Topological Torsion and Thermodynamic Irreversibility.

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Abstract: Cartan's differential topology is used to construct a method for determining if a process applied to a physical system is thermodynamically irreversible.

Introduction

For physical systems that can be described by a 1-form of Action, A, and processes that can be described by a vector field, $\mathbf{V}$, it is possible, using Cartan's magic formula [1], to determine if the process is thermodynamically reversible or irreversible, without resorting to the assumptions of statistical mechanics. In classical thermodynamics, a process acting on a physical system will produce a 1-form of Heat Q. If the Heat 1-form, Q, admits an integrating factor, then the process is thermodynamically reversible [2]. Otherwise the process is irreversible. The Frobenius theorem implies that an integrating factor exists if and only if the Heat 1-form satisfies the conditions of integrability, $Q\hat{}dQ = 0$. In such a situation, Q admits a representation of the form $Q = TdS$. From the first law it follows that the Work 1-form, for a reversible process, satisfies the equations,

$$W = TdS - dU, \quad dW = dT\hat{}dS, \quad W\hat{}dW = dU\hat{}dT\hat{}dS, \quad dW\hat{}dW = 0.$$

It follows that,

$$An\ irreversible\ process \supset dW\hat{}dW \neq 0.$$

All that is left to do is to find a relationship between thermodynamics and evolutionary dynamics that will permit the evaluation of $Q\hat{}dQ$ and $dW\hat{}dW$. As Tisza points out [3], this search for a non-statistical relationship between mechanics and thermodynamics has been an open question. This article demonstrates how Cartan's theory of exterior differential systems may be used to solve this problem, and treat thermodynamic problems of topological evolution whereby the topology of the initial state and the topology of the final state are not the same [4]. A major result is that thermodynamic irreversibility is an artifact of topological dimension ≥ 4, a fact that follows immediately from the requirement that an irreversible process satisfies the equation, $dW\hat{}dW \neq 0$. Hence turbulence (which is by agreement thermodynamically irreversible) in 2D+1 dimensions is a myth.

Cartan's Magic Formula

In Cartan's Calculus, a physical system is represented by a 1-form of Action:

$$A = A_\mu dx^\mu - \phi dt \quad \approxeq \quad p_\mu dq^\mu - hdt.$$

and a process is represented by a vector field:

$$\mathbf{V} = [\mathbf{v}, 1] \quad relative\ to\ \{x, y, z, t\}$$

Evolution of the Action, A, relative to the process, $\mathbf{V}$, is described by Cartan's Magic Formula, using the Lie derivative

$$L_{\mathbf{V}}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) \Rightarrow Q.$$

By rewriting this equation of topological evolution (called the homotopy formula by Arnold [5]),

$$L_{\mathbf{V}}A = W + d(U) = Q$$

it is apparent that there is a connection between Cartan's Magic Formula and the first law of electrodynamics. Assume that the relationship is valid. Then for a given physical system, A, it is possible to compute both W and Q. If $dW\hat{}dW \neq 0$, then that process $\mathbf{V}$ acting the system A is thermodynamically irreversible. It is remarkable that on a symplectic manifold of even dimension 2n+2 it is possible to find a unique direction field, defined as the Topological Torsion vector, such that evolution with a component in the direction of the Topological Torsion vector satisfies the conditions of thermodynamic irreversibility. This direction field is defined by the equation,

$$i(\mathbf{T})dx\hat{}dy\hat{}dz\hat{}dt.... = A\hat{}dA....$$

Evolution of the Action in the direction of the Torsion vector yields

$$L_{\mathbf{T}}A = \Gamma\ A$$

where $\Gamma = div(\mathbf{T})$. It follows that

$$Q\hat{}dQ = L_{\mathbf{T}}A\hat{}L_{\mathbf{T}}dA = \Gamma^2\ A\hat{}dA$$

This expression is not zero if the divergence of the Torsion vector is not zero. Such a requirement implies that $d(A\hat{}dA...)$ does not vanish and insures that the manifold is symplectic and of even dimension. Evolution in the direction of $\mathbf{T}$ with $div(\mathbf{T}) \neq 0$ is thermodynamically irreversible. Examples are presented below.

