Coherent Structures in Fluids are Deformable Topological Torsion Defects

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Abstract: Cartan's theory of a global 1-form of Action on a projective variety permits the algebraic evaluation of certain useful geometric and topological objects which can be singular. The projective algebraic methods therefore lend themselves to the development of a theory of coherent structures and defects in which the concept of translational shear dislocations and rotational shear disclinations can be put on equal footing. The topological methods not only lead to a precise definition of coherent structures in fluids, but also produce a non-statistical test for thermodynamic irreversibility on a symplectic manifold of dimension 4, and therefore yield a necessary criteria for turbulence.

INTRODUCTION

The objective is to devise non-statistical theoretical methods that will describe the one key feature of a turbulent flow that everyone agrees upon, the feature of irreversibility, and then to show that topological torsion defects in such irreversible regimes have long lived observable consequences that permit the defects to be defined as coherent structures. The intuitive suggestion is that starting from arbitrary initial conditions on 4 dimensional variety $\{x, y, z, t\}$, an irreversible process will decay to one of its (non-unique) "stationary states", a long-lived self-organized state which is defined as a "coherent" structure. The mathematical suggestion is that irreversible processes occur on symplectic manifolds of Pfaff dimension 4 (or topological class [1]), and conformally decay or are attracted into closed sets of measure zero and Pfaff dimension 3. On the 4 dimensional manifold, the anholonomic differential (non-statistical) fluctuations in the classic kinematic formulas, which lead to irreversibility, disappear on the sub-manifolds of measure zero (the long-lived coherent structure).

Most of these ideas are based primarily upon the calculus of variations as extended by Cartan's theory of differential forms [2], and secondly upon Cartan's concept of the Repere Mobile on a projective manifold [3]. Both of these concepts will be described briefly in that which follows. A particular topological feature of the Cartan method that has been ignored by the hydrodynamic community is the concept of the Torsion Current [4], perhaps because the idea involves non-Riemannian manifolds and their implicit non-uniqueness of solutions. It will be demonstrated below that when the Torsion Current has a non-zero space-time divergence, then the associated dynamical system is irreversible in a thermodynamic sense. The associated dynamical system decays to "coherent" states where the divergence of the Torsion Current vanishes, yielding a conservation law for the evolution of the resulting (coherent) structure [5].

It is this Cartan idea of spaces with torsion [6] that is the major theme of this article. It is known that in a space with an affine connection it is possible to have torsion defects produced by shears of translation [7]. In this article, it is emphasized that in a fluid the dominant torsion defect is not induced by translational shears, but instead is induced by rotational shears, and their attendent accelerations. Such rotational torsion defects (disclinations) do not occur in affinely connected manifolds, but are latent in projective manifolds. Affine translational shears preserve parallelism; rotational projective shears do not. In hydrodynamics, such disclinations are observed as circulations without vorticity – a vector field which is closed, not exact, and without curl. They are representatives of deRham period integrals , and are generated by Harmonic vector fields . As Harmonic vector fields do not produce any contributions to the RHS of the Navier-Stokes equations, no matter how large the kinematic viscosity, they do not induce dissipation. They are topological limit sets which will produce the visible wakes or coherent structures often seen in experiments [8]. Moreover, as Harmonic vector fields produce minimal surfaces, these wakes, as coherent structures or topological defects, will appear as tangential discontinuities that, like soap films, are globally stabilized. These concepts have been reported elsewhere [9].

In part 2, some historical background and motivation is provided for the present stage of the theory. In part 3, Cartan's Magic Formula (from the calculus of variations) will be used to to describe topological evolution, and to develop a thermodynamic criteria for irreversibility. In part 4, the development of the Repere Mobile on a projective domain, with the constructive derivation of two species of torsion defects, will be described. More detailed descriptions may be obtained from pdf files and references available on the internet.

