Holonomic and Anholonomic Constraints and Coordinates, Frobenius Integrability and Torsion of Various Types

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This article started out as a reply to an email of J. Sarfatti in which he discussed anholonomic constraints as described by A. Sommerfeld. The email message from Sarfatti stated that ”The distinction between holonomic and non-holonomic is partly that between local and global”. IMO this statement should be discarded for it is only partly correct, and may have the tendency to promote an incorrect conclusion. As I wrote the email response expressing this view, the explanations and discussions grew much too long for an email statement. Also, the mathematical notation limitations of email became severe.

Although I am very pleased that JS and others are taking interest in non-holonomic systems, both holonomic and non-holonomic constraints have both ”local and global” features. A particular reason on my part for promoting non-holonomic systems is that holonomic coordinate systems are always free from topological torsion. An integrable system of equations expressing a map from an initial state to a final state is also free from the affine torsion associated with a connection. I am interested in topological torsion, which implies that at some level there is a constraint of non-integrability. I hope that which follows sorts out the various ideas.

Introduction

Preliminary

The topics described below deal with things called Pfaffian equations and differential forms on a domain. When defined on a differential variety of functions, the differential forms can separate the domain into two sets. Regions of support (where the differential form is not zero) and the zero sets (regions where the differential form vanishes.) Pfaffian equations are of the latter category; they are essentially differential forms constrained to zero. Pfaff's problem is to find solutions to Pfaffian equations. Pfaffian equations as anholonomic constraints appear throughout physical theories. Sometimes in subtle and unappreciated ways.

Coordinate differentials are differential forms which are not zero. Given an explicit mapping from one set of (coordinate) variables $\{y^a\}$ to another set of variables $\{x^k\}$ in terms of differentiable functions

$$\boxed{\phi : y^a \Rightarrow x^k = \phi^k(y^a),}$$

the coordinate differentials are related linearly by the Jacobian mapping:

$$\boxed{d\phi : |dy^a\rangle \Rightarrow |dx^k\rangle = [\partial\phi^k(y^b)/\partial y^a)]\ |dy^a\rangle}$$

Working backwards, the differential equations are said to be integrable to exactness. They have a unique solution whose differentials reproduce the differential equations. The coordinate differentials are said to be holonomic, as they have a unique functional pre-image. A necessary condition for integrability to exactness is that the exterior derivatives of the coordinate differentials must vanish, $d(dx^k) = 0.$ Often the target of the map is presumed to be a (pseudo) euclidean space with a diagonal metric of constants equal to $\pm 1.$ The reverse problem of mapping from a (pseudo) euclidean space can also be considered.

For the integrable situations, the Jacobian matrix of functions

$$\boxed{[F^k_a(y^b)] = [\partial\phi^k(y^b)/\partial y^a)]}$$

can play the role of a basis set for a vector space, at least on subspaces of (y^b) where the determinant of the Jacobian matrix does not vanish. As the Jacobian matrix has an inverse on such domains, the exterior derivatives of the Jacobian matrix lead to the concept of a connection $[C]$ that linearly connects the differentials of the basis functions to linear combinations of the basis functions (see ”The many Faces of Torsion”).

$$\boxed{d[F]+[F][C]=0.}$$

The matrix elements of the right Cartan connection matrix, $[C],$ are differential 1-forms, $C^a_{bc}dy^c.$ For a holonomic mapping, the coefficients of C^a_{bc} are symmetric, in the sense that the torsion, $T^a_{bc}=C^a_{bc}-C^a_{cb}\Rightarrow 0.$ The integrable holonomic system is said to be free of affine torsion, $T^a_{bc}=0.$

The affine torsion (or lack thereof) analysis presented above requires only that the basis frame have an inverse. This does not mean that the inverse mapping is well defined globally. Hence the map ϕ may continuous but not homeomorphic, nor diffeomorphic. The usual coordinate mapping is both continuous and invertible, hence is a differentiable homeomorphism or a diffeomorphism.

