```
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=0,Az=0,C=0,Phi=0,phi=0,theta=0,r=0,a=const,b=const,c=const,Lx=0,Ly=0,Lz=0);
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

Spheres in 3D space and Cartan Connections

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last update February 25, 2000
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Introduction:

The Cartan connection coefficients will be computed for both the map of cartesian coordinates into spherical coordinates, (Part1)
and the map from spherical coordinates into cartesian coordinates (Part2)

Part 1 The map is from {x,y,z} into spherical coordinates {r,theta,phi}.

```
> r:=(x^2+y^2+z^2)^(1/2);cos(theta):=z/(r);cos(phi):=x/(x^2+y^2)^(1/2);sin(theta):=(1-cos(theta)^2)^(1/2);sin(phi):=(1-cos(phi)^2)^(1/2);rho:=(x^2+y^2)^(1/2);
>
```

$$r := \sqrt{x^2+y^2+z^2}$$

$$\cos(\theta) := \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\cos(\phi) := \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin(\theta) := \sqrt{1-\frac{z^2}{x^2+y^2+z^2}}$$

$$\sin(\phi) := \sqrt{1-\frac{x^2}{x^2+y^2}}$$

$$\rho := \sqrt{x^2+y^2}$$

Compute the induced "dribeins" (they are not exact differentials, but they are closed)

```
> d(r);DZ:=solve(d(r)=0,d(z));
```

$$\frac{x\,\mathrm{d}(x)}{\sqrt{x^2+y^2+z^2}}+\frac{y\,\mathrm{d}(y)}{\sqrt{x^2+y^2+z^2}}+\frac{z\,\mathrm{d}(z)}{\sqrt{x^2+y^2+z^2}}$$

$$DZ := -\frac{x\,\mathrm{d}(x)+y\,\mathrm{d}(y)}{z}$$

```
> `dphi`:=factor(d(cos(phi))/sin(phi));
```

$$dphi := \frac{y\,(y\,\mathrm{d}(x) - x\,\mathrm{d}(y))}{(x^2+y^2)^{(3/2)}\sqrt{\frac{y^2}{x^2+y^2}}}$$

```
> `dtheta`:=factor(subs(d(z)=DZ,(d(cos(theta))/sin(theta))));
```

$$dtheta := -\frac{x\,\mathrm{d}(x) + y\,\mathrm{d}(y)}{\sqrt{x^2+y^2+z^2}\; z\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}}$$

The Frame matrix of functions on x,y,z that causes the differential structures dr,dtheta,dphi to be created linearly in terms of dx,dy,dz is the Frame matrix [FF] below:

```
> FF:=(array([[x/r,y/r,z/r],[-z*x/(rho*r^2),-z*y/(rho*r^2),(x^2+y^2)/(rho*r^2)],[-y/rho^2,x/rho^2,0]]));GG:=evalm(inverse(FF)):DETFF:=factor(det(FF));
```

$$FF := \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\frac{z\,x}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} & -\frac{z\,y}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} & \frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{bmatrix}$$

$$DETFF := -\frac{1}{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2+z^2}}$$

Note that the Frame matrix has a singularity along the z axis and at the origin (excluded) points. The next equation checks to see that the specified frame produces the desired differential structures: [FF]|dR>

```
> zz:=evalm(innerprod(FF,[d(x),d(y),d(z)]));zzb:=innerprod(GG,zz);
```

$$zz := \left[\frac{x\,\mathrm{d}(x) + y\,\mathrm{d}(y) + z\,\mathrm{d}(z)}{\sqrt{x^2+y^2+z^2}}, -\frac{z\,x\,\mathrm{d}(x) + z\,y\,\mathrm{d}(y) - \mathrm{d}(z)\,x^2 - \mathrm{d}(z)\,y^2}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}, -\frac{y\,\mathrm{d}(x) - x\,\mathrm{d}(y)}{x^2+y^2}\right]$$

$$zzb := [\mathrm{d}(x), \mathrm{d}(y), \mathrm{d}(z)]$$

Note that each component (except the first) is not exact, but is closed. Each component in terms of xyz has an infinity at the origin and along the Z axis. The induced spherical coordinate "differentials" are not exact , but are closed and obey the Frobenius integrability theorem.

```
> d(zz[1]);d(zz[2]);d(zz[3]);
```

$$0$$

$$0$$

$$0$$

The metric on the domain xyz is the unit matrix of constants, by assumption. As such the Christoffel symbols will be zero. The pushed forward metric on the spherical coordinate range is (but with arguments on the domain x,y,z) is

```
> pushedmetric:=simplify(innerprod(transpose(GG),GG));inducedmetric:=innerprod(transpose(FF),pushedmetric,FF):
```

$$pushedmetric := \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2+y^2+z^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix}$$

Now check to see if the Frame matrix is normal and find the Left and Right representations P2R=[FF][transpose FF]

P2L= [transpose FF] [F]. Conclusion: the Frame matrix is not normal!!! Note that P2R is the pushedforward inverse metric on {r,theta,phi} but with arguments on {x,y,z}

