \varnothing, \tag{2.40}$$
$$A \cap \overline{B} \neq \varnothing, \tag{2.41}$$
$$\overline{A} \cap B \neq \varnothing, \tag{2.42}$$

where $\overline{A}$ is the symbol for the closure of A; $\overline{A} = A \cup dA$, the union of the subset and its limit points.

Chapter 3
THE KOLMOGOROV-CARTAN TOPOLOGY AND THERMODYNAMICS

3.1 Axioms of Topological Thermodynamics

In this section, a topological perspective will be used to deduce those properties of physical systems, and their evolution, that are independent from the geometrical constraints of connections and/or metrics. It is subsumed that the presence of a physical system establishes a *topological structure* on a base space-time differentiable variety. This concept is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space-time differentiable variety, and the dynamics is established in terms of a connection. These are assumptions of constraint on the base space, and are avoided when a topological perspective, not a geometric perspective, is assumed. Note that a given differentaible variety may support many different topological structures; hence a given base may support many different physical systems. A major success of theory is that continuous non-homeomorphic processes of topological evolution establish a logical basis for thermodynamic irreversibility and the arrow of time [91] without the use of statistics.

The fundamental axioms and theorems utilized in the universal theory are:

Axiom 1. *Thermodynamic physical systems can be encoded in terms of a 1-form of Action potentials,* $A_k(x, y, z, t)$*, on a four-dimensional abstract differentiable variety of ordered independent variables,* $\{x, y, z, t, dx, dy, dz, dt\}$*. The variety supports a non-zero differential volume element* $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$*.*

Axiom 2. *Every 1-form of Action,* $A_k(x, y, z, t)dx^k$, has an irreducible number of functions required to encode its topological features. This minimum number is defined as the Pfaff Topological Dimension of the 1-form, A. The largest PTD on a 4D variety is PTD=4, which corresponds to an Open non-equibrium Thermodynamic system. The PTD=4 Open system plays the role of the physical envirionment, or Aether. Open, Closed, Isolated and Equilibrium thermodynamic systems will be associated with 1-forms of PTD=4, PTD=3, PTD=2, and PTD=1.

Axiom 3. *Thermodynamic processes are assumed to be encoded, to within a density factor, $\rho(x,y,z,t)$, in terms of a Vector direction field and/or a complex isotropic Spinor direction field, $\boldsymbol{V}_4(x,y,z,t)$. The density distribution factor can be chosen such that the differential volume element, $dx\hat{}dy\hat{}dz\hat{}dt$ is an invariant of the process.*

Axiom 4. *Continuous topological evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [56]). The Lie differential, relative to a process vector field $\boldsymbol{V}_4$(x,y,z,t), when applied to a exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent abstractly to the first law of thermodynamics.*

$$\begin{aligned}
\text{The First Law} \quad &: \quad \text{of Thermodynamics} && (3.1)\\
\text{Cartan's Magic Formula } L_{(\rho \mathbf{V}_4)}A \quad &= \quad i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A), && (3.2)\\
\text{A statement of Cohomology} \quad &: \quad W + dU = Q, && (3.3)\\
\text{Inexact Heat 1-form } Q \quad &= \quad W + dU = L_{(\rho\mathbf{V}_4)}A, && (3.4)\\
\text{Inexact Work 1-form } W \quad &= \quad i(\rho\mathbf{V}_4)dA, && (3.5)\\
\text{Internal Energy } U \quad &= \quad i(\rho\mathbf{V}_4)A. && (3.6)
\end{aligned}$$

Axiom 5. *Equivalence classes of systems and continuous processes can be defined and refined in terms of the Pfaff Topological Dimension of the 1-forms of Action, A, Work, W, and Heat, Q.*

Axiom 6. *$Q\hat{}dQ \neq 0$ (Pfaff Topological Dimension of Q is ≥ 3) is a necessary and sufficient condition for a process, V, to be thermodynamically irreversible.*

In a perhaps over simplistic comparison, it might be said that ubiquitous tensor methods are restricted to geometric applications, while Cartan's methods can be applied directly to topological concepts as well as geometrical concepts. Cartan's theory of exterior differential systems is a topological theory not necessarily limited by geometrical constraints and the class of diffeomorphic transformations that serve as the foundations of tensor calculus. A major objective of this chapter is to show how limit points, intersections, closed sets, continuity, connectedness and other elementary concepts of modern topology are inherent in Cartan's theory of exterior differential systems. These ideas do not depend upon the geometrical ideas of size and shape. Hence Cartan's theory, as are all topological theories, is renormalizable (perhaps a better choice of words is that the topological components of the theory are independent from scale). In fact the most useful of Cartan's ideas do not depend explicitly upon the geometric ideas of a metric, nor upon the choice of a differential connection between basis frames, as in fiber bundle theories. The theme of this chapter is to explore the physical usefulness of those topological features of Cartan's methods which are independent from the constraints and refinements imposed by a connection and/or a metric.

In this chapter the Cartan topology will be constructed in terms of an arbitrary 1-form of Action, A. All elements of the Cartan topology will be evaluated, and the limit points, the boundary sets and the closure of every subset will be computed abstractly. Earlier intuitive results, which utilized the notion that Cartan's concept of the exterior product may be used as an intersection operator, and his concept of the exterior differential may be used as a limit point operator acting on differential forms, will be given formal substance in this chapter. A major result of this chapter, with important physical consequences in describing topological evolutionary processes, is the demonstration that the Cartan topology is not necessarily a connected topology. To be connected, the property of Topological Torsion vanishes; the PTD of A is greater that 2. Thermodynamic irreversibility is a natural consequence of Pfaff topological dimension 4.

3.2 A Point Set Topology Example

At first I will discuss a simple point set topology based upon 4 points. Then, later, the "points" will be considered in terms of subsets of exterior differential forms. Consider the set of 4 elements or points,

$$X : \{a, b, c, d\}, \quad \varnothing \tag{3.7}$$

and all possible subsets:

$$\{a\}, \{b\}, \{c\}, \{d\}, \tag{3.8}$$
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \tag{3.9}$$
$$\{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, d\}, \tag{3.10}$$
$$\{a, b, c, d\} = X. \tag{3.11}$$

Select the following subset elements as a topological basis,

$$\text{basis selection} \quad \{a\}, \{a, b\}, \{c\}, \{c, d\}, \tag{3.12}$$

and then compose of what I called a Cartan topology of 4 points, $CT4$, of open sets from all possible unions of the selected basis elements:

$$CT4\{open\} : \varnothing, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}. \tag{3.13}$$

The closed sets are the complements of the open sets:

$$CT4\{closed\} : \{a, b, c, d\}, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \varnothing. \quad (3.14)$$

It is an easy exercise to demonstrate that the collections above indeed satisfy the axioms of a topology. (Warning: this is not the only topology that can be constructed over 4 elements.)

This simple example of a point set topology permits explicit construction of all the topological concepts, which include limit sets, interiors, boundaries, and closures, for the all of subsets of X, relative to the topology, $CT4$. The standard definitions are:

1. A limit point of a subset A is a point p such that all open sets O that contain p also contain a point of A not equal to p. $O\backslash p \cap A \neq 0$.
2. The closure of a subset A is the union of the subset and its limit points, and is the smallest closed set that contains A.
3. The interior of a subset is the largest open set contained by the subset.
4. The exterior of a subset is the interior of its complement.
5. A boundary of a subset is the set of points not contained in the interior or exterior.
6. The closure of a subset is also equal to the union of its interior and its boundary.

The results of applying these definitions to the $CT4$ topology of 4 points are given in Table 1:

Table 1. A (Cartan) CT4 Topology of 4 points

$X = \{a, b, c, d\}$

Basis subsets $\{a\}, \{a, b\}, \{c\}, \{c, d\}$

$CT4\{open\} : \varnothing, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X$

$CT4\{closed\} : X, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \varnothing$

Subset S	Limit Pts	Interior	Exterior	Boundary	Closure
$\varnothing$ *	$\varnothing$	$[\varnothing]$	$[X]$	$\varnothing$	$\varnothing$
$\{a\}$	$\{b\}$	$\{a\}$	$\{c, d\}$	$\{b\}$	$\{a, b\}$
$\{b\}$	$\varnothing$	$[\varnothing]$	$\{a, c, d\}\mathbb{D}$	$\{b\}$	$\{b\}$
$\{c\}$	$\{d\}$	$\{c\}$	$\{a, b\}$	$\{d\}$	$\{c, d\}$
$\{d\}$	$\varnothing$	$[\varnothing]$	$\{a, b, c\}\mathbb{D}$	$\{d\}$	$\{d\}$
$\{a, b\}$ *	$\{b\}$	$\{a, b\}$	$\{c, d\}$	$\varnothing$	$\{a, b\}$
$\{a, c\}$ $\mathbf{d}\mathbb{D}$	$\{b\}, \{d\}$	$\{a, c\}\mathbb{D}$	$[\varnothing]$	$\{b, d\}\mathbb{D}$	X
$\{a, d\}$ $\mathbb{D}$	$\{b\}$	$\{a\}$	$\{c\}$	$\{b, d\}\mathbb{D}$	$\{a, b, d\}\mathbb{D}$
$\{b, c\}$ $\mathbb{D}$	$\{d\}$	$\{c\}$	$\{a\}$	$\{b, d\}\mathbb{D}$	$\{b, c, d\}\mathbb{D}$
$\{b, d\}$ $\mathbb{D}$	$\varnothing$	$\varnothing$	$\{a, c\}\mathbb{D}$	$\{b, d\}\mathbb{D}$	$\{b, d\}$ $\mathbb{D}$
$\{c, d\}$ *	$\{d\}$	$\{c, d\}$	$\{a, b\}$	$\varnothing$	$\{c, d\}$
$\{a, b, c\}$ $\mathbf{d}\mathbb{D}$	$\{b\}, \{d\}$	$\{a, b, c\}$	$[\varnothing]$	$\{d\}$	X
$\{b, c, d\}$ $\mathbb{D}$	$\{d\}$	$\{c, d\}$	$\{a\}$	$\{b\}$	$\{b, c, d\}\mathbb{D}$
$\{a, c, d\}$ $\mathbf{d}\mathbb{D}$	$\{b\}, \{d\}$	$\{a, c, d\}$	$[\varnothing]$	$\{b\}$	X
$\{a, b, d\}$ $\mathbb{D}$	$\{b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{b, c, d\}\mathbb{D}$
$\{a, b, c, d\}$ *	$\{b\}, \{d\}$	$[X]$	$[\varnothing]$	$\varnothing$	X

$\mathbb{D}$: subset is disconnected, *: subset is open and closed, **d**: subset is dense

Boundary of $S : \partial S = X - Interior - Exterior = X - S^0 - S^0_c,$

Closure of $S : \widetilde{S} = S + Limit\ points = S + S'$

Closure of $S : \widetilde{S} = Interior + Boundary = S^0 + \partial S$

This $CT4$ topology is quite interesting for many demonstrable reasons. First note that all of the singletons of the topology are not closed. This implies that the topology is NOT a metric topology, NOT a Hausdorff topology, and even does NOT satisfy the separation axioms that define a T_1 topology*. Note that all closed sets contain all of their limit points. Some open sets can contain limit points, but some open sets do not contain their limit points. Some subsets have boundaries that are composed of their limit points. Some subsets have limit points which are not boundary points. Certain subsets have a boundary, but do not have limit points, and in other cases there are subsets that have limit points, but do not have a boundary. There are certain subsets with a boundary, but without an interior. There are certain subsets with an interior, but without a boundary. These situations, though topologically correct, are not always intuitive to those accustomed to metric based topological concepts, which impose a number of additional constraints on the sets of interest. Yet all of these topological ideas, including the non-intuitive ones, are easy to grasp from the simple example of the $CT4$ point set topology.

