Non-Equilibrium Systems and Irreversible Processes

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Non Equilibrium Thermodynamics

from a Perspective of Continuous Topological Evolution.

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0.1 Preface

Felix Klein, in describing how real research takes place, said

> "You often hear from non-mathematicians, especially from philosophers, that mathematics consists of drawing conclusions from clearly stated premises....... The investigator himself, however, in mathematics, as in every other science, does not work in this rigorous deductive fashion. On the contrary, he makes essential use of his phantasy and proceeds inductively, aided by heuristic expedients" – From his Erlangen program: Elementary Mathematics from an Advanced Standpoint : *Arithmetic, Algebra, Analysis,* Dover, NY, 1962.

The purpose of this monograph is to present features of non equilibrium and irreversible physical systems that can be understood in terms of applied topology. The monograph is not intended to be a text book in Topology nor a textbook in Tensor Analysis and/or Differential Geometry, although the methods developed in these disciplines will be employed in that which follows. The objective is to display and apply techniques of *continuous topological evolution* in order to gain a better understanding of non equilibrium physical systems and irreversible processes. If you are not sure of what continuous topological evolution entails, you can pick up the details and a more formal description in chapter 5. However, it is not necessary to jump to chapter 5, for many of concepts are explained by application and example in the earlier chapters. Classic equilibrium thermodynamics utilizes statistical methods, strongly influenced by properties of deterministic geometric evolution. These classic methods have limited (though useful) success when applied to non equilibrium systems and irreversible processes. In this monograph it is demonstrated that fundamental thermodynamic principles can be extended to describe non equilibrium systems and irreversible processes, when such concepts are described in terms of topological, not geometrical, evolution.

Evolutionary processes of the type that can be described by C2 differentiable maps from initial to final state will be at the foundations of the methods to be developed. When the inverse processes do not exist, or are not continuous, such irreversible processes permit description of topological change. but they do not admit representations in terms of linear groups of motions. A projection from a space of N+M dimensions to a space of M dimensions is an example of a non invertible, but continuous map. These processes of continuous topological change are not diffeomorphisms. Diffeomorphisms, which form a differentiable subset of homeomorphisms, do not describe topological evolution. A basic axiom is that thermodynamic irreversibility requires topological change.

The historical use of a geometric diffeomorphic approach (tensor analysis), with emphasis on uniqueness, symmetries and conservation laws, to solve problems in physics has heretofore constrained, if not eliminated, the stated objective of understanding non equilibrium systems and irreversible processes. However, geometric methods, borrowing the words of Eugene Wigner, have been "unreasonably effective" in understanding physical phenomena - at least for phenomena that can be approximated by isolated-equilibrium systems and statistical averages. The geometric methods developed historically (and based upon geometry) are time reversal invariant. However, thermodynamic irreversible continuous processes require that the topology of the initial state and the topology of the final state are not the same. Paraphrasing Eddington:

> *Aging and the arrow of time have slipped through the net of geometric analysis.*

Most of the references to my earlier publications have been compiled for convenience in Vol 7 "Selected Publications", which is available in paper back form, or in PDF file download format. See www.cartan.pair.com.

0.2 Points of Departure

Herein it is demonstrated that these concepts of thermodynamic irreversibility can be captured in terms of continuous topological evolution. The bottom line is that geometric based, diffeomorphic, processes (describing invariant topology and invariant geometry) and group based symmetry methods (with their inherent inverse operations leading to unique invertible solutions) are not explicitly useful in describing the non equilibrium systems and irreversible processes of interest herein. As the development progresses, it may come as a surprise to many readers to find that the theoretical basis of thermodynamics and electromagnetism indicate that these disciplines are topological, not geometrical, physical theories.

In this monograph, a perspective of topological evolution and change is subsumed from the outset. Topological properties and features of physical systems and processes are emphasized, and their evolutionary change becomes the point of departure from classical physical theories. Physical properties of size and shape (though useful and interesting, indeed) are intentionally suppressed, in favor of topologically coherent deformable features. Such topologically coherent, but deformable structures, appear to self organize themselves during thermodynamically irreversible processes of topological change. It was recognized by Tisza (1961) that metrical based properties can not be used to distinguish between the two classes of intensive and extensive thermodynamic variables. Thermodynamic features appear immediately in terms of the topological properties of isolated-equilibrium, closed, and open physical systems.

