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Covariant tensors and Contravariant tensor densities

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Abstract

Examples of how covariant tensors and contravariant tensor densities are different from contravariant tensors and covariant tensor capacities are given.

I. Introduction

The best way to understand the features of differential forms is to compare their properties with respect to (differentiable) maps between manifolds of different dimension. The two most important cases are the immersion (which is 1-1) and the submersion (which is onto).

However, it is first necessary to review the concepts of tensor analysis which are built upon the features of differentiable maps between manifolds of the same dimension.

A. Contravariant Vectors:

Consider an initial state as a manifold of independent variables fu^k $fu; v; w$: Consider a final state as a manifold of independent variables fx^k $fx; y; z$: A point in the initial state can be considered as a position vector from the origin $f0; 0; 0$ g to the set $fu; v; w$: It is conventional to consider the position vector, as a column vector. Similarly on the final state the position vector can be written as

$$R = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Presume a differentiable map \hat{A} exists between the initial and final state.

$$\hat{A} : fu^k \rightarrow fx^k = f f^k(u^k)g$$

Example:

$$\begin{aligned} x &= u \\ y &= u + v^2 \\ z &= v + uw^3 \end{aligned}$$

The differentiability of the non-linear map leads to a linear relationship between the differentials:

$$d\hat{A} : fdu^k \rightarrow fdx^k = f[\partial f^k(u^k)]du^k g$$

Example:

$$\begin{aligned} dx &= du \\ dy &= du + 2v dv \\ dz &= dv + w^3 du + u 3w^2 dw \end{aligned}$$

The matrix

$$[\partial f^k(u^k)] = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{pmatrix}$$

is defined as the Jacobian matrix of the mapping. Except at the origin on the initial state, this Jacobian map is well defined in the sense that it is invertible at all points of the initial domain except the origin. From the implicit function theorem there are restricted domains for which an inverse map exists, defining local diffeomorphisms.

Note that the linear mapping of the differentials can be written in matrix form

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{pmatrix} \begin{pmatrix} du \\ dv \\ dw \end{pmatrix}$$

an equation that takes a row vector on the initial state and creates a row vector on the final state. This linear mapping (the push forward) is the epitome of a contravariant tensor mapping. That is, those ordered arrays of functions of arguments on the initial state that are linearly mapped into ordered arrays of values on the final state by means of the Jacobian matrix are defined as contravariant tensors of rank 1, or a contravariant vector.

Example: Consider a dynamical system on the initial state as a velocity vector field U of functions on the initial state.

$$U = \begin{pmatrix} U^u \\ U^v \\ U^w \end{pmatrix} = \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \\ \frac{dw}{dt} \end{pmatrix} = \begin{pmatrix} 1 + uv \\ v^3 \\ uw \end{pmatrix}$$

This vector on the initial state can be transformed into a vector on the final state by the Jacobian matrix rule

$$\begin{pmatrix} V^x \\ V^y \\ V^z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 4 & 1 & 2v & 0 \\ 2 & w^3 & 1 & 3uw^2 \end{pmatrix} \begin{pmatrix} U^u \\ U^v \\ U^w \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 4 & 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 & uw \end{pmatrix} \begin{pmatrix} 1 + uv \\ v^3 \\ uw \end{pmatrix} = \begin{pmatrix} 1 + uv \\ 1 + uv + 2v^4 \\ w^3 + w^3uv + v^3 + 3u^2w^3 \end{pmatrix}$$

Note that the ordered array of functions on the final state has arguments on the initial state! In order to obtain an ordered array of functions with arguments on the final state it is necessary to know the inverse mapping such that the $u; v; w$ can be expressed in terms of the $x; y; z$: Given a point on the initial state, a value of the contra-variant vector on the final state is readily computed. However, the functional form of the components of the contra-variant vector on the final state with arguments over the base $x; y; z$ is impossible without knowledge of the inverse mapping.

B. Co-variant Vectors:

Next consider a function defined on the final state, $E(x; y; z)$: Compute the differential of this function using the chain rule:

$$dE(x; y; z) = \frac{\partial E(x; y; z)}{\partial x} dx + \frac{\partial E(x; y; z)}{\partial y} dy + \frac{\partial E(x; y; z)}{\partial z} dz$$

Note that this construction can be written as the linear-row column product:

$$dE(x; y; z) = \begin{pmatrix} \frac{\partial E(x; y; z)}{\partial x} & \frac{\partial E(x; y; z)}{\partial y} & \frac{\partial E(x; y; z)}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

Now use the mapping to rewrite the expression on terms of the variables of the initial state, $u; v; w$:

$$\begin{aligned} dE(x; y; z) &= \begin{pmatrix} \frac{\partial E(x; y; z)}{\partial x} & \frac{\partial E(x; y; z)}{\partial y} & \frac{\partial E(x; y; z)}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \begin{pmatrix} du \\ dv \\ dw \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} A_x(x; y; z) & A_y(x; y; z) & A_z(x; y; z) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2v & 0 \\ w^3 & 1 & 3uw^2 \end{pmatrix} du + dv + dw \\
&= \begin{pmatrix} A_u(x; y; z) & A_v(x; y; z) & A_w(x; y; z) \end{pmatrix} \begin{pmatrix} du \\ dv \\ dw \end{pmatrix} \\
&= \mathbf{D} \begin{pmatrix} \tilde{A}_u(u; v; w) & \tilde{A}_v(u; v; w) & \tilde{A}_w(u; v; w) \end{pmatrix} \begin{pmatrix} du \\ dv \\ dw \end{pmatrix} = d \tilde{\mathbf{E}}(u; v; w)
\end{aligned}$$