*For those not familiar with point set topology, chapter 5 in Schaum's outline [52] can be useful.

One other very important observation is that there are subsets of the $CT4$ topology, $\{a,b\}$ and $\{c,d\}$, (other than $\varnothing$ and X) which are both open and closed. The union of these two subsets $\{a,b\}$ and $\{c,d\}$ is X. Topologies with this property are said to be disconnected topologies. What is important is that it is possible to construct a **continuous** map from a disconnected topology to a connected topology, but it is impossible to construct a **continuous** map from a connected topology to a disconnected topology. If the mapping process is interpreted as an evolutionary process, these facts establish a logical or topological basis for the arrow of time [91]. This idea can be exploited to explain the concept of thermodynamic irreversibility without the use of statistics. Note that the sets $\{a,c\}$, $\{a,b,c\}$ and $\{a,c,d\}$ are dense. They are homeomorphic invariants relative to the $CT4$ topology of the thermodynamic system, but when the topology is augmented (refined) by the constraint of continuous topological evolution and *processes* acting on thermodynamic systems, they can represent topological change.

What is even more remarkable is that properties of the $CT4$ topology can be replicated in terms of the Pfaff sequence of exterior differential sets,

$$\text{Pfaff Sequence}:\{A,dA,A\hat{}dA,dA\hat{}dA...\},\tag{3.15}$$

generated from any given 1-form of Action, A, or W, or Q, on the space time differentalbe variety. The Pfaff sequence is readily computed, and will contain $M\leq N$ elements, where M is defined as the Pfaff Topological Dimension (or class) of the given 1-form, A, W, or Q. The topologies associated with W, or Q, are process dependent. The Topology associated with A is not dependent upon a process; the topology of the 1-form A will be described as intrinsic, of a particular thermodynamic system.

The realization of a $CT4$ topology in terms of exterior differential forms is herein defined as the "Cartan topology", and is detailed in the next section. The Cartan topology has far reaching consequences in applications to physical problems.

3.3 Algebraic and Differential Closure

The concept of closure is one of the most important ideas in Cartan's theory. His methods center on two procedures of closure, one algebraic, and one differential. Both processes are closed in the sense that when they operate on a subset of a set of exterior differential forms, the objects created are also subsets of the set of exterior differential forms. There are no surprises. Cartan utilized the exterior algebra over a variety of dimension N to construct a vector space of exterior differential forms of dimension 2^N. The N subspaces of this (Grassmann) space are vector spaces of dimension equal to N things taken p at a time. The elements of the subspaces are called p-forms. In four dimensions, the subspace sets are one dimensional, N = four dimensional, N(N+1)/2 = six dimensional, N = four dimensional, and one dimensional. The elements of the subspaces are often called scalars (0-forms), vectors

(1-forms), tensors (2-forms), pseudovectors (3-forms), and pseudo-scalars† (4-forms) in relativistic physical theories. The Exterior (Grassmann) algebra has a finite 2^N-basis (equal to 16 elements in a space of 4 independent variables). The concept of closure means that the operations on elements of the 2^N-dimensional space yield results that are contained within the 2^N-dimensional space. When the operations are applied to elements of a subspace, the results usually are not contained in the same subspace, but they are contained within the 2^N-dimensional vector space of p-forms.

The exterior product (with symbol ^) takes elements of the 2^N-base space and multiplies them together in a manner such that the result is contained as an element of the 2^N-base space. This process of exterior multiplication is closed, for the action of the process on any subset of the 2^N-base space produces another subset of the 2^N-base space. However, the exterior product takes a p-form times a q-form into a (p+q)-form. The elements of the product can be from different or from the same vector subspaces, but the resultant is always a linear combination of the subspaces of the Exterior algebra.

Similarly the concept of exterior differentiation (with symbol d) is defined such that the operation produces a (p+1)-form from a p-form. This process of exterior differentiation is "closed", for the action of the process on any subset of the 2^N base space produces another subset of the 2^N base space. A differential ideal is defined as the union of a collection of given p-forms and their exterior differentials.

An "interior" product with respect to a direction field $\mathbf{V}$ (with symbol $i(\mathbf{V})$ and of dimension N) can be defined on the Grassmann algebra of exterior differential forms. The interior product takes a p-form to a (p-1)-form, and in this sense is another operation which is closed within the Grassmann algebra. The resultant product is still an element of the 2^N-base space. Where the exterior differential raises the rank of a p-form to a (p+1)-form, the inner product lowers the rank of a p-form to a (p-1)-form. (There are other useful operators that lower the rank of the exterior differential p-form, and involve integration.)

By composition of the exterior differential and the inner product operators, the Lie differential operator (with symbol $L_{(\mathbf{V})} = i(\mathbf{V})d + di(\mathbf{V})$) can be constructed, such that when the Lie differential operates on an exterior p-form, the resultant object is another p-form. For a 1-form of Action, A, the process reads,

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q. \tag{3.16}$$

The resultant is not only closed relative to the Grassmann algebra, it also remains within the same Grassmann vector subspace. The Lie differential does not depend upon a metric nor upon a connection. When the Lie differential acting on a p-form vanishes, the p-form is said to be an invariant of the process, $\mathbf{V}$. When the Lie differential of a p-form does not vanish, the topological features of the resultant p-form

†Distinctions between differential form Scalars and differential form Densities will modify this terminology

permit the processes, **V**, that produce such a result, to be put into equivalence classes, depending on the Pfaff dimension of the resultant form. For example, if in the formula given above for a 1-form, A, yields a result Q such that $dQ = 0$, then the process **V** belongs to the class of process known as Hamiltonian processes in mechanics, and to the Helmholtz class of processes that conserve vorticity in Hydrodynamics. Of particular interest to this monograph are processes where Q is of Pfaff dimension greater than 2. The Pfaff sequence constructed from Q contains three or more elements. Such processes, V, that produce Heat 1-forms, Q, which are of Pfaff Topological Dimension 3, are thermodynamically irreversible.

The Lie differential will be used extensively in physical applications of Cartan's theory, especially to the study of processes that involve topological evolution. The perhaps more familiar covariant derivative, highly constrained by connection or metric assumptions, is a special case of the Lie differential. The use of the covariant derivative leads to useful, but limited, physical theories for which the description of topological evolution is awkward, if not impossible.

Even more remarkable in a thermodynamic sense is the comment made by Mason and Woodhouse (see p. 49 [59] and also [7]):

Remark 7 *"Then there is a Higgs field ϕ_V associated with each conformal Killing vector $V \in \mathfrak{h}$, (the Lie algebra of H) which measures the difference between the Covariant derivative along V and the Lie derivative along V."*

The implication is that the concept of a Higgs field represents the difference between a process that is not dependent upon the constraint of a gauge group (the Lie differential), and a process that is restricted to a specific choice of a connection defined by some gauge group, (the Covariant differential).

For the cases where $(i(fV)A) = 0$, (which corresponds to processes that do not change the internal energy, U) the two differentials are equivalent. It follows that

$$\begin{aligned} L_{(fV)}A &= f\ L_{(V)}A + d(\ln f)\ (i(fV)A), && (3.17)\\ &= f\ L_{(V)}A = f \cdot i(V)dA = f \cdot \nabla_{(V)}A = f\ Q. && (3.18)\\ \text{But then, } i(V)Q &= f\ i(V)i(V)dA \Rightarrow 0. && (3.19) \end{aligned}$$

Therefor the process, V, is a null orbit of the heat 1-form, $i(V)Q = 0$, which defines an adiabatic process [12]. The general adiabatic condition implies that all exchanges of Heat are transverse to the process. A strong adiabatic condition is defined when there is no heat exchange, $Q = 0$.

Theorem 8 *Hence, all covariant derivatives with respect to an affine connection have an equivalent representation as an adiabatic process!!! (Such covariant adiabatic processes need not be thermodynamically reversible.)*

Suppose that the covariant process satifies the strong adiabatic condition,

$$L_{(V)}A = \nabla_{(V)}A = \ Q \Rightarrow 0. \tag{3.20}$$

Then,

$$d(L_{(V)}A) = L_{(V)}dA = \ dQ \Rightarrow 0, \tag{3.21}$$
$$Q\hat{\ }dQ = 0 \tag{3.22}$$

and it follows that the covariant (adiabatic) process is reversible. However, the strong covariant condition, $Q \Rightarrow 0$, is the equivalent to the condition of parallel transport:

$$L_{(V)}\omega \Rightarrow \nabla_{(V)}\omega = 0. \tag{3.23}$$

Theorem 9 *The remarkable conclusion is that the concept of parallel transport in tensor analysis is - in effect - an adiabatic, reversible process!!!*

3.4 The Kolmogorov-Cartan Topology with sets that are differential forms

Cartan built his theory around an exterior differential system, Σ, which consists of a collection of 0-forms, 1-forms, 2-forms, etc. [22]. He defined the closure of this collection as the union of the original collection of differential forms with those differnential forms which are obtained by forming the exterior differentials of every p-form in the initial collection. It is now appreciated via the KCT topological structure, that the exterior differential is a limit point generator. In general, the collection of exterior differentials will be denoted by $d\Sigma$, and the closure of Σ by the symbol, $K_{Cl}(\Sigma)$, where,

$$\text{Kuratowski Closure operator:} \quad K_{Cl}(\Sigma) = \Sigma_{\cup}d\Sigma. \tag{3.24}$$

For notational simplicity in this monograph the systems of p-forms will be assumed to consist of the single 1-form, A. Then the exterior differential of A is the 2-form $F = dA$, and the closure of A is the union of A and $F : K_{Cl}(A) = A_{\cup}F$. The other logical operation is the concept of intersection, so that from the exterior differential it is possible to construct the set $A\hat{\ }F$ defined collectively as $\mathrm{H} : \mathrm{H} = A\hat{\ }F$. The exterior differential of H produces the set defined as $\mathrm{K} = d\mathrm{H}$, and the closure of H is the union of H and K : $K_{Cl}(\mathrm{H}) = \mathrm{H}_{\cup}\mathrm{K}$.

