Electromagnetic Waves in the Vacuum with Torsion and Spin.

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Abstract: New time dependent wave solutions to the classical homogeneous Maxwell equations in the vacuum have been found. These waves are not transverse; they exhibit both torsion and spin; they have finite magnetic helicity, $\mathbf{A} \circ \mathbf{B} \neq 0$, a non-zero Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0,$ and a non zero second Poincare invariant, $\mathbf{E} \circ \mathbf{B} \neq 0$. Two four component rank 3 tensors, constructed on topological grounds in terms of the Fields and Potentials, are used to define the concepts of torsion and Spin, even in domains with plasma currents. The divergence of the spin pseudo vector generates the Poincare invariant equivalent to the Lagrangian of the field, $(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi)$. The divergence of the Torsion pseudo vector generates the second Poincare invariant, $2\mathbf{E} \circ \mathbf{B}$. The Poincare invariants have closed integrals which are deformation invariants, and therefore can be used to define deformable coherent structures in a plasma. When the second Poincare invariant is non-zero, there can exist solutions that are not time-reversal invariant.

The Domain of Classical Electromagnetism

In terms of the notation and the language of Sommerfeld and Stratton [1], the classic definition of an electromagnetic system is a domain of space-time independent variables, $\{x, y, z, t\}$, which supports both the Maxwell-Faraday equations,

$$curl\,\mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0, \tag{1.1}$$

and the Maxwell-Ampere equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \qquad div\,\mathbf{D} = \rho. \tag{1.2}$$

For the Lorentz vacuum state, the charge-current densities are subsumed to be zero [$\mathbf{J}$, ρ]= 0 and the field excitations, $\mathbf{D}$ and $\mathbf{H}$, are linearly connected to the field intensities, $\mathbf{E}$ and $\mathbf{B}$, by means of the homogeneous and isotropic constitutive relations $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$. It is further subsumed that the classic Maxwell electromagnetic system is constrained by the statement that the field intensities are deducible from a system of twice differentiable potentials, $[\mathbf{A}, \phi]$:

$$\mathbf{B} = curl\,\mathbf{A}, \qquad \mathbf{E} = -grad\,\phi - \partial\mathbf{A}/\partial t. \tag{1.3}$$

This statement is a topological constraint which in effect states that the domain of support for the $\mathbf{E}$ and $\mathbf{B}$ fields *cannot* be compact without boundary, except for the (twisted or flat) torus or Klein bottle.

The Fields of Torsion and Spin

Besides the charge current 4-vector density, $[\mathbf{J}, \rho]$, whose integral over any closed 3 dimensional manifold is a deformation invariant of the Maxwell system, there exist two other algebraic combinations of the fields and potentials that can lead to similar topological quantities. These objects are the rank 3 Spin (pseudo) vector, or current [2], defined in component form as

$$Spin\ :\ \mathbf{S}_4 = [\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi, \mathbf{A}\circ\mathbf{D}] \equiv [\mathbf{S},\sigma], \tag{2.1}$$

and the rank 3 Torsion (pseudo) vector [3] defined in component form as

$$Torsion\ :\ \mathbf{T}_4 = [\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}] \equiv [\mathbf{T},h]. \tag{2.2}$$

These topologically important fields, which are intimately connected with the potentials and their physical significance, have been little studied (if at all) in classical electromagnetic theory. Solutions to the classical Maxwell's equations are given below that demonstrate existence of such fields of spin and torsion, without recourse to any modification of the classical theory [4].

The 4-divergence of these 4-component rank 3 tensor fields leads to the Poincare projective invariants of the Maxwell system:

$$Poincare\,Invariant\ 1 = div_3(\mathbf{A}\times\mathbf{H}+\mathbf{D}\phi) + \partial(\mathbf{A}\circ\mathbf{D})/\partial t \tag{2.3}$$

$$= (\mathbf{B}\circ\mathbf{H}-\mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J}-\rho\phi)$$

$$Poincare\ Invariant\ 2 = div_3(\mathbf{E}\times\mathbf{A}+\mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t \tag{2.4}$$

$$= -2\mathbf{E}\circ\mathbf{B}$$

These Poincare invariants, like the field intensities, are invariant in the vacuum state to any choice of gauge. When the Spin vector is non-zero, and its 4-divergence (the first Poincare invariant) vanishes, integrals over closed three manifolds of the Spin 4 vector lead a topological property equivalent to a deRham period integral [5]:

$$Spin = \iiint_{closed}\{S^x dy\hat{}dz\hat{}dt - S^y dx\hat{}dz\hat{}dt + S^z dx\hat{}dy\hat{}dt - \sigma dx\hat{}dy\hat{}dz\}. \tag{2.6}$$

This result is valid in the plasma state as well as the vacuum state.

This closed integral is a deformation invariant of any evolutionary process that can be described by a singly parameterized vector field, $\beta\mathbf{V}$, independent of the choice of parameterization, $\beta(x,y,z,t)$, for the Lie derivative of the Spin integral vanishes:

$$L_{(\beta\mathbf{V})}\,Spin = 0. \tag{2.7}$$

When the associated Poincare invariant vanishes, the values of the Spin integral form rational ratios, and as topological quantities, they can be used to define deformable topological coherent structures in a plasma.

