```
> restart:with(linalg):with(plots):
```

Warning, new definition for norm

Warning, new definition for trace

Fresnel Kummer WAVE Surfaces

fresnel.mws R.M.Kiehn 11/15/97

fresnel.mws

This program will compute and plot the Fresnel WAVE surfaces for a generalized Hermitean constitutive matrix. The surfaces are specializations of a Kummer quartic surface, and are related to the Clifford algebra Cl(3,3). Specialized forms have been selected for easy visualization. The formulas can be modified to handle the general case.

For details see http://www.uh.edu/~rkiehn/pd2/pd2fre10.htm for a downloadable reference.

The constitutive tensor is a 6 x 6 complex hermitian matrix partitioned into 3x3 matrices.

```
>
```

The on-diagonal upper 3x3 matrix is the epsilon matrix. The epsilon matrix real part describes birefringence. The lower on-diagonal 3x3 matrix is the reciprocal mu matrix. The complex part of the mu matrix represents magnetic Faraday effects. The sign convention of Post is used below:

```
> constitutive_tensor:=matrix([[-epsilon,GD],[GH,1/mu]]);
```

$$constitutive\_tensor := \begin{bmatrix} -\varepsilon & GD \\ GH & \frac{1}{\mu} \end{bmatrix}$$

```
> DV=[-epsilon]*(-E)+[GD]*B;
> HV=[GH]*(-E)+[1/mu]*B;
>
```

$$DV = -[-\varepsilon]\,E + [GD]\,B$$

$$HV = -[GH]\,E + \left[\frac{1}{\mu}\right]B$$

```
>
```

>

The Maxwell Faraday and Maxwell Ampere equations become as an ideal annihilated by the exterior product of a wave vector, nm1 or nm2.

> **MF:=nm1*EV-BV;MA:=nm2*HV+DV;**

$$MF := nm1\ EV - BV$$

$$MA := nm2\ HV + DV$$

> **DV=[-epsilon]*(-E)+[GD]*B;**

> **HV=[GH]*(-E)+[muin]*B; B:=solve(MF,BV);E:=EV;**

>

$$DV = -[-\varepsilon]\, E + [GD]\, B$$

$$HV = -[GH]\, E + [muin]\, B$$

$$B := nm1\ EV$$

$$E := EV$$

> **MA:=subs(DV=epsilon*E+GD*B,HV=-GH*E+(muin)*B,MA);**

$$MA := nm2\ (-GH\ EV + muin\ nm1\ EV) + \varepsilon\ EV + GD\ nm1\ EV$$

> **MATRIXEQ:=factor(MA/EV);restart:with(linalg):with(plots):**

$$MATRIXEQ := -nm2\ GH + nm2\ muin\ nm1 + \varepsilon + GD\ nm1$$

Warning, new definition for norm

Warning, new definition for trace

The matrixeq is to be solved for its real eigen values which represent the effective index of refraction in the chosen direction.

The off-diagonal 3x3 matrix has a real part which represents Fresnel-Fizeau effects, and a complex part which represents optical activity. The fundamental reference for the constitutive matrix is

E.J. Post "The Formal Structure of Electromagnetics" Dover 1997

Typical Constitutive Matrix entries are shown below. The elements have been scaled such that the "position vector" to a Fresnel surface point in this space has a value equal to the "effective" index of refraction in that direction. The "phase velocity" in that direction is c/n, where c is defined as 1/sqrt (epsilon*mu). The "phase velocity" should be viewed as at the speed of the momentum flux (DxB) in the direction of the position vector.

For the Fresnel Ray surface which defines the propagation speed of the energy flux (ExH) (usually described as the "group" velocity) see

http://www22.pair.com/csdc/maple/fresnelr.zip

Note that when the media is dispersive, the energy flux and the momentum flux do not propagate at the same speed, but the product of the group speed times the phase speed is always 1/(epsilon.mu).