This ladder process of constructing exterior differentials, and exterior products, may be continued until a null set is produced, or the largest p-form so constructed is equal to the dimension of the space under consideration. The set so generated is defined as a Pfaff sequence. The largest rank of the sequence determines the Pfaff dimension of the domain (or class of the form, [99]), which is a topological invariant. The idea is that the 1-form A (in general the exterior differential system, Σ) generates on space-time four equivalence classes of points that act as domains of support for the elements of the Pfaff sequence, $A, F, \mathrm{H}, \mathrm{K}$. The union of all such points will be denoted by $X = A_{\cup}F_{\cup}\mathrm{H}_{\cup}\mathrm{K}$. The fundamental equivalence classes are given specific names [78]:

$$\begin{aligned}
\text{Topological ACTION} &: A && (3.25)\\
A &= A_{\mu}dx^{\mu} && (3.26)\\
\text{Topological VORTICITY} &: F = dA && (3.27)\\
dA &= F_{\mu\nu}dx^{\mu}\,\hat{}\,dx^{\nu} && (3.28)\\
\text{Topological TORSION} &: \mathrm{H} = A\hat{}dA && (3.29)\\
A\hat{}dA &= H_{\mu\nu\sigma}dx^{\mu}\,\hat{}\,dx^{\nu}\hat{}\,dx^{\sigma} && (3.30)\\
\text{Topological PARITY} &: \mathrm{K} = dA\hat{}dA && (3.31)\\
dA\hat{}dA &= K_{\mu\nu\sigma\tau}dx^{\mu}\,\hat{}\,dx^{\nu}\hat{}\,dx^{\sigma}\hat{}\,dx^{\tau}. && (3.32)
\end{aligned}$$

The Cartan topology is constructed from a basis of open sets, which are defined as follows. First consider the domain of support of A. Define this "point" by the symbol A. A is the first open set of the Cartan topology. Next construct the exterior differential, $F = dA$, and determine its domain of support. Next, form the closure of A by constructing the union of these two domains of support, $K_{Cl}(A) = A_{\cup}F$. $A_{\cup}F$ forms the second open set of the Cartan topology.

Next construct the intersection $\mathrm{H} = A\hat{}F$, and determine its domain of support. Define this "point" by the symbol H, which forms the third open set of the Cartan topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and $\mathrm{K} = d\mathrm{H}$. The construction forms the fourth open set of the Cartan topology. In four dimensions, the process stops, but for $\mathrm{N} > 4$, the process may be continued.

Now consider the basis collection of open sets that consists of the subsets:

$$B = \{A,\ K_{Cl}(A),\ \mathrm{H},\ K_{Cl}(\mathrm{H})\} = \{A,\ A_{\cup}F,\ \mathrm{H},\ \mathrm{H}_{\cup}\mathrm{K}\}. \quad (3.33)$$

The collection of all possible unions of these base elements, and the null set, $\varnothing$, generate the Cartan topology of open sets:

$$T(open) = \{X,\ \varnothing,\ A,\ \mathrm{H}, A_{\cup}F,\ \mathrm{H}_{\cup}\mathrm{K},\ A_{\cup}\mathrm{H},\ A_{\cup}\mathrm{H}_{\cup}\mathrm{K},\ A_{\cup}F_{\cup}\mathrm{H}\}. \quad (3.34)$$

These nine subsets form the open sets of the Cartan topology constructed from the domains of support of the Pfaff sequence constructed from a single 1-form, A, in four dimensions. The complements of the open sets are the closed sets of the Cartan topology:

$$T(closed) = \{\varnothing,\ X,\ F_{\cup}\mathrm{H}_{\cup}\mathrm{K},\ A_{\cup}F_{\cup}\mathrm{K}, A_{\cup}F,\ \mathrm{H}_{\cup}\mathrm{K}, F_{\cup}\mathrm{K}, F,\ \mathrm{K}\}. \quad (3.35)$$

From the set of 4 "points" $\{A, F, \mathrm{H}, \mathrm{K}\}$ that make up the Pfaff sequence it is possible to construct 16 subset collections by the process of union. It is possible to compute the limit points for every subset relative to the Cartan topology. The classical definition of a limit point is that a point p is a limit point of the subset Y relative to the topology T if and only if for every open set which contains p there exists another point of Y other than p [52]. The results of this and other standard definitions are presented in Table 2, and are to be compared to Table 1.

Table 2. The KCTo Topological Structure of Differential Forms

$$X = \{A, \quad F = dA, \quad \mathrm{H} = A\hat{}F, \quad \mathrm{K} = F\hat{}F\}$$

Basis subsets: Basis subsets $\{A, K_{Cl}(A), \mathrm{H}, K_{Cl}(\mathrm{H})\} = \{A, A_\cup F, \mathrm{H}, \mathrm{H}_\cup \mathrm{K}\}$

$$T(open) = \{X, \varnothing, A, \mathrm{H}, A_\cup F, \mathrm{H}_\cup \mathrm{K}, A_\cup \mathrm{H}, A_\cup \mathrm{H}_\cup \mathrm{K}, A_\cup F_\cup \mathrm{H}\}$$

$$CT4\{open\} : \{X, \varnothing, A, \mathrm{H}, A_\cup F, \mathrm{H}_\cup \mathrm{K}, A_\cup \mathrm{H}, A_\cup \mathrm{H}_\cup \mathrm{K}, A_\cup F_\cup \mathrm{H}\}$$

$$CT4\{closed\} : \{\varnothing, X, F_\cup \mathrm{H}_\cup \mathrm{K}, A_\cup F_\cup \mathrm{K}, \mathrm{H}_\cup \mathrm{K}, A_\cup F, F_\cup \mathrm{K}, F, \mathrm{K}\}$$

Subset S	Limit Pts	Interior -	Exterior	Boundary	Closure
$\varnothing$ *	$\varnothing$	$[\varnothing]$	$[X]$	$\varnothing$	$\varnothing$
$\{A\}$	$\{F\}$	$\{A\}$	$\{H_\cup K\}$	$\{F\}$	$\{A_\cup F\}$
F	$\varnothing$	$[\varnothing]$	$\{A_\cup H_\cup K\}\mathbb{D}$	$\{F\}$	$\{F\}$
H	$\{K\}$	$\{H\}$	$\{A_\cup F\}$	$\{K\}$	$\{H_\cup K\}$
K	$\varnothing$	$[\varnothing]$	$\{A_\cup F_\cup H\}\mathbb{D}$	$\{K\}$	$\{K\}$
$\{A_\cup F\}$ *	$\{F\}$	$\{A_\cup F\}$	$\{H_\cup K\}$	$\varnothing$	$\{A_\cup F\}$
$\{A_\cup H\}$ d$\mathbb{D}$	$\{F\}, \{K\}$	$\{A_\cup H\}\mathbb{D}$	$[\varnothing]$	$\{F_\cup K\}\mathbb{D}$	X
$\{A_\cup K\}$ $\mathbb{D}$	$\{F\}$	$\{A\}$	$\{H\}$	$\{F_\cup K\}\mathbb{D}$	$\{A_\cup F_\cup K\}\mathbb{D}$
$\{F_\cup H\}$ $\mathbb{D}$	$\{K\}$	$\{H\}$	$\{A\}$	$\{F_\cup K\}\mathbb{D}$	$\{F_\cup H_\cup K\}\mathbb{D}$
$\{F_\cup K\}$ $\mathbb{D}$	$\varnothing$	$\varnothing$	$\{A_\cup H\}\mathbb{D}$	$\{F_\cup K\}\mathbb{D}$	$\{F_\cup K\}$ $\mathbb{D}$
$H_\cup K\}$ *	$\{K\}$	$\{H_\cup K\}$	$\{A_\cup F\}$	$\varnothing$	$\{H_\cup K\}$
$\{A_\cup F_\cup H\}$ d$\mathbb{D}$	$\{F\}, \{K\}$	$\{A_\cup F_\cup H\}$	$[\varnothing]$	$\{K\}$	X
$\{F_\cup H_\cup K\}$ $\mathbb{D}$	$\{K\}$	$\{H, K\}$	$\{A\}$	$\{F\}$	$\{F_\cup H_\cup K\}\mathbb{D}$
$\{A_\cup H_\cup K\}$ d$\mathbb{D}$	$\{F\}, \{K\}$	$\{A_\cup H, K\}$	$[\varnothing]$	$\{F\}$	X
$\{A_\cup F_\cup K\}$ $\mathbb{D}$	$\{F\}$	$\{A_\cup F\}$	$\{H\}$	$\{K\}$	$\{F_\cup H_\cup K\}\mathbb{D}$
$\{A_\cup F_\cup H_\cup K\}$ *	$\{F\}, \{K\}$	$[X]$	$[\varnothing]$	$\varnothing$	X

By examining the set of limit points so constructed for every subset of the Cartan system, and presuming that the functions that make up the forms are C2 differentiable (such that the Poincare lemma is true, $dd\omega = 0, any\ \ p-form,\ \omega$), it is easy to show that for all subsets of the Cartan topology every limit set is composed of the exterior differential of the subset thereby proving the conjecture that the exterior differential is a limit point operator relative to the Cartan topology.

Theorem 10 *With respect to the Cartan topology, the exterior differential is a limit point generator.*

For example, the open subset, $A_\cup\mathrm{H}$, has the limit points that consist of F and K. The limit set consists of $F_\cup\mathrm{K}$ which can be derived directly by taking the exterior differentials of the elements that make up $A_\cup\mathrm{H}$; that is, $(F_\cup\mathrm{K}) = d(A_\cup\mathrm{H}) = (dA_\cup d\mathrm{H})$.

Note that this open set, $A_\cup H$, does not contain its limit points. Similarly for the closed set, $A_\cup F$, the limit points are given by F which may be deduced by direct application of the exterior differential to $(A_\cup F) : (F) = d(A_\cup F) = (dA_\cup dF) = (F_\cup \varnothing) = (F)$.

3.5 Topological Torsion, Connected vs. Non-Connected Cartan topologies

Topological torsion of a 1-form, A, is defined as the exterior product of the 1-form and its exterior differential, $\mathrm{H} = A\hat{}dA$. Topological torsion is different from, but can be related to, the Frenet torsion of a space curve and the affine torsion of a connection. If non-zero, Topological torsion has important topological properties. The Cartan topology as given in Table 2 is composed of the union of two sub-sets which are both open and closed:

$$(X = K_{Cl}(A)_\cup K_{Cl}(\mathrm{H}) = \{A_\cup F\}_\cup \{\mathrm{H}_\cup \mathrm{K}\}), \tag{3.36}$$

a result that implies that the Cartan topology is not necessarily a connected topology‡ [52]. An exception exists if the topological torsion, $\mathrm{H} = A\hat{}dA$, and hence its closure, vanishes, for then the Cartan topology is connected. This extraordinary result has broad physical consequences. The connected Cartan topology based on a vanishing topological torsion is at the basis of most physical theories of equilibrium. In mathematics, the connected Cartan topology corresponds to the Frobenius integrability condition for Pfaffian forms. In thermodynamics, the connected Cartan topology is associated with the Caratheodory concept of inaccessible thermodynamic states [40], and the existence of an equilibrium thermodynamic surface. If the non-exact 1-form, Q, of heat generates a Cartan topology of null topological torsion relative to the 1-form Q, $Q\hat{}dQ = \varnothing$, then the Cartan topology built on Q is connected. Such systems are "isolated" in a topological sense, and the Heat 1-form, Q, has a representation in terms of two and only two functions, conventionally written as, $Q = TdS$. Note again that a fundamental physical concept, in this case the idea of equilibrium, is a topological concept independent from geometrical properties of size and shape. Processes that generate the 1-form Q such that $Q\hat{}dQ = \varnothing$ are thermodynamically reversible. If $Q\hat{}dQ \neq \varnothing$, the process that generates Q is thermodynamically irreversible [58].