SOME HISTORICAL MOTIVATION

It is important to understand the motivation behind this article. It started in 1974, when, using Cartan's techniques of exterior calculus, an "Extensions of Hamilton's Principle to include Dissipative Systems" [10] was suggested. In short, it was suggested (on intuitive grounds) to examine evolutionary systems that satisfied the equation:

$$i(\mathbf{V})d\mathrm{A} = \Gamma\mathrm{A} + d\Theta, \qquad 1$$

rather than the classic Cartan-Hamilton (extremal) equation:

$$i(\mathbf{V})d\mathrm{A} = 0. \qquad 2$$

That work was then extended and compared to the projective features of the conformal group. Later, more detailed appplications to hydrodynamics were made that led to a derivation of the Navier-Stokes equations on a 4-dimensional space-time setting [11]. In 1975, Cartan's methods of differential topology were applied to the theory of period integrals, with the introduction of a novel 3-dimensional period integral, the integral of quantized spin [12]. The 3-dimensional spin integral is distinct from, but related to, the 3-dimensional period integral of Topological Torsion, which forms the basis of the current work.

Then in 1977 it was determined that irreversibility could be associated with continuous topological evolution, which, although not deterministically predictive, was deterministically retrodictive on the space of exterior forms (covariant anti-symmetric tensor fields) [13]. A natural logical arrow of time is built into the set of differentiable, but not homeomorphic maps. It was also suggested about that time that the transition to turbulence must involve the failure of the Frobenius integrability theorem, but the details were not clear. It was argued that the streamline state of a fluid implied that the Frobenius condition, $\mathrm{A}\hat{}d\mathrm{A} = 0,$ was satisfied; as the turbulent state was the antithesis of the streamline state, the Frobenius condition must fail in the turbulent regime. The key idea, however, was that the failure of the Frobenius theorem implied the necessity of including the topic of torsion into the analysis. In the current mathematics literature these ideas have migrated into what are called the Chern-Simons forms.

For a vector field that fails the Frobenius condition, the associated dynamical system can not be planar; the space curve has (Frenet) torsion and a helical signature. In 1979 it was determined that parity and time reversal symmetry breaking could occur in macroscopic electromagnetic systems, but the Pfaff dimension had to be 3 or greater [14] (a necessary condition for the failure of the Frobenius theorem). Moreover, such electromagnetic systems can support a new form of propagating discontinuities defined as a Torsion wave. In fact, it was determined that helical or torsional electromagnetic waves propagate with different speeds in different directions (a result verified by experiments in dual polarized ring laser systems)! The Torsional waves could not be represented as functions of a single variable (scalar longitudinal waves), or even an ordered pair of variables (complex transverse polarizable waves), but were irreducibly 4 dimensional [15]. The basis of the Torsional waves was the division algebra of quaternions. Such waves can also appear in fluids, but have been little studied.

Then in 1986, while in Rio de Janeiro, the author became aware of what are now known as Falaco Solitons [16]. They are easily produced - easily observed - long lived topological defects, obviously involving rotational shears, in a dynamical fluid system. Observations of these long lived topological defects gave credence to the theory of topological defects in hydrodynamic systems. These defects are not to be associated with affine translational shears as the Falaco effect is dominated by rotational shears. The two 2D surface defects whose Snell projections produce the black spots on the floor of the swimming pool are connected by a 1D string defect that is not visible in the photograph unless dye is injected into the water. The 1D string connects the vertices of the two dimensional surface dimples, and globally stabilizes the coherent structure. Helical torsion waves will propagate along the guiding center furnished by the invisible string connecting the surface defects, much in the fashion of whistlers along the earth's magnetic field lines. These topological defects will last for more than 15 minutes in a still pool of water. For more details and pictures, see [17]

In 1990, using the ideas of Pfaff reduction, some exact solutions to the Navier Stokes equations were obtained in a rotating frame of reference. The extraordinary feature of such solutions is that they replicated certain features of the Falaco Solitons, and exhibited topological phase changes as certain flow coefficients were varied. In one example, bifurcation into a Torsion bubble was produced as the mean flow speed parameter was increased beyond a critical value; the bifurcation took place at constant vorticity! These results are not too widely known, but should be of interest to those at this conference who study coherent structures in rotating systems. The results offer another alternative to the problem that goes by the name of "Vortex bursting" in the hydrodynamics literature. It is suggested herein that this phenomena has nothing to do with vorticity per se, but is an exhibition of one coherent structure of topological torsion transforming into another [18].