The epsilon (permittivity) matrix

```
> eps:=(matrix([[A*epsilon,I*fd,0],[-I*fd,B*epsilon,0],[0,0,C*epsilon]]));
```

$$eps := \begin{bmatrix} A\,\varepsilon & I\,fd & 0 \\ -I\,fd & B\,\varepsilon & 0 \\ 0 & 0 & C\,\varepsilon \end{bmatrix}$$

The real (symetric) part of the epsilon matrix (above) is the permittivity matrix and leads to bifrefringence;

the complex antisymmetric part represents dielectric Faraday effects.

The Gamma matrix (Real and Imaginary Parts)

```
> GR:=matrix([[0,r,0],[-r,0,0],[0,0,0]]);
```

$$GR := \begin{bmatrix} 0 & r & 0 \\ -r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The real part of the off-diagonal gamma matrix is given above.

The Fresnel-Fizeau and Sagnac effect is contained the real part of the off diagonal gamma matrix.

```
> GI:=matrix([[I*g,0,I*p],[0,I*g,0],[I*p,0,I*g]]);
```

$$GI := \begin{bmatrix} I\,g & 0 & I\,p \\ 0 & I\,g & 0 \\ I\,p & 0 & I\,g \end{bmatrix}$$

The optical activity is due to the complex part of the off diagonal submatrix above.

```
> GD:=evalm(GR+GI);GA:=evalm(transpose(+GR+GI));
```

$$GD := \begin{bmatrix} I\,g & r & I\,p \\ -r & I\,g & 0 \\ I\,p & 0 & I\,g \end{bmatrix}$$

$$GA := \begin{bmatrix} I\,g & -r & I\,p \\ r & I\,g & 0 \\ I\,p & 0 & I\,g \end{bmatrix}$$

The complex conjugate transpose of the Gamma Matrix

```
> GH:=evalm(transpose(GR)-transpose(GI));
```

$$GH := \begin{bmatrix} -I\,g & -r & -I\,p \\ r & -I\,g & 0 \\ -I\,p & 0 & -I\,g \end{bmatrix}$$

The Gamma matrix need not be Hermitian. It can be anti-hermitian as well.

The inverse mu matrix (reciprocal permeability matrix)

```
> mmu:=matrix([[a/(mu),I*fm,0],[-I*fm,b/(mu),0],[0,0,c/(mu)]]);
```

$$mmu := \begin{bmatrix} \frac{a}{\mu} & I\,fm & 0 \\ -I\,fm & \frac{b}{\mu} & 0 \\ 0 & 0 & \frac{c}{\mu} \end{bmatrix}$$

The real (symetric) part of the mu matrix is the magnetic birefringence part;

the complex antisymmetric part represents magnetic Faraday effects.

```
> muinv:=evalm(inverse(mmu));
```

$$muinv := \begin{bmatrix} -\frac{b\,\mu}{-a\,b+fm^2\,\mu^2} & \frac{I\,fm\,\mu^2}{-a\,b+fm^2\,\mu^2} & 0 \\ -\frac{I\,fm\,\mu^2}{-a\,b+fm^2\,\mu^2} & -\frac{a\,\mu}{-a\,b+fm^2\,\mu^2} & 0 \\ 0 & 0 & \frac{\mu}{c} \end{bmatrix}$$

The matrices are now scaled for ease of computation. Values of n = [x.y.z] > 1 imply a phase velocity less than c =1/sqrt(epsilon*mu). Values of n < 1 imply phase speeds greater than c.

The N matrix (Index of Refraction Operator)

```
> cc:=1/simplify((epsilon)^(1/2)*(mu)^(1/2),sqrt);N:=matrix([[0,z/cc,-y/cc],[-z/cc,0,x/cc],[y/cc,-x/cc,0]]);
```

$$cc := \frac{1}{\sqrt{\varepsilon}\,\sqrt{\mu}}$$

$$N := \begin{bmatrix} 0 & z\sqrt{\varepsilon}\,\sqrt{\mu} & -y\sqrt{\varepsilon}\,\sqrt{\mu} \\ -z\sqrt{\varepsilon}\,\sqrt{\mu} & 0 & x\sqrt{\varepsilon}\,\sqrt{\mu} \\ y\sqrt{\varepsilon}\,\sqrt{\mu} & -x\sqrt{\varepsilon}\,\sqrt{\mu} & 0 \end{bmatrix}$$

The N matrix above is the scaled "index of refraction matrix" which acts like a cross product operator. The N matrix has three components which form the position vector to the Kummer surface. The magnitude of N is the "index" of refraction, n, in the direction of the vector N. The phase speed is then related to 1/n.