When the Cartan topology is connected, it might be said that all forces are extendible over the whole of the set, and that these forces are of "long range". Conversely when the Cartan topology is disconnected, the "forces" cannot be extended indefinitely over the whole domain of independent variables, but perhaps only over a single component. The components are not arc connected. In this sense, such forces are said to be of "short range", as they are confined to a specific component.

‡It should be noted that disconnected is not the same as separated, for the disconnected component boundaries could be in contact.

Note that this notion of short or long range forces does not depend upon geometrical size or scale. The physical idea of short or long range forces is a topological idea of connectivity, and not a geometrical concept of "how far".

In an earlier article, these ideas were formulated intuitively in order to give an explanation of the "four forces" of physics. The earlier work was based upon experience with differential geometry [72]. The features of the Pfaff sequence were used to establish equivalence classes for 1-forms constructed from known example metric field solutions, $g_{\mu v}$, to the Einstein field equations. The original ideas, based upon experience with systems in differential geometry, can now be given credence based upon differential topology. The construction of a 1-form, $A = g_{\mu 4} dx^{\mu}$, whose coefficients are the space-time components of a metric tensor, will divide the topology into equivalence classes depending upon the number of non-zero elements of its Pfaff sequence. This number has been defined above as the Pfaff Topological Dimension. Long range parity preserving forces due to gravity (Pfaff dimension 1) and electromagnetism (Pfaff dimension 2) are to be associated with a Cartan Topology that is connected (H $= A\hat{}F = A\hat{}dA = 0$). Both the strong force (Pfaff dimension 3) and the weak force (Pfaff dimension 4) are "short" range (H $\neq 0$) and are to be associated with a disconnected Cartan topology. The strong force is parity preserving (K $= 0$) and the weak force is not (K $\neq 0$). The fact that the Cartan topology is not necessarily connected is the topological (not metrical) basis that may be used to distinguish between short and long range forces. The methods also have applicability to the theory of black holes.

In much of our physical experience with nature, it appears that the disconnected domains of Pfaff dimension 3 or more are often isolated as nuclei, while the surrounding connected domains of Pfaff dimension 2 or less appears as fields of charged or non-charged molecules and atoms. However, part of the thrust of this monograph is to demonstrate that such disconnected topological phenomena are not confined to microscopic systems, but also appear in a such mundane phenomena as the flow of a turbulent fluid. Physical examples of the existence of topological torsion (and hence a non-connected Cartan topology) are given by the experimental appearance of what appear to be coherent structures in a turbulent fluid flow.

To prove that a turbulent flow must be a consequence of a Cartan topology that is not connected, consider the following argument. First consider a fluid at rest. From a global set of unique, synchronous, initial conditions generate a vector field of flow. Such flows must satisfy the Frobenius complete integrability theorem, which requires that H $= A\hat{}dA = 0$. The Cartan topology for such systems is connected, and the Pfaff dimension of the domain is 2 or less. Such domains do not support topological torsion (the 3-form H $= 0$). Such globally laminar flows are to be distinguished from flows that reside on surfaces, but do not admit a unique set of connected sychronizable initial conditions.

Next consider turbulent flows which, as the antithesis of laminar flows, can not be integrable in the sense of Frobenius; such turbulent domains support topological torsion (H $= A\hat{}dA \neq 0$), and therefore a disconnected Cartan topology [74]. The components of the disconnected Cartan topology can be defined as the (topologically) coherent structures induced by the turbulent flow.

Note that a domain can support a homogeneous topology of one component and then undergo continuous topological evolution to a domain with some interior holes. The domain is simply connected in the initial state, and multiply connected in the final state, but still connected. However, consider the dual point of view where the originally connected domain consists of a homogeneous space that becomes separated into multiple components. The evolution to a topological space of multiple components is not continuous. It follows that the case of a transition from an initial laminar state (H $= A\hat{}dA = 0$) to the turbulent state (H $= A\hat{}dA \neq 0$) is a transition from a connected topology to a disconnected topology; therefore the transition to turbulence is NOT continuous. However, note that the decay of turbulence can be described by a continuous transformation from a disconnected topology to a connected topology. Condensation is continuous, gasification is not. It can be demonstrated (see chapter 7 in vol 1, [1]) that relative to the Cartan topology all C2 differentiable, **V**, acting on C2 p-forms by means of the Lie differential are continuous. The conclusion is reached that the transition to turbulence must involve transformations that are not C2, hence can occur only in the presence of shocks or tangential discontinuities.

Chapter 4

SPINTRONICS WITH FALACO SOLITONS AS SURFACE SPIN PAIRS

To summarize the previous discussion: From a topological perspective, a differential 1-form on a differential variety encodes a physical thermodynamic system. The coefficients of a differential N-1=3-form encodes a process vector direction field acting on the thermodynamic system. The dynamics are given in terms of the Lie differential, with respect to a process vector direction field, acting on the system 1-form of Action potentials to produce the differential 1-form of Heat. The resulting formula is an abstract realization of the First Law of Thermodynamics. The 3-forms, to within a density factor, represent Currents in the thermodynamical system.

4.1 Current 3-forms

Process Vector fields are direction fields that can be related to the coefficients of a 3-form on the 4D variety. Vector direction fields $V^k(x,y,z,t)$ are not necessarily global generators of of a 1-parameter group. That is, Kinematic Perfection,

$$dx^k - V^k(x,y,z,t)dt \neq 0 \tag{4.1}$$

may not be true over a neighborhood. In terms of the 1-form,

$$\begin{aligned} A_{kinematic} &= (dx^k - V^k dt), && (4.2)\\ dA_{kinematic} &= dV^k\hat{}dt,\ and && (4.3)\\ A\hat{}dA_{kinematic} &= 0. && (4.4) \end{aligned}$$

Hence kinematic perfection corresponds to a differential system of Pfaff Topological Dimension, PTD($A_{kinematic}$) = 2. This constraint is a highly restrictive topological constraint that denies the viability of kinematic perfection to non-equilibrium systems.

Typically a process vector field has functional components,

$$V = [V^1(\xi^k), V^2(\xi^k), V^3(\xi^k), V^4(\xi^k)], \tag{4.5}$$

relative to the ordered array of base variables, $\{\xi^1, \xi^2, \xi^3, \xi^4\}$, with an oriented differential volume element $\Omega_4 = d\xi^1\hat{}d\xi^2\hat{}d\xi^3\hat{}d\xi^4$. The volume element, or the vector field can be scaled by an arbitrary function, $\rho(\xi^k)$. The N-1=3-form Current, C, can be defined as

$$C = i(\rho V)\Omega_4 = \rho(V^1 d\xi^2\hat{}d\xi^3\hat{}d\xi^4 - V^2 d\xi^1\hat{}d\xi^3\hat{}d\xi^4 \tag{4.6}$$
$$+V^3 d\xi^1\hat{}d\xi^2\hat{}d\xi^4 - V^4 d\xi^1\hat{}d\xi^2\hat{}d\xi^3). \tag{4.7}$$

On the classic 4D differential variety $\{x, y, z, t;\ dx, dy, dz, dt\}$ the process vector direction field can be made homogenous, by dividing each coefficient by V^4.

$$\text{Homogeneous} \quad \mathbf{v} = [V^1/V^4, V^1/V^4, V^1/V^4, 1], \tag{4.8}$$
$$= [\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, 1] \tag{4.9}$$

Then by renaming the 3 spatial components as $\mathbf{J}_3 = \rho\mathbf{v}_3$, the familar format of a current 3-form becomes recognizable:

$$J = i(\rho V)\Omega_4 = (\mathbf{J}^x dy\hat{}dz\hat{}dt - \mathbf{J}^y dx\hat{}dz\hat{}dt \tag{4.10}$$
$$+\mathbf{J}^z dx\hat{}dy\hat{}dt - \rho dx\hat{}dy\hat{}dz. \tag{4.11}$$

The exterior differential of the Current then defines the classic germ of a 4-divergence conservation law, when $dJ \Rightarrow 0$.

$$dJ = di(\rho V)\Omega_4 = \{div(\mathbf{J}_3) + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt. \tag{4.12}$$

However, unlike the Amperian current density (which is closed in the sense that $dJ_{Ampere} = 0$) the general process current is not closed.

There are several important 3-forms of current in a topological perspective:

- **i.** The Amperian Current, (Maxwell 1861).
- **ii.** The Spin Current (1969 [68]).
- **iii.** The Torsion Current (1976 [74]).

The latter two currents have been ignored in most treatments of classical Physics

The Torsion Current is an indicator of non-unique predictability, and if its divergence is not zero, the system process is thermodynamically irreversible and dissipative.

The Spin Current is an indicator of collective states, and if its divergence is not zero, of inertial energy.

The zero divergence conditions of Spin Current and Torsion Current are known as the first and second Poincare conservation theorems in Electromagnetism. From a topological thermodynamic perspective, the concepts are universal.

4.1.1 The Amperian Current

When the current is computed in terms of the Amperian 3-form, $J = dG$, the components of the Current define a process, $\mathbf{J}_4$

$$\begin{aligned}
\text{Amperian Current} \quad \mathbf{J}_4 &= [\mathrm{J}^1, \mathrm{J}^2, \mathrm{J}^3, \rho], && (4.13)\\
dG &= i(\mathbf{J}_4)\Omega_4, && (4.14)\\
ddG &= d(i(\mathbf{J}_4)\Omega_4) = (div4\mathbf{J}_4)\Omega_4 = 0. && (4.15)\\
L_{(\mathbf{T}_4)}A &= i(\mathbf{J}_4)dA + d(i(\mathbf{J}_4)A) && (4.16)\\
&= W_{Lorentz_force} + dU_{interaction_energy.} && (4.17)
\end{aligned}$$

Note that the Amperian Current 3-form is exact, and has Zero 4 Divergence. This Amperian Current charge-current 3-form is ubiquitous in electromagnetic systems.

4.1.2 The Torsion Current

When the current is computed in terms of the Torsion 3-form, $A\hat{}F$, then the components of the Current define a process, $\mathbf{T}_4$

$$\begin{aligned}
\text{Torsion Current} \quad \mathbf{T}_4 &= [\mathrm{T}^1, \mathrm{T}^2, \mathrm{T}^3, \mathrm{T}^4], && (4.18)\\
A\hat{}F &= (i(\mathbf{T}_4)\Omega_4, && (4.19)\\
L_{(\mathbf{T}_4)}A &= i(\mathbf{T}_4)dA + d(i(\mathbf{T}_4)A) && (4.20)
\end{aligned}$$

Note that the 4-divergence (exterior differential) of the Torsion Current can be expressed as,

$$\begin{aligned}
d(i(\mathbf{T}_4)\Omega_4 &= (div4\mathbf{T}_4)\Omega_4 = d(A\hat{}F) = F\hat{}F\ , && (4.21)\\
\text{Poincare II} \quad d(i(\mathbf{T}_4)\Omega_4) &= F\hat{}F = \mathrm{K}\Omega_4. && (4.22)
\end{aligned}$$

which is the Topological Parity 4-form. The coefficient, when zero, is known as the second Poincare invariant.

$$\begin{aligned}
L_{(\mathbf{T}_4)}A\hat{}F &= (i(\mathbf{T}_4)d(A\hat{}F) + d((i(\mathbf{T}_4)(i(\mathbf{T}_4)\Omega_4, && (4.23)\\
&= (i(\mathbf{T}_4)(F\hat{}F). && (4.24)
\end{aligned}$$

$$\begin{aligned}
\text{However} \quad (i(\mathbf{T}_4)A)\Omega_4 &= A\hat{}A\hat{}dA = 0 && (4.25)\\
L_{(\mathbf{T}_4)}A &= i(\mathbf{T}_4)dA = \sigma A && (4.26)\\
L_{(\mathbf{T}_4)}dA &= di(\mathbf{T}_4)dA = d\sigma A + \sigma dA && (4.27)\\
Q\hat{}dQ &= \sigma^2 A\hat{}dA \neq 0 && (4.28)
\end{aligned}$$

Note that the 4-divergence of the Torsion current leads to an expanding or contracting differential volume element, and can be associated with irreversible dissipation (bulk viscosity in the hydrodynamic case, and $\mathbf{E} \circ \mathbf{B} \neq \mathbf{0}$ in the electromagnetic case). Any process that has the direction field of $\mathbf{T}_4$ is an irreversable process, as $Q\hat{}dQ \neq 0$.

