Electromagnetism from the Topological Point of View.

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Many physical systems permit description in terms of a set of constrained exterior differential forms. Topological equivalence classes of evolutionary processes can be defined by the homotopy invariants generated by the action of a vector field on such a given physical system. Two such constraints may be used to define classical electromagnetism. The first topological constraint of classical electromagnetism, $F-dA=0$, leads to the result that closed (global) integrals of electromagnetic flux, F, are deformation invariants with respect to *any* evolutionary process that can be represented by a singly parameterized vector field, $\mathbf{W}_4 = [\mathbf{w}, w^4]$. However, open (local) integrals of electromagnetic flux are evolutionary invariants only with respect to those singly parameterized vector fields which define the symplectic category. The symplectic category is made up of three topological equivalence classes for which the virtual work 1-form, $W = i(\mathbf{W}_4)F$, is zero (extremal), exact or closed. Taking into account the fact that electromagnetic flux contains both a magnetic part and an electric part, symplectic processes can describe the evolution of tubular regions of magnetic field "lines" which apparently "cut and reconnect". Extremal processes exist only when the 4-form $F\hat{}F$ vanishes, and the process is said to be force-free. When valid, this topological refinement, equivalent to $\mathbf{E}\cdot\mathbf{B}=0$, permits the closed (global) integrals of the Torsion-Helicity 3-form, $A\hat{}F$, to be invariant with respect to *any* evolutionary process. The second topological constraint of electromagnetism, $J-dG=0$, leads to the result that closed (global) integrals of charge current density, J, are deformation invariants with respect to *any* singly parameterized vector field. Plasma processes are defined as those evolutionary processes that conserve the open integrals of J . The category of plasma processes is made up from three topological equivalence classes, where the 2-form of Polarization, $P = i(\mathbf{W}_4)J$, is zero, exact or closed.

1. Introduction

1.1. Philosophy and Physics

Galileo's idea that "Laws of Nature are written in the language of geometry" has been the cornerstone of many physical theories. As topology underlies geometry, it seems natural to search for those "Laws of Nature that can be written in the language of topology". It is the thesis of this article that Maxwell's Electromagnetism is just such an artifact. From Felix Klein comes the idea that a geometry is determined in terms of its evolutionary invariants; similarly, so should a topology be determined from its invariants. For elementary geometry, the primitive processes are rigid body rotations and translations. For elementary topology, the primitive processes are continuous deformations or homotopies [1]. In particular, the study of homotopic invariants seems to cover and extend the classical theories of continuum mechanics and electromagnetism.

In order to implement such a program , Galileo's methods again can be used to advantage. First, assume that a certain minimal set of topological constraints describes a physical system. Next, derive the mathematical consequences of these assumptions. Finally study the system's deformation or homotopic invariants, and see if the results agree with observation. Then, refine the topological constraints and repeat the analysis. Evolutionary processes described by a vector field, $\mathbf{V}$, will act on a physical system, defined by a system of exterior differential forms, Σ, by means of Cartan's magic formula, [2]. For a given element, A, of the system of forms, Cartan's formula reads:

$$L_{(\mathbf{v})}A = i(\mathbf{v})dA + d(i(\mathbf{v})A). \tag{1.1}$$

Arnold describes this formula as the homotopy formula [3]. In addition, this same formula when applied to a 1-form of Action, A, can be interpreted as the equivalent cohomological statement of the first law of thermodynamics,

$$\begin{aligned} L_{(\mathbf{v})}A &= i(\mathbf{v})dA + d(i(\mathbf{v})A) & (1.2) \\ &= W + dU = Q. & (1.3) \end{aligned}$$

When the Lie derivative (with respect to some arbitrary vector field V) acting on the form, A, vanishes, then the object is to be regarded as a homotopic invariant relative to V. The formula can be applied to integrals of A. If the integration domain is closed, then the integral is defined as a relative integral invariant. If the integration domain is open, then the integral is defined as an absolute integral invariant. Note that these concepts are independent from a metric or a connection which will act as further constraints on the system.

For both mechanical and electromagnetic systems, the minimum topological constraint will be the postulate that the system can be described by an exact (global) 2-form, $F = dA$. The minimum physical system then consists of the set $\Sigma = \{A, dA = F, A\hat{}F, F\hat{}F, ...\}$ constructed from the algebraic intersections of the forms and their exterior derivatives.. In classical electromagnetism, the system is enlarged by a further constraint that there exists a global exact 3-form, $J = dG$, that defines the charge-current density. The electromagnetic system of forms, therefore, includes $\Sigma = \{A, F, A\hat{}F, F\hat{}F, ...G, dG = J, A\hat{}G, F\hat{}G, ...\}$. If all of the forms are invariants with repect to a given process, V, then that process must be a homeomorphism (and the topology is an evolutionary invariant). However, in the more general case, only some of the forms that make up the Pfaff sequence, Σ, will be evolutionary invariants, and others will not. It is these situations that may be use to define topological evolution. An objective is to put the possible evolutionary vector fields into equivalence classes, depending upon which elements of the Pfaff sequence are left invariant. For example, in the mechanical system, all vector fields for which $dW = 0$ generate symplectomorphisms. This statement is equivalent to the conservation of vorticity in hydrodynamics. In electromagnetism, the statement leads to the "Master Equation" of plasma physics. A special subset (the extremal class) of symplectomorphisms will generate the force free plasma.