The elements of the constitutive matrix have been scaled (below) for algebraic reduction purposes.

```
> fd:=fda*epsilon;p:=pa*epsilon^(1/2)/(mu)^(1/2);g:=ga*epsilon^(1/2)/(mu)^(1/2);r:=ra*epsilon^(1/2)/(mu)^(1/2);fm:=fma/mu;
```

$$fd := fda\,\varepsilon$$

$$p := \frac{pa\sqrt{\varepsilon}}{\sqrt{\mu}}$$

$$g := \frac{ga\sqrt{\varepsilon}}{\sqrt{\mu}}$$

$$r := \frac{ra\sqrt{\varepsilon}}{\sqrt{\mu}}$$

$$fm := \frac{fma}{\mu}$$

Now compute the various matrices whose determinant create the Kummer equation.

```
> eps:=(matrix([[A*epsilon,I*fd,0],[-I*fd,B*epsilon,0],[0,0,C*epsilon]])):
```

```
> M:=innerprod(N,mmu):MM:=innerprod(M,N):
```

The Fresnel Hamiltonian Matrix

```
> HHH:=evalm(eps)+innerprod(GD,N)-innerprod(N,GH)+evalm(MM);
```

$$HHH := \begin{bmatrix} A\,\varepsilon & I\,fda\,\varepsilon & 0 \\ -I\,fda\,\varepsilon & B\,\varepsilon & 0 \\ 0 & 0 & C\,\varepsilon \end{bmatrix} + \begin{bmatrix} -ra\,\varepsilon\,z + I\,pa\,\varepsilon\,y & I\,ga\,\varepsilon\,z - I\,pa\,\varepsilon\,x & -I\,ga\,\varepsilon\,y + ra\,\varepsilon\,x \\ -I\,ga\,\varepsilon\,z & -ra\,\varepsilon\,z & ra\,\varepsilon\,y + I\,ga\,\varepsilon\,x \\ I\,ga\,\varepsilon\,y & I\,pa\,\varepsilon\,z - I\,ga\,\varepsilon\,x & -I\,pa\,\varepsilon\,y \end{bmatrix}$$
$$- \begin{bmatrix} ra\,\varepsilon\,z + I\,pa\,\varepsilon\,y & -I\,ga\,\varepsilon\,z & I\,ga\,\varepsilon\,y \\ I\,ga\,\varepsilon\,z - I\,pa\,\varepsilon\,x & ra\,\varepsilon\,z & I\,pa\,\varepsilon\,z - I\,ga\,\varepsilon\,x \\ -I\,ga\,\varepsilon\,y - ra\,\varepsilon\,x & -ra\,\varepsilon\,y + I\,ga\,\varepsilon\,x & -I\,pa\,\varepsilon\,y \end{bmatrix}$$
$$+ \begin{bmatrix} -z^2\,\varepsilon\,b - y^2\,\varepsilon\,c & -I\,z^2\,\varepsilon\,fma + y\,\varepsilon\,c\,x & I\,z\,\varepsilon\,fma\,y + z\,\varepsilon\,b\,x \\ I\,z^2\,\varepsilon\,fma + y\,\varepsilon\,c\,x & -z^2\,\varepsilon\,a - x^2\,\varepsilon\,c & z\,\varepsilon\,a\,y - I\,z\,\varepsilon\,fma\,x \\ \varepsilon\,(-I\,y\,fma + x\,b)\,z & \varepsilon\,(y\,a + I\,x\,fma)\,z & -\varepsilon\,y^2\,a - \varepsilon\,x^2\,b \end{bmatrix}$$