4.1.3 The Spin Current

When the current is computed in terms of the Spin 3-form, $A\hat{}G$, then the components of the Current define a process, $\mathbf{S}_4$

$$\begin{aligned} \text{Spin Current} \quad \mathbf{S}_4 &= [\mathrm{S}^1, \mathrm{S}^2, \mathrm{S}^3, \mathrm{S}^4], \quad (4.29) \\ A\hat{}G &= i(\mathbf{S}_4)\Omega_4, \quad (4.30) \\ \text{Poincare I} \ \ d(i(\mathbf{S}_4)\Omega_4) &= (div4\mathbf{S}_4)\Omega_4 = F\hat{}G - A\hat{}J. \quad (4.31) \end{aligned}$$

It is these features that imply that superfluidity and superconductivity are related to zero values of the Poincare invariants.

4.1.4 The London equation of superconductivity. $J_{London} = \chi A$

It the historical literature, the concept of superconductivity was conjectured to be described by a modification of Maxwell's equations, called London's equation. The fundamental assumption was that the superconductor Charge-Current 3-vector density, $\mathbf{J}$, was proportional to the 3-Vector potential, $\mathbf{A}$,

$$\text{London Supercurrent} \quad \mathbf{J}_{London} = \boldsymbol{\chi}\mathbf{A}. \quad (4.32)$$

The implication is that Maxwell's equations must be modified to include the supercurrent. From a topological perspect, this conclusion is false. That is, the London equation is contained within the topological format of electromagnetism. Several examples are given in vol 4 [1]. For a specific choice(s) of the electrodynamic potential(s) (the thermodynamic 1-form of Action), the Amperian current(s) $\mathbf{J}_4$ can be computed* and demonstrate that the 4 components of the Amperian Current, $\mathbf{J}_4$, are proportional to the 4 components of the electrodynamic potentials, $\mathbf{A}_4$.

$$\mathbf{J}_4 = [\mathrm{J}^x, \mathrm{J}^y, \mathrm{J}^z, \rho] = \chi\mathbf{A}_4 = \chi[A_x, A_y, A_z, -\phi] \quad (4.33)$$

The first 3-components yield the London equation.

4.1.5 The Inertial equation of Spin. $J_{spin} = \lambda W$

Another important equation that can be extracted from the topological perspective of electromagnetism (that does not seem to appear in the historic literature) is that for specific choice(s) of the electrodynamic potential(s) (the thermodynamic 1-form of Action), the Spin currents $\mathbf{S}_4$ can be computed and demonstrate that the 4 components of the Work 1-form, W, are proportional to the 4 components of the Spin current, $\mathbf{S}_4$, is proportional to the 3 components of the Vector potential, $\mathbf{A}$.

$$\mathbf{W}_4 = [W_x, W_y, W_z, P] = \epsilon\chi\mathbf{S}_4 = \epsilon\chi[\mathrm{S}^x, \mathrm{S}^y, \mathrm{S}^z, \mathrm{S}^t] \quad (4.34)$$

Remark 11 *The Spin Current can be proportional to the Work 1-form, while the Amperian Current can be proportional to the Action 1-form. The physical impact of this Spin current formulation is under study.*

*A Maple program was constructed for the somewhat tedious calculations.

4.2 Topological Insulators, Superconductors and the Topological Spin Hall effect

E. J. Post recognized that one of the most remarkable features of electrodynamics of the micro world compared to the macro world is that when written in terms of the MKS system of units, the fine structure constant, α, becomes,

$$\alpha = 2\pi e^2/4\pi\varepsilon\, hc = 1/2(\mu/\varepsilon)^{1/2}/(h/e^2). \tag{4.35}$$

This formula demonstrates that α is a ratio of two fundamental impedances, the free-space impedance, Z_0,

$$Z_0 = (\mu/\varepsilon)^{1/2} = 376.730313\Omega, \tag{4.36}$$

and the Hall impedance, Z_{Hall},

$$Z_{Hall} = h/e^2 = 25812.81491\Omega. \tag{4.37}$$

The relation between the quantum mechanical entities and the free space impedance as given by the equation,

$$\alpha = 1/2(Z_0/Z_{Hall}), \tag{4.38}$$

which elevates the importance of the free space impedance, Z_0.

The objective is to define superconductivity in a topological manner which incorporates the "quantization" features of deRham cohomology theory. These topological defect structures can be evaluated in terms of period integrals, which are integrals over closed domains of closed integrands. Such integrals are topologically quantized in the sense that the values of the integrals are intergers times a scaling constant. If the closed integration domain is a boundary, not a cycle, then the integer is zero, and the closed and bounded set is an invariant set with respect to any thermodynamic process. If the closed integrand is a 1-form, and the integration chain is a cycle, but not a boundary, then the period integral has been denoted as the Berry phase, but it was predicted long before by Bohm and Aharanov.

The idea of forming a ratio of period integrals to define an impedance was an idea of E. J. Post [62]. In fact, Post predicted that the Hall effect would exhibit rational fraction behavior two years before the experimental measurements were made. He did this without the assumption of fractional charge. The implication is that superconductivity is related to topological defect structures. The period integrals are exhibitions of the deRham theory of cohomology, and the quantization result is independent from scales.

There are three ways to construct an impedance Z (with physical dimensions (h/e^2) from period integrals [84]:

Ordinary Superconductors	Impedance $\mathsf{Z}_1 = \oint A/\iint_{z2} G$.
Anyon (High Tc ?)	$\mathsf{Z}_2 = \iiint_{z3} A\hat{\ }G/(\iint_{z2} G)^2$.
Fractional Hall	$\mathsf{Z}_3 = \iiint_{z3} A\hat{\ }F/\iiint_{z3} A\hat{\ }G$.

Post chose Z_1. In order to produce rational fractions, the closed integrals must be period integrals, where the integrands are closed in an exterior differential sense over the closed domains (cycles of one, two, or three dimensions) of integration. The closure condition on the first impedance Z_1 requires that $dA \Rightarrow 0$, which implies that the domain excludes the field intensities. This constraint is in agreement with the experimentally measured Meissner repulsion of the **B** field in ordinary superconductors, and the Bohm-Aharanov effect. The closure condition on the third impedance, Z_3 requires that both Poincare 4-forms must vanish, but **E** and **B** fields are permitted in the domain of integration (as is observed experimentally in the Hall effect). I believe, counter to Post, that Z_3 is the proper expression for the Hall effect, not Z_1. Z_3 is the ratio of the closed integrals of Topological Torsion, $A\hat{}F$, divided by the closed integrals of Topological Spin, $A\hat{}G$. Moreover, on thermodynamic grounds, the non-zero value of Topological Torsion, $A\hat{}F$, indicates that the Hall effect is an artifact of a non-equilibrium system.

Note that Ordinary Superconductors are such that the field intensities are expelled from the interior, and $dA = F \Rightarrow 0$. This criteria implies that this class of electromagnetic systems is of Pfaff Topological Dimension 1, which is an equilibrium system. The Fractional Hall superconductor admits a non-zero Topological Torsion, $A\hat{}F \neq 0$, and a non-zero Topological Spin, $A\hat{}G$. Hence such Hall effect superconductors are not equilibrium or isolated systems, but instead are non-equilibrium electromagnetic systems. However, the Poincare 4-forms vanish, enabling the fractional quantization, which implies that such systems are of Pfaff Topological Dimension 3. They are closed, not open, thermodynamic configurations.

4.2.1 Interaction Energy density and Topological Superconductivity

The conjecture to be explored herein is that a supercurrent corresponds to the case where the electromagnetic interaction energy density, $A\hat{}J$, vanishes in a topological sense. The motivation for such an assumption is founded upon the observation that if the 3-form of charge-current density, J, was proportional to either the 3-form of Topological Torsion, $J_{Torsion} = \gamma \cdot A\hat{}F$, or the 3-form of Topological Spin, $J_{spin} = \eta \cdot A\hat{}G$, then it follows that the interaction energy density of classical field theory will vanish, $A\hat{}J \Rightarrow 0$:

$$A\hat{}J_{Torsion} = A\hat{}(\gamma \cdot A\hat{}F) = \gamma \cdot (A\hat{}A\hat{}F) = 0, \tag{4.39}$$

$$A\hat{}J_{Spin} = A\hat{}(\eta \cdot A\hat{}G) = \eta \cdot (A\hat{}A\hat{}G) = 0. \tag{4.40}$$

Assume that a supercurrent contains components proportional to Topological Torsion 3-form, $\gamma \cdot A\hat{}F$, and the Topological Spin 3-form, $\eta \cdot A\hat{}G$. In order for *each* of the supercurrent components to be conserved,

$$d(J_{Torsion}) = d\gamma\hat{}A\hat{}F + \gamma \cdot F\hat{}F \Rightarrow 0, \tag{4.41}$$

$$d(J_{Spin}) = d\eta\hat{}A\hat{}G + \eta \cdot F\hat{}G \Rightarrow 0. \tag{4.42}$$

Simple solutions are found if both Poincare 4-forms vanish, and the functions γ and η are constants. More complicated solutions can occur if the sum of the two supercurrents has a zero divergence,

$$d(J_{\text{supercurrent}}) = d(J_{Torsion} + J_{Spin}) \Rightarrow 0. \tag{4.43}$$

In other words, the divergence of the torsion current and the spin current could compensate to form a total supercurrent which is conserved.

Another case would be to consider those situations where the 3-form charge-current density has components proportional to those components of the 1-form of potentials which are elements of a spinor, $J = \lambda J_{spinor}$. The 3-form can always be multiplied by an integrating factor such that the rescaled spinor current has zero divergence. Similarly, suppose the 1-form A_{spinor} (to within a factor) also has the same spinor component functions. Then the interaction density vanishes, as,

Spinor London current

$$J_{spinor} = i(\lambda A_{spinor})\Omega, \tag{4.44}$$

Interaction Energy density

$$A\hat{}J = \lambda \langle A_{spinor} \circ A_{spinor} \rangle \ \Omega_4 \Rightarrow 0. \tag{4.45}$$

Hence, a charge-current 3-form composed of three parts, such that,

Total Supercurrent

$$J_{\text{supercurrent}} = J_{spinor} + A\hat{}F/\lambda + A\hat{}G/\eta, \tag{4.46}$$

With Interaction Energy density

$$A\hat{}J = A\hat{}J_{\text{supercurrent}} \Rightarrow 0, \tag{4.47}$$

is a candidate for a superconducting current, which intuitively has no interaction energy density.

