2D Turbulence is a Myth

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Fundamental Assumption:
Turbulence is
thermodynamically irreversible

Primary Objectives

- Differential Topology can be used to determine when a flow process is irreversible - without statistics!
- Cartan’s exterior calculus describes Continuous Topological Evolution.
- Continuous Topological Evolution can be used to describe the decay, but not the creation, of turbulence.
- Cartan’s Magic formula of Continuous Topological Evolution is a dynamical equivalent to the First Law of Thermo.
- **Thermodynamic Irreversibility is an artifact of 4 topological dimensions**.

As Turbulence is irreversible (4D), time dependent 2D (2+1=3D) turbulence is a myth

Secondary Objectives

- Importance of the
  Topological Torsion Vector
  (helicity density is a 4th component)
- Examples of Irreversible Evolution in the direction of the
  Topological Torsion Vector

THERMODYNAMICS

Physical Systems and Processes

To a topologist the First Law is a statement of Cohomology

$$\mathbf{Q} - \mathbf{W} = \mathbf{dU}$$

Definition: A process that produces a 1-form of Heat, **Q**, is reversible when **Q** admits an integrating factor. $\mathbf{Q} = \mathbf{TdS}$.

2 independent functions $\supset$ topological dimension =2

Frobenius: **Q** admits an integration factor iff

$$\mathbf{Q\wedge dQ} = \mathbf{0}$$

Hence $\mathbf{dQ\wedge dQ} = \mathbf{0}$ for a reversible process.

If $dQ\wedge dQ \neq 0$, then the process is thermodynamically IRREVERSIBLE

Cartan's Magic Formula

In Cartan's Calculus, a Physical system is represented by a 1-form of Action:

$$A = A_\mu dx^\mu - \phi dt \quad or \quad p_\mu dq^\mu - hdt$$

A process is represented by a vector field:

$$V = [\mathbf{v}, 1] \quad relative\,to \; \{x, y, z, t\}$$

Evolution of the Action, A, relative to the process, V, is described by Cartan's Magic Formula using the Lie derivative

$$\mathbf{L}_V(\mathbf{A}) = \mathbf{i}(\mathbf{V})\mathbf{dA} + \mathbf{d}(\mathbf{i}(\mathbf{V})\mathbf{A}) \Rightarrow \mathbf{Q}.$$

Rewriting:

$$\boldsymbol{L}_V(\boldsymbol{A}) = \boldsymbol{W} + \boldsymbol{d}(\boldsymbol{U}) = \boldsymbol{Q}$$

Cartan's Magic Formula is a dynamical equivalent to the First Law of Thermodynamics!

Note independence from a metric or a connection!

An Electromagnetic Example

The Action 1-form is composed from the electromagnetic vector and scalar potentials:

$$A = \mathbf{A} \bullet d\mathbf{r} - \phi dt$$

with

$$\mathbf{B} = curl\,\mathbf{A} \quad \text{and} \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$$

Then the virtual work $W = i(V)dA$ becomes:

$$W = \{\mathbf{E} + \mathbf{v} \times \mathbf{B}\} \bullet d\mathbf{r} - (\mathbf{E} \bullet \mathbf{v})dt$$

where the spatial part is the Lorentz force times the differential displacement - the classical definition of differential work.

If $W = 0$ then have a ”Force Free Plasma”

The Cartan Topology

The 1-form of Action used to represent a physical system induces a (Cartan) Topology.

The Pfaff (topological) dimension is defined as the mininum number of functions required to define the Action 1-form.

The Pfaff dimension is easy to compute.

From $A,$ construct $dA,$ and then the Pfaff Sequence:

$$\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$$

The sequence terminates at an integer N, which is the Pfaff dimension of the domain.

Theorem:
Potential and Streamline Flows are possible iff the Pfaff dimension is less than 3, or
$A\hat{}dA = 0.$

THE CARTAN TOPOLOGY

$A = v_k dx^k - \mathcal{H} dt$

$F = dA,\ H = A \hat{}\, dA,\ K = dA \hat{}\, dA$

Basis = { A, A^c, H, H^c } = { A, A∪ F, H, H∪K}

Open Sets: = {X, 0, A, H, A^c, H^c, A∪ H, A∪ H^c, A^c∪H}.

