

TOPOLOGICAL PARITY AND THE TURBULENT STATE OF A NAVIER-STOKES FLUID

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Abstract A global 1-form of Action, A , on a space-time variety not only acts as a Lagrange-Euler integrand for a variational principle, but also induces a fluctuation topology on $\{x,y,z,t\}$. The extremal solutions to the variation problem based on such an action, mod fluctuations, lead to a necessary set of partial differential equations of evolution that can be put into correspondence with the Navier-Stokes equations. The Euler characteristic of the topology induced by such extremal vector field solutions is generated by the 4-form $K = dA \wedge dA$ of Topological Parity. As domains of non-zero Euler index are necessary to break parity and time reversal symmetries at a macroscopic level, it follows that $K \neq 0$ is a necessary condition for those vector fields that describe a irreversible or turbulent process of evolution.

1. Introduction

Recently, topological arguments have been used [Kiehn 1990, 1991, 1992a] to suggest that those vector fields that represent the turbulent state in a fluid flow must be of Pfaff dimension 4. This idea is based upon the fact that when the Euler characteristic of a manifold is not zero, then *every* vector field on the manifold, which is assumed to be continuous in the forward direction, must have at least one singular point or self intersection. If that specific forward trajectory is reversed in an evolutionary sense, then, as the singularity is approached, the return bifurcation branch cannot be chosen unambiguously. Hence, the inverse of the continuous mapping generated by the vector field is not continuous, the mapping is not a homeomorphism, and the associated vector field is *irreversible* in a deterministic sense. As all researchers in hydrodynamics will agree that irreversibility is a key feature of a turbulent flow, it follows that if the Euler characteristic of a space-time domain is not zero, then all flows on such domains of space-time are not globally reversible, and must be turbulent.

Recall that a given variety $\{x,y,z,t\}$ can support many topologies, and the question arises as to the source of a physically useful topology. A key premise of this article is that the very existence of a global covariant dynamical vector field on a variety $\{x,y,z,t\}$ generates a useful topology on the set, and the Euler characteristic of that induced topology may be computed in several different ways. For hydrodynamic purposes, that dynamical vector field will be taken to be the flow field defined by some solution to the Navier-Stokes equations. The solution vector field and its singularities can be used, along with the methods of Cartan, to define a topology on the domain. If in addition a *compatible* metric or connection can be deduced from the vector field, then the classical methods of differential geometry may be used to compute the associated Gaussian curvature. Integration of the curvature, and use of a generalization of the Gauss-Bonnet theorem, will yield, ultimately, the Euler characteristic. In the simpler cases, if it can be shown that if the Gauss curvature is everywhere zero, then its integral must be zero, and

the Euler characteristic vanishes. Vector fields that represent hydrodynamic flows on such domains can be chaotic, but are intrinsically reversible, and therefore can not be turbulent. However, certain topological and intrinsic properties for a constrained variety can be more simply evaluated using the methods of Cartan's exterior calculus. These methods do not depend explicitly on a metric or a connection, or the now fashionable methods of fiber bundles. As Cartan's simpler methods are exploited in this article, and it is anticipated that many researchers in hydrodynamics do not yet have much experience with the Cartan techniques, a detailed expose showing the comparisons and differences between the Cartan methods and the classical methods is presented in Appendix A. This Appendix is also of significance to researchers in the theory of defects, for it presents for the first time a projective theory of defects that includes both dislocations and disclinations on an equal footing. It may be shown that the usual contravariant Affine treatments cannot include the shears of rotation that often dominate hydrodynamics. The present author has long been interested in the production and destruction of defects, and other evidences of topological evolution, in hydrodynamics [Kiehn, 1992a]. This interest stems from a proof [Kiehn, 1976b] that dissipative irreversible processes must be associated with a change in topology from the initial to final state.

All vector fields on a variety $\{x,y,z,t\}$ can be put into four general equivalence classes, each class having different topological properties. This topological property is defined as the Pfaff class, or Pfaff dimension, of the vector field. The idea is that each vector field of a given equivalence class of Pfaff dimension, n , requires a minimum of n functions, and their derivatives, for its description. The concept is at the basis of the Darboux reduction theorems, and the theory of Clebsch functions. For a four dimensional variety, the Pfaff dimension of a domain is either 1, 2, 3 or 4. Only vector fields that belong to the equivalence class of Pfaff dimension 4 will induce a non-zero Gauss curvature on the four dimensional variety, $\{x,y,z,t\}$. This equivalence class is characterized by the non-null property defined as topological parity. For vector fields of this equivalence class, there does not exist a mapping to a variety of less than four functions which will adequately describe the field. In particular, if the integral of the topological parity is non-zero, then all such vector fields of Pfaff dimension 4 must have singularities and self-intersections. This conclusion implies that an unambiguous retrace of the lines generated by the vector field is impossible, hence the vector field is not reversible past the singularity or self-intersection.

Note that the classic representation of a vector field in terms of three Clebsch functions does *not* produce a vector field of Pfaff dimension 4, but can be a vector field of Pfaff dimension 3. The argument of this article is that vector fields of Pfaff dimension 3 on a space time variety can be chaotic, but not turbulent. The objective of this article is to classify the vector field solutions to the Navier-Stokes equations, and in particular to determine criteria when such flows are of Pfaff dimension 4. Pfaff dimension 4 is a necessary requirement for intrinsic irreversibility, and therefore a necessary requirement for the existence of the turbulent process.

It is not generally appreciated that the existence of a global vector field induces a topology on a variety. In particular, the methods of how to evaluate the Euler characteristic of the induced topology get bogged down by historical techniques that rely

heavily on coordinate geometry, and tensor analysis. Fortunately, a simpler method can be developed in terms of Cartan's calculus of exterior differential forms, a method which is not only coordinate free, but does not depend explicitly on a metric or connection constraint imposed upon the variety. In fact, the existence of a global 1-form, A , on the variety $\{x^1, x^2, \dots, x^n, s\}$ may be used to construct a basis frame, and a set of compatible connection coefficients on the variety, for which the classical methods may be used to deduce the Gaussian curvature and its integral. Such methods are described in Appendix A. However, if the 1-form A is "renormalized" (or as Chern says, projectivized) then the components of the 1-form may be used to define an *adjoint surface* of N dimensions in the $N+1$ dimensional space. The important geometrical quantities, including the Gaussian curvature, of this *adjoint* N dimensional subspace, are determined by the invariant functions of the Jacobian matrix constructed from the first partial derivatives of the components of the normalized covariant vector field, A . This Jacobian matrix is equivalent to the *shape* matrix of classical surface theory. The mean curvature of this subspace is related to the trace of the Jacobian, or more vividly, is equal to the divergence of the projectivized adjoint vector field. A variational principle on the $N+1$ space produces a *minimal* surface in the space of N dimensions, when the divergence of the normalized adjoint field is set to zero. The Gauss curvature is related to the determinant of the Jacobian matrix, and its integral yields the Euler characteristic.

However, the existence of a global Cartan 1-form of Action, A , on the variety $\{x^1, x^2, \dots, x^n, s\}$ is equivalent to constructing a line bundle [Chern, 1957] over the base, $\{x, y, z, t\}$. Hence, following Chern, a *Top Pfaffian* may be constructed from algebraic products of the matrix of curvature 2-forms. Chern has proved [Chern, 1954] that the integral of this form yields the Euler characteristic. The matrix of curvature 2-forms is computed classically by differentiating a set of connection 1-forms which constrain the variety and the derivatives of the vector field. However, comparing the projective line bundle and Cartan's methods, it may be shown that the matrix of curvature 2-forms can be computed in an alternate algebraic way that does not depend upon a second differentiation of the vector field. These methods are given in detail in Appendix A. The result is that for the normalized 1-form, A , and its exterior derivatives, $F=dA$, the forms, F and $K = F \wedge F$, define the elements of the first and second Chern classes for this line bundle. For a four dimensional variety, the *Top Pfaffian* is equal to $F \wedge F$, and its integral is associated with the Euler characteristic induced by the 1-form A . As $F \wedge F$ must not be zero if the Euler characteristic is not to vanish, it follows that the vector field must be of Pfaff dimension 4. In this article, the complex machinery of fiber bundles is not needed, for the Cartan procedure is much more easy to use in engineering applications.

If the top Pfaffian is equal to zero everywhere, then the associated covariant vector field, interpreted as a hydrodynamic flow vector field, is without self intersections and therefore is unambiguously reversible; i.e., not turbulent. For hydrodynamics, the problem is to determine simple expressions which classify those vector fields that are solutions to the Navier-Stokes equations. The final result is that the *Top Pfaffian*, or Topological Parity, for flow fields constrained to be solutions of the Navier-Stokes equations, is equal to

$$K = -2 \nu \boldsymbol{\omega} \bullet \text{curl } \boldsymbol{\omega} \, dx^{\wedge} dy^{\wedge} dz^{\wedge} dt.$$

When $K = 0$, the Euler characteristic vanishes (at least over compact oriented manifolds) and it follows that such flows are **not** turbulent. As $\boldsymbol{\omega} = \text{curl } \mathbf{v}$ is the vorticity of the fluid flow, and ν the kinematic viscosity, it follows that Navier-Stokes solutions for which the vorticity vector field is completely integrable in the sense of Frobenius can never be turbulent. It is to be remarked that if the enstrophy of the flow has a gradient orthogonal to the vorticity, then the integrable vorticity field is proportional to the normal field of a minimal surface [Kiehn, 1993]. This minimal surface, like a Julia set, can bound an interior in which the velocity field is chaotic.