If the Action 1-form is divided by a suitable quadratic Holder norm, then the Jacobian matrix of the Action 1-form can be computed. The matrix has a determinant zero, if the homogeneity index is 1. The matrix defines the equivalent to the Shape matrix in differential geometry, and its Cayley–Hamilton similarity invariants define the curvatures generated by the zero set of the Cayley-Hamilton characteristic polynomial. If the current $J_{adjoint}$ is defined as the product of the adjoint of the shape matrix times the homogeneous coefficients of the 1-form of Action, then the homogeneous interaction energy,

$$\text{Homogeneous Interaction energy}: A_{\text{homogeneous}}\hat{}J_{\text{adjoint}}, \tag{4.48}$$

is exactly equal to the cubic curvature similarity invariant.

4.2.2 Irreversible Evolutionary Processes (Pfaff Topological Dimension 4)

Assume that the Pfaff Topological Dimension of the domain of interest is 4, hence the space is symplectic. However, consider evolutionary fields that are not constrained to be symplectic such that $dW \neq 0$. Direct evaluation of the virtual work 1-form, $W = i(\mathbf{V}_4)dA$ yields (the Lorentz force),

$$W = i(\mathbf{V}_4)dA = -(\{\rho\mathbf{E} + \mathbf{J}x\mathbf{B}\}_k dx^k - \{\mathbf{J} \bullet \mathbf{E}\}dt). \tag{4.49}$$

The obvious first choice for the evolutionary vector field has been based on the classic assumption that $\mathbf{V}_4$=$[\mathbf{J};\rho] \Rightarrow \rho[\mathbf{V};1]$. The expression for virtual work becomes,

$$W = -\rho(\{\mathbf{E} + \mathbf{V}x\mathbf{B}\}_k dx^k - \{\mathbf{V} \bullet \mathbf{E}\}dt). \tag{4.50}$$

However, another perhaps not so obvious a candidate for a solution vector field is the expression for the Torsion current. That is, examine the evolution along the unique four-dimensional vector field,

$$\mathbf{T}_4 = -\{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \bullet \mathbf{B}\}. \tag{4.51}$$

The expression for virtual work becomes

$$\begin{aligned} W &= i(\sigma\mathbf{T}_4)dA && (4.52)\\ &= \sigma(\{(\mathbf{A} \bullet \mathbf{B})\mathbf{E} + (\mathbf{E} \times \mathbf{A}) \times \mathbf{B}\}_k dx^k - \{\mathbf{E} \bullet \mathbf{B}\phi\}dt) && (4.53)\\ &= \sigma(\mathbf{E} \bullet \mathbf{B})A. && (4.54)\end{aligned}$$

The torsion current is an associated field relative to the 1-form of Action, in the sense that

$$i(\sigma\mathbf{T}_4)A \Rightarrow 0. \tag{4.55}$$

Evolution in the direction of the Torsion vector does not produce any internal energy of interaction, even though the process is not extremal. The $\mathbf{T}_4$ process is thermodynamically locally[†] adiabatic. In Pfaff Topological Dimension 4, the Torsion vector is not extremal, but adiabatic. It is amazing that $\mathbf{T}_4$ can decay to a process which is characteristic; i.e., a process which is adiabatic and homogeneous of degree zero.

It follows that the Lie differential of the Action along the direction of the Torsion current is a special isovector process in the sense that

$$L_{(\sigma\mathbf{T}_4)}A = \Gamma A = \sigma(\mathbf{E} \bullet \mathbf{B})A = Q. \tag{4.56}$$

By direct computation,

$$L_{(\sigma\mathbf{T}_4)}dA = d\Gamma\hat{}A + \Gamma dA = dQ \tag{4.57}$$

[†]Locally adiabatic means along a flow line, but not necessarily from flow line to flow line.

from which it follows that

$$Q\hat{}dQ = \Gamma^2 A\hat{}dA. \tag{4.58}$$

If the topological parity $\Gamma = \sigma(\mathbf{E} \bullet \mathbf{B})$ does not vanish, then the Torsion current $\sigma\mathbf{T}$ represents an irreversible non-conservative process. For such processes the Heat 1-form, Q, does not admit an integrating factor.

The formula $L_{(\sigma\mathbf{T}_4)}A = \Gamma A$ was, in effect, the fundamental equation that I used in 1974 to describe "An Extension of Hamilton's Principle to Include Dissipative Systems" [69] [73, RMK 1975]. It was not known at that time the such processes implied the existence of a symplectic structure, nor the fact that these processes were not symplectomorphisms.

4.3 Hedgehog fields, Rotating plasmas, Accretion discs, Spin currents

It is possible to find a modification of a closed 1-form solution to Maxwell's equations that makes the magnetic field lines appear like the spines of a Hedgehog. It is also possible to demonstrate how such modifications of closed 1-forms make the z=0 plane of a rotating plasma a chiral attractor.

Example 1. Plasma Accretion disc from Hedgehog B field solutions

When the **B** field is radial in all directions, and either inbound or outbound, the field is called a "hedgehog" magnetic field. At first glance it would appear that there is a magnetic charge monople similar to an electric charge monopole This interpretation is inconsistent with the Maxwell-Faraday postulate, $F - dA = 0$. Hence, the topology generated by the potentials is not simply connected. If the Topological Torsion tensor vanishes, the thermodynamically the domain is an isolated-equilibrium system.

Many such structures can be generated by multiplying a closed, but not exact, 1-form of the type,

$$A_{closed} = \varpi[-y, x, 0]/(x^2 + y^2), \quad \phi = 0 \tag{4.59}$$
$$dA_{closed} = 0, \quad \varpi = \pm 1 \tag{4.60}$$

by a function $\Gamma(x, y, z)$. For example, the following functions all lead to hedgehog magnetic fields:

$$\Gamma(x, y, z) = \alpha z/\sqrt{x^2 + y^2 + cz^2}, \tag{4.61}$$
$$\Gamma(x, y, z) = \alpha(x^2 + y^2)/(x^2 + y^2 + cz^2) \tag{4.62}$$
$$\Gamma(x, y, z) = \alpha z^2(x^2 + y^2)/(x^2 + y^2 + cz^2)^2, \tag{4.63}$$

The closed 1-form, A_{closed}, represents a rotation, without **E** and **B** fields, and is singular along the z axis. In this example presented below the function Γ is chosen to be:

$$\Gamma(x,y,z) = \alpha\varpi z/\sqrt{x^2+y^2+cz^2}, \tag{4.64}$$
$$\text{with } \lambda = \sqrt{x^2+y^2+cz^2} \quad \text{and} \quad \varpi = \pm 1. \tag{4.65}$$

The coefficient c is chosen in order to examine the anisotropy effects of oblateness or prolateness relative to the rotation anout the z axis. The formulation presented above corresponds to a counter clockwise rotation (for positive ϖ). The singularity is the origin for a positive signature (prolate), and on a cone for the negative signature (oblate). If however, $\Gamma = \varpi z$, then the singularity is along the z axis, and is remindful of the Dirac monopole with a string.

These potentials induce the field intensities and charge-current densities below: Note that the **B** field is inbound for counterclockwise rotation, and outbound for clockwise rotation (negative ϖ). Also note that the Lorentz constitutive charge current is proportion to the 1-form of Action, $J = \chi A$. (This is to be recognized as the London current, $\mathbf{J} = \chi\mathbf{A}$).

$$\text{Hedgehog } \mathbf{B} = -\alpha\varpi[x,y,z]/(\lambda)^3, \tag{4.66}$$
$$\mathbf{E} = [0,0,0], \tag{4.67}$$
$$\mathbf{J}_4 = [J,\rho_{em}] \quad \text{Lorentz Constitutive Current} \tag{4.68}$$
$$\text{Current } \mathbf{J} = (3\alpha z\varpi(c-1)/\mu)[-y,x,0]/(\lambda^5), \qquad \rho_{em} = 0, \tag{4.69}$$
$$= 3\alpha(c-1)(x^2+y^2)/(\mu\lambda^4)\cdot\mathbf{A} = \chi\mathbf{A} \tag{4.70}$$
$$\text{London coefficient} \quad \chi = (c-1)(x^2+y^2)/(\mu\lambda^4) \tag{4.71}$$
$$\text{Top. Spin } A\hat{\ }G = i(\mathbf{S}_4)\Omega_4, \qquad \mathbf{S}_4 = [\mathbf{S},\rho_{spin}] \tag{4.72}$$
$$\mathbf{S} = (z\alpha^2\varpi^2/\mu\zeta)[-zx,-zy,(x^2+y^2),0], \ \rho_{spin} = 0. \tag{4.73}$$
$$\zeta = \{(x^2+y^2)(\lambda)^4\} \tag{4.74}$$
$$\text{Poincare I } d(A\hat{\ }G) = (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi), \tag{4.75}$$
$$= (\alpha^2\varpi^2/\mu)(x^2+y^2+(4-3c)z^2)/(\mu\lambda^6), \tag{4.76}$$
$$\mathbf{D}\circ\mathbf{E} = 0, \quad \mathbf{B}\circ\mathbf{H} = \alpha^2\varpi^2(x^2+y^2+z^2)/(\mu\lambda^6) \tag{4.77}$$
$$\rho\phi = 0 \qquad \mathbf{A}\circ\mathbf{J} = 3z^2\alpha^2\varpi^2(c-1)/(\mu a\lambda^6) \tag{4.78}$$
$$\text{Top. Torsion } A\hat{\ }F = i(\mathbf{T}_4)\Omega_4 = 0, \tag{4.79}$$
$$\mathbf{T}_4 = [0,0,0,0], \text{ PTD} = 2 \text{ isolated equilibirium} \tag{4.80}$$
$$\text{Poincare II } d(A\hat{\ }F) = 2(\mathbf{E}\circ\mathbf{B}) = 0, \tag{4.81}$$

The Lorentz force and the dissipative terms become:

$$
\begin{aligned}
-\{\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\} &= \text{Lorentz force}, && (4.82)\\
&= (3(c-a)z\varpi^2/\mu)[zx, zy, -(x^2+y^2)]/(\lambda)^8, && (4.83)\\
&= 3\chi\cdot\mathbf{S}_4, \quad \text{Spin Inertia !!!} && (4.84)\\
\text{Power } \mathbf{J}\circ\mathbf{E} &= 0 && (4.85)\\
\text{Poynting Vector } \mathbf{E}\times\mathbf{H} &= [0,0,0], && (4.86)\\
\text{Interaction Energy } A\hat{}J &= 3z^2m^2(c-1)/(\mu\lambda^3), && (4.87)\\
\text{Lagrange Field } F\hat{}G &= z^2m^2(c-1)/(\mu\lambda^3) && (4.88)\\
\text{Poincare II } \quad d(A\hat{}F) &= 2(\mathbf{E}\circ\mathbf{B}) = 0, && (4.89)\\
\text{Helicity } \mathbf{A}\circ\mathbf{B} &= 0 \quad \text{(TTM mode as A\^{}F = 0)}, && (4.90)\\
\text{Chirality } \mathbf{A}\circ\mathbf{D} &= 0 \quad \text{(but not a TTE mode as A\^{}G is not zero)} && (4.91)
\end{aligned}
$$

Note that the current density is proportional to the 1-form of Action with a London order parameter equal to χ, and the Lorentz Force is proportional to the Spin current with the same factor times 3!