```
> P2R:=simplify(innerprod(FF,transpose(FF)));P2L:=simplify(innerprod(transpose(FF),FF));
```

$$P2R := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{x^2+y^2+z^2} & 0 \\ 0 & 0 & \frac{1}{x^2+y^2} \end{bmatrix}$$

$P2L :=$

$$\left[\frac{x^8+3x^6y^2+3x^4y^4+x^2y^6+x^6z^2+2x^4z^2y^2+x^2z^2y^4+z^2x^4+3z^2x^2y^2+x^4y^2+2x^2y^4+y^6+2z^2y^4+y^2z^4}{(x^2+y^2+z^2)^2(x^2+y^2)^2},\right.$$
$$\left.\frac{xy(x^6+3x^4y^2+3x^2y^4+y^6+z^2x^4+2z^2x^2y^2+z^2y^4-z^2x^2-y^2z^2-x^4-2x^2y^2-y^4-z^4)}{(x^2+y^2+z^2)^2(x^2+y^2)^2}, \frac{zx(x^2+y^2+z^2-1)}{(x^2+y^2+z^2)^2}\right]$$

$$\left[\frac{xy(x^6+3x^4y^2+3x^2y^4+y^6+z^2x^4+2z^2x^2y^2+z^2y^4-z^2x^2-y^2z^2-x^4-2x^2y^2-y^4-z^4)}{(x^2+y^2+z^2)^2(x^2+y^2)^2},\right.$$
$$\frac{x^6y^2+3x^4y^4+3x^2y^6+y^8+x^4z^2y^2+2x^2z^2y^4+y^6z^2+3z^2x^2y^2+z^2y^4+x^6+2x^4y^2+2z^2x^4+x^2y^4+x^2z^4}{(x^2+y^2+z^2)^2(x^2+y^2)^2},$$
$$\left.\frac{zy(x^2+y^2+z^2-1)}{(x^2+y^2+z^2)^2}\right]$$

$$\left[\frac{zx(x^2+y^2+z^2-1)}{(x^2+y^2+z^2)^2}, \frac{zy(x^2+y^2+z^2-1)}{(x^2+y^2+z^2)^2}, \frac{z^2x^2+y^2z^2+z^4+x^2+y^2}{(x^2+y^2+z^2)^2}\right]$$

P2L is complicated algebracially. But the bottom line is that it is NOT equal to P2R,
hence the Jacobian matrix is NOT NORMAL
Now any matrix with an inverse can be composed as a product of a symmetric matrix and an orthogonal matrix.
There are in general two ways to construct this representation, which will be denoted as the Lefthanded and the Right handed formulations. [SR] is the symmetric matrix of the "right handed" formulation, and [OR] is the orthogonal matrix for the right handed formulation. (If the matrices are complex, the notions symmetric and orthogonal translate to Hermitean and Unitary)
The two formats are:
[F]= [SR][OR]=[OL] [SL]
If [G][F]=1, then
[SR]={[FF].transpose[FF]}^(1/2)
and
[OR]=[SR].transpose[G]
The Left handed representations
[SL]={transpose[FF].[FF]}^(1/2)
and
[OL]=transpose[G].[SL]
The representations are distinct if the Frame [FF] is not a normal matrix.
**
So the first step is to find the square roots of these matrices P2R and P2L above.
This is easy to do for the Right Handed P2R, for it is diagonal.
The symmetric component of the right handed representation: F = [SR][OR]

```
> SR:=array([[1,0,0],[0,1/r,0],[0,0,1/rho]]);
```

$$SR := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{x^2+y^2+z^2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{x^2+y^2}} \end{bmatrix}$$

Compute the orthogonal factor [OR] of the right handed representation.

```
> GS:=transpose(inverse(FF)):OR:=innerprod(SR,GS);simplify(innerprod(transpose(OR)
,OR)):`Should_be_zero`:=evalm(innerprod(SR,OR)-FF):
```

$$OR := \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\frac{z\,x}{\sqrt{x^2+y^2+z^2}\,\sqrt{x^2+y^2}} & -\frac{z\,y}{\sqrt{x^2+y^2+z^2}\,\sqrt{x^2+y^2}} & \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{bmatrix}$$

Find Omega, the right Cartan matrix of [OR], which should be an anti-symmetric matrix
and Delta the left Cartan matrix for [SR]

```
> dOR:=d(OR):
> Omega_RR:=simplify(innerprod(transpose(OR),dOR));Delta_RL:=innerprod(d(SR),inver
se(SR));
```

$$Omega\_RR := \begin{bmatrix} 0\,, & \frac{-y\,\mathrm{d}(x)+x\,\mathrm{d}(y)}{x^2+y^2}\,, & \frac{(-z\,x\,\mathrm{d}(x)-z\,y\,\mathrm{d}(y)+\mathrm{d}(z)\,x^2+\mathrm{d}(z)\,y^2)\,x}{(x^2+y^2)\,(x^2+y^2+z^2)} \\ -\frac{-y\,\mathrm{d}(x)+x\,\mathrm{d}(y)}{x^2+y^2}\,, & 0\,, & \frac{y\,(-z\,x\,\mathrm{d}(x)-z\,y\,\mathrm{d}(y)+\mathrm{d}(z)\,x^2+\mathrm{d}(z)\,y^2)}{(x^2+y^2)\,(x^2+y^2+z^2)} \\ -\frac{(-z\,x\,\mathrm{d}(x)-z\,y\,\mathrm{d}(y)+\mathrm{d}(z)\,x^2+\mathrm{d}(z)\,y^2)\,x}{(x^2+y^2)\,(x^2+y^2+z^2)}\,, & -\frac{y\,(-z\,x\,\mathrm{d}(x)-z\,y\,\mathrm{d}(y)+\mathrm{d}(z)\,x^2+\mathrm{d}(z)\,y^2)}{(x^2+y^2)\,(x^2+y^2+z^2)}\,, & 0 \end{bmatrix}$$

$$Delta\_RL := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{x\,\mathrm{d}(x)+y\,\mathrm{d}(y)+z\,\mathrm{d}(z)}{x^2+y^2+z^2} & 0 \\ 0 & 0 & -\frac{x\,\mathrm{d}(x)+y\,\mathrm{d}(y)}{x^2+y^2} \end{bmatrix}$$

Or one can find the Left Cartan matrix for [OR] and the right Cartan matrix for [SR]

```
> Omega_RL:=simplify(innerprod(dOR,transpose(OR)));Delta_RR:=innerprod(d(SR),inver
se(SR));
```

$$Omega\_RL := \begin{bmatrix} 0 & \frac{-z\,x\,\mathrm{d}(x)-z\,y\,\mathrm{d}(y)+\mathrm{d}(z)\,x^2+\mathrm{d}(z)\,y^2}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} & \frac{-y\,\mathrm{d}(x)+x\,\mathrm{d}(y)}{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2+z^2}} \\ -\frac{-z\,x\,\mathrm{d}(x)-z\,y\,\mathrm{d}(y)+\mathrm{d}(z)\,x^2+\mathrm{d}(z)\,y^2}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} & 0 & -\frac{(-y\,\mathrm{d}(x)+x\,\mathrm{d}(y))\,z}{(x^2+y^2)\,\sqrt{x^2+y^2+z^2}} \\ -\frac{-y\,\mathrm{d}(x)+x\,\mathrm{d}(y)}{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2+z^2}} & \frac{(-y\,\mathrm{d}(x)+x\,\mathrm{d}(y))\,z}{(x^2+y^2)\,\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix}$$

$$Delta\_RR := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{x\,d(x)+y\,d(y)+z\,d(z)}{x^2+y^2+z^2} & 0 \\ 0 & 0 & -\frac{x\,d(x)+y\,d(y)}{x^2+y^2} \end{bmatrix}$$

Note that the matrix elements of Delta are perfect exact differentials, and the matrix elements of Omega are closed but not exact differentials.

It is possible to write the differential of the Frame field in several ways:

d[FF] = [FF][CR] = [CL][FF]

= {[DeltaRL][FF] + [FF][OmegaRR]} = {[OmegaL][FF]+[FF][DeltaL]}

= d{[SR][OR]} = d[SR][OR] + [SR]d[OR] = [SR]{[DeltaRR] + [OmegaLR]}[OR]

-

Note that appropriate linear combinations can be constructed to represent d[FF] such as is done in polarization of light.

*

Now Compute the Right Cartan Matrix [CR]