The concept that a 1-form of Action for a fluid system, when constrained with anholonomic differential fluctuations, would lead to a derivation of the Navier-Stokes equations was presented at the 1992 SECTAM conference in Tennessee [19]. The idea was to define a hydrodynamic action as the 1-form constructed from a classical Lagrange Action, but with possibly non-holonomic differential fluctuations $(d\mathbf{r} - \mathbf{v}dt) \neq 0$ included as constraints on the kinematics. In the following equation, the coefficients, $\mathbf{p}$, are to be considered as Lagrange multipliers.

$$\mathrm{A} = L(\mathbf{r}, \mathbf{v}, t)dt + \mathbf{p} \circ (d\mathbf{r} - \mathbf{v}dt) \qquad 3$$

If all of the variables are independent, the domain of definition is 10 dimensional, $\{\mathbf{r}, t, \mathbf{v}, \mathbf{p}\}$. For the 10 dimensional velocity vector $\mathbf{V} = \{\mathbf{v}, 1, \mathbf{a}, \mathbf{f}\}$, the virtual work 1-form becomes

$$W = i(\mathbf{V})d\mathrm{A} = (\mathbf{f} - \partial L/\partial \mathbf{r}) \circ (d\mathbf{r} - \mathbf{v}dt) - (\mathbf{p} - \partial L/\partial \mathbf{v}) \circ (d\mathbf{v} - \mathbf{a}dt) \neq 0 \qquad 4$$

The fundamental result is that if the system under consideration is without differential fluctuations $((d\mathbf{r} - \mathbf{v}dt) \Rightarrow 0,\ (d\mathbf{v} - \mathbf{a}dt) \Rightarrow 0)$, then the virtual work must vanish. But this can happen only on a manifold of odd Pfaff dimension! In contrast, if the system is a symplectic system of even Pfaff dimension, then the virtual work 1-form can never vanish. The key feature is that if the Pfaff dimension is even, then differential fluctuations are to be expected, and these lead to dissipation. The result implies that the evolution is described only imperfectly by a single parameter group of a dynamical system on the symplectic space. In the SECTAM reference explicit expressions were given for a Navier-Stokes system, for which the criteria of irreversibility required that

$$curl\mathbf{v} \circ curl\, curl\mathbf{v} \neq 0. \qquad 5$$

Now it is known a Lagrange system constrained by non-holonomic differential kinematic fluctuations leads to a non-compact symplectic manifold of dimension 2n+2. This (thermodynamic) manifold will not admit unique extremal vector fields that will leave the action integral stationary as a relative integral invariant (the virtual work must vanish for extremal fields, which is impossible on the sympectic manifold). There do exist non-extremal vector fields on the symplectic manifold that leave the Action integral invariant, but they are non unique and are dependent upon initial conditions that may require closed additions to be imposed on the Action 1-form. In modern language, the vector fields that produce stationary states (Bernoulli-Casimir functions) in a symplectic system are not gauge invariant. However, it has been observed that there does exist a unique, gauge independent, vector field on the symplectic manifold that would leave the Action integral a conformal, but not a stationary, invariant; this unique vector field, the Torsion vector field, satisfies the thermodynamic criteria of irreversibility defined below.

Differential Topology - Pfaff Dimension

A basic tool of the first method is Cartan's magic formula [20], in which it is presumed that a physical (hydrodynamic) system can be described adequately by a 1-form of Action, A, and that a physical process can be represented by a contravariant vector field, **V**, which can be used to generate a dynamical system or flow.

$$\mathrm{L}_{(\mathbf{V})} \int_a^b A = \int_a^b \mathrm{L}_{(\mathbf{V})} A = \int_b^a \{i(\mathbf{V})dA + d(i(\mathbf{V})A)\} \ = \int_b^a \{W + d(U)\} \ = \int_a^b Q \qquad 6$$

The base manifold will be the 4-dimensional variety $\{x, y, z, t\}$ of engineering practice, but no metrical features are presumed a priori. In fact, the defect analysis is based upon a projective space in which concept of length has been abrogated away.

From the point of view of differential topology, the key idea is that the Pfaff dimension, or class, of the 1-form of Action specifies topological properties of the system [21]. Given the Action 1-form, A, the Pfaff sequence, $\{\mathrm{A}, d\mathrm{A}, \mathrm{A}\hat{}d\mathrm{A}, d\mathrm{A}\hat{}d\mathrm{A}, ...\}$ will terminate at an integer number of terms ≤ the number of dimensions of the domain of definition. On a 2n+2=4 dimensional domain, the top Pfaffian, $d\mathrm{A}\hat{}d\mathrm{A}$, will define a volume element with a density function whose singular zero set (if it exists) reduces the symplectic domain to a contact manifold of dimension 2n+1=3. This (defect) contact manifold supports a unique extremal field that leaves the Action integral "stationary", and leads to the Hamiltonian conservative representation for the Euler flow in hydrodynamics.