The potentials, the charge-current and the **B** field are sensitive to the rotation direction (orbital angular momentum), which is governed by the parameter, ϖ. The Topological Spin and the Lorentz force are not sensitive to the circulation sense, ϖ, as they are quadratic in the rotation parameter. For the isotropic case, $a = c$, there is no induced current density, no Lorentz force, and yet the Topological Spin density is non-zero. For any c/a ratio, the dissipative power, and the Poynting vector, vanish. The first Poincare invariant is not zero

It is important to note that the zero value for the charge density, ρ_{em}, implies no "net" charge. This will be defined as charge pairing; the volume contains the same number of plus charges and minus charges. The zero value for the charge density, ρ_{spin}, implies no net spin. This will be defined as spin pairing; the volume contains the same number of plus spins as minus spins. The arguement establishes a topological foundation for the concept of Cooper pairs (spin pairing of electrons) and massive photons (spin pairing of Bosons).

The singularities for the potentials (as given in the example) are the z-axis and the origin (for positive anisotropy coefficients). If c is negative and a is positive, then the singular set for the denominator, $ax^2+ay^2-cz^2 \Rightarrow 0$, generates a cone. The cone is centered on the z-axis, with its vertex at the origin. The cone is oblate when the ratio $|a/c|$ is very small, and prolate when $|a/c|$ is very large. The prolate cone is remindful of jets, and the oblate cone is remindful of flat spirals and Cherenkov radiation.

The magnetic field in the example is "hedgehog radial". Although its divergence is zero everywhere, the lines of the field do not close on themselves. The **B** field starts and stops on points of a topological boundary of a not simply connected isolated-equilibrium themodynamic system. Note that the rotational orientation of the flux circulation integral (whether ϖ is positive or negative) determines whether or not the **B** field is pointing inbound or outbound relative to the origin. The topological (radial) orientation is related to the rotational sense in this example. It is also true that the induced current **J** (linear in ϖ) depends upon the rotational orientation, and it is proportional to the vector potential, but the field is opposite in a rotational sense. However the Lorentz force, the Topological Torsion and the Interaction Energy do not depend upon the sense of the rotational circulation, as they are quadratic in ϖ.

The Hedgehog **B** field in this example has zero divergence everywhere except at the zeros of the denominator, which herein are interpreted as topological obstructions, or defects, to be excised from consideration.

In this example, the non-zero plasma current density, **J** , has a sense of "circulation" about the z-axis, and is proportional to the vector potential representing a rotation. This deduced result has the format of a London current, $\mathbf{J} = \chi \cdot \mathbf{A}$. The "London" parameter , χ, due to the rotation is,

$$\chi = (3(c-1)/\mu) \cdot (x^2 + y^2)/(x^2 + y^2 + cz^2)^2. \tag{4.92}$$

The London formula depends strongly upon the anisotropy, which is confirmed in superconductivity experiments. Indeed the hedgehog example in an anisotropic situation generates a non-dissipative current density with some of the attributes of a super current. Also note that the spatial components of the Topological Spin are proportional to the direction field of the spatial components of the Lorentz force. The proportionality factor is exactly equal to 3 times the (same!) London parameter, χ. It appears that the Spin current is related to inertia. Note that the Lorentz charge current density is closed, but the Spin current density is not closed: $d(A\hat{}G) \neq 0$. This result can be interpreted as the production of an energy density due to production of spin-paired Bosons (massive photons). The combination leads to the speculation that dynamic inertia can be associated with massive photons.

The formula for the Lorentz force demonstrates that the system of circulating currents is directed radially away (centrifugally) from the rotational axis, and yet is such that the plasma is attracted to the $z = 0, xy$ plane. Independent of the sense of rotation, but dependent upon the anisotropy, the Lorentz force is divergent in the radial plane and convergent in the direction of the z-axis, towards the $z = 0$ plane. The conjecture is that this electromagnetic field for the rotating plasma would have the tendency to form an accretion disk of plasma in the presence of a central gravitational field.

It is apparent that the helicity density and the second Poincare 4-form are zero. In fact, the 3-form of Topological Torsion vanishes identically. Although the 3-form of Topological Torsion vanishes identically, the 3-form of Topological Spin is not zero. It is also true that the divergence of the 3-form of the Topological Spin, $A\hat{}G$, is not zero, for the first Poincare 4-form is not zero.

If the system is spherical, then the deformation parameter vanishes, $a = c$, the

Lorentz force vanishes, the charge-current 3-form vanishes, but the Topological Spin does not vanish and the first Poincare 4-form does not vanish. There is a residual field energy related to $\mathbf{B} \circ \mathbf{H}$. and a potential massive photon (spin paired Boson).

Example 2. Other B field Hedgehog solutions

The famous Dirac monopole is considered to be a magnetic hedgehog solution similar to the ubiquitous Hedgehog **E** field distribution from an isolated charge. The Dirac radial **B** field solution given in the literature [38] decays with the inverse cube of the radial distance, r. In the previous example the Hedgehog **B** field decays with the inverse square of a length. The Dirac Hedgehog potentials were presumed to be imaginary, but that is not necessary.

For this example which leads to the inverse cube hedgehog, consider the choice

$$\begin{aligned} \mathbf{A} &= \Gamma(x,y,z)[-y,x,0]\ , \quad \text{CCW rotation} && (4.93)\\ \textit{with} \quad \Gamma &= (\alpha\varpi/2)/\{\lambda\}, \quad \varpi = \pm 1, && (4.94)\\ \phi &= 0, \quad \textit{and} \quad x^2+y^2+cz^2 = \lambda. && (4.95) \end{aligned}$$

The only difference between this example and the previous hedgehog example is the divisor is $\lambda = x^2+y^2+cz^2$, not $\lambda = (x^2+y^2)\sqrt{x^2+y^2+cz^2}$. These potentials induce the field intensities and charge-current densities below:

$$\begin{aligned} \text{Hedgehog} \quad \mathbf{B} &= 2\alpha\varpi zc[x,y,z]/(\lambda)^2, && (4.96)\\ \mathbf{E} &= [0,0,0], && (4.97)\\ \mathbf{D} &= \epsilon\mathbf{E} \qquad \mathbf{H} = \mathbf{B}/\mu, && (4.98)\\ \text{London Current} \quad \mathbf{J} &= 2\alpha mc(\zeta)[-y,x,0] = \chi\mathbf{A}, \quad \rho_{em} = 0, && (4.99)\\ \zeta &= \{(4-3c)z^2+x^2+y^2)\}/(\mu\lambda^3) && (4.100)\\ \text{London coefficient} \quad \chi &= 2c\{(4-3c)z^2+x^2+y^2)\}/(\mu\lambda^2) && (4.101)\\ \text{Top. Spin} \quad A\hat{\ }G &= i(\mathbf{S}_4)\Omega_4, \qquad \mathbf{S}_4 = [\mathbf{S},\ \rho_{spin}] && (4.102)\\ \mathbf{S} &= (z\alpha^2/2\mu r^6)[zx, zy, -(x^2+y^2), 0], && (4.103)\\ \mathbf{D}\circ\mathbf{E} &= 0, \quad \mathbf{B}\circ\mathbf{H} = 4c^2z^2\alpha^2/(\mu\lambda^3) && (4.104)\\ \rho_{em}\phi &= 0 \qquad \mathbf{A}\circ\mathbf{J} = (\alpha^2/4)(x^2+y^2)\chi/(\lambda^2) && (4.105)\\ \text{Top. Torsion} \quad A\hat{\ }F &= i(\mathbf{T}_4)\Omega_4, && (4.106)\\ \mathbf{T}_4 &= [0,0,0,0], && (4.107)\\ \text{Poincare II} \quad d(A\hat{\ }G) &= 2\alpha^2c\{(x^2+y^2)5cz^2 - 4z^2(x^2+y^2) && (4.108)\\ &\quad +2cz^4 - (x^2+y)^2\} && (4.109)\\ \text{Poincare I} \quad d(A\hat{\ }F) &= 2(\mathbf{E}\circ\mathbf{B}) = 0, && (4.110) \end{aligned}$$

Note that the current density is proportional to the 1-form of Action with a London order parameter equal to χ, and the Lorentz Force is proportional to the Spin current with the same factor!

$$\begin{aligned}
\text{Poynting Vector } \mathbf{E}\times\mathbf{H} &= [0,0,0], && (4.111)\\
-\{\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\} &= \text{Lorentz force}, && (4.112)\\
&= \chi\mathbf{S} = \boldsymbol{\chi}\{(z\alpha^2/2\mu r^6)[zx, zy, -(x^2+y^2), 0]\} && (4.113)\\
\text{Dissipative power } \mathbf{J}_{amperian}\circ\mathbf{E} &= 0 && (4.114)\\
\text{Lagrange Field energy } F\hat{}G &= 4c^2z^2\alpha^2/(\mu\lambda^3) && (4.115)\\
\text{Interaction Energy } A\hat{}J &= (1/4)(x^2+y^2)\chi/(\lambda^2). && (4.116)\\
\text{Helicity } \mathbf{A}\circ\mathbf{B} &= 0 \quad \text{(TTM mode)}, && (4.117)\\
\text{Chirality } \mathbf{A}\circ\mathbf{D} &= 0 \quad \text{(but not TTE mode)}. && (4.118)
\end{aligned}$$

Example 3. Electromagnetic Pump without current flow

A simple electromagnetic pump consists of an insulated square section of steel pipe inserted into a circuit of pipe that contains liquid metal. The square boundary sections of the pipe will be labeled x and y, the liquid metal direction will be labeled z. An external field **B** is applied to the x direction, and an external electric field **E** is applied to the y direction. Voila: the liquid metal moves in the direction of $\mathbf{E}\times\mathbf{B}$. Why?? The usual answer given is that if the applied **E** field induces a current flow, then the $\mathbf{J}\times\mathbf{B}$ component of the Lorentz force causes flow of the conductive material in the direction of $\mathbf{J}\times\mathbf{B}$. However, suppose the applied **E** field is prevented from producing an ohmic current, as in the field of a capacitor? In other words the system behaves like an insulator.