```
> cartan:=simplify(innerprod(inverse(FF),d(FF))):
```

The matrix elements of the Right Cartan connection matrix using the matrix methods:

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);
```

$$\Gamma 11 := -\frac{(2y^4x^3+3y^2x^3z^2+y^6x+y^2xz^4+y^2x^5+x^5z^2+2y^4xz^2)\,d(x)}{(x^2+y^2)^2(x^2+y^2+z^2)^2} - \frac{(y^3z^4+y^7+y^3x^4+yx^4z^2+2y^5x^2+3x^2y^3z^2+2y^5z^2)\,d(y)}{(x^2+y^2)^2(x^2+y^2+z^2)^2} - \frac{(x^4z^3+x^2z^3y^2)\,d(z)}{(x^2+y^2)^2(x^2+y^2+z^2)^2}$$

$$\Gamma 12 := \frac{(-yx^4z^2-y^3x^4-3x^2y^3z^2-2y^5x^2-y^3z^4-y^7-2y^5z^2)\,d(x)}{(x^2+y^2)^2(x^2+y^2+z^2)^2} + \frac{(2x^5z^2+4y^2x^5+x^7+5y^2x^3z^2+5y^4x^3+2y^6x+x^3z^4+3y^4xz^2+2y^2xz^4)\,d(y)}{(x^2+y^2)^2(x^2+y^2+z^2)^2} + \frac{(-xy^3z^3-x^3z^3y)\,d(z)}{(x^2+y^2)^2(x^2+y^2+z^2)^2}$$

$$\Gamma 13 := -\frac{z^3x^2\,d(x)}{(x^2+y^2)(x^2+y^2+z^2)^2} - \frac{yz^3x\,d(y)}{(x^2+y^2)(x^2+y^2+z^2)^2} + \frac{(x^4+2x^2y^2+2z^2x^2+2y^2z^2+y^4)x\,d(z)}{(x^2+y^2)(x^2+y^2+z^2)^2}$$

```
> Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(cartan[2,3]);
```

$$\Gamma 21 := -\frac{(-5y^3x^4-5x^2y^3z^2-2yz^4x^2-4y^5x^2-3yx^4z^2-2yx^6-y^3z^4-y^7-2y^5z^2)\,d(x)}{(x^2+y^2)^2(x^2+y^2+z^2)^2} - \frac{(2x^5z^2+2y^2x^5+x^7+y^4xz^2+y^4x^3+x^3z^4+3y^2x^3z^2)\,d(y)}{(x^2+y^2)^2(x^2+y^2+z^2)^2} - \frac{(xy^3z^3+x^3z^3y)\,d(z)}{(x^2+y^2)^2(x^2+y^2+z^2)^2}$$

$$\Gamma 22 := -\frac{(2x^5z^2+2y^2x^5+x^7+y^4xz^2+y^4x^3+x^3z^4+3y^2x^3z^2)\,d(x)}{(x^2+y^2)^2(x^2+y^2+z^2)^2} - \frac{(yx^6+y^5z^2+y^5x^2+2yx^4z^2+yz^4x^2+3x^2y^3z^2+2y^3x^4)\,d(y)}{(x^2+y^2)^2(x^2+y^2+z^2)^2} - \frac{(x^2z^3y^2+z^3y^4)\,d(z)}{(x^2+y^2)^2(x^2+y^2+z^2)^2}$$

$$\Gamma23 := -\frac{y\,z^3\,x\,d(x)}{(x^2+y^2)\,(x^2+y^2+z^2)^2} - \frac{y^2\,z^3\,d(y)}{(x^2+y^2)\,(x^2+y^2+z^2)^2} + \frac{(x^4+2\,x^2\,y^2+2\,z^2\,x^2+2\,y^2\,z^2+y^4)\,y\,d(z)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

```
> Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(
  cartan[3,3]);
```

$$\Gamma31 := -\frac{(-2\,z\,x^3-2\,z\,x\,y^2-z^3\,x)\,x\,d(x)}{(x^2+y^2)\,(x^2+y^2+z^2)^2} - \frac{(-2\,z\,y\,x^2-2\,z\,y^3-y\,z^3)\,x\,d(y)}{(x^2+y^2)\,(x^2+y^2+z^2)^2} - \frac{(x^4+2\,x^2\,y^2+y^4)\,x\,d(z)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\Gamma32 := -\frac{(-2\,z\,x^3-2\,z\,x\,y^2-z^3\,x)\,y\,d(x)}{(x^2+y^2)\,(x^2+y^2+z^2)^2} - \frac{(-2\,z\,y\,x^2-2\,z\,y^3-y\,z^3)\,y\,d(y)}{(x^2+y^2)\,(x^2+y^2+z^2)^2} - \frac{(x^4+2\,x^2\,y^2+y^4)\,y\,d(z)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\Gamma33 := -\frac{(x\,y^2+x^3)\,d(x)}{(x^2+y^2+z^2)^2} - \frac{(y\,x^2+y^3)\,d(y)}{(x^2+y^2+z^2)^2} - \frac{(z\,y^2+x^2\,z)\,d(z)}{(x^2+y^2+z^2)^2}$$