The four-form of topological parity, K is always exact, and can be generated from the 3-form, $H = A^{\wedge} dA$, defined as the topological torsion. On the set $\{x,y,z,t\}$ the topological torsion generates a contravariant vector field of four components (defined herein as the torsion current $\{\mathbf{T}, h\}$) that may or may not have a zero divergence. The exploitation of this vector field in hydrodynamics is almost non-existent. Mathematicians have known about this vector field and its applications to problems where the solutions to systems of differential equations are not unique for more than 100 years [Forsythe, 1953].

In this article, this topological perspective is given further credence by developing a theory of topological evolution, in which it may be demonstrated that the Navier-Stokes equations form a subset of necessary conditions equivalent to constraints that refine the Cartan topology induced by the global 1-form, A . The point of departure is to recognize that the usual kinematic statement of rigid body dynamics, $\Delta \mathbf{x} = d\mathbf{x} - \mathbf{v} dt = 0$, is a topological constraint on a domain that need not be true for the evolution of deformable media. The non-zero values of $\Delta \mathbf{x}$ will be defined as deformation-fluctuations of position. Higher order fluctuations also may be defined. In this article, the space time variety of dimension 4 will be prolonged to higher dimensions to accommodate the possible fluctuations. Then, following Cartan's method, which is equivalent to a variational principle, certain topological constraints, not equivalent to the null fluctuation constraints of rigid body dynamics, can be placed on this higher dimensional space. Finally, these results can be pulled back to space-time by functional substitution, yielding the Navier-Stokes equations as necessary conditions to satisfy the topological constraints. For certain classes of solutions, the resultant geometry is non-Riemannian, but contains many of the features developed by Finsler [Chern 1948] and Cartan for spaces that support torsion, and non-integrable vector fields.

The two key results of this article are given by engineering formulas that represent the Topological Torsion 3-form, $H = A^{\wedge} F$, as axial vector current of four components on the domain of space-time, and the 4-form of Topological Parity, K , as a density distribution on the space-time variety, $\{x,y,z,t\}$. The axial vector current has been completely ignored by both experimental and theoretical hydrodynamicists, perhaps because it is evanescent in a deterministic, non-chaotic, integrable regime. If the Pfaff dimension of the domain is less than 4, then the axial vector torsion current satisfies a conservation law, the Topological Parity 4-form vanishes, and the Euler characteristic of the induced Cartan topology is zero. The flow is reversible, not turbulent, but perhaps

describes a chaotic process. If the Pfaff dimension is less than 3, then **both** the 3-form of Topological Torsion and the 4-form of topological parity vanish, and the flow field admits a deterministic and non-chaotic realization. Reversing the argument, in the chaotic regime, the axial current vector cannot be zero. In order for the flow field to be irreversible on the variety $\{x,y,z,t\}$, it is necessary (but not sufficient) that the Pfaff dimension be 4.

In section 2, the features of the Cartan method and the topology induced by a systems of differential forms are described. The concept of Pfaff dimension is used to define equivalence classes of vector fields.. In section 3, the abstract Finsler space and its associated global Cartan-Hilbert 1-form of Action (constructed on a ten dimensional prolongation of the original space in order to accommodate possible kinematic and dynamical fluctuations and dissipation) is described. In section 4, equations capable of describing topological evolution and irreversible phenomena in a non-statistical manner are constructed in terms of the Lie derivative acting on a system of differential forms generated from the Cartan-Hilbert global 1-form of Action. In section 5 it is demonstrated that the dissipative irreversible Navier-Stokes equations are deduced by applying topological constraints to the general equations of topological evolution generated in section 4. In addition, the topological constraints that yield the Navier-Stokes equations are used to determine a necessary criteria for the turbulent state. In Appendix A a comparison is made between the Cartan method and the Line Bundle technology. In Appendix B a development is given for an electromagnetic system.

2. The Euler Index by the Cartan Method

The Euler index is one of the key topological properties of a domain. For example in two-space, the Euler index of a sphere is 2, the Euler index of a torus is 0, and the Euler index of a two hole button is -2. A vector field can exist globally without singularities (or zeros) only on a space of zero Euler index. The Euler index on space-time depends on the integral of a certain 4-form defined to be the Topological Parity (see below).

By studying the neighborhoods of "singular points" of a vector field on a variety, certain features of the underlying topology may be determined. The vector field "diverges" in these neighborhoods in ways that yield topological information. The vector field on a sphere must have two of these "singular" neighborhoods, that can be shrunk to "points". The vector field on a torus (Euler index = 0) can be without singularities. The idea that $\text{div}\mathbf{V} = 0$ in the interior of a domain means that the vector lines associated with vector field can never stop or start, except perhaps at "boundary points". These concepts have been used by hydrodynamicists to study certain features of flow separation on airfoils [Su 1990]. The general question is: What is the topology induced by those vector fields that represent fluid flows on a domain of space time? In particular, the thesis to be examined herein is that as a turbulent flow is irreversible, it induces a non zero Euler index on a domain of space-time, and this fact implies that the Pfaff dimension of the domain is 4. If the Pfaff dimension is less than 4, Parity and Time-reversal symmetry [Kiehn, 1991c] are not broken, and the system is reversible.

Although the singularities of contravariant vector fields can be used to characterize the induced topology on the variety $\{x,y,z,t\}$, there are some problems that are better handled in terms of covariant vector fields. To quote Arnold: "The notion of the index of a singular point of a vector field does not generalize directly to the case of manifolds with boundary. However, it turns out that a generalization to the case of boundary singularities becomes possible if vector fields are replaced by differential forms. For a 1-form on a manifold with boundary we define singular points and their indices so that the sum of the indices of all (interior and boundary) singular points of any 1-form with isolated singularities on a compact manifold with boundary is equal to Euler index of the manifold." [Arnold 1981] The idea is that a globally defined 1-form, A , induces a topology on a variety, and its singularities may be used to compute the topological property called the Euler index. This statement by Arnold, and the significance of singularities as stagnation and separation points in fluid flows, is the *raison d'être* for a development of hydrodynamics in terms of Cartan's exterior differential calculus [Flanders 1963].

In this article the principle object of interest is the Cartan 1-form of Action, A , defined with its singularities globally over a domain of a space-time variety, $\{x,y,z,t\}$:

$$A = A_{\mu}dx^{\mu}.$$

The functional properties of this 1-form may be used to construct a topology on the set $\{x,y,z,t\}$. Guided by the fact that a differential constraint, $xdx+dy+zdz = 0$, imposed on the 3 dimensional variety leads to the 2 dimensional topology of a sphere, this 1-form, A , on space time will be prolonged to a space of higher dimensions. Topological constraints on the higher dimensional space will then be used to deduce the Navier-Stokes equations as a set of necessary conditions on the prolonged 1-form. Hence, solutions to the Navier-Stokes equations fall into classes that will generate the Cartan topology.

Suppose that such a 1-form, A , is given. Then, the exterior derivative of A produces a 2-form, $F = dA$, with components,

$$F = dA = F_{\mu\nu}dx^{\mu}dx^{\nu}.$$

The 1-form, A , and the 2-form, $F=dA$, form a "closed" exterior differential system, $\{A, F\}$. Using these two elements, a sequence of higher order sets can be constructed algebraically by forming all possible exterior products of A and F : The resulting system of forms is defined as the Pfaff sequence: $\{A, F=dA, H=A^{\wedge}dA, K=dA^{\wedge}dA\dots\}$.

On space-time of 4 dimensions, there are only 4 possibilities for the Pfaff sequence, and these sets are defined as:

TOPOLOGICAL ACTION

$$A = A_{\mu}dx^{\mu}$$

TOPOLOGICAL TORSION

$$F = dA = F_{\mu\nu}dx^{\mu}dx^{\nu}$$

TOPOLOGICAL TORSION

$$H = A \wedge dA = H_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho$$

TOPOLOGICAL PARITY

$$K = dA \wedge dA = K_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma.$$

The largest non-null element of the Pfaff sequence defines the Pfaff dimension of the domain. Certain flows have Pfaff dimension 1; others, Pfaff dimension 2, 3, and 4. Examples are given in Kiehn (1990).

The domain of support of each element of the Pfaff sequence may be considered as a "point" of what will be called the Cartan topology. In this sense the "point", A , and its closure, $A \cup F$ may be used as base elements to define "open sets" of the Cartan topology. If the domain is such that H and K are null, then that is the end of the construction, and the Cartan topology is a connected topology. However, suppose that there exists a domain of support for the exterior product of A and F , such that H is not empty. Then construct the closure of H as the union, $H \cup K$, and use these 4 "points", $\{A, A \cup F, H, H \cup K\}$, as a basis of open sets for the Cartan topology. The resulting topology is NOT connected! [Baldwin, 1991] This choice for a topology is extraordinary in that the Cartan exterior derivative may be interpreted as a "limit point operator" relative to the Cartan topology. Given any set of the topology, if the exterior derivative vanishes, then the set has no limit points. In this sense, the vorticity field, as the exterior derivative of the covariant velocity field, generates the limit sets for the covariant velocity field (relative to the Cartan topology). For currents constructed in terms of contravariant vector fields (or $N-1$ forms), then the vanishing of the exterior derivative (equivalent to the vanishing of the divergence of the contravariant vector field) implies that no limit points for these currents exist within the domain of support. If the divergence vanishes, then the associated axial vector current satisfies a conservation law. The "lines" associated with this vector field do not stop or start within the domain.

It is of fundamental interest to determine the value of the Euler index for this induced Cartan topology, as the Euler index not only is one of most fundamental of topological invariants, it is also crucial in determining if a flow field is unambiguously reversible. Chern has given an approach to evaluating the Euler index [Greub 1973] in terms of an integral over a certain $2N$ form which he calls the "Top Pfaffian". Also see Appendix A. On the 4-dimensional variety $\{x,y,z,t\}$, this "Top Pfaffian" is proportional to 4-form of Topological Parity, $K = dA \wedge dA$. Hence, for those domains where $K = 0$, an immediate result is that the Euler index must vanish. According to the definition, the Pfaff dimension is less than 4 on domains for which $K = 0$.