The irreversible regime will be on an irreducible symplectic manifold of Pfaff dimension 4, where $d\mathrm{A}\hat{}d\mathrm{A} \neq 0$, with topological defects (or coherent structures) appearing as singularities of lesser Pfaff (topological) dimension, $d\mathrm{A}\hat{}d\mathrm{A} = 0$. On the non-singular symplectic space there do not exist unique extremal stationary states, but there can exist Bernoulli-Casimir functions, Θ, that generate non-extremal but stationary states that are invariants of Hamiltonian processes generated from the Casimirs. Recall that in order to be Extremal, the process, **V**, must satisfy the equation

$$Extremal--Hamiltonian \quad i(\mathbf{V})d\mathrm{A} = 0; \qquad 7$$

in order to be Hamiltonian the process must satisfy the equation

$$Bernouilli--Casimir--Hamiltonian \quad i(\mathbf{V})d\mathrm{A} = d\Theta; \qquad 8$$

in order to be Symplectic, the process must satisfy the equation

$$Helmholtz--Symplectic \quad di(\mathbf{V})d\mathrm{A} = 0. \qquad 9$$

Extremal processes cannot exist on the symplectic domain, for a anti-symmetric matrix of maximal rank on space of even dimensions does not have null eigenvectors. The Bernoulli processes correspond to energy dissipative symplectic processes, but they, as well as the Symplectic processes are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ, are evolutionary invariant(s), and may be used to describe non-unique stationary state(s).

A crucial idea is the recognition that irreversible processes must support Topological Torsion, $A\hat{}dA \neq 0$, with its attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles. The existence of Topological Torsion leads to the realization that the classical constraints of kinematic perfection, $\Delta\mathbf{x} = (d\mathbf{r} - \mathbf{v}dt) \Rightarrow 0$, *and* $\Delta\mathbf{v} = (d\mathbf{v} - \mathbf{a}dt) \Rightarrow 0$ put severe restrictions on the topology of the evolutionary process, restrictions that need not be realized in nature. Indeed, it appears that such constraints of null anholonomic differential fluctuations, such as $\Delta\mathbf{x} = 0, \Delta\mathbf{v} = 0$, are not realized during the irreversible phase of a process, and such differential fluctuations can cause the lifetime of a ”stationary state” to be finite. Anholonomic differential fluctuations may be viewed as multiple parameter topological replacements for Langevin noise.

Although there does not exist a unique gauge independent stationary state on the symplectic manifold, remarkably there does exist a unique vector field on the symplectic domain, with components that are generated by the 3-form $\mathrm{A}\hat{}d\mathrm{A}$. This unique (to within a factor) vector field is defined as the Torsion Current, **T**, and satisfies (on the 2n+2=4 dimensional manifold) the equation,

$$i(\mathbf{T})dx\hat{}dy\hat{}dz\hat{}dt = \mathrm{A}\hat{}d\mathrm{A} \qquad 10$$

This vector field, **T**, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form $d\mathrm{A}\hat{}d\mathrm{A}$ vanishes, and the domain is no longer a symplectic manifold! The Torsion vector, **T**, can be used to generate a dynamical system that will decay to the stationary states ($div(\mathbf{T}) \Rightarrow 0$) starting from arbitrary initial conditions. These processes are irreversible in the thermodynamic sense.

To understand what is meant by thermodynamic irreversibility, realize that Cartan’s magic formula of topological evolution is equivalent to the first law of thermodynamics.

$$\mathrm{L}_{(\mathbf{v})}\mathrm{A} = i(\mathbf{V})d\mathrm{A} + d(i(\mathbf{V})\mathrm{A}) = W + dU = Q. \qquad 11$$

A is the ”Action” 1-form that describes the hydrodynamic system. **V** is the vector field that defines the evolutionary process. *W* is the 1-form of (virtual) work. *Q* is the 1-form of heat.