1.2. The Topology of Classical Electromagnetism

In the language of exterior differential systems, classical electromagnetism is equivalent to two postulates of topological constraint:

$$The\,Postulate\;of\;Potentials \quad : \quad F - dA = 0 \tag{1.4}$$

$$The\;Postulate\;of\;Currents \quad : \quad J - dG = 0 \tag{1.5}$$

The Maxwell-Faraday equations are a consequence of the postulate of potentials, where A is a 1-form of Action, with twice differentiable coefficients, which induce a 2-form, F, of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). On a four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (representing the postulate of potentials) can be written in the format

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A} \circ d\mathbf{r} - \phi dt. \tag{1.6}$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes:

$$F = dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\} dx^j\hat{}dx^k \tag{1.7}$$

$$= F_{jk} dx^j\hat{}dx^k = \mathbf{B}_z dx\hat{}dy ... + \mathbf{E}_x dx\hat{}dt ... \tag{1.8}$$

where the coefficients of the 2-form F, may be replaced by the familiar engineering notations,

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\;\mathbf{A} = \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{1.9}$$

The closure of the exterior differential system, $dF = 0$,

$$dF = ddA = \{curl\;\mathbf{E} + \partial \mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B} dx\hat{}dy\hat{}dz\} \Rightarrow 0, \tag{1.10}$$

generates the Maxwell-Faraday partial differential equations, as each coefficient of the 3-form dF, must vanish:

$$\{curl\;\mathbf{E} + \partial \mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}. \tag{1.11}$$

For future reference it is important to note that

$$F\hat{}F = -2(\mathbf{E}\circ\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt. \tag{1.12}$$

Hence $F\hat{}F = 0$ is equivalent to $\mathbf{E}\circ\mathbf{B} = 0$.

The Maxwell Ampere equations are a consequence of the postulate of currents, where G is an N-2 form *density* of field excitations ($\mathbf{D}$ and $\mathbf{H}$, related to sources), and J is the N-1 form of charge-current densities. This N-2 form density given by the expression,

$$\begin{aligned} G &= G^{34}(x,y,z,t)dx\hat{}dy... + G^{12}(x,y,z,t)dz\hat{}dt... && (1.13)\\ &= \mathbf{D}^z dx\hat{}dy... + \mathbf{H}^z dz\hat{}dt... && (1.14)\end{aligned}$$

Exterior differentiation produces an N-1 form,

$$J = \mathbf{J}^z(x,y,z,t)dx\hat{}dy\hat{}dt... - \rho(x,y,z,t)dx\hat{}dy\hat{}dz. \tag{1.15}$$

Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \quad and \quad div\,\mathbf{D} = \rho. \tag{1.16}$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0$, or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = 0. \tag{1.17}$$

The two postulates, and the constraint of C2 differentiable functions, implies that $dF = 0$ and $dG = 0$.

Recall that a system of differential forms may be used to define a coarse topology on a domain. The two topological postulates of classical electromagnetism restrict the domains of support for F and J such that, in general, such domains cannot be compact without boundary. The domains of support are compact with boundary, or are open (the only exceptions for the 2-form F are the torus and the Klein bottle).

1.3. Evolutionary Processes

Cartan's magic formula will be used to describe the evolution of exterior differential forms with respect to vector fields. The method does not require the extra constraints of a metric or a connection. Topological evolution will be determined by examining the change (or lack of change) exhibited by the action of the Lie derivative, with respect to a singly parameterized vector field (the process), acting on the set of exterior differential forms (the system) that define the topology of the classical electromagnetic system. If all of the exterior differential forms that define the system topology are evolutionary invariants, then such evolutionary processes are homeomorphisms. In the more general case, for a specific process, some of the exterior differential forms will change and others will be conserved, indicating topological evolution.

The two electromagnetic postulates, and the constraint of C2 differentiable functions, implies that $dF = 0$ and $dG = 0$. It follows that the integrals of F and J over closed domains are evolutionary invariants with respect to *all* evolutionary processes that can be described by a singly parameterized vector field. The proof follows from Cartan's magic formula:

$$L_{(\mathbf{W}_4)}F = i(\mathbf{W}_4)dF + d(i(\mathbf{W}_4)F = d(i(\mathbf{W}_4)F \quad (1.18)$$

$$L_{(\mathbf{W}_4)}J = i(\mathbf{W}_4)dJ + d(i(\mathbf{W}_4)J = d(i(\mathbf{W}_4)J \quad (1.19)$$

The integrals over cycles (closed domains) of the RHS vanish in each case, as the RHS is a perfect differential. Hence both electromagnetic flux, defined as $\iint_{closed} F$, and current, defined as $\iiint_{closed} J$, are evolutionary deformation invariants for *any* process represented by the 4-vector field, $\mathbf{W}_4 = [\mathbf{w}, w^4]$. In fact, the integration domain can be deformed arbitrarily, by multiplying the evolutionary vector field, $\mathbf{W}_4$, by a new parametrization factor $\beta(x, y, z, t)$, and the invariance of the closed integral still holds. The result demonstrates the global topological features of these closed integrals. When the integration domains are over boundaries, the formulas lead to the conservation laws for net flux and net current (Kirchoff's Laws).