Subsets (σ)	Limit Pts (dσ)	Interior	Boundary (∂σ)	Closure (σ∪dσ)
0	0	0	0	0
A	F	A	F	A∪F
F	0	0	F	F
H	K	H	K	H∪K
K	0	0	K	K
A∪F	F	A∪F	0	A∪F
A∪H	F,K	A∪H	F∪K	X
A∪K	F	A	F∪K	A∪F∪K
F∪H	K	H	F∪K	F∪H∪K
F∪K	0	0	F∪K	F∪K
H∪K	K	H∪K	0	H∪K
A∪F∪H	F,K	A∪F∪H	K	X
F∪H∪K	K	H∪K	F	F∪H∪K
A∪H∪K	F,K	A∪H∪K	F	X
A∪F∪K	F	A∪F	K	A∪F∪K
X	F,K	X	0	X

TOPOLOGICAL CONCLUSIONS:

Pfaff dimension D implies the existence of a (submersive) map to a space of D dimensions.

Turbulent flows must have a Pfaff dimension greater than 2! The suggestion is:

$$Chaos \supset Pfaff\ Dimension\ D = 3$$

$$Turbulence \supset Pfaff\ Dimension\ D = 4$$

More importantly,can show:

$$A \hat{} dA = 0 \supset Pfaff\ Dimension\ D < 3$$

$\supset$ a Connected Topology

$$A \hat{} dA \neq 0 \supset Pfaff\ Dimension\ D > 2$$

$\supset$ a Disconnected Topology

Anholonomic Fluctuations

Consider a Physical system represented by a Lagrange 1-form of Action:

$$A = L(q^{\mu}, V^{\mu}, t)dt + \quad p_{\mu} \bullet (dq^{\mu} - V^{\mu}dt)$$

Anholonomic fluctuations are defined as the deviations from kinematic perfection:

$$\Delta^{\mu} = (dq^{\mu} - V^{\mu}dt) \neq 0$$

The p_{μ} are Lagrange multipliers

QUESTION

What is the Pfaff dimension of A?

Note that A has **3N+1** independent variables

ANSWER: D = 2N + 2

Anholonomic fluctuations
produce
symplectic manifolds

HOWEVER, if the Lagrange multipliers are constrained to be

CANONICAL MOMENTA

$$p_{\mu} - \partial L(q^{\mu}, V^{\mu}, t)/\partial V^{\mu} = 0$$

Then

$$\mathbf{D} = \mathbf{2N} + \mathbf{1}$$

and the physical manifold is of odd dimension (state space) and admits a

Unique Extremal

Reversible Hamiltonian

Direction Field.

IN THIS SENSE

ANHOLONOMIC

FLUCTUATIONS

Are the

SOURCE OF

IRREVERSIBILITY

Note:

$$W = i(V)dA = \sigma_\mu \bullet (dq^\mu - V^\mu dt) + d\Theta$$

such that

$$i(V)W = i(V)i(V)dA = 0 \supset i(V)d\Theta = 0$$

Work is transversal, **Heat is NOT**.

Transition to/from Turbulence

Theorem: Continuous Evolution from a streamline flow (connected topology), $A\wedge dA = 0$, to a turbulent flow, (disconnected topology), $A\wedge dA \neq 0$, is impossible. However, the converse is possible.

Practical Result

Continuous decay of Turbulence

$$\mathbf{A\wedge dA = 0 \Leftarrow A\wedge dA \neq 0}$$

is possible.

Continuous creation of Turbulence

$$\mathbf{A\wedge dA = 0 \Rightarrow A\wedge dA \neq 0}$$

is impossible.

When is a Dynamical System Irreversible?

For those physical systems that can be represented by a 1-form of Action, A, and those processes that can be represented by a vector field, V, Cartan's Magic formula yields:

$$L_{(V)} \int_a^b A = \int_a^b \{ i(V)dA + d(i(V)A) \} \quad = \int_a^b Q$$

- **Those adiabatic solutions, V, such that $L_{(V)} \int_a^b A = 0$ are equivalent to those paths in the calculus of variations that leave the integral stationary.**
- **The equation may be interpreted as an equation describing Continuous Topological Evolution. By defining the 1-form of virtual work, $W = i(V)dA$, and the internal energy as $U = i(V)A$, Cartan's Magic formula becomes the first law of Thermodynamics.**

Cartan's Magic Formula

may be used to test if a

Dynamical System, V,

represents a

Reversible Process.