Without knowledge of the unique integral solutions (which may not exist) it often possible to construct (or impose) over a domain a Frame Field, $[F]$, or basis matrix of linearly independent functions with arguments on the variety, $y^a.$ (Methods for doing this are described in ”The Many Faces of Torsion”). The question then can be asked, given holonomic differentials, $dy^a,$ does the Frame matrix induce holonomic differentials on the variety, x^k? Explicitly

$$\boxed{\text{If }[F^k_a]\bullet|dy^a\rangle\Rightarrow\sigma^k\qquad\text{does }d\sigma^k\overset{?}{=}0}$$

An alternate question is: Are there non-holonomic coordinate differentials σ^a on y^a such that

$$\boxed{[F^k_a]\bullet|\sigma^a\rangle\overset{?}{\Rightarrow}dx^k}$$

As will be seen (below) when the system is not holonomic, and no unique integral equivalent exists, then the spaces involved can have connections which are not free of affine torsion. The bottom line is that there is a correspondence between integrability and no- torsion. This correspondence will be demonstrated below.

Suggested Reading

There are three books on Pfaffian systems that cover the topics in some detail. The first is by

Schouten, J. A. and Van der Kulk, W., ”Pfaff’s Problem and its Generalizations”, (Oxford Clarendon Press, 1949)

Another is by

M. Zhitomirski, ”Typical Singularities of differential 1-forms and Pfaffian Equations” Mathematical Monographs 113, AMS 1992.

Neither of these texts is easy reading.

A third book of interest in ”Exterior Differential Systems” by Griffiths, Bryant, Chern et al. (I will update this reference later.)

A paper by P. Fiziev and H. Kleinert gr-qc/9605046 May 1996 does a detailed job explaining ”Anholonomic Transformations of Mechanical Action Principle(s)”

A paper by Manual del Leon and David de Diego ”On the geometry of non-holonomic Lagrangian systems” J Math Phys. **37** (7) 1996, p.3389 gives some examples, and a set of references, but is mostly restricted to what are called semi-holonomic constraints below. Also see ”Solving non-Holonomic Lagrangian Dynamics....” Extract Mathematica **11**, 2 1996 p.325, by the same authors

Also see David C. Robinson and W. F. Shadwick ”The Griffiths-Bryant algorithm and the Dirac theory of Constraints” Fields Institute Communications **7** 1996, p189

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Basic Ideas

First a few definitions.

Definition An exterior differential form, ϖ, is a function of independent variables (functions) and the differentials of the independent variables. (There are also a few restrictions of anti-symmetry and homogeneity in the differentials that lead to algebraic closure. These conditions need not be discussed now. See Flanders ”Exterior Differential Forms”)

$$\boxed{A\ differential\ form: \qquad \varpi = f(x^k, dx^k)}$$

Definition

Example: The classic Cartan Hilbert Action is a (linear) exterior differential form

$$\boxed{\varpi = f(x^k, dx^k) \Rightarrow \omega = p_k dq^k - H(p_k, q^k, t)dt.}$$

Definition A constraint is an exterior differential form set equal to zero.

$$\boxed{\varpi = f(x^k, dx^k) \Rightarrow 0.}$$

Definition The zero set of a differential form (which is homogeneous in the differentials) defines a constraint called a Pfaffian equation. (The differential form, especially when linear in the differentials, often is referred to as the ”Pfaffian”).

$$\boxed{\varpi \Rightarrow \omega = f(x^k, dx^k) = \sum_k A_k(x^k)dx^k \Rightarrow 0. \quad Pfaffian\ Equation}$$

Classic examples of Pfaffian equations of constraint are given by the kinematic and dynamic expressions subsumed as axioms of freshman mechanics:

$$\boxed{dx^k - V^k dt = 0, \quad dV^k - A^k dt = 0, \quad dp_k - F_k dt = 0.}$$

Definition A constraint that does not depend upon differentials is said to be a holonomic constraint. A constraint that involves the differentials is said to be an anholonomic or a non-holonomic constraint. If only one differential is involved, then the constraint is called a semi-holonomic constraint.

Example 1 A Holonomic constraint: $f(x^k) - c \Rightarrow x^2 + y^2 + z^2 - 1 = 0.$ This constraint, as a function of coordinate (independent) variables, if applied to a mechanical system, implies the motion must be confined to a sphere of unit radius. Holonomic constraints define hypersurfaces which are oriented, global, and often of more than one component. A Mobius band cannot be represented by a holonomic constraint (submersion) on a three dimensional space (It can be represented by a two dimensional immersion into a three dimensional space.)