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=3;coord:=[x,y,z];GG:=inverse(FF);
```

$$dim := 3$$

$$coord := [x, y, z]$$

$$GG := \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & -\frac{z\,x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & -\frac{z\,y}{\sqrt{x^2+y^2}} & x \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & \sqrt{x^2+y^2} & 0 \end{bmatrix}$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k]
  := (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for
  m from 1 to dim do ss := ss+(d1GG[a,m,k]*FF[m,b]); CC[a,b,k]:=simplify(-ss) od
  od od od ;
>
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if
  CC[a,b,k]=0 then else print(`CCabk`(a,b,k)=factor(CC[a,b,k])) fi od od od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients.

CC(abk) index (1,-1,-1)

$$CCabk(1, 1, 1) = -\frac{(x^4\,y^2+z^2\,x^4+2\,x^2\,y^4+3\,z^2\,x^2\,y^2+2\,z^2\,y^4+y^2\,z^4+y^6)\,x}{(x^2+y^2)^2\,(x^2+y^2+z^2)^2}$$

$$CCabk(1, 1, 2) = -\frac{y\,(x^4\,y^2+z^2\,x^4+2\,x^2\,y^4+3\,z^2\,x^2\,y^2+2\,z^2\,y^4+y^2\,z^4+y^6)}{(x^2+y^2)^2\,(x^2+y^2+z^2)^2}$$

$$CCabk(1, 1, 3) = -\frac{z^3\,x^2}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CCabk(2, 1, 1) = \frac{y\,(2\,x^6+5\,x^4\,y^2+4\,x^2\,y^4+3\,z^2\,x^4+5\,z^2\,x^2\,y^2+2\,x^2\,z^4+y^6+2\,z^2\,y^4+y^2\,z^4)}{(x^2+y^2)^2\,(x^2+y^2+z^2)^2}$$

$$CCabk(2, 1, 2) = -\frac{(x^6 + 2 x^4 y^2 + 2 z^2 x^4 + 3 z^2 x^2 y^2 + x^2 y^4 + x^2 z^4 + z^2 y^4) x}{(x^2 + y^2)^2 (x^2 + y^2 + z^2)^2}$$

$$CCabk(2, 1, 3) = -\frac{z^3 x y}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(3, 1, 1) = \frac{x^2 z (2 x^2 + 2 y^2 + z^2)}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(3, 1, 2) = \frac{z x y (2 x^2 + 2 y^2 + z^2)}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(3, 1, 3) = -\frac{(x^2 + y^2) x}{(x^2 + y^2 + z^2)^2}$$

$$CCabk(1, 2, 1) = -\frac{y (x^4 y^2 + z^2 x^4 + 2 x^2 y^4 + 3 z^2 x^2 y^2 + 2 z^2 y^4 + y^2 z^4 + y^6)}{(x^2 + y^2)^2 (x^2 + y^2 + z^2)^2}$$

$$CCabk(1, 2, 2) = \frac{x (4 x^4 y^2 + 5 x^2 y^4 + 2 y^6 + 5 z^2 x^2 y^2 + 3 z^2 y^4 + 2 y^2 z^4 + x^6 + 2 z^2 x^4 + x^2 z^4)}{(x^2 + y^2)^2 (x^2 + y^2 + z^2)^2}$$

$$CCabk(1, 2, 3) = -\frac{z^3 x y}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(2, 2, 1) = -\frac{(x^6 + 2 x^4 y^2 + 2 z^2 x^4 + 3 z^2 x^2 y^2 + x^2 y^4 + x^2 z^4 + z^2 y^4) x}{(x^2 + y^2)^2 (x^2 + y^2 + z^2)^2}$$

$$CCabk(2, 2, 2) = -\frac{(x^6 + 2 x^4 y^2 + 2 z^2 x^4 + 3 z^2 x^2 y^2 + x^2 y^4 + x^2 z^4 + z^2 y^4) y}{(x^2 + y^2)^2 (x^2 + y^2 + z^2)^2}$$

$$CCabk(2, 2, 3) = -\frac{z^3 y^2}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(3, 2, 1) = \frac{z x y (2 x^2 + 2 y^2 + z^2)}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(3, 2, 2) = \frac{z y^2 (2 x^2 + 2 y^2 + z^2)}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(3, 2, 3) = -\frac{(x^2 + y^2) y}{(x^2 + y^2 + z^2)^2}$$

$$CCabk(1, 3, 1) = -\frac{z^3 x^2}{(x^2 + y^2) (x^2 + y^2 + z^2)^2}$$

$$CCabk(1, 3, 2) = -\frac{z^3 x y}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(1, 3, 3) = \frac{x (2 z^2 + x^2 + y^2)}{(x^2 + y^2 + z^2)^2}$$

$$CCabk(2, 3, 1) = -\frac{z^3 x y}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$CCabk(2, 3, 2) = -\frac{z^3 y^2}{(x^2+y^2+z^2)^2 (x^2+y^2)}$$

$$CCabk(2, 3, 3) = \frac{y (2 z^2 + x^2 + y^2)}{(x^2+y^2+z^2)^2}$$

$$CCabk(3, 3, 1) = -\frac{(x^2+y^2) x}{(x^2+y^2+z^2)^2}$$

$$CCabk(3, 3, 2) = -\frac{(x^2+y^2) y}{(x^2+y^2+z^2)^2}$$

$$CCabk(3, 3, 3) = -\frac{(x^2+y^2) z}{(x^2+y^2+z^2)^2}$$

These results agree with matrix method.

Next check for Affine Torsion using the tensor methods:

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (CC[i,j,k]-CC[i,k,j])/2; CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  CCTTS[i,j,k]=0 then else print(`RIGHT_AffineTorsion`(i,k,j)=CCTTS[i,k,j]) fi od
  od od ;
```

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

```
>
```

Now compute the CARTAN LEFT CONNECTION

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[a,j,k]
  := simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrix product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do  ss:=0;for
  m to dim do ss := ss+FF[i,m]*(d1GG[m,j,k]); DD[i,j,k]:=simplify(ss) od od od od
  ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  DD[i,j,k]=0 then else print(`Cartan_LEFT`(i,j,k)=DD[i,j,k]) fi od od od ;
```

$$Cartan\_LEFT(1, 2, 1) = \frac{z x}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}$$

$$Cartan\_LEFT(1, 2, 2) = \frac{z y}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}$$

$$Cartan\_LEFT(1, 2, 3) = -\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$Cartan\_LEFT(1, 3, 1) = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$Cartan\_LEFT(1, 3, 2) = -\frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$Cartan\_LEFT(2, 1, 1) = -\frac{z x}{(x^2+y^2+z^2)^{(3/2)}\sqrt{x^2+y^2}}$$

$$\text{Cartan\_LEFT}(2, 1, 2) = -\frac{z\, y}{(x^2+y^2+z^2)^{(3/2)} \sqrt{x^2+y^2}}$$

$$\text{Cartan\_LEFT}(2, 1, 3) = \frac{\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(2, 2, 1) = \frac{x}{x^2+y^2+z^2}$$

$$\text{Cartan\_LEFT}(2, 2, 2) = \frac{y}{x^2+y^2+z^2}$$

$$\text{Cartan\_LEFT}(2, 2, 3) = \frac{z}{x^2+y^2+z^2}$$

$$\text{Cartan\_LEFT}(2, 3, 1) = -\frac{z\, y}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}$$

$$\text{Cartan\_LEFT}(2, 3, 2) = \frac{z\, x}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}$$

$$\text{Cartan\_LEFT}(3, 1, 1) = -\frac{y}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)}$$

$$\text{Cartan\_LEFT}(3, 1, 2) = \frac{x}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)}$$

$$\text{Cartan\_LEFT}(3, 2, 1) = \frac{z\, y}{(x^2+y^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(3, 2, 2) = -\frac{z\, x}{(x^2+y^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(3, 3, 1) = \frac{x}{x^2+y^2}$$

$$\text{Cartan\_LEFT}(3, 3, 2) = \frac{y}{x^2+y^2}$$

The anti-symmetric part of the LEFT CARTAN Connection appear above.
Check for assymetry (LEFT Torsion)

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (DD[i,j,k]-DD[i,k,j])/2; TTS[i,j,k]:=simplify(ss) od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  TTS[i,j,k]=0 then else print(`LEFT_Torsion`(i,k,j)=TTS[i,k,j]) fi od od od ;
```