From classical thermodynamics, a process is irreversible when the heat 1-form Q does not admit an integrating factor. From the Frobenius theorem, the lack of an integrating factor implies that $Q\hat{}dQ \neq 0$. Hence a simple test may be made for any process, **V**, relative to a physical system described by an Action 1-form, A:

If $L_{(\mathbf{v})}A\hat{}L_{(\mathbf{v})}dA \neq 0$ *then the process is irreversible.*

This definition implies that symplectic processes are reversible (as $L_{(\mathbf{S})}dA = dQ = 0$), but vectors in the direction of the Torsion vector are irreversible. For the Torsion vector, the fundamental equations are given by the constraints of conformal invariance,

$$L_{(\mathbf{T})}A = \sigma A \quad and \quad i(\mathbf{T})A = 0, \qquad 12$$

such that

$$L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = Q\hat{}dQ = \sigma^2 A\hat{}dA \neq 0. \qquad 13$$

Turbulent flows must have a component along the Torsion vector to be irreversible. A coherent structure is the end result of an irreversible decay process that forms a set of measure zero on space time, $\iiiint dA\hat{}dA = 0$, but such that the integral over a closed 3-dimensional hypersurface, $\iiint_{closed} A\hat{}dA \neq 0$, is a relative integral invariant of the remainder of the evolution. For a system constrained by the Navier-Stokes equations, the Torsion vector has 4 space time components:

$$\mathbf{T} = \{(\mathbf{v} \circ curl\,\mathbf{v})\mathbf{v} - (\mathbf{v} \circ \mathbf{v}/2)\,curl\,\mathbf{v} - \nu\, curl\, curl\,\mathbf{v};\ \ (\mathbf{v} \circ curl\,\mathbf{v})\}, \qquad 14$$

and a 4 divergence given by the expression, $div_4\mathbf{T} = -2\nu\ curl\,\mathbf{v} \circ curl\,curl\,\mathbf{v}$. The set of measure zero implies that for the Navier-Stokes fluid the vorticity field must be proportional to a gradient.

It would appear that the concept of two dimensional turbulence is paradoxical, for it requires four dimensions to support an irreversible flow according to the definitions above. It should be remarked that the definition of irreversibility, $Q\hat{}dQ \neq 0$, implies that there are two topological classes of irreversibility. Either $dQ\hat{}dQ = 0$, implying that the "heat current" does not stop or start in the interior, or $dQ\hat{}dQ \neq 0$, implying internal sources of heat current (pinch points).

Differential Geometry - the Projective Repere Mobile

The objective is to construct Cartan's Repere Mobile, and call attention to the fact that two types of torsion defects (both rotational and translational) can be generated on a projective manifold. Recall that such geometries (with torsion defects) are non-Riemannian Finsler spaces [22]. Although the affine translational torsion has a growing literature, the projective rotational torsion has been ignored. Yet, the suggestion of this article is that rotational torsion, intuitively, seems to be of more importance for hydrodynamic situations.

Consider a 1-form of Action on a 2n+2= 4D domain of definition given by the expression,

$$A = \{v_k(x,y,z,t)dx^k - cdt\}/\phi(x,y,z,t) \qquad 15$$

At any point p of the domain, there exists 2n+1=3 vectors $\mathbf{e}_m$ of four components that are orthogonally transversal to the form in the sense that $i(\mathbf{e}_m)A = 0$. These vectors (to within an arbitrary factor) may be used as column vectors of a basis frame at the point p. The coefficient functions of the one form itself (to within an arbitrary factor) form the 2n+2 elements of a basis frame at the point p. A useful but not unique choice for a basis set at the point p is given by the expression,

$$\mathrm{F} = [\mathbf{e}_k, \mathbf{n}] = \mathrm{F} = \begin{bmatrix} 1 & 0 & 0 & -v_x/\phi \\ 0 & 1 & 0 & -v_y/\phi \\ 0 & 0 & c & -v_z/\phi \\ v_x/c & v_y/c & v_{/cz} & +c/\phi \end{bmatrix}. \qquad 16$$

The determinant of this matrix is equal $\det \mathrm{F} = (c^2 + A_x^2 + A_y^2 + A_z^2)/c\phi$, which is never zero for bounded coefficients. Hence this basis frame has an inverse almost everywhere.