Example 2 An Anholonomic constraint: $g(x, y, z, dx, dy, dz) = 0$ is a non-holonomic or anholonomic constraint. Such an equation, if linear in the differentials, is defined as a Pfaffian equation. This definition is a bit more general than that definition used in many control theory texts. (see the definition for semi-holonomic constraints given below where the differentials are replaced by velocity functions). The constraint of rolling without slipping is a anholonomic constraint.

$$\boxed{\lambda d\theta - dx = 0}$$

Often it is not appreciated that the kinematic definitions given above are indeed anholonomic constraints on the domain.

Example 3 A Semi Holonomic constraint: $g(x, y, z; V^x, V^y, V^z) = 0.$ It is usually assumed that V^x, V^y, V^z are defined as differentials of x,y,z with respect to time. Note that this assumption implies that the system admits the additional anholonomic kinematic constraints mentioned above. Under the assumption that $d\theta - \Omega dt = 0$ and $dx - Vdt = 0,$ the anholonomic constraint of rolling without slipping becomes a semi-holonomic constraint.

$$\boxed{(\lambda\Omega - V)dt = 0 \supset (\lambda\Omega - V) = 0.}$$

Particle motion subsumes the kinematic constraints. Wave (or fluid motion) does **not**. Particles and waves are equivalent on domains where the kinematic equations are valid for the velocity field. These domains form a submanifold of the space of coordinates and velocity functions. The space of ”waves” is much larger than the space of ”particles”. (It turns out that when the system is integrable, the two developments are alias-alibi related, and then can be said to be equivalent. A Hamiltonian formulation and a Lagrange formulation are strictly equivalent only when they are integrable. A Lagrange formulation with anholonomic constraints is equivalent to a Hamiltonian formulation if they are of the same class or Pfaff dimension. (see below) As will be shown below, integrability implies that either the differential 1-form can be represented in terms of at most two independent functions of the original independent variables: $\omega = \varphi d\psi,$ or $\omega = d\psi.$ The associated Pfaffian equations of constraint are the same for either representation. However, the exterior derivative of the first possibility is not zero, where the exterior derivative of the second possibility is zero.

Every differentiable holonomic constraint, $\phi(x^j) - c = 0,$ leads (by differentiation) to a non-holonomic Pfaffian equation:

$$\boxed{d\phi(x^j) = \sum_k (\partial\phi/\partial x^k)dx^k \Rightarrow \sum_k A_k(x^j)dx^k = 0}$$

but not conversely. Differential forms, $\omega,$ not equal to zero define a module of vector fields such that $i(\mathbf{V})\omega = 1$. Pfaffian equations (differential forms constrained to zero) define a module of vector fields such that $i(\mathbf{V})\omega = 0$

Definition A differential 1-form, $\omega = (\sum_k A_k(x^j)dx^k)$ is said to be integrable on domains where either $\sum_k A_k(x^j)dx^k = d\phi(x^j)$ or $\Theta(x^j)(\sum_k A_k(x^j)dx^k) = d\phi(x^j).$ The function $\Theta(x^j)$ is defined as an integrating factor.

A Pfaffian equation of constraint constructed from an integrable 1-form has a unique direction field, in the sense that the coefficient functions, $A_k(x^j),$ are proportional to the gradient of a single function, $\phi(x^j).$ The hypersurface, $\phi(x^j) = 0,$ defines the subspace (N-1 dimensional domain) of points which are isolated from the larger (N dimensional) domain.

Definition (Frobenius) An integrable differential 1- form $\omega = A_k(x^j)dx^k$ satisfies the exterior differential equation (an equation of constraint!)

$$\boxed{\omega \wedge d\omega = 0.}$$

The kinematic constraints mentioned above satisfy the Frobenius theorem. As the differentials dx are presumed to be exact, their exterior derivatives vanish. Hence the exterior derivative of the differential form $\omega = dx - Vdt,$ whose zero set represents the Pfaffian constraint, has only one component equal to the 2-form, $d\omega = dV^x \wedge dt.$ If the Pfaffian is integrable, it must be true that $d\omega = 0.$ Hence this single 2-form must vanish, which implies that $V^x = V^x(t)$ a function of t alone. Consider a more general case, where there exists a differentiable map from $\{t, y^m\} \Rightarrow x^k = f^k(t, y^m).$ Then

$$\boxed{\omega = dx^k - \{\partial f^k(t,y^m)/\partial t\}\, dt = \sum \{\partial f^k(t,y^m)/\partial y^j\}\, dy^j}$$