The Kummer Fresnel WAVE Quartic Polynomial

```
> HAMILTONIAN:=factor(det(HHH)/epsilon^3);
```

$$HAMILTONIAN := A\,x^2\,c\,y^2\,a + A\,x^4\,c\,b - A\,z^2\,fma^2\,x^2 + 4\,A\,ga\,x^2\,z\,fma + 4\,A\,ga\,x\,pa\,z - 2\,A\,pa$$

$$-2\,ra\,z\,B\,C + 2\,ra\,z\,B\,y^2\,a + 2\,ra\,z^3\,a\,C + z^2\,b\,ra^2\,y^2 - y^2\,c\,B\,C + y^4\,c\,B\,a + y^2\,c\,B\,x^2\,b + 2\,y^2\,c\,ra$$

$$+y^2 c z^2 a C + y^2 c pa^2 z^2 + ra^2 x^2 z^2 a + 4\, ra\, x^3 pa\, ga - 2\, ra\, x^2 pa^2 z + 2\, ra\, x^2 fda\, z\, fma - 4\, fda$$

$$+ 2\, fda\, pa\, x\, C - 2\, fda\, pa\, x\, y^2 a - 2\, fda\, pa\, x^3 b - 2\, pa\, x\, z\, fma\, y^2 ra + 4\, pa\, x\, ga\, y^2 ra + pa^2 x^2 y^2 a$$

$$- 2\, pa\, x\, z^2 fma\, C + 4\, ga\, z\, pa\, x\, C + 4\, ga\, z^3 fma\, C - 4\, fda\, ga\, y^2 ra + 2\, fda\, z^2 fma\, C - 4\, ga^2 y^2 B +$$

$$- 4\, A\, ga^2 x^2 - A\, pa^2 z^2 - A\, ra^2 y^2 + 4\, ra^2 z^2 C + 2\, ra\, z^3 pa^2 + 2\, ra^3 z\, y^2 + z^4 b\, pa^2 + y^4 c\, ra^2 +$$

$$- ra^2 x^2 B + 2\, ra^3 x^2 z - z^4 fma^2 C - pa^2 x^2 C + pa^2 x^4 b - 4\, ga^2 z^2 C + fda^2 y^2 a - 2\, z^2 b\, x^2 pa$$

$$+ 2\, ra\, x\, fda\, pa\, z - 4\, ra\, x^2 fda\, ga - 2\, ra\, x^3 pa\, z\, fma + 2\, y^2 c\, x^2 ra^2 + 2\, fda\, z\, fma\, y^2 ra - z^2 fma^2 y^2$$

$$+ 4\, ga\, y^2 z\, fma\, B + 2\, z^2 b\, x\, fda\, pa + 2\, ra\, z\, x^2 c\, C - 4\, ra\, z^2 ga\, x\, pa + 2\, ra\, z^3 pa\, fma\, x - z^2 b\, B\, C +$$

$$+ 2\, z^3 b\, ra\, C + z^4 b\, a\, C + z^2 b\, x^2 c\, C - A\, B\, y^2 a - A\, B\, x^2 b - 2\, A\, ra\, z\, C + 2\, A\, ra\, z\, x^2 b - A\, z^2 a\, C -$$

$$- A\, x^2 c\, C + fda^2 x^2 b - fda^2 C$$

Now choose values for the matrix elements. A,B,C are numeric factors times epsilon. a,b,c are numeric factors times mu. The Optical Activity part is scaled by the impedance of "free space" or sqrt (epsilon/mu). The algebraic method presented permits symbolic factorization. The dielectric Faraday fd is scaled by epsilon. The magnetic faraday is scaled by mu. The example below is for a chiral Optical Activity coefficient gamma of sqrt(2)/2. Note that the phase velocity of one of the

polarization states is faster than the speed of light and the other is slower.!!!

Select the effects to be studied algebraically by elimating all effects but one or two of the scalars,

fma,fda,ra,pa,ga

The example studies optical activity algebraically, by setting all factors to zero, except ga:

```
> HAMIL:=subs(fma=0,fda=0,pa=0,ra=0,HAMILTONIAN);eval(%);
```

$$HAMIL := A\, x^2 c\, y^2 a + A\, x^4 c\, b - y^2 c\, B\, C + y^4 c\, B\, a + y^2 c\, B\, x^2 b + y^2 c\, z^2 a\, C - 4\, ga^2 y^2 B + A\, B$$