Consider that the vector and scalar potentials are given by the expressions:

$$A = [0, -(\mathbf{B}_x/2)z, +(\mathbf{B}_x/2)y, \mathbf{E}_y y], \tag{4.119}$$

The induced fields (assuming $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$) are:

$$
\begin{aligned}
\mathbf{E} &= [0,-\mathbf{E}_y,0] \quad \mathbf{B}=[\mathbf{B}_x,0,0], && (4.120)\\
\text{Lorentz Current } \mathbf{J}_4 &= [0,0,0,0], \quad \rho_{em}=0, && (4.121)\\
\text{Topological Spin } \mathbf{S}_4 &= [\mathbf{S},\rho_{spin}], && (4.122)\\
\mathbf{S} &= [0,(\mathbf{B}_x^2-2\mathbf{E}_y^2\varepsilon\mu)y,\mathbf{B}_x^2 z,]/2\mu, && (4.123)\\
\rho_{spin} &= (\mathbf{E}_y\mathbf{B}_x\varepsilon\mu)z/2\mu, && (4.124)\\
\textit{Poincare I} &= (\mathbf{B}_x^2-\mathbf{E}_y^2\varepsilon\mu), && (4.125)\\
\text{Top. Torsion } \mathbf{T}_4 &= (\mathbf{E}_y\mathbf{B}_x/2)[-y,0,0,0], && (4.126)\\
\textit{Poincare II} &= 2(\mathbf{E}\circ\mathbf{B})=\mathbf{0}, \quad (\mathbf{J}\circ\mathbf{E})=0, && (4.127)\\
\text{Lorentz Force} &= [0,0,0], && (4.128)\\
\text{Poynting vector } \mathbf{E}\times\mathbf{H} &= (\mathbf{E}_y\mathbf{B}_x/\mu)[0,0,1], && (4.129)\\
\textbf{Helicity } (\mathbf{A}\circ\mathbf{B}) &= 0, \text{ but not } \textbf{TTM as } \mathbf{T}\neq\mathbf{0}. && (4.130)\\
\textbf{Chirality } (\mathbf{A}\circ\mathbf{D}) &= 0, \text{ but not } \textbf{TTE as } \mathbf{S}\neq\mathbf{0}\ . && (4.131)
\end{aligned}
$$

The first thing to note is that there is no Lorentzian-Maxwell-Amperian current, $\mathbf{J}_4$. Is their any induced motion of say a gas or fluid within an insulated tube? It is apparent that there are both finite Topological Torsion currents and Topological Spin Currents, and that Spin Pairing effects can take place, especially if the region is such that the first Poincare 4-form is not zero, The thermodynamics of the system is then of PTD=3; the on-shell photons occur when the First Poincare 4-form vanishes:

$$\text{On shell photons} \qquad \textit{Poincare } I = (\mathbf{B}_x^2-\mathbf{E}_y^2\varepsilon\mu) \Rightarrow 0 \tag{4.132}$$

When the offshell photons are present they are propagated in the direction of the Poynting vector. Do these "heavy photons", which are embedded in the fluid or the gas , make it move. It appears that Current Quantum Field Theory and proponents of dark energy would have us to believe it is so. Come on, experimenters, give us an answer. Could this be an example of a fluid with coupled spins, or of an insulator in the presence of of a n E field, where there is no Amperian current, but there is both a Torsion current and a Spin current.

Example 4. Coulomb singularity

Consider the Potentials for the Coulomb 1/r potential:

$$
\begin{aligned}
A &= [0,0,0,-(1/4\pi\varepsilon)q/r], && (4.133)\\
r &= \sqrt{x^2+y^2+z^2}. && (4.134)
\end{aligned}
$$

The induced fields (assuming $\mathbf{D}=\varepsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$) are:

$$\begin{aligned}
\mathbf{E} &= (1/4\pi\varepsilon)q[x,y,z]/r^3, && (4.135)\\
\mathbf{B} &= [0,0,0], && (4.136)\\
\mathbf{J}_4 &= [0,0,0,0], && (4.137)\\
\rho_{em} &= 0, && (4.138)\\
\text{Hedgehog Top. Spin } \mathbf{S}_4 &= (1/4\pi\varepsilon)^2\varepsilon q^2/r^4[x,y,z,0], \qquad \rho_{spin}=0, && (4.139)\\
\mathit{Poincare\ I} &= -(1/4\pi\varepsilon)^2\varepsilon q^2/r^4, && (4.140)\\
\text{Top. Torsion } \mathbf{T}_4 &= (1/4\pi\varepsilon)q\varpi[-zx,-zy,(x^2+y^2),0]/(r^3(x^2+y^2)), && (4.141)\\
\mathit{Poincare\ II} &= 2(\mathbf{E}\circ\mathbf{B}) = \mathbf{0}, \quad (\mathbf{J}\circ\mathbf{E}) = 0, && (4.142)\\
\mathbf{E}\times\mathbf{H} &= [0,0,0], && (4.143)\\
\textbf{Helicity } (\mathbf{A}\circ\mathbf{B}) &= 0, \text{ but not } \textbf{TTM as T} \neq \mathbf{0}. && (4.144)\\
\textbf{Chirality } (\mathbf{A}\circ\mathbf{D}) &= 0, \text{ but not } \textbf{TTE as S} \neq \mathbf{0}\ . && (4.145)
\end{aligned}$$

This example demonstrates how the addition of a closed but not exact contribution to a 1-form of Action can influence the topological features of an electromagnetic system. The Topological Torsion dependes upon the "rotation" or angular momentum term, ϖ, with a strength determined by the coefficient ϖ. Note that the Topological Spin density term does not vanish. The combined system is not in thermodynamic equilibrium as $A\hat{}G$ is not zero.

Example 12. Bateman-Whittaker solutions

In the modern language of differential forms, Bateman [10] (and Whittaker) determined that if two *complex* functions $\alpha(x,y,z,t)$ and $\beta(x,y,z,t)$ are used to define the 1-form of Action,

$$A = \alpha d\beta - \beta d\alpha \Rightarrow \mathbf{A} = \alpha\nabla\beta - \beta\nabla\alpha, \quad \phi = -(\alpha\partial\beta/\partial t - \beta\partial\alpha/\partial t), \tag{4.146}$$

then the derived 2-form $F = 2d\alpha\hat{}d\beta$ generates the complex field intensities,

$$\begin{aligned}
\mathbf{E} &= (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha, && (4.147)\\
\mathbf{B} &= \nabla\alpha\times\nabla\beta, && (4.148)
\end{aligned}$$

which of course satisfy the Maxwell-Faraday equations. If in addition, the functions α and β satisfy the complex Bateman constraints:

$$\nabla\alpha\times\nabla\beta = \pm(i/c)[(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha], \tag{4.149}$$

then the complex field excitations, computed from the Lorentz Vacuum constitutive constraints, will satisfy the Maxwell-Ampere equations for the vacuum, without charge-currents. It is apparent immediately that the second Poincare 4-form is identically zero for such solutions. It is also apparent immediately that the Torsion vector is identically zero. What is not immediately apparent is that first Poincare 4-form and the Spin 4-vector vanish identically as well. In fact, the constrained complex solutions of the Bateman type are examples of topologically transverse (**TTEM**) waves. The Bateman solutions, like TTEM waveguide solutions, do not radiate!

As an explicit example, consider,

$$\alpha = (x \pm iy)/(z - r), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2)}. \tag{4.150}$$

These functions satisfy the Bateman conditions (and, it should be mentioned, the Eikonal equation subject to the dispersion relation $\varepsilon\mu c^2 = 1$). The **E** and the **B** fields are complex (and complicated algebraically):

$$\begin{aligned}
\mathbf{B} &= [yx + \sqrt{-1}(z^2 + y^2 - rz), & (4.151)\\
&\quad -(z^2 + x^2 - rz) - \sqrt{-1}xy, \\
&\quad (r^2 + z^2 - 2rz)/(r - z)\,)(y - \sqrt{-1}x)]2/(r(z - r)^2), \\
\mathbf{E} &= [-\sqrt{-1}yx + (y^2 + z^2 - rz), & (4.152)\\
&\quad \sqrt{-1}(x^2 + z^2 - rz) - xy, \\
&\quad (z - r\,)(x + \sqrt{-1}y)]2c/(r(z - r)^2), \\
\text{Top. Spin } \mathbf{S}_4 &= [0, 0, 0, 0], & (4.153)\\
\text{Top. Torsion } \mathbf{T}_4 &= [0, 0, 0, 0], & (4.154)\\
\mathbf{E} \times \mathbf{H} &= [0, 0, 0], \qquad \mathbf{D} \times \mathbf{B} = [0, 0, 0], & (4.155)\\
\mathbf{E} \circ \mathbf{E} &= 0, \qquad \mathbf{B} \circ \mathbf{B} = 0, & (4.156)\\
(\mathbf{E} \circ \mathbf{B}) &= 0, \qquad (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) = 0\,. & (4.157)
\end{aligned}$$

The functions α and β that satisfy the Bateman condition may be used to construct an arbitrary function, $F(\alpha, \beta)$, and remarkably enough, that arbitrary function $F(\alpha, \beta)$ satisfies the Eikonal equation,

$$(\nabla F)^2 - \varepsilon\mu(\partial F/\partial t)^2 = 0. \tag{4.158}$$

From experience with Eikonal solutions and wave equations, it might be thought that Eikonal solutions are sufficient. However, the Bateman conditions are necessary, for both the candidate solutions,

$$\alpha = (x \pm iy)/(z - ct), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2)}, \tag{4.159}$$

satisfy the Eikonal equation, but not the Bateman conditions. They (for each sign, $\pm$) do not generate TTEM modes in the vacuum. For arbitrary functions the algebra can become quite complex. A Maple symbolic mathematics program for computing the various terms is available.

Chapter 5

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5.1 About the Author

Professor R. M. Kiehn, B.Sc. 1950, Ph.D. 1953 (Physics, Course VIII, MIT), started his career working (during the summers) at MIT, and then at the Argonne National Laboratory on the Navy's nuclear powered submarine project. Argonne was near his parents home in the then small suburban community known as Elmhurst, Illinois. At Argonne, Dr. Kiehn was given the opportunity to do nuclear experiments using Fermi's original reactor, CP1. The experience stimulated an interest in the development of nuclear energy. After receiving the Ph. D. degree as the Gulf Oil Fellow at MIT, Dr. Kiehn went to work at Los Alamos, with the goal of designing and building a plutonium powered fast breeder reactor, a reactor that would produce more fissionable fuel then it consumed. He was instrumental in the design and operation of LAMPRE, the Los Alamos Molten Plutonium Reactor Experiment. He also became involved with diagnostic experiments on nuclear explosions, both in Nevada on shot towers above ground, and in the Pacific from a flying laboratory built into a KC-135 jet tanker. He is one of the diminishing number of people still alive who have witnessed atmospheric nuclear explosions.

Dr. Kiehn has written patents that range from AC ionization chambers, plutonium breeder reactor power plants, to dual polarized ring lasers and down-hole oil exploration instruments. He is active, at present, in creating new devices and processes, from the nanometer world to the macroscopic world, which utilize the features of Non-Equilibrium Systems and Irreversible Processes, from the perspective of Continuous Topological Evolution.

Dr. Kiehn left Los Alamos in 1963 to become a professor of physics at the University of Houston. He lived about 100 miles from Houston on his Pecan Orchard - Charolais Cattle ranch on the banks of the San Marcos river near San Antonio. As a pilot, he would commute to Houston, and his classroom responsibilities, in his Cessna 172 aircraft. He was known as the "flying professor".

He is now retired, as an "emeritus" professor of physics, and lives in a small villa at the base of Mount Ventoux in the Provence region of southeastern France. He maintains an active scientific website at

(http://www.cartan.pair.com).

5.2

Many of the examples used to describe the theoretical concepts of non-equilibrium systems involve overwhelming algebra and calculus computations. These computations can be made tractable by the use of symbolic mathematics programs such as Maple. As Maple permits the rapid visualization of many different cases, it has been my experience that unsuspected trends can be deduced, leading to unsuspected new theorems. A number of useful Maple programs have been compiled and can be downloaded from http://www22.pair.com/csdc/maple