Caratheodory pointed out that a thermodynamic physical system in isolated-equilibrium admitted description in terms of a Pfaffian form constructed from at most two independent functions (but with arguments over perhaps N geometric variables and parameters). Such Pfaff systems are said to be of (Pfaff) topological dimension two, and are uniquely integrable in the sense of Frobenius. Such uniquely integrable systems consist of a single topologically connected and topologically coherent component. In other words they are systems of a single phase. Once the integrating factor for an isolated system is specified, the Pfaff topological dimension is reduced from 2 to 1, which defines the state of equilibrium. It is remarkable that the Cartan topology constructed from an integrable Pfaffian 1- form is a connected (but not necessarily simply connected) isolated topology.

On the other hand, irreducible, non equilibrium thermodynamic systems are of Pfaff topological dimension three or more. The Frobenius theorem of unique integrability fails. Even more remarkably, the Cartan topology for such systems, of Pfaff topological dimension greater than 2, is a disconnected topology and may have many components (mixed phases). Another way of describing such a topologically disconnected system is that if solutions exist, there may be more than one solution (non uniqueness) at any geometric point, leading to the notion of envelopes, Huygen wavelets, and edges of regression representing stability limits and the possibility of thermodynamic phase change. Pfaff topological dimension three (or more) systems are non equilibrium systems of multiple topological components. Pfaff dimension three systems can be chaotic, but the chaotic processes can be reversible in a thermodynamic sense. However, Pfaff dimension 3 (in general, 2n+1) systems, always admit a unique extremal vector direction field which can be interpreted as long-lived kinematic evolution - neglecting topological fluctuations. Such extremal fields do not exist in domains that are of Pfaff topological dimension four (or 2n+2). Such four dimensional (2n+2) topological spaces are the domain of thermodynamic irreversible processes. Self organized topologically coherent structures are the domains of Pfaff topological dimension 2n+1.

The topological perspective of thermodynamics used in this monograph is based upon Cartan's theory of exterior differential forms, which can be utilized to describe continuous topological evolution. A fundamental example of continuous topological evolution is described by the evolutionary change of Pfaff topological dimension. The topological perspective is founded on the idea that thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_k(x, y, z, t)$, on a 4 dimensional abstract variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports a volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$. It is also assumed that thermodynamic processes can be encoded, to within a factor, $\rho(x, y, z, t)$, in terms of contravariant vector direction fields, $\mathbf{V}_4(x, y, z, t)$. Variational principles are not used to define "equations" of motion. Instead, continuous topological evolution of the thermodynamic system and its system of differential forms is encoded in terms of Cartan's magic formula (see p. 122 in [131]),

$$L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A). \tag{1}$$

The motivation for this departure from classical theories is that the Lie differential, when applied to a exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent *abstractly* to the first law of thermodynamics. Hence, the first law of thermodynamics is a topological, not a geometrical idea. Remarkably, physical systems and processes can be put into equivalence classes defined by the concept of Pfaff topological dimension. These concepts will be presented in detail in Chapter 2, with applications to be found in Chapter 3.

Discontinuous processes and statistical methods are, more or less, ignored. However, it is important to remember (and for some - a surprising fact) that continuous evolution in a topological sense can cause discrete changes in the topological properties of a given system. Indeed, an important topological property is the number of disconnected parts, which in this treatment of thermodynamics will be related to the mole number n.

0.3 Results

The original motivation for this monograph was based upon the goal of developing analytical methods which can decide if a given physical system was an equilibrium system or a non equilibrium system. If a specific analytic process was applied to the physical system the methods should be able to decide if that process was thermodynamically reversible or irreversible. It is remarkable that by using a topological perspective and the axioms for continuous processes, given in detail below, these goals have been achieved without the use of probability or statistical methods, and without the use of metric constraints and linear connections. The topological method, constructed on a Cartan system of exterior differential forms which are inherently anti-symmetric, emphasizes the anti-symmetric properties of a physical system, where the more geometric and statistical methods, based upon quadratic metric forms and symmetric averages, tend to obscure the anti-symmetry properties.

It is further remarkable that the Jacobian matrix of the coefficients of the 1-form of Action - for those non equilibrium turbulent physical systems of Pfaff topological dimension 4 - leads to a universal thermodynamic phase function represented by a polynomial equation of 4th degree. The universality is related to the singularity theory of non degenerate systems which are equivalent under (small) deformations. The Phase function is constructed in terms of the symmetric similarity invariants of the Jacobian matrix of the component functions that encode the 1-form of Action, A. The resultant Phase function brings attention to thermodynamic phases that have equivalent (symmetry) structures other than those depending upon size and shape. In general, the exterior differential form method focuses attention on thermodynamic phases that have equivalent deformable topological structures (equivalent Pfaff topological dimension), and which are the result of continuous topological evolution.