$$- 4\, A\, ga^2 x^2 - 4\, ga^2 z^2 C - z^2 b\, B\, C + z^2 b\, B\, y^2 a + z^4 b\, a\, C + z^2 b\, x^2 c\, C - A\, B\, y^2 a - A\, B\, x^2 b -$$

$$+ A\, z^2 a\, x^2 b - A\, x^2 c\, C$$

$$A x^2 c y^2 a + A x^4 c b - y^2 c B C + y^4 c B a + y^2 c B x^2 b + y^2 c z^2 a C - 4\, ga^2 y^2 B + A B C - 4 A g$$

$$- 4\, ga^2 z^2 C - z^2 b B C + z^2 b B y^2 a + z^4 b a C + z^2 b x^2 c C - A B y^2 a - A B x^2 b - A z^2 a C + A$$

$$- A x^2 c C$$

Reduced Fresnel Kummer quartic polynomial

> **HAMRED:=HAMIL;**

$$HAMRED := A x^2 c y^2 a + A x^4 c b - y^2 c B C + y^4 c B a + y^2 c B x^2 b + y^2 c z^2 a C - 4\, ga^2 y^2 B +$$

$$- 4 A\, ga^2 x^2 - 4\, ga^2 z^2 C - z^2 b B C + z^2 b B y^2 a + z^4 b a C + z^2 b x^2 c C - A B y^2 a - A B x^2 b -$$

$$+ A z^2 a x^2 b - A x^2 c C$$

Set anisotropic coefficients:

> **HAMA:=subs(A=1,B=1,C=1,a=1,b=1,c=1,HAMRED):eval(%):HAMA:=factor(HAMA);**

>

$$HAMA := 2 y^2 x^2 + x^4 - 2 y^2 + y^4 + 2 z^2 y^2 - 4\, ga^2 y^2 + 1 - 4\, ga^2 x^2 - 4 z^2\, ga^2 - 2 z^2 + z^4 + 2 z^2$$

Set the numeric values for the coefficients

> **HAM:=subs(fma=0,fda=0,ra=0,pa=0,ga=2^(1/2)/2,HAMA);eval(%);**

$$HAM := 2 y^2 x^2 + x^4 - 4 y^2 + y^4 + 2 z^2 y^2 + 1 - 4 x^2 - 4 z^2 + z^4 + 2 z^2 x^2$$

$$2 y^2 x^2 + x^4 - 4 y^2 + y^4 + 2 z^2 y^2 + 1 - 4 x^2 - 4 z^2 + z^4 + 2 z^2 x^2$$

Coefficients were chosen to display chiral vacuum effect.

> **KUMM:=factor(HAM);LORENTZ:=(x^2+y^2+z^2)^2-1;factor(KUMM-LORENTZ);**

$$KUMM := 2 y^2 x^2 + x^4 - 4 y^2 + y^4 + 2 z^2 y^2 + 1 - 4 x^2 - 4 z^2 + z^4 + 2 z^2 x^2$$

$$LORENTZ := (y^2 + x^2 + z^2)^2 - 1$$

$$-4\,y^2 + 2 - 4\,x^2 - 4\,z^2$$

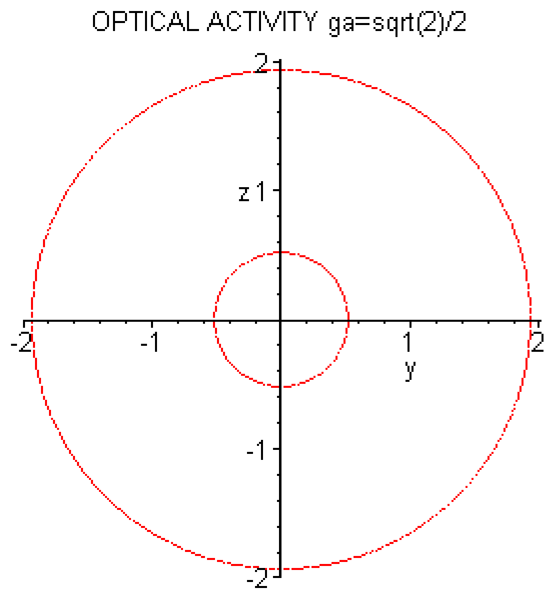
Now for an x=0 section to get more detail.