The existence of the inverse matrix, of course, guarantees that the differential of every basis vector is decomposable into a linear combination of the original elements of the basis set, independent of the partition. The matrix of coefficients of this linear expansion defines the (right) Cartan matrix of connection 1-forms, C_r,

$$d\mathrm{F} = \mathrm{F} \circ \{d\mathrm{F} \circ \mathrm{F}^{-1}\} = \mathrm{F} \circ \{-d\mathrm{F}^{-1} \circ \mathrm{F}\} = \mathrm{F} \circ \mathrm{C}_r \qquad 17$$

over the domain of support for the basis frame (where F^{-1} exists). (An alternate development would use the left Cartan matrix representation, $d\mathrm{F} = \mathrm{C}_l \circ \mathrm{F}$).

It is convenient to partition the (arbitrary) basis frame F in terms of the *associated* (horizontal, interior, coordinate or transversal) vectors, $\mathbf{e}_k$, and the *adjoint* (normal, exterior, parametric or vertical) field, $\mathbf{n}_p$,

$$\mathrm{F} = [\mathbf{e}_k, \mathbf{n}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}]. \qquad 18$$

The corresponding Cartan matrix has the partition,

$$d\mathrm{F} = d\begin{bmatrix} e_1^1 & e_2^1 & \dots & n^1 \\ e_1^2 & e_2^2 & \dots & n^2 \\ \dots & \dots & \dots & \dots \\ e_1^n & e_2^n & \dots & n^n \end{bmatrix} = \mathrm{F} \circ \mathrm{C} = \mathrm{F} \circ \begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & \dots & \gamma^1 \\ \Gamma_1^2 & \Gamma_2^2 & \dots & \gamma^2 \\ \dots & \dots & \dots & \dots \\ h_1 & h_2 & \dots & \Omega \end{bmatrix} \qquad 19$$

The Cartan matrix, C, is a matrix of differential 1-forms which can be evaluated explicitly from the functions that make up the basis frame if they admit first partial derivatives. Moreover, the differential of the position vector can be expanded in terms of the same basis frame and a set of Pfaffian 1-forms:

$$d\mathbf{R} = \mathrm{I} \circ \left| \begin{matrix} d\mathbf{x} \\ \dots \\ d\mathbf{t} \end{matrix} \right\rangle = \mathrm{F} \circ \mathrm{F}^{-1} \circ \left| \begin{matrix} d\mathbf{x} \\ \dots \\ d\mathbf{t} \end{matrix} \right\rangle = \mathrm{F} \circ \left| \begin{matrix} \sigma \\ \dots \\ \omega \end{matrix} \right\rangle, \qquad 20$$

where the vector $\left| \begin{matrix} \sigma \\ \dots \\ \omega \end{matrix} \right\rangle$ is a (4 component) vector of 1-forms that can be computed explicitly.

By the Poincare lemma, it follows that

$$dd\mathrm{F} = d\mathrm{F}\wedge\mathrm{C} + \mathrm{F}\wedge d\mathrm{C} = \mathrm{F}\circ\{\mathrm{C}\wedge\mathrm{C} + d\mathrm{C}\} = 0, \qquad 21$$

and

$$dd\mathbf{R} = d\mathrm{F}\wedge\left|\begin{matrix}\sigma\\ \omega\end{matrix}\right\rangle + \mathrm{F}\circ\left|\begin{matrix}d\sigma\\ d\omega\end{matrix}\right\rangle = \mathrm{F}\circ\{\mathrm{C}\wedge\left|\begin{matrix}\sigma\\ \omega\end{matrix}\right\rangle + \left|\begin{matrix}d\sigma\\ d\omega\end{matrix}\right\rangle\} = 0. \qquad 22$$