Of more immediate interest to this article are the possibilities that the *open* integrals of electromagnetic flux and current exhibit evolutionary invariance. In such cases it must be true that, with respect to the evolutionary process, $\mathbf{W}_4$,

$$L_{(\mathbf{W}_4)}F = d(i(\mathbf{W}_4)F = 0 \quad (1.20)$$

$$L_{(\mathbf{W}_4)}J = d(i(\mathbf{W}_4)J = 0 \quad (1.21)$$

Hence only special classes of vector fields, $\mathbf{W}_4$, are admissable for local evolutionary invariance of Flux and Current. The vector fields that satisfy $d(i(\mathbf{W}_4)F = 0$ are defined as Symplectic processes, and the vector fields that satisfy $d(i(\mathbf{W}_4)J = 0$ are defined as Plasma processes. An objective of this article is to study these special classes of vector fields.

1.4. Spin and Chirality in the classical electromagnetic field.

The two postulates that form the topological basis of classical electromagnetic theory enable the derivation of two other exterior differential systems.

$$d(A\hat{}G) - (F\hat{}G - A\hat{}J) = 0, \qquad d(A\hat{}F) - F\hat{}F = 0. \quad (1.22)$$

These equations introduce the (apparently novel to many researchers) 3-forms of Spin Current density, $A\hat{}G$,[4] and Topological Torsion-Helicity, $A\hat{}F$ [5]. By direct evaluation of the exterior product, on a domain of 4 independent variables, each 3-form will have 4 components that can be symbolized by the 4-vector arrays,

$$Spin - Current \ : \ \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}] = [\mathbf{S}, \sigma], \quad (1.23)$$

$$Torsion - vector \ : \ \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] = [\mathbf{T}, h], \quad (1.24)$$

which are to be compared with the charge current 4-vector density:

$$\text{Charge-Current} : \mathbf{J}_4 = [\mathbf{J}, \rho], \tag{1.25}$$

Note that the ubiquitous helicity density is merely the fourth component of the topological torsion 3-form, $A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt$. For future developments, observe that the torsion vector $\mathbf{T}_4$ and the Spin vector $\mathbf{S}_4$ are associated 4-vectors to the 1-form of Action, in the sense that

$$i(\mathbf{T}_4)A = 0 \quad and \quad i(\mathbf{S}_4)A = 0. \tag{1.26}$$

The two distinct concepts of Spin Current density and the Torsion vector have had almost no utilization in applications of classical electromagnetic theory, for they are explicitly dependent upon the potentials, A. The two 3-forms are not gauge-invariant, but gauge invariance is not compatible with topological evolution, which is of interest herein. Examples, both novel and well-known, of vacuum and plasma solutions to the electromagnetic system which satisfy (and which do not satisfy) these topological constraints appear below.

The two 4-forms, $(F\hat{}G - A\hat{}J)$, and $F\hat{}F$ have coefficients that are equivalent to the first and second Poincare invariants of classical theory. The closed integrals of these 4-forms are deformation invariants with respect to *all* singly parametrizeable evolutionary processes.

When $\mathbf{E} \circ \mathbf{B} = 0$ (equivqlent to $d(A\hat{}F) \Rightarrow 0$) it is also possible to define the topological integral of chirality (related to the helicity integral in the static case) as

$$Chirality = \iiint_{closed} A\hat{}F \tag{1.27}$$

Chirality is a global evolutionary invariant for all processes, when $F\hat{}F = 0$.

Similarly, when $(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi) = 0$ (equivalent to $d(A\hat{}G) \Rightarrow 0$) it is also possible to define the topological integral of Spin as

$$Spin = \iiint_{closed} A\hat{}G \tag{1.28}$$

Spin is a global evolutionary invariant for all processes, when $F\hat{}G - A\hat{}J = 0$.

2. Evolution of Electromagnetic Flux

Relative to the postulates of topological constraints that define an electromagnetic system in terms of the differential systems, $F - dA = 0$ and $J - dG = 0$, all evolutionary processes that can be described by singly parameterized C2 vector fields conserve the closed (global) integrals of flux, $\iint_{closed} F$, and current, $\iiint_{closed} J$. This set of all evolutionary processes has one subset, defined as the Symplectic category, which consists of all those processes which conserve the open (local) integrals of flux. Another subset, defined as the Plasma category, consists of all those processed that conserve the open (local) integrals of current.

2.1. Symplectic Processes

The Symplectic category of evolutionary processes, $\mathbf{W}_4 = [\mathbf{w}, w^4]$, is composed from three equivalence classes of vector fields. Each equivalence class is defined by constraints on the evolutionary vector field such that the virtual work 1-form, W, defined as,

$$W = i(\mathbf{W}_4)F = \{w^4\mathbf{E} + \mathbf{w} \times \mathbf{B}\}_k dx^k - (\mathbf{w} \circ \mathbf{E})dt, \tag{2.1}$$

satisfies one of three conditions:

$$\begin{aligned} Extremal \quad W &= i(\mathbf{W}_4)F = 0 & (2.2) \\ Exact \quad W &= i(\mathbf{W}_4)F = d\Theta & (2.3) \\ Closed \quad W &= i(\mathbf{W}_4)F = d\Theta + \gamma,\ d\gamma = 0. & (2.4) \end{aligned}$$