$$\text{LEFT\_Torsion}(1, 2, 1) = \frac{1}{2}\frac{z\, x}{\sqrt{x^2+y^2+z^2}\,\sqrt{x^2+y^2}}$$

$$\text{LEFT\_Torsion}(1, 3, 1) = \frac{1}{2}\frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\text{LEFT\_Torsion}(1, 1, 2) = -\frac{1}{2}\frac{z\, x}{\sqrt{x^2+y^2+z^2}\,\sqrt{x^2+y^2}}$$

$$\mathrm{LEFT\_Torsion}(1,3,2) = -\frac{1}{2}\frac{-\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2+z^2}}$$

$$\mathrm{LEFT\_Torsion}(1,1,3) = -\frac{1}{2}\frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\mathrm{LEFT\_Torsion}(1,2,3) = \frac{1}{2}\frac{-\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2+z^2}}$$

$$\mathrm{LEFT\_Torsion}(2,2,1) = \frac{1}{2}\frac{x\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}+z\,y}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^{(3/2)}}$$

$$\mathrm{LEFT\_Torsion}(2,3,1) = -\frac{1}{2}\frac{x^2+y^2+z\,y\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^{(3/2)}}$$

$$\mathrm{LEFT\_Torsion}(2,1,2) = -\frac{1}{2}\frac{x\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}+z\,y}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^{(3/2)}}$$

$$\mathrm{LEFT\_Torsion}(2,3,2) = \frac{1}{2}\frac{z\,(-\sqrt{x^2+y^2}+x)}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}$$

$$\mathrm{LEFT\_Torsion}(2,1,3) = \frac{1}{2}\frac{x^2+y^2+z\,y\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)^{(3/2)}}$$

$$\mathrm{LEFT\_Torsion}(2,2,3) = -\frac{1}{2}\frac{z\,(-\sqrt{x^2+y^2}+x)}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}$$

$$\mathrm{LEFT\_Torsion}(3,2,1) = \frac{1}{2}\frac{-x\sqrt{x^2+y^2}+z\,y\sqrt{x^2+y^2+z^2}}{(x^2+y^2)^{(3/2)}\sqrt{x^2+y^2+z^2}}$$

$$\mathrm{LEFT\_Torsion}(3,3,1) = \frac{1}{2}\frac{x}{x^2+y^2}$$

$$\mathrm{LEFT\_Torsion}(3,1,2) = \frac{1}{2}\frac{x\sqrt{x^2+y^2}-z\,y\sqrt{x^2+y^2+z^2}}{(x^2+y^2)^{(3/2)}\sqrt{x^2+y^2+z^2}}$$

$$\mathrm{LEFT\_Torsion}(3,3,2) = \frac{1}{2}\frac{y}{x^2+y^2}$$

$$\mathrm{LEFT\_Torsion}(3,1,3) = -\frac{1}{2}\frac{x}{x^2+y^2}$$

$$\mathrm{LEFT\_Torsion}(3,2,3) = -\frac{1}{2}\frac{y}{x^2+y^2}$$

For this example from cartesian to spherical coordinates, there is no assymetry for the [CR], but there is assymetry for [CL]
(The physical implication is not clear to me)

Next the Christoffel symbols will be computed for the metric on the initial state.
As the metric on {x,y,z} is presumed to be the unit matrix, all the Christoffel symbols should be zero

Christoffel Connection coefficients from the induced metric

```
>
> metric:= array([[1,0,0],[0,1,0],[0,0,1]]);
```

$$metric := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k]
  := 0  od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to
  dim do C1S[i,j,k] := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od
  od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0;
  for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then  else print(`Christoffel_Gamma2`(i-1,j-1,k-1)=C2S[i,j,k]) fi
  od od od;
```

The non zero Christoffel Connection coefficients 2nd kind   on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

If no entries appear above the Christoffel symbols on the domain space vanish
The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) = ChristoffelGamma(ijk) + T(ijk)

Compute the T(i,j,k):

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else print(`T`(i,j,k)=simplify(SHIPTR[i,j,k]))
  fi od od od ;
```

T(ijk) index (1,-1,-1)

$$T(1,1,1) = -\frac{(x^4 y^2 + z^2 x^4 + 2 x^2 y^4 + 3 z^2 x^2 y^2 + 2 z^2 y^4 + y^2 z^4 + y^6)\, x}{(x^2+y^2)^2 (x^2+y^2+z^2)^2}$$

$$T(1,1,2) = -\frac{y\,(x^4 y^2 + z^2 x^4 + 2 x^2 y^4 + 3 z^2 x^2 y^2 + 2 z^2 y^4 + y^2 z^4 + y^6)}{(x^2+y^2)^2 (x^2+y^2+z^2)^2}$$

$$T(1,1,3) = -\frac{z^3 x^2}{(x^2+y^2)(x^2+y^2+z^2)^2}$$

$$T(1,2,1) = -\frac{y\,(x^4 y^2 + z^2 x^4 + 2 x^2 y^4 + 3 z^2 x^2 y^2 + 2 z^2 y^4 + y^2 z^4 + y^6)}{(x^2+y^2)^2 (x^2+y^2+z^2)^2}$$

$$T(1,2,2) = \frac{x\,(4 x^4 y^2 + 5 x^2 y^4 + 2 y^6 + 5 z^2 x^2 y^2 + 3 z^2 y^4 + 2 y^2 z^4 + x^6 + 2 z^2 x^4 + x^2 z^4)}{(x^2+y^2)^2 (x^2+y^2+z^2)^2}$$