Most of the known closed form solutions to dynamical systems are to be associated with domains for which $K = 0$. In fact, most known global solutions are of Pfaff dimension 2, which implies that both H and K are null. If the Cartan-Pfaff 1-form, A , is completely integrable in the sense of Frobenius, then $H = A \wedge dA = 0$, $K = 0$ and the Euler index for the space-time domain is zero. Complete integrability means that there exists a unique function which for constant values forms a surface through which the trajectories of the associated vector field pass transversely. This means that the points on

neighboring trajectories are synchronizable globally over the whole domain. Such vector fields can admit vorticity, but the vorticity must be orthogonal to the velocity at each point. The Topological Torsion, H , must vanish for a completely integrable 1-form.

Another result of Frobenius complete integrability (Pfaff dimension 2 or less) is that neighboring points are isolated in the Caratheodory sense that they reside on the equivalent of a thermodynamic equilibrium hypersurface [Hermann 1968]. In the hydrodynamic case, a completely integrable action always can be mapped to a space of at most two dimensions, and this representation is the epitome of a laminar (to be in layers) flow. If the system does not satisfy the Frobenius integrability theorem, then there may exist local domains that are laminar, but they are isolated. Examples of such systems with Pfaff dimension 3, that support Topological Torsion, are presented in reference [Kiehn, 1991a]. In this article, systems of non-zero K are of interest, and it follows that such systems necessarily support Topological Torsion, H .

If a 1-form of Action is completely integrable in the sense of Frobenius, then the associated vector field never generates chaotic trajectories [Schuster, 1984]. A completely integrable vector field always admits a smooth global transverse hypersurface such that there does *not* exist "an extreme sensitivity" to initial conditions for neighboring trajectories; the initial conditions, and the subsequent evolution of these points, reside on a sequence of globally transverse hypersurfaces. On the other hand, if the Frobenius integrability condition is not satisfied, and suppose that the Pfaff dimension is 3, then there exists a global submersion to a space of 3 dimensions, and a unique smooth transversal hypersurface does not exist over the whole domain. Although there can exist local sets of hypersurfaces that connect local neighborhoods of trajectories, there exist "nearby" trajectories which cannot be so connected. Points on these nearby, but not connected, trajectories may exhibit that "extreme sensitivity" to initial conditions, that has become the definition of deterministic chaos. Points that start out in a nearby domain will not necessarily evolve into a transversely connected set. In fact, this is one of the obvious results of the Cartan topology, for when H and K are zero it is easy to show that the Cartan topology is connected, but when H and K are not zero, the Cartan topology is disconnected [Baldwin, 1991].

It follows that a necessary condition for a chaotic velocity field is that the Pfaff dimension of the 1-form of Action shall be greater than 2. But chaos is not the same as turbulence, for the chaotic flow is reversible along each nearby, but perhaps disconnected, space time trajectory, without ambiguity: the chaotic trajectories never intersect in space time. On the other hand, a turbulent flow is presumed to be dissipative and irreversible. If the flow lines intersect in space time, it is impossible to reverse the evolution unambiguously (uniquely) through the obstruction, or bifurcation singularity; such flows are defined to be irreversible. In order for the flow lines to have irreducible self-intersections in space time, it is necessary that the 4-form of topological parity, K , be non-zero. A non-zero Euler index is a signal that the vector field admits such singularities, and the associated evolution is therefore irreversible. It follows that a necessary condition for irreversibility is that the Pfaff dimension of the domain be 4. The conclusion reached is that reversible chaos is of Pfaff dimension 3, and can be distinguished from turbulence, which is irreversible and must be of Pfaff dimension 4.

These ideas support the definition that the turbulent state must be irreducibly time dependent, irreducibly 3 dimensional in space, and dissipatively irreversible. Although time-dependent two dimensional flows can exhibit chaos, they can never be turbulent, for they are of maximum Pfaff dimension 3. At least 4 parameters are required to produce a domain with Pfaff dimension 4. It is the opinion of this author that the concept of "two" dimensional turbulence is not logical. The Topological Parity must not vanish globally in the turbulent state, a fact that would preclude the concept of turbulence on a two space-one time dimensional domain.

For this author the appreciation that Topological Torsion and Topological Parity are important physical concepts came first from applications of the Cartan technique to problems in electromagnetism. An abstract example of the Cartan technique for electromagnetic systems is given in Appendix B, where the 1-form of Action, A , is based on the electromagnetic potentials. A most influential result of this analysis is the fact that the Topological Parity 4-form for a Maxwell system is given by the expression,

$$K = -2 \mathbf{E} \bullet \mathbf{B} dx^{\wedge}dy^{\wedge}dz^{\wedge}dt.$$

In Lagrangian field theories of *microscopic* elementary particles, non-zero values of $\mathbf{E} \bullet \mathbf{B}$ are signals that parity and time reversal symmetries are not preserved [Callan 1976]. Examples of Parity and Time-Reversal symmetry breaking at a *macroscopic* level in electromagnetic systems are presented in [Kiehn, 1991]. These results imply that the associated evolutionary processes are not reversible, and that irreversibility is a topological idea independent from size and scales. The ideas led this author to conjecture that similar properties are to be associated with irreversible, dissipative, turbulent hydrodynamic flows. A first step in this direction was the suggestion that the transition to turbulence involved a topological transition from a laminar state without Topological Torsion, $H=A^{\wedge}dA = 0$, to a state where $H \neq 0$. This transition was expressed by saying that the flow system went from a state that was completely integrable, in the sense of Frobenius, to a system that was not integrable [Kiehn 1976a]. At that time the possibility of an intermediate non-integrable chaotic state of Pfaff dimension 3 occurring before a irreversible, irreducibly time-dependent and three dimensional non-integrable turbulent state of Pfaff dimension 4 was not appreciated. If the Pfaff dimension is less than 4, the Euler index relative to the Cartan topology must vanish on space time.

The creation of turbulence from a laminar state must involve the topological evolution from a connected Cartan topology (Pfaff dimension < 3) to a disconnected Cartan topology (Pfaff dimension > 2). Relative to the Cartan topology, it may be shown that all C^2 functional descriptions of vector fields are continuous. As a consequence it is possible to state that the creation of a turbulent state cannot be accomplished by a continuous transformation, but the decay of turbulence is amenable to analysis by means of continuous but irreversible transformations. [Kiehn, 1992] The implication is that numerical simulations which impose strict continuity conditions on the evolution are not capable of describing the *creation* of turbulence. On the other hand, continuous numerical simulations can describe the *decay* of turbulence.

In this article, the technique will be to define a 1-form of action, A , similar to that variational integrand proposed by Finsler [Chern 1948] and Cartan in their studies of non-Riemannian geometries that admit torsion. The Finsler methods motivated this author to treat the general problem of topological evolution as an extremal problem on a variety of higher dimensions. In effect, the classic kinematic constraints of rigid body motion are considered to be "overly severe" topological constraints on fluid motion, and are relaxed such that the new topology admits "fluctuations". The fluctuations can be interpreted in many ways; in particular they can represent deviations from the classic kinematic constraints, $d\mathbf{x} - \mathbf{v}dt = 0$. These deviations may be interpreted as variations of the initial conditions, or they may be interpreted as an uncertainty of the origin, but for this author they do not imply that a *statistical* analysis is necessary. A refinement, or specialization, of this "fluctuation" topology leads to the set of necessary partial differential equations which are recognized as the Navier-Stokes equations for a compressible, viscous flow. Of particular importance is the recognition that the dynamics of a deformable system are to be associated with a field of axial "current", whose 4 components form the 3-form of Topological Torsion on (x,y,z,t) . This axial-vector current is a completely anti-symmetric third rank tensor field, and is evanescent in rigid body systems that satisfy the topological constraints of a perfect kinematics. The divergence of this axial current may or may not vanish for deformable systems, and it may or may not be an evolutionary invariant. When the divergence is anomalous, the Torsion Current can stop or start in the interior, thereby generating a topological defect in the domain.

3. The Fluctuation 1-form

The format of the Cartan 1-form, A , will be that of the Cartan-Hilbert invariant integrand,

$$A = L(\mathbf{x},t; \mathbf{v})dt + \mathbf{p} \cdot (d\mathbf{x} - \mathbf{v}dt). \quad (1)$$

Note that the original space-time, $\{\mathbf{x}, t\}$ has been extended to a 10 dimension space of functions, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$. On this set, it is convenient to define a vector field, \mathbf{V} , with 10 components given by the functions, $\{\mathbf{v}, 1; \mathbf{a}, \mathbf{f}\}$. Differential and functional constraints will be imposed on this 10 dimension space thereby defining a topology. Further note that the Cartan-Hilbert action involves a classic Lagrange function, $L(\mathbf{x},t; \mathbf{v})$, and a linear combination of non-zero position "fluctuation or deformation" 1-forms, defined as:

$$\Delta\mathbf{x} = d\mathbf{x} - \mathbf{v}dt \neq 0. \quad (2)$$

The covariant array, \mathbf{p} , of coefficients of the fluctuation 1-forms in (1) may be described as set of Lagrange multipliers. It will be demonstrated below that this covariant field, \mathbf{p} , dual to the contravariant velocity field, \mathbf{v} , plays the role of the canonical momentum, when subjected to additional, but classic, constraints that are equivalent to the constraint of zero temperature.

The 1-forms, $\Delta \mathbf{x}$, are defined as fluctuation-deformation 1-forms for they represent deviations from the pure kinematic point of view associated with a rigid body dynamics or the evolution of a point particle in terms of a single parameter group of transformations. Although not always true, these deviations are often small corrections to the kinematic constraints, $\Delta \mathbf{x} = 0$, and have the appearance of "fluctuations" about the "kinematic" lines that act as guiding centers for the evolution.