> **KUMMX:=subs(x=0,KUMM);**

$$KUMMX := 1 - 4\,y^2 + y^4 + 2\,z^2\,y^2 - 4\,z^2 + z^4$$

X=0 section of Fresnel Kummer Wave Vector Surface

> **implicitplot(KUMMX=0,y=-2.0..2.0,z=-2..2,numpoints = 5000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/2`);**



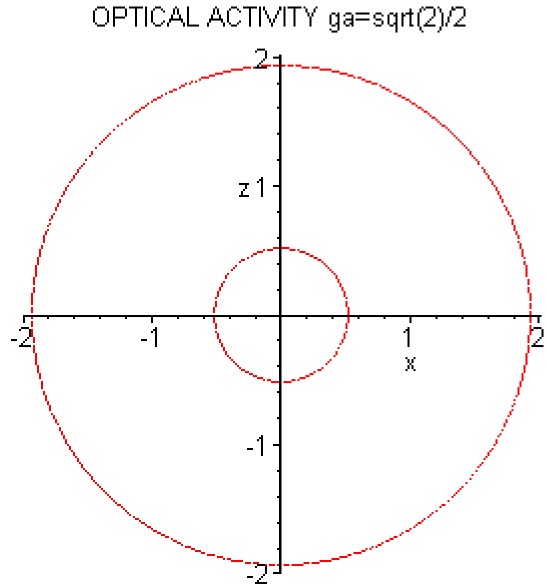
Now for a y=0 section.

> **KUMMY:=subs(y=0,KUMM);**

$$KUMMY := 1 + x^4 - 4\,x^2 - 4\,z^2 + z^4 + 2\,z^2\,x^2$$

Y=0 section of Fresnel Kummer Wave Vector Surface

> **implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints = 5000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/2`);**



Now for a Z=0 section

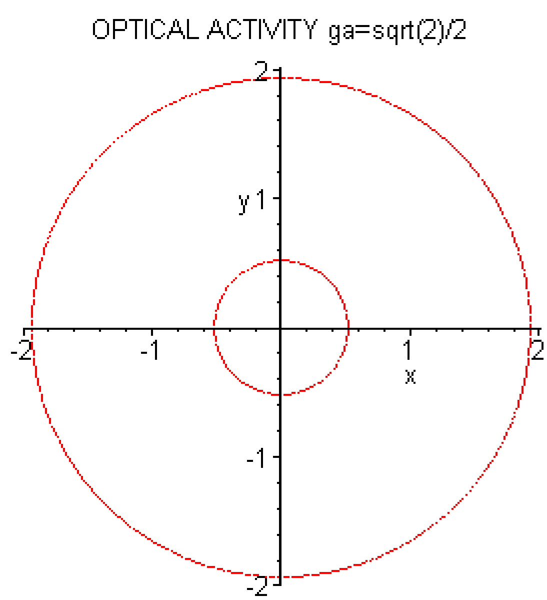
> **KUMMZ:=subs(z=0,KUMM);**

>

$$KUMMZ := 2\,y^2\,x^2 + x^4 - 4\,y^2 + y^4 + 1 - 4\,x^2$$

Z=0 section of Fresnel Kummer Wave Vector Surface

> **implicitplot(KUMMZ=0,x=-2.0..2.0,y=-2..2,numpoints = 5000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/2`);**

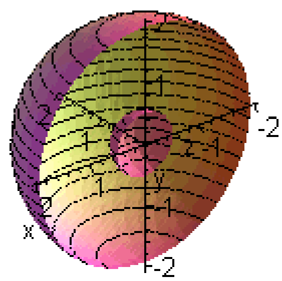


It would appear that the propagation velocity exceeds c for the inner sphere?

3-D Plot of Fresnel Wave Vector Surface

> **implicitplot3d(KUMM=0,x=-0..2,y=-2..2,z=-2..2,shading=zgreyscale,lightmodel=light4,axes=NORMAL,style=PATCHCONTOUR,scaling=CO ACTIVITY ga=sqrt(2)/2`,numpoints=9000,orientation=[142,62]);**

OPTICAL ACTIVITY ga=sqrt(2)/2



>

>

>