$$T(1,2,3) = -\frac{z^3 x y}{(x^2+y^2+z^2)^2 (x^2+y^2)}$$

$$T(1,3,1) = -\frac{z^3 x^2}{(x^2+y^2)(x^2+y^2+z^2)^2}$$

$$T(1,3,2) = -\frac{z^3 x y}{(x^2+y^2+z^2)^2 (x^2+y^2)}$$

$$T(1,3,3) = \frac{x\,(2 z^2 + x^2 + y^2)}{(x^2+y^2+z^2)^2}$$

$$T(2,1,1) = \frac{y\,(2 x^6 + 5 x^4 y^2 + 4 x^2 y^4 + 3 z^2 x^4 + 5 z^2 x^2 y^2 + 2 x^2 z^4 + y^6 + 2 z^2 y^4 + y^2 z^4)}{(x^2+y^2)^2 (x^2+y^2+z^2)^2}$$

$$T(2,1,2) = -\frac{(x^6 + 2 x^4 y^2 + 2 z^2 x^4 + 3 z^2 x^2 y^2 + x^2 y^4 + x^2 z^4 + z^2 y^4)\, x}{(x^2+y^2)^2 (x^2+y^2+z^2)^2}$$

$$T(2,1,3) = -\frac{z^3 x y}{(x^2+y^2+z^2)^2 (x^2+y^2)}$$

$$T(2,2,1) = -\frac{(x^6 + 2 x^4 y^2 + 2 z^2 x^4 + 3 z^2 x^2 y^2 + x^2 y^4 + x^2 z^4 + z^2 y^4)\, x}{(x^2+y^2)^2 (x^2+y^2+z^2)^2}$$

$$T(2,2,2) = -\frac{(x^6 + 2 x^4 y^2 + 2 z^2 x^4 + 3 z^2 x^2 y^2 + x^2 y^4 + x^2 z^4 + z^2 y^4)\, y}{(x^2+y^2)^2 (x^2+y^2+z^2)^2}$$

$$T(2,2,3) = -\frac{z^3 y^2}{(x^2+y^2+z^2)^2 (x^2+y^2)}$$

$$T(2,3,1) = -\frac{z^3 x y}{(x^2+y^2+z^2)^2 (x^2+y^2)}$$

$$T(2,3,2) = -\frac{z^3 y^2}{(x^2+y^2+z^2)^2 (x^2+y^2)}$$

$$T(2,3,3) = \frac{y\,(2 z^2 + x^2 + y^2)}{(x^2+y^2+z^2)^2}$$

$$T(3,1,1) = \frac{x^2 z (2x^2 + 2y^2 + z^2)}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$T(3,1,2) = \frac{z x y (2x^2 + 2y^2 + z^2)}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$T(3,1,3) = -\frac{(x^2 + y^2) x}{(x^2 + y^2 + z^2)^2}$$

$$T(3,2,1) = \frac{z x y (2x^2 + 2y^2 + z^2)}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$T(3,2,2) = \frac{z y^2 (2x^2 + 2y^2 + z^2)}{(x^2 + y^2 + z^2)^2 (x^2 + y^2)}$$

$$T(3,2,3) = -\frac{(x^2 + y^2) y}{(x^2 + y^2 + z^2)^2}$$

$$T(3,3,1) = -\frac{(x^2 + y^2) x}{(x^2 + y^2 + z^2)^2}$$

$$T(3,3,2) = -\frac{(x^2 + y^2) y}{(x^2 + y^2 + z^2)^2}$$

$$T(3,3,3) = -\frac{(x^2 + y^2) z}{(x^2 + y^2 + z^2)^2}$$

Right Cartan(ijk) = Christoffel Gamma(ijk) + T(ijk)

These computations agree with Shipov on page 217, except for the T(3,3,2), T(3,3,1) and the T(2,2,1) terms given above.

****

```
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=0,A
  z=0,C=0,Phi=0,phi=0,theta=0,r=0,a=const,b=const,c=const,Lx=0,Ly=0,Lz=0);
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

Part 2: The map is from {r,theta,phi} into {x,y,z}

```
> x:=r*sin(theta)*cos(phi);y:=r*sin(theta)*sin(phi);z:=r*cos(theta);
>
```

$$x := r \sin(\theta) \cos(\phi)$$

$$y := r \sin(\theta) \sin(\phi)$$

$$z := r \cos(\theta)$$

```
> R:=[x,y,z];FF:=jacobian(R,[r,theta,phi]);DR:=d(R):`dx`:=DR[1];`dy`:=DR[2];`dz`:=
  DR[3];
```

$$R := [r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)]$$

$$FF := \begin{bmatrix} \sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{bmatrix}$$

$$dx := \sin(\theta)\cos(\phi)\,d(r) + r\cos(\phi)\cos(\theta)\,d(\theta) - r\sin(\theta)\sin(\phi)\,d(\phi)$$

$$dy := \sin(\theta)\sin(\phi)\,d(r) + r\sin(\phi)\cos(\theta)\,d(\theta) + r\sin(\theta)\cos(\phi)\,d(\phi)$$

$$dz := \cos(\theta)\,d(r) - r\sin(\theta)\,d(\theta)$$

```
> GG:=evalm(inverse(FF)):DETFF:=simplify(det(FF));
```

$$DETFF := \sin(\theta)\,r^2$$

Note that the Frame matrix has a singularity at values of theta equal to multiples of pi, and at r=0. The next equation checks to see that the specified frame produces the desired differential structures: [FF]|dR>

```
> zz:=simplify(evalm(innerprod(FF,[d(r),d(theta),d(phi)])));zzb:=innerprod(GG,zz);
```

$$zz := [\sin(\theta)\cos(\phi)\,d(r) + r\cos(\phi)\cos(\theta)\,d(\theta) - r\sin(\theta)\sin(\phi)\,d(\phi),$$
$$\sin(\theta)\sin(\phi)\,d(r) + r\sin(\phi)\cos(\theta)\,d(\theta) + r\sin(\theta)\cos(\phi)\,d(\phi), \cos(\theta)\,d(r) - r\sin(\theta)\,d(\theta)]$$

$$zzb := [d(r), d(\theta), d(\phi)]$$

Note that each component is exact (by construction)