In each case, $d(i(\mathbf{W}_4)F) = 0$,which is the defining equation of a symplectic process [6] relative to the 2-form, F. (A symplectic process is different from a symplectic manifold. A symplectic manifold is always even dimensional). Note that the spatial components of the virtual work 1-form, W, have the format of the Lorentz force. In the extemal case, the process is often said to be "force-free", as the Lorentz force vanishes. Relative to a given 1-form of Action, A, the evolutionary vectors, $\mathbf{W}_4$, that belong to either of the three classes defined above, leave the open integrals of electromagnetic flux invariant.

$$L(\mathbf{W}_4) \iint F = \iint \mathbf{B}_z dx\hat{}dy ... + \mathbf{E}_x dx\hat{}dt.. = \iint d(i(\mathbf{W}_4)F \Rightarrow 0 \tag{2.5}$$

The contribution to the electromagnetic flux consists of two separate parts, the magnetic flux, $\iint \mathbf{B} \circ d\mathbf{n}$, and the electroflux, $\iint \mathbf{E} \circ d\mathbf{r}\hat{}dt$. The suggestion of Hornig [7] is that the total electromagnetic flux can be conserved locally by a symplectic process, and yet the magnetic flux component can evolve from a finite value to a zero value and then back to a finite value, giving the appearance that the magnetic field "lines" undergo a "cut and reconnect" process. The fundamental idea is that conservation of flux is not equivalent to conservation of lines [8]. Examples are presented below.

Symplectic processes also preserve the open integrals of $F\hat{}F$ as evolutionary (local) invariants. (This statement is the basis for the derivation of the second Poincare invariant).

2.2. Extremal Symplectic processes, $\mathbf{E} \circ \mathbf{B} = 0$

An extremal process requires that $W = i(\mathbf{W}_4)F = 0$. This constraint is the equivalent to the defintion of a "force-free" plasma. As the coefficients of the 2-form F form an anti-symmetric matrix on $[x, y, z, t]$, the matrix has a rank equal to 0, 2, or 4. If $F\hat{}F \neq 0$ or equivalently, $\mathbf{E} \circ \mathbf{B} \neq 0$, then the rank of the matrix is 4, and no null eigenvectors can exist. In otherwords, extremal fields **cannot exist** for domains where the topological parity 4-form, $F\hat{}F$,does not vanish. In domains where $\mathbf{E} \circ \mathbf{B} = 0$, extremal fields (null eigenvectors) can exist (and the plasma can be "force free"). In fact, two linearly independent null eigenvectors exist for the 2-form F on the 4-dimensional domain. One of these extremal vectors is the Torsion vector, which is not only extremal, but also is associative with respect to A, when the Pfaff dimension is not 4. Vector fields which are both associative and extremal form a set of charateristics. The other extremal vector field is an extension of the Poynting vector. These two extremal fields are not equivalent.

2.2.1. The Topological Torsion extremal-vector

One of these null eigenvectors is very easy to construct, for it is proportional to 4-vector of Topological Torsion, $\mathbf{T}_4$, derived algebraically from the 3-form, $A\hat{}F$:

$$Torsion - vector \quad : \quad \mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] = [\mathbf{T}, h] \Rightarrow [\mathbf{w}, w^4]. \tag{2.6}$$

$$where \quad : \quad i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}F \tag{2.7}$$

Note that the Helicity density, $\mathbf{A} \circ \mathbf{B}$, becomes the timelike component, w^4. Direct computation demonstrates that

$$W = i(\mathbf{T}_4)F = i(\mathbf{T}_4)dA = \{\mathbf{E} \circ \mathbf{B}\}A, \tag{2.8}$$

vanshes, subject to the single constraint, $\mathbf{E} \circ \mathbf{B} = 0$. Hence, the Topological Torsion vector is extremal when the Pfaff dimension is not 4. It is also possible to show that the Topological orsion vector is associative in that

$$i(\mathbf{T}_4)A = 0. \tag{2.9}$$

Extremal motion in the direction of $\mathbf{T}_4$ is special. Not only is such an extremal process adiabatic in a thermodynamic sense, as $L_{(\mathbf{T}_4)}A = Q = 0$, but also such a process leaves the 3-form, $A\hat{}F$, a local invariant. In otherwords, extremal evolutionary processes in the direction of $\mathbf{T}_4$ preserve chirality locally.

It is extraordinary that in the case where $F\hat{}F \neq 0$, then evolution in the direction of $\mathbf{T}_4$ is not extremal, but such that $L_{(\mathbf{T}_4)}A = Q = (\mathbf{E} \circ \mathbf{B})\, A$ and $Q\hat{}dQ \neq 0$. The heat 1-form, Q, therefore does not admit an integrating factor, and the process in the direction of the Topological Torsion vector is thermodynamically irreversible [9].

2.2.2. The Poynting energy flux as an extremal vector.

A second null eigenvector can be constructed in terms of a timelike modification of the Poynting 4-vector representing the energy flux:

$$Null\ \ Poynting - vector : \mathbf{N}_4 = [\mathbf{E} \times \mathbf{H}, (\mathbf{B} \circ \mathbf{H})] \Rightarrow [\mathbf{w}, w^4] \tag{2.10}$$

Direct computation demonstrates that

$$W = i(\mathbf{N}_4)F = \mathbf{E} \circ \mathbf{B}\{\mathbf{H} \circ d\mathbf{r}\} - 0dt, \tag{2.11}$$

which vanishes when $\mathbf{E} \circ \mathbf{B} = 0$. It is important to note that the work one form (hence the force) has no time-like component. Motion in the direction of the Poynting vector has no power term.