This equation has the appearance of the kinematic formula, if the function $\{\partial f^k(t,y^m)/\partial t\}$ is identified with $V^k(t,y^m),$ and if the RHS is zero: $\sum\{\partial f^k(t,y^m)/\partial y^j\}\, dy^j = 0.$ If the matrix $[F(t,y^m)] = \{\partial f^k(t,y^m)/\partial y^j\}$ is of maximal rank (det $[F] \neq 0)$ then the only possibility is that the y^m are fixed constants in the sense that $dy^m = 0$. If the y^m are not constants (constant initial conditions) then the kinematic equations will have fluctuations where the RHS is not zero.

Definition A Pfaff sequence of a differential 1-form A consists of a finite number n of ordered non-zero elements, constructed as {A, dA, A^dA, ...}. The class, or Pfaff dimension, of the 1-form A at a point $\{x^k\}$ is equal to n, the number of non-zero elements of the sequence.

Example: The differential form $A = y^1dy^2 + dy^3$ defined on the 4 dimensional domain, $\{y^1,y^2,y^3,y^4\}$ generates the Pfaff sequence,

$$\boxed{\{A = y^1dy^2 + dy^3, dA = dy^1\wedge dy^2, A\wedge dA = dy^1\wedge dy^2\wedge dy^3, dA\wedge dA = 0\}.}$$

The Pfaff dimension of the example is 3. The first zero element in the sequence is the 4 form $dA\wedge dA$. The class or Pfaff dimension of an integrable form is 2 or less. The proof follows from the Frobenius theorem. The Pfaff dimension in effect determines the minimum number of independent functions which are required to describe the 1-form. Integrable 1-forms can be written in the format $A = \varphi d\psi$ where $\varphi(x^k)$ and $\psi(x^k)$ are independent functions of the independent variables.

Definition An Anholonomic Kinematic Fluctuation is defined as

$$\boxed{dx^k - V^k dt = \Delta x^k \neq 0. \quad dV^k - A^k dt = \Delta V^k \neq 0.}$$

The Cartan-Hilbert invariant integral has an Action integrand, A, which is based upon Anholonomic Kinematic Fluctuations. The ”momenta” $p_k,$ play the role of Lagrange multipliers.

$$\boxed{\begin{aligned} Action &= L(x,V,t)dt + p_k\Delta x^k = L(x,V,t)dt + p_k(dx^k - V^k dt) \\ &= p_kdx^k - (p_kV^k - L)dt = p_kdx^k - H(p_k,V^k,x^k,t)dt \end{aligned}}$$

Note that the ”Hamiltonian” function $H(p_k,V^k,x^k,t)$ depends, in general, on the velocity functions, $V^k,$ as well as the Lagrange multipliers, $p_k,$ and the coordinates and time, $x^k,t.$ The Maximum Pfaff dimension of the Action is 2n+2 even though there are what appear to be 3n+1 independent functions used in its construction.

The exterior derivative of the Cartan Hilbert Action is given by the expression

$$\boxed{dA = (p_k - \partial L/\partial V^k)(\Delta V^k)\wedge dt + (dp_k - \partial L/\partial x^k dt)\wedge(\Delta x^k)}$$

which demonstrates the influence of the fluctuations (in both velocity and position) on the 2-form, dA. It is apparent that the anholonomic fluctuations in velocity, (ΔV^k), are unimportant if the momenta are presumed to be canonical, $(p_k - \partial L/\partial V^k) \Rightarrow 0.$ Intuitively, fluctuations in Velocities are attributed to temperature, where fluctuations in position are associated with pressure.

Definition Topological torsion of a 1-form (on a 4 dimensional domain) is defined as the 3 form $A\wedge dA$

A differential 1-form with non-zero topological torsion is of Pfaff dimension 3 or more. The 1-form can be expressed in terms of not less than 3 independent functions. Therefore, according to the Frobenius theorem, the1-form A which supports a non-zero Topological Torsion 3 form is not uniquely integrable.