An order array of functions on the final state that behaves like the components of the gradient of a function (a row vector) producing an ordered array of functions as a row vector on the initial state is the epitome of what is defined classically as a co-variant vector. In the usual realm of tensor analysis, the gradient is defined on the initial state, and the Jacobian maps are presumed to be invertible. Then the covariant tensor field is defined as the push forward with respect to the inverse Jacobian matrix transposed. It is usual to restrict the mappings such that the Jacobians are elements of the orthogonal group. In which case the inverse and the transpose are identical. It is much more illuminating to consider the covariants as defined on the final state and consider the functional substitution and the pullback as the proper definitions of covariant tensor fields. The reason is that the co-tensors then are well defined with respect to pullbacks whether or not the Jacobian is invertible.

Further note that the process is deduced by the behavior of the gradient, but the mapping and pullback concepts can start with the set

$$\begin{pmatrix} A_x(x; y; z) & A_y(x; y; z) & A_z(x; y; z) \end{pmatrix}$$

whether or not it can be constructed from a gradient operation. In other words the differential form

$$A = \sum_{k=1}^n A_k(x; y; z) dx^k$$

need not be exact. Objects that behave as under the pullback transform as the Jacobian transpose are defined as covariant tensor fields.

Example:
Define

$$E(x; y; z) = 1 + x^2 + yxz$$

Then

$$\begin{aligned}
&\begin{pmatrix} \frac{\partial E(x; y; z)}{\partial x} & \frac{\partial E(x; y; z)}{\partial y} & \frac{\partial E(x; y; z)}{\partial z} \end{pmatrix} \\
&= \begin{pmatrix} A_x(x; y; z) & A_y(x; y; z) & A_z(x; y; z) \end{pmatrix} \\
&= \begin{pmatrix} 2x + yz & xz & yx \end{pmatrix}
\end{aligned}$$

and

$$\mathbf{D} \begin{pmatrix} \tilde{A}_u(u; v; w) & \tilde{A}_v(u; v; w) & \tilde{A}_w(u; v; w) \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} A_x(x; y; z) & A_y(x; y; z) & A_z(x; y; z) \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 4 & 1 & 2v & 0 \\ & w^3 & -1 & 3uw^2 \end{pmatrix} \\
&= \begin{pmatrix} 2x + yz + xz + yxw^3 & xz2v + yx & yx3uw^3 \end{pmatrix}
\end{aligned}$$

Finally the components of the co-vector field on the initial state can be expressed as three functions with arguments on the initial state, as the pull-back:

$$\begin{aligned}
\tilde{A}_u(u; v; w) &= 2u + (u + v^2)(v + uw^3) + uu(v + uw^3) + (u + v^2)uw^3 \\
\tilde{A}_v(u; v; w) &= u(v + uw^3)2v + (u + v^2)u \\
\tilde{A}_w(u; v; w) &= (u + v^2)u3uw^3
\end{aligned}$$

Everything with respect to the pullback is well defined without need for a inverse mapping or for a inverse jacobian mapping.

C. Contra-tensor densities

In addition to the pullback of 1-forms there is another construction that can pullback properly and yields contravariant tensor densities (not tensors). Consider the volume element on the final state, with perhaps some measure function, $\omega(x; y; z)$.

$$\omega = \omega(x; y; z) dx \wedge dy \wedge dz$$

Contract the volume element with a contravariant vector on the final state to form the N-1 form density:

$$J = i(V^k(x; y; z))\omega = \omega(x; y; z) fV^x(x; y; z) dy \wedge dz + V^y(x; y; z) dx \wedge dz + V^z(x; y; z) dx \wedge dy$$

Now substitute for each differential is expression in terms of differentials of the variables of the initial state, and collect terms. The ordered array is equivalent to multiplying the contravariant vector by the adjoint matrix (matrix of cofactors transposed) of the original jacobian matrix. This adjoint matrix is well defined and exists even though the determinant of the Jacobian vanishes. Finally substitute in the pulled back ordered array of functions the values of x,y,z in terms of u,v,w to create a contravariant tensor density field on the initial state.

In otherwords, there are two species of differential forms that have functionally well defined formats on the initial state given functional data on the final state, and without knowledge of an inverse mapping or and inverse Jacobian mapping. One species defines covariant tensors (Field intensities) and the other defines contravariant tensor densities (Field excitations) Not only can values of such objects be retrodicted to the initial state from the final data, but also the functional forms on the initial state are well defined in terms of the functional data on the final state.

The duals of these concepts, contravariant tensors and co-variant capacities are not well behaved with respect to non-invertible mappings. That is values may be predicted, but functional formats cannot.