```
> d(zz[1]);d(zz[2]);d(zz[3]);
```

$$0$$
$$0$$
$$0$$

The metric on the target xyz is the unit matrix of constants, by assumption. As such the Christoffel symbols will be zero. The induced pulled back metric on the spherical coordinate range is

```
> inducedmetric:=simplify(innerprod(transpose(FF),FF));
```

$$inducedmetric := \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 - r^2\cos(\theta)^2 \end{bmatrix}$$

Now check to see if the Frame matrix is normal and find the Left and Right representations P2R=[FF][transpose FF]
P2L= [transpose FF] [F]. Conclusion: the Frame matrix is not normal!!! Note that P2R is the pushedforward inverse metric on {r,theta,phi} but with arguments on {x,y,z}

```
> P2R:=simplify(innerprod(FF,transpose(FF)));P2L:=simplify(innerprod(transpose(FF)
  ,FF));
```

$P2R :=$

$$[\cos(\phi)^2 - \cos(\theta)^2\cos(\phi)^2 + 2r^2\cos(\theta)^2\cos(\phi)^2 + r^2 - \cos(\phi)^2 r^2 - r^2\cos(\theta)^2,$$
$$\cos(\phi)\sin(\phi) - \cos(\phi)\sin(\phi)\cos(\theta)^2 + 2r^2\cos(\theta)^2\cos(\phi)\sin(\phi) - r^2\sin(\phi)\cos(\phi),$$
$$\sin(\theta)\cos(\phi)\cos(\theta) - r^2\cos(\theta)\cos(\phi)\sin(\theta)]$$
$$[\cos(\phi)\sin(\phi) - \cos(\phi)\sin(\phi)\cos(\theta)^2 + 2r^2\cos(\theta)^2\cos(\phi)\sin(\phi) - r^2\sin(\phi)\cos(\phi),$$
$$1 - \cos(\phi)^2 - \cos(\theta)^2 + \cos(\theta)^2\cos(\phi)^2 + r^2\cos(\theta)^2 - 2r^2\cos(\theta)^2\cos(\phi)^2 + \cos(\phi)^2 r^2,$$
$$\sin(\theta)\sin(\phi)\cos(\theta) - r^2\cos(\theta)\sin(\phi)\sin(\theta)]$$
$$[\sin(\theta)\cos(\phi)\cos(\theta) - r^2\cos(\theta)\cos(\phi)\sin(\theta), \sin(\theta)\sin(\phi)\cos(\theta) - r^2\cos(\theta)\sin(\phi)\sin(\theta),$$
$$\cos(\theta)^2 + r^2 - r^2\cos(\theta)^2]$$

$$P2L := \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 - r^2\cos(\theta)^2 \end{bmatrix}$$

P2R is complicated algebracially. But the bottom line is that it is NOT equal to P2L, hence the Jacobian matrix is NOT NORMAL

Now any matrix with an inverse can be composed as a product of a symmetric matrix and an orthogonal matrix.

There are in general two ways to construct this representation, which will be denoted as the Lefthanded and the Right handed formulations. [SR] is the symmetric matrix of the "right handed" formulation, and [OR] is the orthogonal matrix for the right handed formulation. (If the matrices are complex, the notions symmetric and orthogonal translate to Hermitean and Unitary)

The two formats are:

[F]= [SR][OR]=[OL] [SL]

If [G][F]=1, then

[SR]={[FF].transpose[FF]}^(1/2)

and

[OR]=[SR].transpose[G]

The Left handed representations

[SL]={transpose[FF].[FF]}^(1/2)

and

[OL]=transpose[G].[SL]

The representations are distinct if the Frame [FF] is not a normal matrix.

**

So the first step is to find the square roots of these matrices P2R and P2L above.

This is easy to do for the Right Handed P2R, for it is diagonal.

The symmetric component of the Left handed representation: F = [OL] [SL]

```
> SL:=array([[1,0,0],[0,r,0],[0,0,(r*sin(theta))]]);
```

$$SL := \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin(\theta) \end{bmatrix}$$

Compute the orthogonal factor [OL] of the Left handed representation.

```
> GS:=simplify(transpose(inverse(FF))):OL:=innerprod(GS,SL);simplify(innerprod(transpose(OL),OL)):`Should_be_zero`:=evalm(innerprod(OL,SL)-FF):
```

$$OL := \begin{bmatrix} \sin(\theta)\cos(\phi) & \cos(\phi)\cos(\theta) & -\sin(\phi) \\ \sin(\theta)\sin(\phi) & \sin(\phi)\cos(\theta) & \cos(\phi) \\ \cos(\theta) & -\sin(\theta) & 0 \end{bmatrix}$$

Find Omega, the right Cartan matrix of [OL], which should be an anti-symmetric matrix and Delta the left Cartan matrix for [SL]

```
> dOL:=d(OL);
```

$$dOL := \begin{bmatrix} \cos(\phi)\cos(\theta)\,d(\theta) - \sin(\theta)\sin(\phi)\,d(\phi) & -\cos(\theta)\sin(\phi)\,d(\phi) - \cos(\phi)\sin(\theta)\,d(\theta) & -\cos(\phi)\,d(\phi) \\ \sin(\phi)\cos(\theta)\,d(\theta) + \sin(\theta)\cos(\phi)\,d(\phi) & \cos(\theta)\cos(\phi)\,d(\phi) - \sin(\phi)\sin(\theta)\,d(\theta) & -\sin(\phi)\,d(\phi) \\ -\sin(\theta)\,d(\theta) & -\cos(\theta)\,d(\theta) & 0 \end{bmatrix}$$

```
> Omega_R:=simplify(innerprod(transpose(OL),dOL));Delta_R:=innerprod(d(SL),inverse(SL));
```

$$Omega\_R := \begin{bmatrix} 0 & -d(\theta) & -\sin(\theta)\,d(\phi) \\ d(\theta) & 0 & -\cos(\theta)\,d(\phi) \\ \sin(\theta)\,d(\phi) & \cos(\theta)\,d(\phi) & 0 \end{bmatrix}$$

$$Delta\_R := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{d(r)}{r} & 0 \\ 0 & 0 & \frac{\sin(\theta)\, d(r) + r\cos(\theta)\, d(\theta)}{r\sin(\theta)} \end{bmatrix}$$