Subtleties

Exactness and Closure

A constrained differential 1-form, $A,$ usually does not have a unique primitive function, $\phi,$ whose total differential generates a Pfaffian equation to within a factor, Θ. A necessary but not sufficient condition for the existence of the unique global holonomic function is that the exterior derivative of $A,$ or at least $\{\Theta A\},$ must vanish. For $\Theta = 1,$ the covariant vector field of components, A_k, must have zero ”curl”. If A represents a vector potential, then such potentials produce no field intensities, $B.$ In a fluid, one would say that there is no vorticity. (The integrating factor sometimes can be considered as a conformal factor)

When the exterior differential of a form vanishes, $d\omega = 0,$ the form is said to be closed, and the union of the form and its closure forms a differential ideal. If the differential form is exact then there is a unique pre-image (in the example, a function ϕ) such that $\omega = d\phi.$ What is the difference between and exact form and a closed form? The answer is that on a simply connected domain, there is no difference. However, if the domain is not simply connected (think of holes in a piece of paper) then there is a difference. For each hole there is a closed *but not exact component* to the differential form. As an example consider the 1-form which as it stands is neither exact nor closed.

$$\boxed{\sigma = (ydx - xdy) \quad with \quad d\sigma = 2dy\text{^}dx}$$

Multiply the form σ by the closure factor $1/(\pm x^2 \pm y^2).$

$$\boxed{\gamma = \sigma/(\pm x^2 \pm y^2) = (ydx - xdy)/(\pm x^2 \pm y^2) \quad with \quad d\gamma = 0}$$

The resulting 1-form is now closed but not exact. The domain of support must exclude a small set where the denominator goes to zero. If the signs are the same, the excluded set is a point at the origin. The original euclidean plane now has a hole, and is no longer simply connected. The form γ is called a harmonic form (in the sense of deRham).

What is remarkable is that the integral of a closed and exact form on a cycle, $z1$, is zero, but the integral of a closed but not exact form on a cycle is an integer multiple of some constant.

$$\boxed{\int_{z1} d\phi = 0 \qquad but \qquad \int_{z1} \gamma = n\ 2\pi}$$

Gauge conditions as exact differential additions to a 1-form, are trivial. Gauge conditions of the closed but not exact type are NOT trivial. They contain topological information (such as the hole count in a non-simply connected domain. (Bohm-Aharanov, Joukowski airfoil, Meissner expulsion, Sommerfeld quantum conditions, etc.)

These same concepts work for differential forms that are not linear in the differentials. Hence the postulate of electromagnetism that $F - dA = 0$ is a strong topological anholonomic constraint, that says over the domain of support, the 2-form of F (E and B) is exact; that is, the 2-form does not have harmonic parts (although the 1-form, $A,$ can have harmonic parts which are the ”flux quanta”). The second postulate of Maxwell electrodynamics is the statement $J - dG = 0.$ The idea again is that the 3-form J is exact without harmonic parts. The 2-form G can have harmonic parts, which serve as the charge quanta.

Now consider the topological torsion for the 1-form A which is defined as the 3-form $H = A\text{^}F = A\text{^}dA.$ If $dH = 0,$ then the question arises: Does H have harmonic parts? If the answer is yes then the harmonic parts serve as ”topological” torsion quanta. A necessary condition for existence of such quanta is that the second Poincare invariant must be zero.

Similarly for the 3-form of topological spin $S = A\text{^}G.$ The necessary condition for existence of EM spin quanta is that the $dS = 0,$ or in other words that the First Poincare invariant must vanish. These points are exemplified at

http://www22.pair.com/csdc/car/carhomep.htm

For an interesting solution to the Maxwell postulates, further constrained by the Lorentz vacuum conditions. See

http://www22.pair.com/csdc/maple/reed21.html

Here, the Torsion field is not closed, but the Spin field is closed. In fact the Spin field in the example is the torsion field multiplied by an integrating factor. One would be led to say torsion is source of spin. However, the solution is a special case and the conclusion is not general, for the next example demonstrates that you can have finite topological torsion with zero topological spin. See

http://www22.pair.com/csdc/maple/reed31.html

Equations of Motion

Given an arbitrary 1-form of Action, which is not closed, Cartan has shown that the equations of motion generating a vector field V are of a Hamiltonian form if the Lie derivative of the Action, A, with respect to the vector field V is exact.