This resultant universal fourth order Phase function result matches the concepts of Landau Ψ^4 mean field theory and phase transitions on one hand, and on the other hand makes contact with the non equilibrium expansion of the universe described by "inflation" and dark matter and dark energy concepts due to a "Higgs" quartic potential below the critical point of a deformable van der Waals gas. The concepts of surface tension (or string theory) can be related to the mean curvature (induced by the molar density) of the universal phase surface, the concepts of temperature and entropy are related to the quadratic or Gauss curvature (induced by the molar density), while the concepts of pressure (of either sign) and interactions are related to the cubic curvatures (induced by the molar density). The theory as presented herein is far from being complete, yet the methods offer a new perspective for analyzing thermodynamic problems. Moreover, the techniques appear to solve the problem of making a marriage between mechanical dynamics and thermodynamics; the methods can be quite useful in the design of new applications previous excluded by assumptions of equilibrium and uniqueness.

The historical limitations of geometric (metric-size-and-shape) and topological (deformation) invariance usually imposed upon theoretical descriptions of nature (especially in relativity theories) are abandoned herein in favor of studying those properties that are homeomorphic invariants in odd topological dimensions, and yet permit description of topological, as well as geometric, change relative to continuous transformations in even topological dimensions. The methods which are presented herein are based upon Cartan's calculus of exterior differential forms [63], [34]. Exterior differential forms are objects, which, in contrast to tensors, are well behaved with respect to differentiable (continuous) mappings that do not have an inverse (and therefor do not preserve topological properties), and are also well behaved with respect to diffeomorphisms, which are differentiable invertible continuous mappings (and which preserve topological properties). Evolutionary processes will be defined in terms of the action of the Lie differential with respect to vector direction fields acting on differential forms [131]. The Lie differential acting on differential forms is not confined by the diffeomorphic constraints of tensor analysis, and can treat problems of topological change. The method goes beyond the more standard "extremal" techniques based upon the calculus of variations. In most of that which follows, the functions used to define the physical systems will be assumed to be C2 differentiable. The functions that describe processes most often will be assumed to be C2 differentiable as well, but certain C1 processes (inducing tangential discontinuities and wakes) and C0 processes(inducing shocks and first order phase transitions) are of physical interest.

A fundamental result can be expressed by the statement:

> *Topological change is a necessary condition for a continuous thermodynamic process to be irreversible. .*

Irreversible processes, related to the arrow of time and the biological aging process, require topological evolution and topological change. Current physical theories that describe evolutionary processes (for example, Hamiltonian or Unitary dynamics) usually are formulated in terms of homeomorphisms that emphasize geometrical properties, but do not permit topological change. Hence all such homeomorphic continuous processes are thermodynamically reversible.

0.4 Monograph Site Map

The monograph starts with a rather long introductory Chapter 1 describing how a "Topological Perspective" can be useful to the applied sciences. Terms that may be new to some readers, and a few equations, are introduced without apology or tutorial description. A terse appendix covers most of the topological features utilized, as well as the basic features and notation of Cartan's theory of exterior differential forms. A number of textbooks are available for those who want more detail [63], [10], [122], [6]. Chapter 2 goes directly to the heart of the theory of Topological Thermodynamics, and uses the concept of Pfaff topological dimension to distinguish between equilibrium and non equilibrium systems, and, along with the concept of Frobenius integrability, to distinguish between reversible and irreversible processes. Certain topological features of non equilibrium thermodynamics lead to subtle differences relative to the concepts presented in the study of equilibrium thermodynamics. For example, from the topological point of view of non equilibrium thermodynamics, internal energy, as well as work and heat, are properties of *both* the physical system and its evolutionary dynamics. A non equilibrium adiabatic process is not necessarily an isentropic process, but describes a path that resides on a surface of constant internal energy. Chapter 3 describes applications to non equilibrium systems and irreversible processes in electrodynamics, hydrodynamics, and mechanics. The details of Cartan's Topological Structure are presented in Chapter 4 in a more or less self contained manner. That is, you can read chapter 4, without reference to the rest of the text. Similarly, Chapter 5 contains a more formal and detailed presentation of the general theory of Continuous Topological Evolution. Chapter 6 develops the topic of closed and homogeneous p-forms, which form the basis of topological "quantum" numbers. The Appendix contains a bit of philosophy, a series of terse examples demonstrating the machinery of the Cartan methods and point set topology, and includes a development of Cartan's structural equations in terms of the connections associated with a line bundle. A subsection is devoted to the generally theory of envelopes, which are important to the study of non equilibrium systems, in terms of exterior differential forms.