The fluctuation 1-forms given by (2) are not necessarily zero for deformable media. If it is assumed that the "points", \mathbf{x} , evolve in terms of a map, f , from a set of initial conditions, \mathbf{y} , then the map

$$x^k = f^k(\mathbf{y}, t),$$

leads to the differential expression,

$$\Delta x^k = dx^k - v^k dt = (\partial f_k / \partial y^i) dy^i \neq 0 . \quad (2b)$$

If the parameters, \mathbf{y} , which could be interpreted either as initial conditions or as the coordinates of the origin, are not constants, then the RHS of (2b) is not zero, and the system does not evolve according to the kinematic rules associated with a single parameter group. The basic idea is that the statement, $d\mathbf{x} - \mathbf{v}dt = 0$, must be interpreted as a topological constraint, just as the statement $x dx + y dy + z dz = 0$ is a topological constraint on Euclidean 3-space that produces the topology of a spherical surface.

In classical hydrodynamics, the non-zero fluctuations given by (2) are usually constrained by topological conditions such that their associated 3-form admits an integrating factor, ρ . This topological constraint means that there must exist a function, $\rho = \rho(x, y, z, t)$ such that the non-zero 3-form,

$$\Omega = \rho(dx - v^x dt)^{\wedge} (dy - v^y dt)^{\wedge} (dz - v^z dt) , \quad (3)$$

has a vanishing exterior derivative,

$$d\Omega = \{ \text{div } \rho \mathbf{v} + \partial \rho / \partial t \} dx^{\wedge} dy^{\wedge} dz^{\wedge} dt = 0. \quad (4)$$

This topological constraint is usually called the "equation of continuity" for deformable media. It may be shown that this topological constraint makes Ω an absolute invariant of the evolution. If the flow lines are retraceable, implying that the Jacobian determinant of the assumed mapping is of rank 3, then the topological constraint may be interpreted as the "conservation of mass". However, it is not apparent that nature always insists on the assumed topological constraint among the fluctuation-deformation 1-forms. Such a constraint is a matter for test, especially in the case of a turbulent, irreversible, evolutionary process.

The Cartan 1-form will be used not only to generate the Cartan topology, but also to generate, by means of a procedure equivalent to a variational principle, a set of partial

differential equations of evolution with solution vector fields, \mathbf{V} . The continuous smooth curves tangent to the vector fields, \mathbf{V} , in the higher dimensional geometry of the partial differential system may pullback, or intersect, or project, discontinuously in the lower dimensional geometry. The ordinary kinematic differential equations based on \mathbf{v} , without fluctuations, yield solution curves that act as "guiding centers" for the fluctuation fields, \mathbf{V} , in the limit that the fluctuations are small. The projections of the continuous curves in the geometry of the higher dimensional space may have gaps and tangential discontinuities on space-time. The discontinuities would be interpreted as defects or fluctuations in an otherwise homogeneous and continuous system. These ideas may be compared to the concept of Poincare sections in the theory of non-linear dynamics. The Cartan method permits the concepts of discontinuous fluctuations to be put on a continuous basis in a space of higher dimension. This topological idea of removing apparent discontinuities by embedding in a space of higher dimensions is similar to the geometric idea where a curved space may be embedded in a higher dimensional euclidean flat space.

Physicists often recognize the Cartan Action in the format,

$$A = \mathbf{p}d\mathbf{x} - (\mathbf{p} \bullet \mathbf{v} - L(\mathbf{x},t; \mathbf{v}))dt = \mathbf{p}d\mathbf{x} - H(\mathbf{x},t; \mathbf{p},\mathbf{v})dt , \quad (5)$$

but do not seem to appreciate that this composition may be interpreted in terms of a fluctuation geometry on a space of 10 dimensions, as given by (1). In current physical theories, it is often assumed that the function, $H(\mathbf{x},t; \mathbf{p},\mathbf{v})$, can be written entirely in terms of the variables $(\mathbf{x},t; \mathbf{p})$ alone. In such cases the function $H^*(\mathbf{x},t; \mathbf{p})$ becomes the Hamiltonian function of classical mechanics, and the Lagrange multipliers, \mathbf{p} , would be identified as the canonical momentum. This assumption corresponds to a functional relationship or constraint between the variables \mathbf{v} and \mathbf{p} such that $\partial H/\partial \mathbf{v} = 0$. If the relationship is linear, then there would exist a constitutive or metrical relationship between the dual fields, \mathbf{v} and \mathbf{p} . Such assumptions are NOT made *a priori* in this article.

Consider first a Cartan 1-form of action where the fluctuations vanish over a domain. Then the 2-form of limit points is $F = dA = dL \wedge dt$. It follows that $H = A \wedge dA = 0$, and $K = 0$. The Pfaff dimension of such systems is 2 at most. Such systems can have vorticity but are without helicity, or Topological Torsion. However, in this article, systems of non-zero H and non-zero K are of interest. Examples of systems that do support Topological Torsion are presented in reference [Kiehn 1991a]. From this point of view, both Topological Torsion and Topological Parity are to be associated with the concept of non-null kinematic fluctuations which are not transversal to the system momentum, $\mathbf{p} \bullet \Delta \mathbf{x} \neq 0$.

When fluctuations are permitted, then the exterior derivative of the Cartan action on the 10 dimensional space becomes explicitly,

$$\begin{aligned} dA &= (\partial L/\partial \mathbf{v} - \mathbf{p}) \bullet d\mathbf{v} \wedge dt + d\mathbf{p} \bullet (d\mathbf{x} - \mathbf{v}dt) + \partial L/\partial \mathbf{x} \bullet (d\mathbf{x} \wedge dt) \\ &= (\partial L/\partial \mathbf{v} - \mathbf{p}) \bullet \Delta \mathbf{v} \wedge dt + \Delta \mathbf{p} \wedge \Delta \mathbf{x} \end{aligned} \quad (6)$$

The term, $\Delta\mathbf{v}$, represents the non-zero 1-forms of velocity fluctuations, defined as,

$$\Delta\mathbf{v} = d\mathbf{v} - \mathbf{a}dt, \quad (7)$$

and, $\Delta\mathbf{p}$, is defined as the non-zero 1-form of Lagrange multiplier fluctuations,

$$\Delta\mathbf{p} = d\mathbf{p} - (\partial L/\partial\mathbf{x}) dt \neq 0. \quad (8)$$

The functions, \mathbf{a} , are defined to be to the contravariant acceleration vector field (with velocity fluctuations) in the same extremal sense that \mathbf{v} is defined as the contravariant velocity vector field (with position fluctuations).

The Hiesenberg like notation, $(\Delta\mathbf{p} \wedge \Delta\mathbf{x})$, stands for the sum of 2-forms,

$$(\Delta\mathbf{p} \wedge \Delta\mathbf{x}) = \sum_k (dp_k - \{\partial L/\partial x^k\}dt) \wedge (dx^k - v^k dt) \quad (9)$$

which is similar to the dot product of two vectors, but here the combinatorial action is through the exterior product, \wedge . Although closely related to an expectation value generated by an inner product, or to the integrand of a cross-correlation integral, no statistical or ensemble averaging of (9) is assumed in this article. The beauty of the Cartan analysis is that it is retrodictively deterministic and well defined in a pullback sense, even when unique, deterministic prediction is impossible [Kiehn,1976b].

The bracket factor, $(\partial L/\partial\mathbf{v} - \mathbf{p}) = -\partial H/\partial\mathbf{v}$ in equation (6) will be defined as the scaled covariant vector field, \mathbf{k}/S . The topological constraint $\mathbf{k} = 0$ permits the Lagrange multipliers to be uniquely determined as the canonical momenta of classical mechanics, $\mathbf{p} = \partial L/\partial\mathbf{v}$.

A direct computation of the Topological Torsion, H , on the 10 dimensional space yields,

$$H = A \wedge dA = L dt \wedge (\Delta\mathbf{p} \wedge \Delta\mathbf{x}) + (\mathbf{k}/S \bullet \Delta\mathbf{v}) \wedge (\mathbf{p} \bullet \Delta\mathbf{x}) \wedge dt. \quad (10)$$

which may be evaluated in principle on 4 dimensional space time by functional substitution. A similar direct computation in the higher dimensional geometry of variables $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$ of the exterior derivative, $K = dH$, produces a 4-form that also can be pulled back to $\{x,y,z,t\}$ by functional substitution. The Topological Parity 4-form becomes,

$$K = dA \wedge dA = 2\{(\partial L/\partial\mathbf{v} - \mathbf{p}) \bullet \Delta\mathbf{v}\} \wedge (\Delta\mathbf{p} \wedge \Delta\mathbf{x}) \wedge dt + 2(\Delta\mathbf{p} \wedge \Delta\mathbf{x}) \wedge (\Delta\mathbf{p} \wedge \Delta\mathbf{x}). \quad (11)$$

On a space-time variety, this 4-form becomes Chern's top Pfaffian whose integral gives information concerning the Euler characteristic of space-time. It is apparent that K

depends on the exterior product of the fluctuations of position, Lagrange multipliers, and velocity, as well as the bracket factor, $(\partial L/\partial \mathbf{v} - \mathbf{p}) = -\partial H/\partial \mathbf{v}$, and dt .

The classic first variation of the Action integral, \mathbb{Z}_A , is an extremal principle which for the action specified by (1) generates a Finsler geometry. According to Chern, the Finsler variation is equivalent to setting $dA = 0, \text{ mod } \Delta \mathbf{x}$. In addition, for Finsler geometries, the Lagrange function is presumed to be homogeneous of degree 1 in the \mathbf{v} , and this constraint is used by Chern to construct a "projectivized" tangent bundle. The homogeneity condition implies that the variable, t , can be reparametrized, and the vector \mathbf{v} forms the elements of a projective geometry.