The null Poynting vector, as do all vector fields, admits an a large number of integrating factors, ρ_m, such that the "mass" current density, ρ_m $[\mathbf{v}, 1]$,satisifies the equation of continuity:

$$\partial\rho_m/\partial t + div_3(\rho_m \mathbf{v}) = 0 \tag{2.12}$$

A useful formula for creating such integrating factors on 4 dimensional domains is given by the expression:

$$\rho_m = 1/\{a(w^1)^p + b(w^2)^p + c(w^3)^p + d(w^4)^p\}^{4/p} \tag{2.13}$$

2.2.3. The Momentum flux as a possible extremal vector.

It is also of interest to consider the momentum flux in terms of the evolutionary field, $\mathbf{P}_4 = \rho_m[\mathbf{v},1]$, with components defined by the equations,

$$\rho_m c^2 = (1/2)\{\mathbf{B} \circ \mathbf{H} + \mathbf{D} \circ \mathbf{E}\} \quad \textit{such that } \rho_m \mathbf{v} = \mathbf{D} \times \mathbf{B}. \tag{2.14}$$

For an isotropic constitutive relationship, $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, an extension of Bateman's result [10] leads to and equation of continuity with a defect, or source:

$$\partial(1/2c^2)\{\mathbf{B} \circ \mathbf{H} + \mathbf{D} \circ \mathbf{E}\}/\partial t + div_3(\mathbf{D} \times \mathbf{B}) = (\mathbf{J} \circ \mathbf{E}). \tag{2.15}$$

The hydrodynamic equation of continuity is satisfied by $\mathbf{P}_4$ in virtue of the Maxwell-Faraday and the Maxwell-Ampere equations, if $(\mathbf{J} \circ \mathbf{E}) = 0$. Such an extremal constraint implies that the current density, $\mathbf{J}$, if it is non-zero, is a Hall current, orthogonal to the $\mathbf{E}$ field.

It follows that the virtual work induced by $\mathbf{P}_4$ is equal to

$$W = i(\mathbf{P}_4)F = \{(1/2)(-\mathbf{B} \circ \mathbf{H} + \mathbf{D} \circ \mathbf{E})\mathbf{E} + (\mathbf{E} \circ \mathbf{B})\mathbf{H}\}/c^2. \tag{2.16}$$

The momentum process is extremal, $W = i(\mathbf{P}_4)F = 0$,only when the magnetic energy density and the electric energy density are equal, and the chirality integral is conserved, $\mathbf{E} \circ \mathbf{B} = 0$. Recall that these are the conditions that define the classical plane wave solutions of the electromagnetic field in a vacuum.

2.2.4. An extremal process with Torsion but without Helicity density ($\mathbf{A} \circ \mathbf{B} = 0$)

As an example, consider a slight generalization of the field suggested by Hornig [11], where the 1-form of Action can be defined as:

$$A = (y^2 - \alpha x^2)dz - \eta z dt. \tag{2.17}$$

The system is of Pfaff dimension 3, because $F\hat{}F = 0$, hence extremal processes exist. The fields are given by the formulas,

$$\mathbf{E} = [0, 0, \eta] \quad \mathbf{B} = [2y, 2\alpha x, 0], \tag{2.18}$$

from which it is apparent that indeed $\mathbf{E} \circ \mathbf{B} = 0$. The parameter α and its sign have been chosen deliberately to emphasize the "hyperbolic" regime. If $\alpha < 0$ the "lines of B" are closed ellipsoids about the z axis. If $\alpha > 0$ then the "lines of B" form hyperbolas.

The charge density vanishes, as $\rho = \varepsilon \, div_3\mathbf{E} \Rightarrow 0$, and yet there is a plasma current, when $\alpha \neq 1$.

$$\mathbf{J} = [0, 0, 2(\alpha - 1)/\mu]. \tag{2.19}$$

The 2-form of F has two null eigenvectors, hence two extremal direction fields,

$$e1 = [-\eta, 0, 0, 2\alpha x] \quad and \quad e2 = [0, \eta, 0, 2y]. \tag{2.20}$$

The Null-Poynting vector is

$$\mathbf{N}_4 = [-(2\eta/\mu)\alpha x, (2\eta/\mu)y, 0, 4(y^2 + \alpha^2 x^2)], \tag{2.21}$$

and is a particular combination of the two null eigenvectors described above.

The system is without helicity density as $\mathbf{A} \circ \mathbf{B} = 0$,but the Topological Torsion vector is not zero:

$$\mathbf{T}_4 = -2z\eta[y, \alpha x, 0, 0] \tag{2.22}$$

As remarked above, the Topological Torsion vector is also an null eigen vector of the 2-form, F, as long as $\mathbf{E} \circ \mathbf{B} = 0$. The Torsion vector in the example is parallel to the $\mathbf{B}$ field lines, and is orthogonal to the Null-Poynting vector. As $div_4\mathbf{T}_4 = 0$,the open 3-dimensional chirality integral, $Chirality = \iiint A\hat{}F$, is an evolutionary invariant.