Similar statements hold for the closed integrals of the Torsion vector. However, non-zero values of the second Poincare invariant are important to the generation of thermodynamic irreversible evolutionary processes. These concepts will be exhibited with more detail elsewhere. [6]

Earlier Work

In earlier articles, Chu and Ohkawa [7] developed a standing wave example that led Khare and Pradhan [8] to construct a free space electromagnetic wave which had non-zero Poincare Invariants. Braunstein [9] mentioned that these developments were technically flawed and further argued that the existence of a bonafide (spatially bounded) electromagnetic wave in free space with non-zero Poincare invariants was impossible.

The solution counter examples to Braunstein's claim, as given below, were inspired by the work of Ranada [10] who investigated the applications of the Hopf map [11] to the problem of finding knotted solutions to the Maxwell equations.

Ranada suggested the 4-potential (based on the Hopf map)

$$\mathbf{A} = [y,-x,-1](2/\pi)/\lambda^4\ ,\quad \phi = 0/\lambda^4,\quad where\ \lambda^2 = 1+x^2+y^2+z^2, \tag{3.1}$$

which will generate the fields

$$\mathbf{E} = [0,0,0] \qquad \mathbf{B} = [-2(y+zx), +2(x-yz), +(-1+x^2+y^2-z^2)](4/\pi)/\lambda^6. \tag{3.2}$$

Unfortunately, the Ranada 4-potential does not satisfy the Maxwell-Ampere equation for the vacuum with a zero charge current 4-vector, and therefore is not a suitable vacuum solution.

Example Vacuum Solutions with Torsion and Spin

Modifications of the Hopf map suggest consideration of the system of potentials given by the equations

$$\mathbf{A} = [+y, -x, +ct]/\lambda^4\ , \quad \phi = cz/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{4.1}$$

which yield the real field intensities,

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6 \tag{4.2}$$

and

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), +(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6. \tag{4.3}$$

Subject to the dispersion relation, $\varepsilon\mu c^2 = 1$ and the Lorentz constitutive conditions, these time dependent wave functions satisfy the homogeneous Maxwell equations without charge currents, and are therefore acceptable vacuum solutions. The extensive algebra involved in these and other computations in this article were checked with a Maple symbolic mathematics program [12]. It is to be noted that when the substitution $t \Rightarrow -t$ is made in the functional forms for the potentials, the modified potentials fail to satisfy the vacuum Lorentz conditions for zero charge-currents. It appears that the valid vacuum solution presented above is not time-reversal invariant.

The Spin current density for this first non-transverse wave example is evaluated as:

$$Spin : \mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), \\ t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}, \tag{4.4}$$

and has zero divergence. The Torsion current may be evaluated as

$$Torsion : \mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8. \tag{4.5}$$

and has a non-zero divergence equal to the second Poincare invariant

$$Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} = +8c/\lambda^8. \tag{4.6}$$

The solution has magnetic helicity as $\mathbf{A} \circ \mathbf{B} \neq 0$ and is radiative in the sense that the Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$.

It is to be noted that the example solution given above is but one of a class of vacuum wave solutions that have similar non transverse properties. As a second example, consider the fields that can be constructed from the potentials,

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4\ , \quad \phi = cx/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{4.7}$$

These potentials will generate the field intensities

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6 \tag{4.8}$$

and

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6. \tag{4.9}$$

As before, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for the first example:

$$Torsion : \mathbf{T}_4 = +[x,y,z,t]2c/\lambda^8, \qquad -2\mathbf{E}\circ\mathbf{B} = -8c/\lambda^8 . \tag{4.10}$$

When the two examples are combined by addition (or subtraction), the resulting wave is transverse magnetic (in the topological sense that $\mathbf{A}\circ\mathbf{B} = 0$). Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector current does not vanish, hence the example superposition is a wave that is not transverse electric (in the topological sense that $\mathbf{A}\circ\mathbf{D} \neq 0$). For the examples presented above and their superposition, the first Poincare invariant vanishes, which implies that the Spin integral remains a conserved topological quantity for the superposition, with values proportional to the integers. The Spin current density for the combined examples is given by the formula:

$$\begin{aligned} Spin : \mathbf{S}_4 = [&-2x(y+ct)^2, (y+ct)(x^2-y^2+z^2-2cty-c^2t^2), -2z(y+ct)^2, \\ &-(y+ct)(x^2+y^2+z^2+2cty+c^2t^2)](4/\mu)/\lambda^{10}, \end{aligned} \tag{4.11}$$

while the Torsion current is a zero vector

$$Torsion : \mathbf{T}_4 = [0,0,0,0]. \tag{4.12}$$

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E}\times\mathbf{H} = \gamma\,\mathbf{S}, \quad with\ \gamma = -(x^2+y^2+z^2+2cty+c^2t^2)/2c(y+ct)\lambda^2. \tag{4.13}$$

These results seem to give classical credence to the Planck assumption that vacuum state of Maxwell's electrodynamics supports quantized angular momentum, and that the energy flux must come in multiples of the spin quanta. In other words, these combined solutions to classical electrodynamics have some of the qualities of the photon.

References

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