In classical field theory, the Finsler constraint is often imposed arbitrarily:

$$\mathbf{k}/S = (\partial L/\partial \mathbf{v} - \mathbf{p}) = -\partial H/\partial \mathbf{v} = 0. \quad (12)$$

As mentioned above, such a constraint uniquely defines the Lagrange multipliers, \mathbf{p} , as the components of the canonical momentum. The Topological Parity 4-form is then dependent on the exterior product of "fluctuations" in position and momentum only, and has the same physical dimensions as the square of Planck's constant. From a qualitative point of view, fluctuations in velocity correspond to the property of temperature.

4. Equations of Topological Evolution

In the Cartan method, an evolutionary process relative to a vector field, \mathbf{V} , is described by the action of the Lie derivative on the p-forms of interest [Slebodzinsky, 1970]. The action of the Lie derivative on 1-forms is equivalent to the "convective" derivative in Cartesian hydrodynamics, but the Lie derivative is defined without the geometric constraints of a metric or a connection. In the present article, the p-forms of evolutionary interest are those p-forms that make up the Pfaff sequence. If any p-form is invariant with respect to the evolutionary process, \mathbf{V} , then its Lie derivative vanishes: $L(\mathbf{V})A = 0$. If all p-forms that make up the topological base of the Cartan topology are invariant, then the topology is invariant, and the process is a homeomorphism: ($L(\mathbf{V})A = 0$ and $L(\mathbf{V})dA = 0$). Such processes are both continuous and reversible, and are to be ignored in this article.

A more general set of evolutionary processes that can admit topological evolution is described by the topological constraint that the limit sets, $dA=F$, are evolutionary invariants. No such constraint is placed on the evolution of A . Such processes are described by the statements:

$$L(\mathbf{V})A = Q \neq 0, \quad L(\mathbf{V})dA = 0. \quad (13)$$

In terms of the Cartan topology, such processes, or vector fields that satisfy (13), are said to be uniformly continuous. Direct computation on C^2 functions indicates that (13) is equivalent to the constraint that 1-form of work, defined as $W = i(\mathbf{V})dA$, is closed (and therefore has no limit points with respect to the Cartan topology):

$$d(i(\mathbf{V})dA) = 0. \quad (14)$$

It may be demonstrated that in a hydrodynamic format such (13) is equivalent to Helmholtz' theorem of the conservation of vorticity [Kiehn 1975, Tur 1991]. In an electrodynamic format, (14) corresponds to the "master" equation for the "perfect plasma" of magneto-hydrodynamics:

$$\partial \mathbf{B} / \partial t = \text{curl}(\mathbf{v} \times \mathbf{B}).$$

This constraint of uniform continuity is satisfied by the more stringent sufficient condition,

$$i(\mathbf{V})dA = 0, \quad (15)$$

an equation of historical significance. For a given Lagrange Action, A , Cartan has demonstrated that the first variation of the Action integral is equivalent to the search for those vector fields, \mathbf{V} , that satisfy equation (15) given above. Such vector fields are called extremal vector fields [Klein 1962, Kiehn 1975]. Given A , the resultant equations deduced from (15) are a set of partial differential equations that represent extremal evolution, \mathbf{V} . Note that the extremal conditions are insensitive to any renormalization of the vectors, \mathbf{V} . That is, if \mathbf{V} satisfies the equations, then $\rho \mathbf{V}$ also satisfies the equations. Such a result is commonplace in the projective geometry of lines, and does not require the Riemannian or euclidean concept of an inner product or a metric [Meserve 1983]. Cartan [1958] has shown that this projective extremal condition is necessary and sufficient for the dynamical system, \mathbf{V} , relative to the action, A , to be Hamiltonian [Kiehn 1974]. Such Hamiltonian systems are not dissipative, and guarantee the existence of a single parameter group for the velocity field generated by the solutions of the necessary partial differential equations that describe the topological constraint. Such evolutionary systems cannot describe the creation of turbulence.

However, on the fluctuation space of 10 dimensions, $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$, contraction of dA with the vector field, $\mathbf{V} = \{\mathbf{v}, 1; \mathbf{a}, \mathbf{f}\}$ explicitly yields,

$$i(\mathbf{V})dA = -\mathbf{F} \bullet (d\mathbf{x} - \mathbf{v} dt) - \mathbf{k}/S \bullet (d\mathbf{v} - \mathbf{a} dt) = -\mathbf{F} \bullet \Delta \mathbf{x} - \mathbf{k}/S \bullet \Delta \mathbf{v}, \quad (16)$$

where $\mathbf{F} = \mathbf{f} - \partial L / \partial \mathbf{x}$ represents the dissipative components of the "force". The RHS of (16) is not necessarily zero, so such evolutionary processes are not necessarily Hamiltonian, but are candidates for a turbulent evolutionary process. The RHS of (16) depends explicitly upon the fluctuations in position and velocity, and is explicitly independent from fluctuations in momentum. The two covariant vector fields, $\mathbf{F} = i(\mathbf{V})\Delta \mathbf{p}$, and \mathbf{k}/S , can be identified with the irreversible dissipative mechanisms of friction and radiation in the system fluctuation dynamics. These dissipative terms are not included explicitly in the usual Lagrange theory, and represent fluctuation interactions with the environment. Note that the Finsler-Chern condition, $dA = 0 \text{ mod } \Delta \mathbf{x}$, implies that $\mathbf{k}/S =$

0, a constraint that on physical grounds implies that the system does not depend on velocity fluctuations (temperature). Further note that the Cartan-Hamilton constraint given by (15) is satisfied by several distinct classes of sufficient conditions, depending on choices for the four factors in the RHS of (16).

5. The Navier Stokes equations

For the liquid state, guided by the physical idea that fluctuations in velocity are related to the concept of temperature, assume that the equations of topological evolution given by (16) are refined by the constraints,

$$\mathbf{k} \neq 0, \quad \mathbf{k} \bullet \Delta \mathbf{v} = S d(kT), \quad \mathbf{F} = \nu \text{curl curl } \mathbf{v}, \quad \mathbf{F} \bullet \Delta \mathbf{x} \neq 0. \quad (17)$$

In this example, the constraint of canonical momentum ($\mathbf{k}/S = 0$) is relaxed to the alternate constraint that the deviation from canonical momentum, \mathbf{k}/S , be transversal to the fluctuations in velocity, if the system is at constant temperature, then $d(kT) = 0$. The equations of motion (16) become

$$i(\mathbf{V})dA = -\nu(\text{curl curl } \mathbf{v}) \bullet \Delta \mathbf{x} + d(kT) \quad (18)$$

For the case of a fluid, define the Cartan fluctuation 1-form, A , as,

$$A = \mathbf{v} \bullet d\mathbf{x} - H dt, \quad (19)$$

with the "barotropic Hamiltonian" function specified as,

$$H = \mathbf{v} \bullet \mathbf{v} / 2 + \int dP / \mathbf{r} + I \text{div } \mathbf{v} + kT \quad (20)$$

Substitution into (16) yields a necessary system of partial differential equations for the constrained topological evolution:

$$\partial \mathbf{v} / \partial t + \text{grad}(\mathbf{v} \bullet \mathbf{v} / 2) - \mathbf{v} \times \text{curl } \mathbf{v} = -\text{grad } P / \rho + \lambda \text{grad div } \mathbf{v} - \nu \text{curl curl } \mathbf{v}. \quad (21)$$

Equations (21) are exactly the Navier-Stokes partial differential equations for the evolution of a compressible viscous irreversible flowing fluid. In other words, the Navier-Stokes equations of hydrodynamics have been deduced by imposing a set of topological refinements on the general equations of topological evolution. However, the vector field which is a solution to the partial differential system may not be the generator of a single parameter group of transformations, and thereby may exhibit fluctuations and tangential discontinuities when projected to space time.

For the Navier-Stokes system, and by direct computation, the 2-form $F = dA$ has components

$$F = dA = \omega_z dx^{\wedge} dy + \omega_x dy^{\wedge} dz + \omega_y dz^{\wedge} dx + (\mathbf{a} \bullet d\mathbf{r})^{\wedge} dt$$

where

$$\boldsymbol{\omega} = \text{curl } \mathbf{v} \quad \text{and} \quad \mathbf{a} = -\partial \mathbf{v} / \partial t - \text{grad } H \quad (22)$$

The 3-form of Helicity or Topological Torsion, H , is constructed from the exterior product of A and dA as,

$$\begin{aligned} H &= A \wedge dA = H_{ijk} dx^i \wedge dx^j \wedge dx^k \\ &= -\mathbf{T}_x dy \wedge dz \wedge dt - \mathbf{T}_y dz \wedge dx \wedge dt - \mathbf{T}_z dx \wedge dy \wedge dt + h dx \wedge dy \wedge dz, \end{aligned} \quad (23)$$

where \mathbf{T} is the fluidic Torsion axial vector current,

$$\mathbf{T} = (\mathbf{a} \times \mathbf{v}) + H \boldsymbol{\omega} \quad , \quad (24)$$

and h is the torsion (helicity) density,

$$h = \mathbf{v} \cdot \boldsymbol{\omega}. \quad (25)$$

The torsion axial vector, \mathbf{T} , consists of two parts. The first term represents the shear of translational accelerations, $(\mathbf{a} \times \mathbf{v})$, and the second part represents the shear of rotational accelerations, $H \boldsymbol{\omega}$. Tubular domains, or singular lines of \mathbf{T} , in a hydrodynamic system can be interpreted as "defect" structures that can be put into correspondence with dislocation and disclination defects in continuous media, depending on whether or not the field is dominated by translational or rotational shears. The topological torsion tensor, $H_{ijk} \equiv \{\mathbf{T}, h\}$, is a third rank completely anti-symmetric covariant tensor field, with four components on the variety $\{x, y, z, t\}$. It transforms as a third rank tensor field with respect to all diffeomorphisms on space-time, including the Galilean transformation.