When $F\hat{}F \Rightarrow 0$, the two null eigenvectors of the 2-form F are the Null-Poynting vector and the Torsion vector. The spatial direction field of the force induced by the Poynting vector is along the $\mathbf{H}$ field lines, while the spatial direction field of the force induced by the Torsion vector is along the $\mathbf{A}$ field lines. The cyclic work induced by the Poynting vector is related to the Amperian currents, $\oint\{(\mathbf{E} \circ \mathbf{B})/(\mathbf{H} \circ \mathbf{B})\}\mathbf{H} \circ d\mathbf{l}$, and the cyclic work induced by the Torsion vector is related to the circulation $\oint\{(\mathbf{E} \circ \mathbf{B})/(\mathbf{A} \circ \mathbf{B})\}\mathbf{A} \circ d\mathbf{l}$. Both contributions to the cyclic work vanish when $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$.

The Spin vector becomes

$$\mathbf{S}_4 = (y^2 - \alpha x^2)[-(2/\mu)\alpha x, (2/\mu)y, \varepsilon\eta^2 z/(y^2 - \alpha x^2), \varepsilon\eta] \tag{2.23}$$

with a non-zero divergence that implies that the Spin integral,

$$Spin = \iiint_{closed} A\hat{}G, \tag{2.24}$$

is not an evolutionary invariant.

2.3. Exact Bernoulli - Casimir processes

This second class of symplectic processes can exist on domains of Pfaff dimension 3 ($F\hat{}F = 0$) or Pfaff dimension 4 ($F\hat{}F \neq 0$). For example, any extremal process, $[\mathbf{e}, e^4]$ can be modified by the map $[\mathbf{e}, f(\phi) + e^4] \Rightarrow [\mathbf{w}, w^4]$ to the yield the desired equation,

$$i(\mathbf{W}_4)F = \{w^4\mathbf{E} + \mathbf{w} \times \mathbf{B}\}_k dx^k - (\mathbf{w} \circ \mathbf{E})dt = d\Theta \tag{2.25}$$

$$\Theta = \int f(\phi)d\phi \tag{2.26}$$

Again there are two situations, depending on whether or not the chirality integral is conserved. In the situation where ($F\hat{}F \neq 0$), given the Bernouilli-Casimir function $\Theta(x, y, z, t)$, the 4-vector field can be uniquely determined. As the gradient of Θ is transversal to the evolutionary process, $i(\mathbf{W}_4)d\Theta = 0$, which implies that Θ is a constant along each flow line, $L_{(\mathbf{W}_4)}\Theta = 0$. Like the Bernoulli constant in hydrodynamics, Θ may have values that differ from evolutionary flow line to flow line.

3. Evolution of Current

3.1. The Plasma Process

Arguments similar to the ones used to define Symplectic processes relative to the exact 2-form F can be used to define Plasma processes relative to the exact 3-form J. Those evolutionary processes that preserve charge current density locally (open integration domains) can be put into three equivalence classes or categories of vector fields, $\mathbf{W}_4 = [\mathbf{w}, w^4]$. The three distinct classes of Plasma processes are defined herein by constraints on the evolutionary vector field such that the virtual Polarization 2-form, P, satisfies one of three conditions,

$$\textit{Extremal}\ P = i(\mathbf{W}_4)J = 0 \tag{3.1}$$

$$\textit{Exact}\ P = i(\mathbf{W}_4)J = d\omega \tag{3.2}$$

$$\textit{Closed}\ P = i(\mathbf{W}_4)J = d\omega + \gamma,\ d\gamma = 0. \tag{3.3}$$

The Virtual Polarization 2-form is defined as,

$$P = i(\mathbf{W}_4)J = i(\mathbf{W}_4)dG = \{w^4\mathbf{J} - \rho\mathbf{w}\}_{kj} dx^k\hat{}dx^j - (\mathbf{J} \times \mathbf{w})_i dx^i\hat{}dt, \tag{3.4}$$

and is the analogue of the virtual work 1-form, W, defined in preceeding paragraphs as $W = i(\mathbf{W}_4)F = i(\mathbf{W}_4)dA$.

In the three constrained cases stated above, $d(i(\mathbf{W}_4)J) = 0$,which is taken to be the defining equation of a Plasma process relative to the 3-form, J. The evolution of the 3-form (density) J is given by the equation,

$$\begin{aligned} L_{(\mathbf{W}_4)}J &= \{i(\mathbf{W}_4)dJ + di(\mathbf{W}_4)J\} && (3.5) \\ &= \{0 + dP\} \Rightarrow 0 \ \ Plasma \ \ process && (3.6) \end{aligned}$$

for a "Plasma" process. Hence a plasma process conserves the charge-current density locally. This result is the analog of the Helmholtz theorem for the conservation of vorticity by symplectic processes:

$$\begin{aligned} L_{(\mathbf{W}_4)}F &= \{i(\mathbf{W}_4)dF + di(\mathbf{W}_4)F\} && (3.7) \\ &= \{0 + dW\} \Rightarrow 0 \quad Synplectic \ \ process && (3.8) \end{aligned}$$

In the extremal case, the equations of constraint indicate that the current density is proportional to the velocity,

$$\{w^4\mathbf{J} - \rho\mathbf{w}\} = 0 \qquad \mathbf{J} \times \mathbf{w} = 0. \tag{3.9}$$

For non-zero charge density, the current density has the ubiquitous form, $\mathbf{J} = \rho\mathbf{v}$. If the charge density vanishes, $\rho \Rightarrow 0$, then either $\mathbf{J} \Rightarrow 0$, or $w^4 \Rightarrow 0$. Evolutionary Plasma processes for which $w^4 \Rightarrow 0$, will be recognized as the transport of coherent states.