Theorem Solutions to the equation $L_{(V)}A = d(\Theta)$ are Hamiltonian vector fields. This is equivalent to the statement that the closed integrals of the harmonic components of A are constants of the motion. The number of holes does not change.

Writing out the theorem shows that it is a statement in the form of an anholonomic differential constraint

$$\boxed{i(V)dA - d(\Theta - i(V)A) = 0}$$

Use the symbols $W = i(V)dA$ defined as the work 1-form, and $U = i(V)A$ defined as the ”internal energy”. The work 1-form W is closed and exact. Hamiltonian systems are systems where the Pfaff dimension of the Work 1-form is 1.

Equations of motion for Non-Hamiltonian dynamics

It is apparent that to find equations of motion for non-Hamiltonian systems, the fundamental anholonomic constraint

$$\boxed{i(V)dA = d(\Theta - i(V)A)}$$

must be modified to include harmonic parts, and non-closed parts.

$$\boxed{W = i(V)dA = d(\Theta - i(V)A) + \gamma + Z}$$

$$\boxed{dW = dZ \neq 0}$$

The last equation destroys the Helmholtz theorem, and the Poincare even dimensional integrals are no longer evolutionary invariants. An example of such a non-Hamiltonian mechanics was suggested in 1974. See

http://www22.pair.com/csdc/pd2/pd2fre5.htm

The formula is the anholonomic constraint

$$\boxed{W - \Gamma A = i(V)dA - \Gamma A = 0.}$$

It is know that this equation requires that the Pfaff dimension of the Action 1-form be even (2n+2). Hence the Pfaff space supports the topological torsion 3-form. Moreover, a unique solution vector V does exist for this problem. In a space of 4 variables this vector is equivalent to the Torsion current (with components proportional to those of the 3-form of non-zero topological torsion, $A\hat{}dA$). Evolution in the direction of the Torsion current is thermodynamically irreversible, as the heat 1-form, Q, does not satisfy the Frobenius integrability theorem, and therefor does not admit an integrating factor.

Extremals and the Calculus of variations.

For integrals of the Action around closed loops, the values at the ”endpoints” cancel out. Similar constraints are often placed on open integrals, forcing the cancellation of contributions at boundary points. The solutions of the problem are then given by vector fields that generate paths such that

$$\boxed{L_{(V)} \int_{z1} A = \int_{z1} i(V)dA + \int_{z1} di(V)A \Rightarrow \int_{z1} i(V)dA = 0}$$

It is apparent that the if equation is satisfied (giving the equation for an extremal as the ”Lie derivative” of the integral must vanish) then the work 1-form must vanish. The bottom line is that Extremals are associated with anholonomic constraints, $W = f_k dx^k - Pdt = 0.$ Extremal solutions say that there are vector fields such that $f_k V^k - P = 0.$ This equation is the freshman definition of power as the product of force times velocity.

What is even more remarkable is that this equation, $W = 0,$ has solutions only in spaces (as defined by the Pfaff sequence for the Action 1-form) of odd Pfaff dimension, 2n+1. (e.g. State Space).

Theorem Unique extremals, defined as solutions to the equation i(V)dA=0 for a given A, do not exist on domains of Pfaff dimension 2n+2

The theorem is easy to prove, for if the Pfaff space is a symplectic manifold of even dimension then the 2-form dA has an anti-symmetric matrix representation with no zero eigenvalues. On the other hand if the Pfaff space is an odd dimensional contact manifold, then the anti-symmetric matrix representation of dA has a unique eigen vector with eigen value zero. Hence on 2n+1 Pfaff space, the extremal exists and is unique.

to be completed later

1. Discuss the torsion induced by A as compared to the torsion induced by W.
2. Discuss the affine torsion associated with an integrable but not exact coordinate system and how the idea intertwines with conformal maps and dilatations and chirality.
3. Compare Cartan Torsion 2-forms, affine torsion, topological torsion and anholonomic constraints.