Categories of Processes

Processes, $\mathbf{V}$, can be put into equivalence classes determined by the deRham decomposition theorems and Pfaff dimension of the Action and Work 1-forms. Consider the sequence $\{A, dA, A\hat{}dA, dA\hat{}dA...\}$ This Pfaff sequence will terminate with D terms. The number of elements, D, in the sequence determines the Pfaff dimension or class [6] of a particular 1-form of Action, and represents the minimum number of functions required to describe the 1-form, from a topological point of view. The idea is that there exists a submersive map from the original domain of definition to the domain of minimal Pfaff dimension, D. The condition for thermodynamic irreversibility, $dW\hat{}dW \neq 0$, implies that the minimum dimension for the existence of irreversible work is equal to (Pfaff dimension) 4.

First, consider the more familiar classes of processes that belong to (but do not exhaust) the reversible category. The work 1-form is then zero, $W = 0$, exact, $W = -d\Theta$, or closed, $dW = 0$. When the Work 1 form is null, the 1-form of Action of maximal rank must be of odd dimension 2n+1, and defines a contact manifold. On such manifolds there exists a unique extremal field that is generated by a Hamiltonian function [7]. The closed integrals of the Action are then deformation invariants of the extremal process.

Classes of Reversible Processes

Extremal Hamiltonian

Extremal processes are defined by the constraint that the Virtual Work 1-form vanishes:

$$W = i(V)dA = 0$$

There is always a Hamiltonian formulation for the extremal process. In hydrodynamics extremal processes are to be associated with the Euler equations. There are various classes of solutions to the Euler equations:

- 1. Potential flows, which have no lift, no drag, no circulation and no vorticity, and are of Pfaff dimension 1.

$$A \neq 0,\ dA = 0,\ \oint A = 0\,.$$

2. Joukowski flows which have lift but no drag, circulation, but no vorticity, and are of Pfaff dimension 1.

$$\supset A \neq 0,\ dA = 0,\ \oint A \neq 0$$

The translational flow combined with the circulation produces lift on an airfoil.

3. Lamb Flows which admit vorticity but are completely integrable, and are of Pfaff dimension 2.

$$\supset A \neq 0,\ dA \neq 0,\ A\hat{}dA = 0$$

Such processes are without helicity.

Bernoulli-Casimir-Hamiltonian:

In Bernoulli processes the Work is exact, not zero, and the cyclic work is zero. Examples are the Eulerian fluid with vorticity, and barotropic flows.

$$W = i(V)dA = d\Theta, \qquad \oint W = 0$$

The Bernoulli-Casimir function Θ is a generator of a Hamiltonian process, and is a flow invariant in the sense that Θ is constant along a flow line, $L_{\mathbf{V}}\Theta = 0.$ However, the flow invariant is not necessarily the same from flow line to flow line.

Helmholtz-Symplectic:

The Work 1-form is closed, but the cyclic work need not be zero!

$$dW = 0, \quad \oint W \neq 0$$

All of the above examples satisfy the Helmholtz theorem on the conservation of vorticity, for

$$L_{\mathbf{V}}dA = dW + ddU \Rightarrow 0 + 0 = dQ.$$

The work 1-form is at most of Pfaff dimension 1. Hence all such classes of processes satisfy the criteria of thermodynamic irreversibility in that $Q\hat{}dQ = 0.$

Non-Barotropic Flows.

In non-barotropic process the Work 1-form is of Pfaff dimension 2, hence is integrable and of the form

$$W = dP/\rho, \qquad dW = -d\rho\hat{}dP/\rho^2, \qquad W\hat{}dW = 0.$$

Presuming irreversibility, leads to $dP/\rho = TdS - dU,\ dW = dT\hat{}dS = -d\rho\hat{}dP/\rho^2$ and implies the existence of a functional relation between $\{P,S,T\}, \{\rho,S,T\}, \{\rho,P,T\}, \{\rho,S,P\},$ as well as $\{U,S,T\}.$ However as $dW\hat{}dW = 0,$ the process is thermodynamically reversible.