3.2. A rotating plasma with an accretion disk (Finite Spin current - Zero Torsion)

A very interesting time independent set of functions that satisfy the Maxwell system is given by the Potentials,

$$\mathbf{A} = \{f(x, y, z, t)\}Kelvin = \{z/\sqrt{(\delta z^2 + x^2 + y^2)}\}[-y, x, 0]/(x^2 + y^2) \tag{3.10}$$

which generates the Hedge-Hog B field,

$$\mathbf{B} = -[x, y, z]/(\delta z^2 + x^2 + y^2)^{3/2}, \tag{3.11}$$

and (assuming the Lorentz constitutive relations) the London-like Current density,

$$\mathbf{J} = \Lambda(x, y, z, t)\mathbf{A} = -3(1 - \delta)(x^2 + y^2)/(\delta z^2 + x^2 + y^2)^2\mathbf{A}. \tag{3.12}$$

It is apparent that the Helicity density $\mathbf{A} \circ \mathbf{B}$ vanishes identically, as do all components of the Topological Torsion tensor: $A\hat{\ }F \Rightarrow 0$. There exists a Lorentz force,

$$\mathbf{J} \times \mathbf{B} = \{3(1 - \delta)/(\delta z^2 + x^2 + y^2)^4\}[yz^2, xz^2, -z(x^2 + y^2)]/\mu \tag{3.13}$$

which has the remarkable features that for an "oblate" situation ($\delta < 1$), the "plasma" is forced away from the rotation z axis, but is attracted to the $z = 0$ plane to form an accretion disk. There are no Amperian currents and no Lorentz force unless the system is anisotropic (oblate or prolate).

When the 3-form of topological Spin is evaluated, $A\hat{}G \neq 0$, it is remarkable that the Lorentz force is proportional to the Spin current

$$\mathbf{J} \times \mathbf{B} = -3(1-\delta)\mathbf{S}/(x^2 + y^2). \tag{3.14}$$

In this example, the spin current is to be considered as a back reaction to the Lorentz force. The usual interpretation in MHD theory is to resolve $\mathbf{J} \times \mathbf{B}$ in terms of a "magnetic pressure and a magnetic tension":

$$\mathbf{J} \times \mathbf{B} = -\nabla(\mathbf{B} \circ \mathbf{H})/2 + \mathbf{H} \circ \nabla \mathbf{B} = \text{"}\textit{magnetic pressure} \;\; + \;\; \textit{magnetic tension}\text{"}. \tag{3.15}$$

However, the Spin interpretation would imply that the effects are more related to rotational deformations, rather than to translational deformations.

3.3. A plasma with Topological Torsion. (but Zero Spin)

Consider the Beltrami potentials (related to a Heisenberg exterior differential system of Pfaff dimension 3)

$$\mathbf{A} = [-y, x, -a]/(a^2 + x^2 + y^2), \tag{3.16}$$

which generate the Fields, and the Amperian currents,

$$\mathbf{B} = 2a[-y, x, -a]/(a^2 + x^2 + y^2)^2, \tag{3.17}$$

$$\mu\mathbf{J} = 4a[-2ay, 2ax, -a(-a^2 + x^2 + y^2)]/(a^2 + x^2 + y^2)^3. \tag{3.18}$$

The Torsion vector has one component (the helicity density) and the Spin vanishes:

$$\mathbf{T}_4 = -2a[0, 0, 0, 1]/(a^2 + x^2 + y^2)^2, \qquad \mathbf{S}_4 = [0, 0, 0, 0] \;\; . \tag{3.19}$$

3.4. A time dependent irreversible vacuum wave (with $\mathbf{E} \circ \mathbf{B} \neq \mathbf{0}$).

Modifications of the Hopf map suggest consideration of the system of potentials given by the equations

$$\mathbf{A} = [+y, -x, +ct]/\lambda^4 \;, \quad \phi = cz/\lambda^4, \quad \textit{where } \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2, \tag{3.20}$$

which yield the real field intensities,

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6 \tag{3.21}$$

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), -(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6. \tag{3.22}$$

Subject to the dispersion relation, $\varepsilon\mu c^2 = 1$ and the Lorentz constitutive conditions, these time dependent wave functions satisfy the homogeneous Maxwell equations without charge currents, and are therefore acceptable vacuum solutions, $\mathbf{J}_4(+t) = 0$.

The extensive algebra involved in these and other computations in this article were checked with a Maple symbolic mathematics program [12].