The Navier-Stokes constraint given by (21) may be used to express the acceleration term, \mathbf{a} , kinematically; i.e.,

$$\mathbf{a} = -\text{grad } H - \partial \mathbf{v} / \partial t = -\mathbf{v} \times \text{curl } \mathbf{v} + \nu \{ \text{curl curl } \mathbf{v} \}. \quad (26)$$

By substituting (26) into (24), the torsion axial vector current becomes expressible in terms of the helicity density, h , the Lagrangian function, L , and the viscosity as:

$$\mathbf{T} = \{ h \mathbf{v} - L \text{curl } \mathbf{v} \} - \nu \{ \mathbf{v} \times (\text{curl curl } \mathbf{v}) \} \quad (27)$$

Note that the torsion axial vector current persists even for Euler flows, where $\nu = 0$. The measurement of the components of the Torsion axial vector current have been completely ignored by experimentalists in hydrodynamics.

The Topological Parity 4-form can be evaluated by exterior differentiation as,

$$K = dH = dA \wedge dA = -2 (\mathbf{a} \cdot \boldsymbol{\omega}) dx \wedge dy \wedge dz \wedge dt. \quad (28)$$

The identity given by eq. (24) is in the form of a divergence when expressed on $\{x,y,z,t\}$,

$$\text{div } \mathbf{T} + \partial h / \partial t = -2(\mathbf{a} \cdot \boldsymbol{\omega}) , \quad (29)$$

and yields the helicity-torsion current conservation law if the anomaly, $-2(\mathbf{a} \cdot \boldsymbol{\omega})$, on the RHS vanishes. It is to be observed that when $K = 0$, the Euler index is zero, and the integral of H over a boundary of support vanishes by Stokes theorem. This idea is the generalization of the conservation of the integral of helicity density in an Eulerian flow. Note the result is true for a *viscous* fluid, subject to the constraint of zero Euler index. However, if the topological parity 4-form, K , does not vanish, then the lines of the torsion current, \mathbf{T} , do not satisfy a conservation law and can start or stop within the fluid interior. The parity 4-form, K , is the source for the production or destruction of anomalous topological defects in the evolutionary process.

By a similar substitution of (26) into (28), the topological parity pseudo-scalar becomes expressible in terms of engineering quantities for a Navier-Stokes flow as,

$$K = -2 \nu \{ \text{curl } \mathbf{v} \cdot (\text{curl curl } \mathbf{v}) \} = -2 \nu \{ \boldsymbol{\omega} \cdot \text{curl } \boldsymbol{\omega} \}. \quad (30)$$

The integral of K over $\{x,y,z,t\}$ gives the Euler index of the flow. It is to be observed that the Topological Parity pseudo-scalar, K , is always zero for non-viscous Eulerian flows, and can be zero for viscous Navier-Stokes flows if the vorticity vector, $\boldsymbol{\omega}$, satisfies the Frobenius integrability condition, $\{ \boldsymbol{\omega} \cdot \text{curl } \boldsymbol{\omega} \} = 0$. From the result given by (30), it is apparent that K , and therefore, the Euler index of the domain of support, is not necessarily zero, unless the vorticity field, $\text{curl } \mathbf{v}$, admits a foliation of codimension 1. See Arnold [1981]. If the flow is to be irreversible, the flowlines must have at least one point of intersection in space-time, and therefore the Euler index of the domain of support cannot be zero. It follows that K for the domain of support cannot be zero, and therefore the Pfaff dimension of the turbulent state must be 4, and the vorticity field of a turbulent Navier-Stokes fluid is not integrable.

If the RHS of (16) goes to zero, then the constraint becomes equivalent to the Euler equations of motion of a non-viscous fluid. The criteria that $i(\mathbf{V})dA = 0$ has been shown by Cartan [Cartan 1958] to be necessary and sufficient for the solution vector to have a Hamiltonian representation relative to the Action, A . This result corresponds to the well known Clebsch potential representation for an Euler flow [Lamb 1945]. Note that even in a viscous flow, if $\mathbf{F}_{(\text{viscous})} = \nu \text{curl curl } \mathbf{v} = 0$, the system has a Hamiltonian representation! If the kinematics are without fluctuations, then the conclusion is again reached that the system is Hamiltonian.

Analytically, if the RHS of (16) vanishes or is closed, then the solution vector fields are uniformly continuous, for the even dimensional elements of the Pfaff sequence, (the limit sets, $F=dA$ and $K = dH = F \wedge F$) are invariants of the flow. However, the odd dimensional intersections of the Pfaff sequence need not be invariant, implying the possibility of a continuous but changing topology. If topology changes, such flows can not be homeomorphisms, but are uniformly continuous; therefore, they must be

irreversible, in the sense that a continuous inverse does not exist. Time reversal symmetry is broken.

The constraint of uniform continuity may be relaxed, for a flow is continuous if the limit points of the flow permute into the closure of the evolving topology [Hocking, 1984]. Such a flow is said to be continuous, but not necessarily uniformly continuous. It is possible to show that relative to the Cartan topology, all evolutionary processes generated by the Lie derivative with respect to C^2 vector fields are continuous. The equations of continuous, but not uniformly continuous, evolution must satisfy transversal constraints of the form [Kiehn 1974],

$$\begin{aligned} i(\mathbf{V})dA &= \Gamma A + \{\text{closed 1-forms to fit boundary conditions}\} \\ &= \Gamma\{L(\mathbf{x},t; \mathbf{v})dt + \mathbf{p} \bullet (d\mathbf{x} - \mathbf{v}dt)\} + \{\text{closed 1-forms}\} . \end{aligned} \quad (31)$$

This result implies that the viscous forces, \mathbf{F} , in (16) must be proportional to the canonical momentum, \mathbf{p} , a result typical of empirical viscous friction assumptions. In other words, solutions to the Navier-Stokes equations (21), which are of the form (31), may be continuous but irreversible relative to the Cartan fluctuation topology. It is this class of processes that can be used to describe the decay of turbulence, but not its creation.

Summary

It has been argued that a necessary condition for an evolutionary process to be irreversible in space time is that the Euler characteristic of Cartan topology induced on space time by the process is non-zero. It follows that Topological Parity associated with such processes is non zero, and the evolutionary field is of Pfaff dimension 4. For hydrodynamic applications, a flow vector field of Pfaff dimension 4 must be time-dependent, and spatially three dimensional. A turbulent flow is therefore irreversible, time dependent, and irreducibly three dimensional. For a Navier-Stokes fluid, if the vorticity field is integrable in the sense of Frobenius, the flow is not turbulent, but can be chaotic.

APPENDIX A The Cartan 1-form of Action as a Line Bundle

This appendix is used to describe the details of how to compute topological and geometrical properties of generalized spaces using the Cartan exterior methods. It is presented in support of the arguments that appear in the preceding article, and compares the perhaps more well known classic methods with the Cartan technique. The general Cartan approach is simplified by specializing to Chern's "projectivized" line bundle as the expository example. This choice is of particular applicability to hydrodynamic systems.

The geometrical development of the "projectivized" line bundle on a manifold of dimension N can be included in the Cartan-Hilbert theory of a global 1-form of Action on a variety of dimension $N+1$. In this theory, the existence of a global 1-form,

$$A = \sum_{\alpha=1}^n f_{\alpha} dx^{\alpha} - ds \quad (\text{A1})$$

may be considered as a form of a topological constraint on the variety $\{x^1, x^2, \dots, x^n, s\}$. The set of vectors, $\mathbf{V} = \{v^1, v^2, \dots, v^n, c\}$ which are *associated* with (algebraically orthogonal to) this form in the sense,

$$i(\mathbf{V})A = 0, \quad (\text{A2})$$

can be used to create the first N columns of a basis frame, \mathbb{F} . Each of the associated vectors, \mathbf{V} , is of arbitrary "magnitude" in the sense that each component may be multiplied by an arbitrary function, and the resulting vector is still a solution of (A2). The concept is intrinsically independent of an inner product, or metric, structure that might be imposed upon the variety. The concept is typical of projective geometries, in which the concept of metric is not required.

The last column vector of the basis frame, \mathbb{F} , may be constructed algebraically as the vector field whose line of action is adjoint to the first N columns. It consists of component functions which are the determinantal cofactors of the first N vectors. The space of the associated vectors is a Finsler space with an arc length defined as

$$ds = \sum_{\alpha=1}^n f_{\alpha} dx^{\alpha} \quad (\text{A3})$$

Note that the Finsler definition of arclength is a differential form constraint (or Pfaffian field) on a projective space of dimension $N+1$. The concept of Finsler arclength then can be interpreted as a constraint to a subspace of dimension N . As Finsler spaces -- with additional constraints -- are known to include Riemannian spaces, the interesting implication is that the concept of distance can be considered to be a topological constraint that produces a subspace in a space of higher dimension.