- **From thermodynamics and Frobenius, if** $Q \wedge dQ = 0$, **then the process is reversible**
- **Hence, use Cartan's Magic formula to compute**

$$\mathbf{Q \wedge dQ = L_{(\mathit{V})} A \wedge L_{(\mathit{V})} dA}$$

for a given physical system, A, **and a given process,** V. **The process is reversible if**

$$\mathbf{L_{(\mathit{V})} A \wedge L_{(\mathit{V})} dA = 0}.$$

EQUATIONS OF MOTION
and topological
EQUIVALENCE CLASSES
of PROCESSES

Topological properties of the 1-form of Virtual Work $W = i(V)dA$, form 2 categories of processes**.**

Either dW $=$ 0 or dW $\neq$ 0.

THEOREM: All processes such that **dW $=$ 0** are thermodynamically reversible.

The category $dW = 0$ includes extremal, Hamiltonian, Bernoulli-Casimir and all Symplectic processes.

Proof:

$$\mathbf{L}_{(V)}\mathbf{dA} = \mathbf{dW} + \mathbf{0} = \mathbf{dQ}$$

If $dW = 0$, it follows that

$$\mathbf{L}_{(V)}\mathbf{A}\hat{}\mathbf{L}_{(V)}\mathbf{dA} = \mathbf{Q}\hat{}\mathbf{dQ} = \mathbf{0}.$$

Hence an integrating factor exists for the Heat 1-form Q and the process is thermodynamically reversible

Classes of

Reversible Processes

- Extremal - Unique Hamiltonian: Virtual Work is zero.

$$\mathbf{W} = \mathbf{i}(\mathbf{V})\mathbf{dA} = \mathbf{0}$$

1. Potential flows $\supset\ dA = 0,\ \int_{cyclic} A = 0$

   no lift.

2. Joukowski flow

   $\supset\ dA = 0,\ \int_{cyclic} A \neq 0$

   Circulation without vorticity is the primary source of Lift on an airfoil

3. Lamb Flows $\supset\ A\hat{}dA = 0$

   No Helicity

- Bernoulli-Casimir-Hamiltonian: Virtual Work is exact, Cyclic work is zero (Eulerian fluid, barotropic fluids).

$$\mathbf{W} = \mathbf{i}(\mathbf{V})\mathbf{dA} = \mathbf{d}\Theta, \quad \int_{cyclic} W = 0$$

- Helmholtz-Symplectic: Virtual Work is closed, but cyclic work is not zero!

$$\mathbf{dW} = \mathbf{0}, \quad \int_{cyclic} W \neq 0$$

Topological Quirks

- Extremal (Hamiltonian) processes are unique on domains of 2n+1 dimensions
- Hamiltonian processes are not unique on domains of dimension 2n+2.

REMARK:
If D = 3, then there exists a unique extremal field which nature selects by the principal of Least Action.
An extremal field is reversible, hence cannot be used to represent turbulence.

Conclusion:

Turbulence must be an artifact of Pfaff dimension 4 (or more)

IRREVERSIBLE PROCESSES

and the

TOPOLOGICAL TORSION VECTOR

Irreversible processes exist only on domains that support a disconnected Cartan topology,

$$\mathbf{A\hat{}dA \neq 0},$$

(with non-uniqueness, envelopes, regressions, and projectivized tangent bundles) Least Action hypotheses $\supset$ Pfaff D = 4, such that for thermodynamic irreversibility:

$$\mathbf{dA\hat{}dA \neq 0}.$$

$\therefore$ The irreversible Action Manifold of support is symplectic of rank 4

On the symplectic domain there exists a unique vector direction field, $\mathbf{T}_4$, with components determined by the functions used to define the physical system (the 1-form of Action, A).

$$\mathbf{i}(\mathbf{T}_4)\mathbf{dx}\hat{}\mathbf{dy}\hat{}\mathbf{dz}\hat{}\mathbf{dt} = \mathbf{A}\hat{}\mathbf{dA}$$