As the Frame matrix and the Cartan matrix are partitioned relative to the tangent (or interior) vectors **e** and the normal (or exterior) vectors, **n**, the Poincare lemma breaks up into linearly independent factors, each of which must vanish. The results are:

$$dd\mathbf{R} = \mathbf{e}\{d|\sigma\rangle + [\Gamma]\wedge|\sigma\rangle - \omega\wedge|\gamma\rangle\} + \mathbf{n}\{d\omega + \Omega\wedge\omega + \langle\mathbf{h}|\wedge|\sigma\rangle\} = 0 \qquad 23$$

$$dd\mathbf{e} = \mathbf{e}\{d[\Gamma]+[\Gamma]\wedge[\Gamma]+|\gamma\rangle\wedge\langle\mathbf{h}|\} + \mathbf{n}\{d\langle\mathbf{h}| + \Omega\wedge\langle\mathbf{h}| + \langle\mathbf{h}|\wedge[\Gamma]\} = 0 \qquad 24$$

$$dd\mathbf{n} = \mathbf{e}\{d|\gamma\rangle + [\Gamma]\wedge|\gamma\rangle - \Omega\wedge|\gamma\rangle\} + \mathbf{n}\{d\Omega + \Omega\wedge\Omega + \langle\mathbf{h}|\wedge|\gamma\rangle\} = 0 \qquad 25$$

By reasons of linear independence, each of the curly bracket factors must vanish, leading to the results on the interior domain (coefficients of **e**):

$$d|\sigma\rangle + [\Gamma]\wedge|\sigma\rangle = \omega\wedge|\gamma\rangle \equiv |\Sigma\rangle = \left|\begin{matrix}\omega\wedge\gamma^1\\ \omega\wedge\gamma^2\\ \omega\wedge\gamma^3\end{matrix}\right\rangle \qquad 26$$

$|\Sigma\rangle =$ the interior torsion vector of dislocation 2-forms.

$$d[\Gamma]+[\Gamma]\wedge[\Gamma] = -|\gamma\rangle\wedge\langle\mathbf{h}| \equiv [\Theta] = \begin{bmatrix}\gamma^1\wedge h_1 & \gamma^1\wedge h_2 & \gamma^1\wedge h_3\\ \gamma^2\wedge h_1 & \gamma^2\wedge h_2 & \gamma^2\wedge h_3\\ \gamma^3\wedge h_1 & \gamma^3\wedge h_2 & \gamma^3\wedge h_3\end{bmatrix} \qquad 27$$

$[\Theta] =$ the matrix of interior curvature 2-forms

$$d|\gamma\rangle + [\Gamma]\wedge|\gamma\rangle = \Omega\wedge|\gamma\rangle \equiv |\Psi\rangle = \left|\begin{matrix}\Omega\wedge\gamma^1\\ \Omega\wedge\gamma^2\\ \Omega\wedge\gamma^3\end{matrix}\right\rangle \qquad 28$$

$|\Psi\rangle =$ the exterior torsion vector of disclination 2-forms.

The first two equations are precisely Cartan's equations of structure (on an affine domain). It is the last equation of exterior disclination 2-forms, $d|\gamma\rangle + [\Gamma]\wedge|\gamma\rangle = |\Psi\rangle$, that appears to be a new equation of structure valid on a projective domain, when $\Omega \neq 0$.

The purpose of this section was to prove constructively the existence of $|\Psi\rangle$, the vector of torsion 2-forms which, it is suggested herein, should be put into correspondence with disclination defects, rotational shears and coherent structures in hydrodynamics. This vector is zero on euclidean orthonormal or affine manifolds. These fundamental formulas of projective differential geometry, and their extensions are displayed in more detail, and with examples, in the internet references.

Epilogue

It is a rare thing to attend a conference where on one day a new theoretical prediction is made, and then on the following day of the conference experimental evidence is presented to support the abstract theory. During the presentation of the material described above on May 27 of the SIMFLO conference, it was stated that in an irreversible turbulent flow there should exist a 4 dimensional defect of topological torsion. For a Navier-Stokes fluid, the signature of such a defect would be a curve of vorticity in the form of a twisted helix, but the basic requirement for the existence of the 4 dimensional symplectic manifold is given by the condition, $curl\mathbf{v} \circ curl\, curl\mathbf{v} \neq 0$. The following day Kuibin and Okulov presented experimental observations with a detailed analysis of a dynamical helical curve of vorticity in a swirling fluid. On the following day, they determined that their independent analysis supported the idea that $curl\mathbf{v} \circ curl\, curl\mathbf{v} \neq 0$, thereby giving credence to the abstract theory of Topological Torsion defects.

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The entire article in expanded form with hot linked references in the form of pdf files can be found at http://www.uh.edu/~rkiehn/pdf/pdflocal.pdf

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