Irreversible Processes and Anholonomic Fluctuations

Anholonomic fluctuations are used to define the deviations from kinematic perfection for a vector field. They carry topological content in the sense that

$$\Delta^\mu = (dq^\mu - V^\mu dt) \neq 0.$$

Cartan's Magic formula of topological evolution can handle anholonomic fluctuations: Consider a physical system represented by a Lagrange 1-form of Action:

$$A = L(q^\mu, V^\mu, t)dt + \quad p_\mu \circ (dq^\mu - V^\mu dt)$$

For now, treat the p_μ as Lagrange multipliers What is the Pfaff topological dimension D (the minimum number of independent functions) of this action 1-form, A? At first glance A has 3N+1 independent variables, but by direct computation the Pfaff dimension = 2n+2. The Top Pfaffian is

$$(d\mathcal{A})^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k) \bullet dv^k\}\wedge dp_1\wedge ...dp_n\wedge dq^1\wedge ..dq^n\wedge dt,$$

If, however, the constraints of canonical momenta are subsumed, such that $\partial L/\partial v^k - p_k = 0,$ then the 2-form $d\mathcal{A}$ is not symplectic on its maximal dimension 2n+2, but instead the top Pfaffian defines a contact manifold on a state space of topological dimension 2n+1 with the formula

$$\mathcal{A}\wedge(d\mathcal{A})^n = n!\{p_k v^k - L(t,q,v\}dp_1\wedge ...dp_n\wedge dq^1\wedge ..dq^n\wedge dt$$

On the even dimensional symplectic manifold, the Topological Torsion, $\mathbf{T},$ vector is defined by the equation

$$i(\mathbf{T})dx\wedge dy\wedge dz\wedge dt.... = A\wedge dA....$$

It is possible to show that $i(\mathbf{T})A = 0$ and $L_{\mathbf{T}}(A) = \Gamma A$ with Γ equal to the divergence of the Torsion vector. As shown above, this leads to thermodynamic irreversibility as $Q\wedge dQ \neq 0.$ In this sense, anholonomic fluctuations from kinematic perfection could be considered as a source of thermodynamic irreversibility.

Examples of $\mathbf{T}_4$

ELECTROMAGNETISM

The Action 1-form of potentials:

$$A = \mathbf{A} \circ d\mathbf{r} - \phi dt$$

with

$$\mathbf{B} = curl\,\mathbf{A} \;\; and \;\; \mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$$

forming the components of $dA.$ By direct computation,

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\boldsymbol{\phi},\;\; \mathbf{A} \circ \mathbf{B}]$$

$$\mathbf{\Gamma} = div_4\mathbf{T}_4 = 2\;\mathbf{E} \circ \mathbf{B} \neq 0.$$

Note 1: The Helicity density $\mathbf{A} \circ \mathbf{B}$ is the 4th component of the Topological Torsion Vector, $\mathbf{T}_4.$

Note 2: $\mathbf{\Gamma}$ is the second Poincare Invariant for an electromagnetic system.

HYDRODYNAMICS

Consider an Action 1-form of potentials:

$$A = \mathbf{v} \circ d\mathbf{r} - Hdt$$

with H defined as $H = (\mathbf{v} \circ \mathbf{v}/\mathbf{2} - \lambda div\mathbf{v} + \int dP/\rho) = \mathbf{v} \circ \mathbf{v} - L$ and Vorticity defined as

$$\boldsymbol{\omega} = curl\mathbf{v}.$$

The equivalence class of solutions that satisfy the topological constraint on the virtual work W,

$$W = i(\mathbf{V})dA = \nu\{curlcurl\;\mathbf{v} \circ (d\mathbf{r} - \mathbf{v}dt)\},$$

are solutions to the Navier Stokes equations of motion.

By direct computation, the topological torsion vector is:

$$\mathbf{T}_4 = [h\mathbf{v} - L\boldsymbol{\omega} - \nu\;curl\;\boldsymbol{\omega},\;\; h]$$

where $h = helicity = \mathbf{v} \circ \boldsymbol{\omega}$ and

$$\Gamma = div_4 \mathbf{T}_4 = 2\nu \ \{\boldsymbol{\omega} \circ curl \boldsymbol{\omega}\} \neq 0.$$

Hence, for a Navier Stokes fluid, domains where the vorticity field does not satisfy the Frobenius integrability condition

$$\boldsymbol{\omega} \circ \ curl \ \boldsymbol{\omega} \neq 0$$

are domains of thermodynamic irreversibility, and are therefore are regions of turbulent flow.

REFERENCES

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http://www22.pair.com/csdc/car/carhomep.htm

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