As $\mathbf{dA}\hat{}\mathbf{dA} = \{\mathbf{div}_4\mathbf{T}_4\}\mathbf{dx}\hat{}\mathbf{dy}\hat{}\mathbf{dz}\hat{}\mathbf{dt} \neq \mathbf{0}$

this vector field has a non-zero divergence almost everywhere except on regions (defect ”holes”) where the manifold is no longer symplectic. $(D \neq 4)$

Evolution in the direction of $\mathbf{T}_4$

is

Thermodynamically IRREVERSIBLE

Proof:

Define (on the symplectic 4D domain)

$$\Gamma = \mathbf{div}_4\mathbf{T}_4 \neq \mathbf{0}$$

Then it is possible to show that

$$\mathbf{W} = \mathbf{i}(\mathbf{T}_4)\mathbf{dA} = \Gamma \bullet \mathbf{A}$$

$$\mathbf{L}_{(\mathbf{T}_4)}\mathbf{A} = \Gamma \bullet \mathbf{A}$$

and

$$\mathbf{dQ\hat{}dQ} = \Gamma^2 \bullet \mathbf{dA\hat{}dA}$$

Hence, Evolution in the direction of the Topological Torsion vector is

Thermodynamically

IRREVERSIBLE

Examples of $\mathbf{T}_4$

ELECTROMAGNETISM

The Action 1-form of potentials:

$$A = \mathbf{A} \bullet d\mathbf{r} - \phi dt$$

with

$$\mathbf{B} = curl\,\mathbf{A} \quad \text{and} \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$$

forming the components of dA.

By direct computation,

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \;\; \mathbf{A} \bullet \mathbf{B}]$$

$$\Gamma = \mathbf{div}_4\mathbf{T}_4 = 2\;\mathbf{E} \bullet \mathbf{B} \neq \mathbf{0}$$

Note 1: The Helicity density $\mathbf{A} \bullet \mathbf{B}$ is the 4th component of the Topological Torsion Vector, $\mathbf{T}_4$.

Note 2: Γ is the second Poincare Invariant for an electromagnetic system.

Examples of $\mathbf{T}_4$

HYDRODYNAMICS

Consider an Action 1-form of potentials:

$$A = \mathbf{v} \bullet d\mathbf{r} - H dt$$

with H defined as

$$H = (\mathbf{v} \bullet \mathbf{v}/\mathbf{2} - \lambda div\mathbf{v} + \int dP/\rho) = \mathbf{v} \bullet \mathbf{v} - L$$

and Vorticity defined as $\omega = curl\mathbf{v}$.

The equivalence class of solutions that satisfy the topological constraint on the virtual work W,

$$W = i(V)dA = \nu\{curlcurl\ \mathbf{v} \bullet (d\mathbf{r} - \mathbf{v}dt)\},$$

are solutions to the Navier Stokes equations of motion.

By direct computation,

$$\mathbf{T}_4 = [h\mathbf{v} - L\omega - \nu\ curl\ \omega,\ h]$$

where $h = helicity = \mathbf{v} \bullet \omega$

and

$$\Gamma = \mathbf{div}_4\mathbf{T}_4 = 2\nu\ \{\omega \bullet curl\,\omega\} \neq \mathbf{0}$$

Hence, for a Navier Stokes fluid, domains where the vorticity field does not satisfy the Frobenius integability condition

$$\omega \bullet curl\ \omega \neq 0$$

are domains of thermodynamic irreversibility, and are therefore candidates for turbulent flow.

More detail and references can be found at

CARTAN's CORNER

http://www.uh.edu/~rkiehn

CONCLUSIONS:

- **2 D Turbulence is a myth**
- **Irreversibility is an artifact of 4 dimensions**.
- **Cartan's methods may be used to describe continuous topological evolution**.
- **Anholonomic fluctuations in kinematic perfection is a topological source of irreversibility**.
- **Evolution along the direction field of the Topological Torsion is Thermodynamically irreversible**.
- **There exist solutions to the Navier-Stokes equations which are thermodynamically irreversible**.
- **Thermodynamics and dynamics can be unified by topological methods rather than by statistical methods**.

An invited talk to be presented at the

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