For arbitrary N , a suitable choice for the basis frame format (with the adjoint field scaled by the arbitrary function, λ) is given by the matrix (written in a format as if $N=2$):

$$\mathbb{F} = \begin{bmatrix} 1 & 0 & \vdots & -If_a \\ 0 & 1 & \vdots & -If_b \\ \dots & \dots & \vdots & \dots \\ f_a & f_b & \vdots & +I \end{bmatrix}, \quad \text{with determinant } \lambda(1 + f_a^2 + f_b^2 + \dots). \quad (\text{A4})$$

(A slightly more general situation would permit arbitrary functional scaling of each of the associated vectors). As the determinant is non-zero for real f_{α} and non-zero values of λ , an inverse frame matrix exists globally, and may be evaluated as

$$\mathbb{F}^{-1} = (1 + f_a^2 + f_b^2 + \dots)^{-1} \begin{bmatrix} (1 + f_b^2 + \dots) & -f_a f_b & \vdots & f_a \\ -f_a f_b & (1 + f_a^2 + \dots) & \vdots & f_b \\ \dots & \dots & \vdots & \dots \\ -f_a / \mathbf{I} & -f_b / \mathbf{I} & \vdots & 1 / \mathbf{I} \end{bmatrix}. \quad (\text{A5})$$

Accordingly, the Cartan matrix of connection 1-forms, \mathbb{C} , may be constructed as

$$d\mathbb{F} = \mathbb{F} \bullet \mathbb{C} = \mathbb{F} \bullet \{-d\mathbb{F}^{-1} \bullet \mathbb{F}\}. \quad (\text{A6})$$

Direct computation yields the Cartan matrix of connection 1-forms:

$$d\mathbb{F} = \mathbb{F} \bullet \begin{bmatrix} f_a df_a & f_a df_b & \vdots & -\mathbf{I}[(f_b df_a - f_a df_b)f_b + df_a] \\ f_b df_a & f_b df_b & \vdots & -\mathbf{I}[(f_a df_b - f_b df_a)f_a + df_b] \\ \dots & \dots & \vdots & \dots \\ df_a / \mathbf{I} & df_b / \mathbf{I} & \vdots & +d \ln[\mathbf{I} \sqrt{1 + f_a^2 + f_b^2 + \dots}] \end{bmatrix} / (1 + f_a^2 + f_b^2 + \dots). \quad (\text{A7})$$

By successive differentiation of the basis frame format, the geometrical theory of the projectivized frame bundle is retrieved. The general format of the basis frame in $N+1$ dimensions may be partitioned in terms of the associated vectors, \mathbf{e}_k , and the adjoint normal field, \mathbf{n} ,

$$\mathbb{F} = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \mathbf{n}]. \quad (\text{A8})$$

with the Cartan connection matrix of 1-forms in partitioned in the format

$$d\mathbb{F} = \mathbb{F} \bullet \begin{bmatrix} \Gamma_a^a & \Gamma_b^a & \dots & \mathbf{g}^a \\ \Gamma_a^b & \Gamma_b^b & \dots & \mathbf{g}^b \\ \dots & \dots & \dots & \dots \\ h_a & h_b & \dots & \Omega \end{bmatrix}. \quad (\text{A9})$$

For a simple 2 dimensional subspace in 3 dimensions, the functions that make up the vector array, \mathbf{h} , of differential 1-forms can be compared to the coefficients of the classical second fundamental form. The coefficients of the vector array, $\boldsymbol{\gamma}$, form the components of the classical shape matrix. It is the trace and determinant of the shape matrix that often are used to compute the mean curvature, H , and the Gaussian curvature, G , of the subspace, $\{\alpha, \beta\}$, without ever directly computing the connection coefficients given by the

matrix of 1-forms, Γ . This result is valid because the matrix of two forms of curvature, Θ , are given by the matrix expression:

$$d\Gamma + \Gamma \wedge \Gamma = -\gamma \wedge \mathbf{h} = \Theta. \quad (\text{A10})$$

The matrix of curvature two forms can be constructed from the knowledge of connection coefficients, Γ , or directly from the "outer-exterior" product of the vector-valued second fundamental form, \mathbf{h} , and the vector-valued shape matrix, γ . For the case $N = 2$, the curvature matrix of two forms is given by the expression:

$$\Theta = \begin{bmatrix} -f_a f_b df_a \wedge df_b & -(1+f_b^2)df_a \wedge df_b \\ +(1+f_a^2)df_a \wedge df_b & f_a f_b df_a \wedge df_b \end{bmatrix} / (1+f_a^2 + f_b^2)^2 \quad (\text{A11})$$

It is to be noted the matrix of 2-forms has the common factor $df_\alpha \wedge df_\beta$. These functions are precisely the components of the 2-form, $F = dA$, constructed in terms of the exterior derivative acting on the Cartan-Hilbert global 1-form of Action. The polynomial products of F , similar to the polynomial products of Θ , lead to the Chern classes.

The *abnormality*, Ω , is often assumed to be zero. For simple subspaces represented by a smooth surface with no intersections or pinch points, such a result is globally true. However, it is apparent that in the general formulation given above, Ω is not necessarily zero, but Ω is always integrable; i.e., Ω is proportional to a total differential and can be written in the form, TdS . The abnormality allows for a scale change of the normal field as it is displaced along trajectories in the associated space. Physically, a non-zero abnormality might be considered as an expansion or contraction of the variety and could be used to account for, or to minimize, dissipation effects. The choice of notation TdS was deliberate. For minimal or null dissipative systems associated with the Navier-Stokes equations, see [Lamb, 1945, p.618, 646].

The complete expression for the differential of the adjoint normal field is given by the expression:

$$d\mathbf{n} = \mathbf{e}_\alpha \gamma^\alpha + \mathbf{e}_\beta \gamma^\beta + \mathbf{n} \Omega. \quad (\text{A12})$$

Similarly, for the differentials of the associated vectors (not necessarily an orthonormal set, and also not necessarily the *tangent* generators of a smooth surface),

$$d\mathbf{e}_\sigma = \mathbf{e}_\alpha \Gamma_\sigma^\alpha + \mathbf{e}_\beta \Gamma_\sigma^\beta + \mathbf{n} h_\sigma. \quad (\text{A13})$$

By repeated exterior differentiation of (A12) and (A13) all of the classic formulae of differential geometry can be generated. For example,

$$dde = \mathbf{e} \{d\Gamma + \Gamma \wedge \Gamma + \gamma \wedge \mathbf{h}\} + \mathbf{n} \{d\mathbf{h} + \mathbf{h} \wedge \Gamma + \Omega \wedge \mathbf{h}\} = 0 \quad (\text{A14})$$

$$dd\mathbf{n} = \mathbf{e} \{d\boldsymbol{\gamma} + \Gamma^{\wedge}\boldsymbol{\gamma} + \boldsymbol{\gamma}^{\wedge}\boldsymbol{\Omega}\} + \mathbf{n} \{d\boldsymbol{\Omega} + \mathbf{h}^{\wedge}\boldsymbol{\gamma} + \boldsymbol{\Omega}^{\wedge}\boldsymbol{\Omega}\} = 0.$$

As dde_{σ} and $dd\mathbf{n}$ must vanish by the Poincare lemma, each of the bracket factors must vanish by reason of linear independence:

$$\begin{aligned} d\Gamma + \Gamma^{\wedge}\Gamma &= -\boldsymbol{\gamma}^{\wedge}\mathbf{h} \equiv \Theta & d\mathbf{h} + \boldsymbol{\Omega}^{\wedge}\mathbf{h} &= -\mathbf{h}^{\wedge}\Gamma \equiv \mathbf{j} \\ d\boldsymbol{\gamma} + \Gamma^{\wedge}\boldsymbol{\gamma} &= -\boldsymbol{\gamma}^{\wedge}\boldsymbol{\Omega} \equiv \boldsymbol{\Psi} & d\boldsymbol{\Omega} + \boldsymbol{\Omega}^{\wedge}\boldsymbol{\Omega} &= -\mathbf{h}^{\wedge}\boldsymbol{\gamma} \equiv \mathbf{f} \end{aligned} \quad (\text{A15})$$

The term Θ is equivalent to the expression $-\boldsymbol{\gamma}^{\wedge}\mathbf{h}$, which is an outer product of the two vectors, $\boldsymbol{\gamma}$ and \mathbf{h} , producing a matrix of 2-forms called the *curvature* 2-forms in the theory of fiber bundles. The matrix array of 1-forms, Γ , are called the connection 1-forms in the fiber bundle theory, and the expression $d\Gamma + \Gamma^{\wedge}\Gamma = \Theta$ is used to compute the curvature. The combination of operators that repeats itself on the left side of the expressions above, that is the operator $d + \Gamma$, is equivalent to the covariant derivative relative to a connection in classical tensor analysis.

The point to be emphasized, and a major purpose of this appendix, is to demonstrate that the matrix of 2-forms can be evaluated in two different, and equivalent ways: the fiber bundle method (given some connection) computes $\Theta = d\Gamma + \Gamma^{\wedge}\Gamma$, while the Cartan procedure (given a global 1-form) computes, $\Theta = -\boldsymbol{\gamma}^{\wedge}\mathbf{h}$. The first method requires another differentiation of the vector field, while the second method is algebraic.

The term $-\boldsymbol{\gamma}^{\wedge}\boldsymbol{\Omega}$, given the equivalent symbol, $\boldsymbol{\Psi}$, is a column vector array of two forms, which is the source of abnormal or *defect* structures in the variety. It is apparent that if $\boldsymbol{\Omega} \equiv TdS$ vanishes, then the formulas (A15) are simplified considerably. In all cases the product $\boldsymbol{\Omega}^{\wedge}\boldsymbol{\Omega} = 0$, algebraically, hence the 2-form \mathbf{f} is exact.