It is to be noted that when the substitution $t \Rightarrow -t$ is made in the functional forms for the potentials, the fields computed from the new functional forms fail to satisfy the vacuum Lorentz conditions for zero charge-currents, $\mathbf{J}_4(-t) \neq 0$. The $\mathbf{E}_{(-t)}$ field calculated from the potentials $\mathbf{A}_4(-t)$ is not equal to the $\mathbf{E}_{(+t)}$ field computed from $\mathbf{A}_4(+t)$, and exhibits a non-zero divergence; *e.g.*,

$$\mathbf{E}_{(-t)} = [-2(cty - xz), +2(ctx + yz), (c^2t^2 + z^2)]2c/\lambda^6 \tag{3.23}$$

$$\rho/\varepsilon = div_3\{\mathbf{E}_{(-t)}\} = -8cz(x^2 + 5c^2t^2 + y^2 + z^2)/\lambda^8. \tag{3.24}$$

In this sense, the valid vacuum $(+t)$ solution presented above is not time-reversal invariant. (Of course the differential form for the Action transforms properly).

The Spin current density for this first non-transverse wave $(+t)$ example is evaluated as:

$$\begin{aligned} Spin \ : \ & \mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), \\ & t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}, \end{aligned} \tag{3.25}$$

and has zero divergence. Hence its global integral (Spin) is quantized. The Torsion current may be evaluated and leads to:

$$Torsion : \mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8 \qquad Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} = +8c/\lambda^8. \tag{3.26}$$

The solution has magnetic helicity as $\mathbf{A} \circ \mathbf{B} \neq 0$ and is radiative in the sense that the Poynting vector, $\mathbf{E} \times \mathbf{H} \neq 0$, and the wave is not transverse. The Chirality integral is not quantized.

It is to be noted that the example solution given above is but one of a class of vacuum wave solutions that have similar non transverse properties. As a second example, consider the fields that can be constructed from the potentials,

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4 \ , \quad \phi = cx/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{3.27}$$

These potentials will generate the field intensities,

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6 \tag{3.28}$$

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6. \tag{3.29}$$

As before, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for the first example:

$$Torsion : \mathbf{T}_4 = +[x, y, z, t]2c/\lambda^8 , \qquad -2\mathbf{E} \circ \mathbf{B} = -8c/\lambda^8 \ . \tag{3.30}$$

3.5. A radiating vacuum wave with Quantized Spin (finite Poynting Vector - Zero Chirality)

When the two examples above are combined by addition (or subtraction), the resulting wave is transverse magnetic (in the topological sense that $\mathbf{A} \circ \mathbf{B} = 0$). Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector current does not vanish, hence the example superposition is a wave that is not transverse electric (in the topological sense that $\mathbf{A} \circ \mathbf{D} \neq 0$). For the superposed example, the first Poincare invariant vanishes, which implies that the Spin integral remains a conserved topological quantity, with values proportional to the integers. The non-zero Spin current density for the combined examples is given by the formula:

$$\begin{aligned} Spin \quad : \quad & \mathbf{S}_4 = [-2x(y+ct)^2, (y+ct)(x^2 - y^2 + z^2 - 2cty - c^2t^2), -2z(y+ct)^2, \\ & -(y+ct)(x^2 + y^2 + z^2 + 2cty + c^2t^2)](4/\mu)/\lambda^{10}, \end{aligned} \tag{3.31}$$

while the Torsion current is a zero vector, $A\hat{}F \Rightarrow 0$.

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E} \times \mathbf{H} = \gamma\ \mathbf{S}, \quad with\ \gamma = -(x^2 + y^2 + z^2 + 2cty + c^2t^2)/2c(y+ct)\lambda^2. \tag{3.32}$$

These results seem to give classical credence to the Planck assumption that the vacuum state of Maxwell's electrodynamics supports quantized angular momentum, and that the energy flux must come in multiples of the spin quanta. In other words, these combined solutions to classical electrodynamics have some of the experimental qualities of the quantized photon.

4. Summary

Many of the chiral and spin features of the quantized photon have their basis in the topological properties of classical electromagnetism, and the 3-forms $A\hat{}F$ and $A\hat{}G$.

5. References

[1] D. H. Gottlieb, "Skew Symmetric Bundle Maps on Space-Time" (1997),

[2] J. E. Marsden and T. S. Riatu, Introduction to Mechanics and Symmetr (Springer-Verlag, 1994) p.122

[3] V.I. Arnold, Mathematical Methods of Classical Mechanics(Springer-Verlag, 1989) p.198

[4] R. M. Kiehn and J. F. Pierce, "An Intrinsic Transport Theorem" Phys. Fluids **12**, 1971 (1969)

[5] R. M. Kiehn, "Topological Torsion, Pfaff Dimension and Coherent Structures", in: H. K. Moffatt and T. S. Tsinober eds, Topological Fluid Mechanics (Cambridge University Press, Cambridge, 1990) p. 449

[6]

[7] G. Hornig and K. Schindler, K. "Magnetic topology and the problem of its invaraint definition" Physics of Plasmas, **3**, p.646 (1996).

[8] N. E. Kochin, I. A. Kibel, N. N. Roze, Theoretical Hydrodynamics, (Interscience, NY 1964) p.157.

R. M. Kiehn, "Intrinsic hydrodynamics with applications to space time Fluids", Int. J. Engng Sci **13**, 941 (1975).

[9] P. M. Morse, Thermal Physics, (Benjamin,1964) p. 60.

[10]

[11] H. Bateman, Electrical and Optical Wave Motion, (Dover, New York, 1914, 1955) p.12.

[12] http://www.uh.edu/~rkiehn/pdf/emhopf.pdf