An application of the exterior derivative to the equations of the set (A15) lead to constraints on systems of 3-forms. Exterior differentiation of each of the four brackets yields:

$$\begin{aligned} d\Theta + \Gamma^{\wedge}\Theta &= \Theta^{\wedge}\Gamma & d\mathbf{j} + \boldsymbol{\Omega}^{\wedge}\mathbf{j} &= \mathbf{f}^{\wedge}\mathbf{h} \\ d\boldsymbol{\Psi} + \Gamma^{\wedge}\boldsymbol{\Psi} &= \Theta^{\wedge}\boldsymbol{\Psi} & d\mathbf{f} + \boldsymbol{\Omega}^{\wedge}\mathbf{f} &= \mathbf{f}^{\wedge}\boldsymbol{\Omega} \end{aligned} \quad (\text{A16})$$

The first expression involving $d\Theta$ is the equivalent to the Bianchi identities in classical tensor analysis. The last expression involving $d\mathbf{f}$, leads to $d\mathbf{f} = 0$, as $\boldsymbol{\Omega}^{\wedge}\mathbf{f} = \mathbf{f}^{\wedge}\boldsymbol{\Omega}$. The process of repetitive exterior differentiation can proceed until a complete set of constraints on a system of $N+1$ forms is generated. From then on, the process of exterior differentiation leads to no new information.

If the adjoint field, $\mathbf{n} = \lambda\mathbf{N}$, is constrained by the normalization factor, λ , then for a special choice of λ such that

$$\lambda = \text{constant} / \sqrt{(1 + f_{\alpha}^2 + f_{\beta}^2)},$$

the abnormality, $\boldsymbol{\Omega}$, vanishes globally. All of the formulas above simplify dramatically, but from a physical point of view, the system loses its defect structures and becomes

adiabatic; $TdS = 0$. This constraint, however, is one of the key assumptions in classical Weingarten surface theory. In this special case, the scaled adjoint vector, $\mathbf{n} = \lambda \mathbf{N}$, can be considered as the *normal* field to a subspace of the variety (defined as the space *adjoint* to the 1-form, A), and this adjoint subspace is composed of those *associate* vectors that annihilate the 1-form A . The 1-form A is not necessarily integrable, in the sense of Frobenius, but the 1-form still represents a subspace. This subspace is not necessarily a smooth surface in the non-integrable case, but can be represented as a surface with self-intersections and pinch points. The *normal* field in such non-integrable cases is not representable by the gradient of a single scalar field, globally. The extensions of the formulas to N dimensions is inductive.

The choice of λ such that $\Omega = 0$ is a constraint that causes the Cartan matrix (not the frame matrix) to be a member of the projective group, with $(n+1)^2 - 1$ parameters. Also note that the vector of forms, \mathbf{h} , is not the negative of the transpose of the vector of forms, $\boldsymbol{\gamma}$, as is the case if the Frame matrix were a member of the orthogonal group. No such group constraints have been assumed for the frame matrix in this article. Written out in terms of the differentials, the elements of \mathbf{h} form the components of the second fundamental form, while the components of $\boldsymbol{\gamma}$ form the shape matrix of classical surface theory.

As mentioned above, the matrix elements of the curvature 2-forms, Θ , are constructed from the components of the 2-form F , and these functions are completely specified in terms of the anti-symmetric components of the Jacobian matrix of the normalized adjoint field, $\mathbb{J}(\mathbf{n})$. In fact, the Jacobian matrix of the normalized adjoint field, $\mathbb{J}(\mathbf{n})$ is precisely equivalent to the shape matrix of classical differential geometry. That is, explicit computation will show that the coefficient functions, $\gamma_k^j dx^k$, that make up the shape matrix (or the vector of 1-forms, $\boldsymbol{\gamma}$) are equivalent to those coefficient functions that make up the Jacobian matrix of the unit normal field, $\mathbb{J}(\mathbf{n})$. This result, that the shape matrix and the Jacobian matrix, $\mathbb{J}(\mathbf{n})$, are synonymous, permits the computation of topological and intrinsic geometrical information to be carried out in a manner more simple than that exemplified by fiber bundle methods.. It is the shape matrix which contains the interesting information about the surface defined by the normalized adjoint field. The trace of $\mathbb{J}(\mathbf{n})$ is n times the mean curvature, H , of the adjoint surface, and the determinant of $\mathbb{J}(\mathbf{n})$ yields K , the Gauss curvature of the adjoint surface [Flanders].

Note that there is no need to construct the Frame field and the associated matrix of connections, or the Riemann curvature tensor following the classical methods, for all of the pertinent topological information comes directly from the computation of the Jacobian matrix of the unit adjoint field. This result is why it may be said that the Cartan 1-form of global Action contains the pertinent topological information on the space. The beauty of the Cartan method is that it is explicitly coordinate free, it is explicitly metric free, it is explicitly connection free. The classical methods, built on an assumed metric and/or a connection, may be used to determine Gauss and mean curvature, as well as a number of other geometrical ideas, but this classical baggage is not needed to determine

certain key topological properties related to the mean and the Gaussian curvature, if a global 1-form of Action is known.

Any given a matrix, such as $\mathbb{J}(\mathbf{n})$, can be split into its symmetric and antisymmetric parts. The anti-symmetric parts of $\mathbb{J}(\mathbf{n})$ form the components of the 2-form, $F = dA$, constructed from the fundamental global 1-form A by exterior differentiation. The concept of the mean curvature is not contained in the 2-form F . But as the determinant of $\mathbb{J}(\mathbf{n})$ is an anti-symmetric concept (which leads to the Gaussian curvature) it is to be expected that the anti-symmetric 2-form F will play a role in the construction of the curvature invariants. As noted above, the components of the matrix of curvature 2-forms, Θ , depends explicitly on the anti-symmetric components of the Jacobian matrix. As powers of the matrix of curvature 2-forms leads to the Chern classes, it follows that powers of the 2-form, $F = dA$, are also related to the Chern classes. In particular, the Top Pfaffian of Chern for the subspace $\{x,y,z,t\}$ is the 4-form F^4 . The integral of this form (according to Chern) is equivalent to the Euler Characteristic (at least for compact oriented manifolds). It follows that if $F^4 = 0$, such that the Pfaff dimension of the vector field is 3 or less, then the integral of F^4 is also zero, and the Euler characteristic is zero. Intrinsic irreducible self-intersections and self-intersections vanish, and such vector fields do not represent irreversible turbulent flows.

The moral of this appendix is that there usually is more than one way to do things. In the case of a projectivized line bundle over a variety, the Cartan method of computing the significant quantities is equivalent to the methods of fiber bundle theory, but it is much simpler to use and easier to interpret in physically useful ways for engineering applications.

APPENDIX B The Electromagnetic Format

For the electromagnetic case, the Cartan 1-form may be defined in terms of the vector and scalar potentials,

$$A = \mathbf{A} \cdot d\mathbf{r} - \phi dt. \quad (B1)$$

Using the classical notation of Sommerfeld, define the \mathbf{E} and \mathbf{B} field intensities as

$$\mathbf{B} = \text{curl } \mathbf{A}, \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - \text{grad } \phi. \quad (B2)$$

Then the components of the Darboux-Cartan-Maxwell field, $F_{\mu\nu}$, may be written as an anti-symmetric matrix (or as a Sommerfeld six-vector) of components :

$$\begin{array}{ll} F_{12} = B_z & F_{14} = E_x \\ F_{13} = -B_y & F_{24} = E_y \\ F_{23} = B_x & F_{34} = E_z \end{array}$$

The components of $dA = F$ become,

$$F = F_{\mu\nu}dx^\mu dx^\nu \quad (B3)$$

The Topological torsion, H, becomes

$$H = A \wedge dA = -\{ \mathbf{E} \times \mathbf{A} + \phi \mathbf{B}, \mathbf{A} \cdot \mathbf{B} \}_{\mu\nu\sigma} dx^\mu dx^\nu dx^\sigma. \quad (B4)$$

with the torsion current defined as,

$$\mathbf{T} = \mathbf{E} \times \mathbf{A} + \phi \mathbf{B} \quad (B5)$$

and the helicity density,

$$h = \mathbf{A} \cdot \mathbf{B}. \quad (B6)$$

The Topological Parity 4-form becomes the global top Pfaffian on the 4 dimensional space-time variety, and is equal to

$$K = dA \wedge dA = -2 \mathbf{E} \cdot \mathbf{B} dx \wedge dy \wedge dz \wedge dt. \quad (B7)$$

Note that

$$\text{div } \mathbf{T} + \partial h / \partial t = -2 \mathbf{E} \cdot \mathbf{B}.$$

The 3-form of axial current, H, is NOT conserved when $K \neq 0$. This divergence anomaly for electromagnetic systems was demonstrated from other arguments by Berger [1984].

Following Chern, the Euler index becomes the integral

$$X = \text{constant} \iiint 2\mathbf{E} \cdot \mathbf{B} dx \wedge dy \wedge dz \wedge dt \quad (B8)$$

When the electric field is orthogonal to the magnetic field, then the Euler index is zero. The idea that this Poincare invariant might have deeper meaning led Eddington [1963] to state: "It is somewhat curious that the scalar-product of the electric and magnetic forces is of so little importance in classical theory, for ..(eq (B7)) .. would seem to be the most fundamental invariant of the field. Apart from the fact that it vanishes for electromagnetic waves propagated in the absence of any bound electric field (i.e., remote from electrons), this invariant seems to have no significant properties. Perhaps it may turn out to have greater importance when the study of electron-structure is more advanced."

The concept of a domain of non-null Euler index ($K \neq 0$) now appears to be useful to the theory of magnetic reconnection in the electromagnetic case [Otto 1990] and to vortex reconnection [Metcalf 1990] in the hydrodynamic case. The correspondence between the bridging and rib structures produced in numerical simulations of turbulent fluid flows

and the 4-string interaction of superstring theory is remarkable [Kaku 1987]. The concept ($K \neq 0$) appears to be applicable to the understanding of the stretching of lines and surfaces in turbulent flows where time-reversal symmetry is violated [Drummond 1990]. The appearance of large scale structures in certain flows has been associated with the lack of parity invariance [Sulem 1989]. The concepts of macroscopic violations of P and T symmetries appear to have application to the theory of the quantum Hall effect [Wen 1989].

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