```
>
> Omega_L:=simplify(innerprod(dOL,transpose(OL)));Delta_L:=innerprod(d(SL),inverse(SL));
```

$$Omega\_L := \begin{bmatrix} 0 & -d(\phi) & d(\theta)\cos(\phi) \\ d(\phi) & 0 & d(\theta)\sin(\phi) \\ -d(\theta)\cos(\phi) & -d(\theta)\sin(\phi) & 0 \end{bmatrix}$$

$$Delta\_L := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{d(r)}{r} & 0 \\ 0 & 0 & \frac{\sin(\theta)\, d(r) + r\cos(\theta)\, d(\theta)}{r\sin(\theta)} \end{bmatrix}$$

Note that the matrix elements of Delta are perfect exact differentials, and the matrix elements of Omega are closed but not exact differentials.

It is possible to write the differential of the Frame field in several ways:

d[FF] = [FF][CR] ={[DeltaR][FF] + [FF][OmegaR]} = [CL][FF] = {[OmegaL][FF]+[FF][DeltaL]}

Note that appropriate linear combinations can be constructed.

*

Now Compute the Right Cartan Matrix [CR]

```
> cartan:=simplify(innerprod(inverse(FF),d(FF))):
```

The matrix elements of the Right Cartan connection matrix using the matrix methods:

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);
```

$$\Gamma 11 := 0$$

$$\Gamma 12 := -r\, d(\theta)$$

$$\Gamma 13 := (-r + r\cos(\theta)^2)\, d(\phi)$$

```
> Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(cartan[2,3]);
```

$$\Gamma 21 := \frac{d(\theta)}{r}$$

$$\Gamma 22 := \frac{d(r)}{r}$$

$$\Gamma 23 := -\cos(\theta)\, d(\phi)\sin(\theta)$$

```
> Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(cartan[3,3]);
```

$$\Gamma 31 := \frac{d(\phi)}{r}$$

$$\Gamma 32 := \frac{\cos(\theta)\, d(\phi)}{\sin(\theta)}$$

$$\Gamma 33 := \frac{\cos(\theta)\,d(\theta)}{\sin(\theta)} + \frac{d(r)}{r}$$

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=3;coord:=[r,theta,phi];GG:=simplify(inverse(FF));
```

$$dim := 3$$

$$coord := [r, \theta, \phi]$$

$$GG := \begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ \frac{\cos(\phi)\cos(\theta)}{r} & \frac{\sin(\phi)\cos(\theta)}{r} & -\frac{\sin(\theta)}{r} \\ -\frac{\sin(\phi)}{r\sin(\theta)} & \frac{\cos(\phi)}{r\sin(\theta)} & 0 \end{bmatrix}$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k]
  := (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for
  m from 1 to dim do ss := ss+(d1GG[a,m,k]*FF[m,b]); CC[a,b,k]:=simplify(-ss) od
  od od od ;
>
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if
  CC[a,b,k]=0 then else print(`CCabk`(a,b,k)=factor(CC[a,b,k])) fi od od od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients.

CC(abk) index (1,-1,-1)

$$\mathrm{CCabk}(2, 1, 2) = \frac{1}{r}$$

$$\mathrm{CCabk}(3, 1, 3) = \frac{1}{r}$$

$$\mathrm{CCabk}(1, 2, 2) = -r$$

$$\mathrm{CCabk}(2, 2, 1) = \frac{1}{r}$$

$$\mathrm{CCabk}(3, 2, 3) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\mathrm{CCabk}(1, 3, 3) = r\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)$$

$$\mathrm{CCabk}(2, 3, 3) = -\cos(\theta)\sin(\theta)$$

$$\mathrm{CCabk}(3, 3, 1) = \frac{1}{r}$$

$$\mathrm{CCabk}(3, 3, 2) = \frac{\cos(\theta)}{\sin(\theta)}$$

These results agree with matrix method.

Next check for Affine Torsion using the tensor methods:

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (CC[i,j,k]-CC[i,k,j])/2; CCTTS[i,j,k]:=ss od od od  ;
>
```

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  CCTTS[i,j,k]=0 then else print(`RIGHT_AffineTorsion`(i,k,j)=CCTTS[i,k,j]) fi od
  od od ;
```

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

```
>
```

Now compute the CARTAN LEFT CONNECTION

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[a,j,k]
  := simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrix product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do  ss:=0;for
  m to dim do ss := ss+FF[i,m]*(d1GG[m,j,k]); DD[i,j,k]:=simplify(ss) od od od od
  ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  DD[i,j,k]=0 then else print(`Cartan_LEFT`(i,j,k)=DD[i,j,k]) fi od od od ;
```

$$\text{Cartan\_LEFT}(1, 1, 1) = -\frac{\cos(\theta)^2 \cos(\phi)^2 + 1 - \cos(\phi)^2}{r}$$

$$\text{Cartan\_LEFT}(1, 1, 2) = -\frac{\sin(\theta)\,(-1 + \cos(\phi)^2)\cos(\theta)}{-1 + \cos(\theta)^2}$$

$$\text{Cartan\_LEFT}(1, 2, 1) = -\frac{\cos(\phi)\sin(\phi)\,(-1 + \cos(\theta)^2)}{r}$$

$$\text{Cartan\_LEFT}(1, 2, 2) = -\frac{\sin(\theta)\sin(\phi)\cos(\phi)\cos(\theta)}{-1 + \cos(\theta)^2}$$

$$\text{Cartan\_LEFT}(1, 2, 3) = 1$$

$$\text{Cartan\_LEFT}(1, 3, 1) = \frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$\text{Cartan\_LEFT}(1, 3, 2) = -\cos(\phi)$$

$$\text{Cartan\_LEFT}(2, 1, 1) = -\frac{\cos(\phi)\sin(\phi)\,(-1 + \cos(\theta)^2)}{r}$$

$$\text{Cartan\_LEFT}(2, 1, 2) = -\frac{\sin(\theta)\sin(\phi)\cos(\phi)\cos(\theta)}{-1 + \cos(\theta)^2}$$

$$\text{Cartan\_LEFT}(2, 1, 3) = -1$$

$$\text{Cartan\_LEFT}(2, 2, 1) = \frac{-\cos(\theta)^2 + \cos(\theta)^2 \cos(\phi)^2 - \cos(\phi)^2}{r}$$

$$\text{Cartan\_LEFT}(2, 2, 2) = \frac{\sin(\theta)\cos(\phi)^2 \cos(\theta)}{-1 + \cos(\theta)^2}$$

$$\text{Cartan\_LEFT}(2, 3, 1) = \frac{\cos(\theta)\sin(\phi)\sin(\theta)}{r}$$

$$\text{Cartan\_LEFT}(2, 3, 2) = -\sin(\phi)$$

$$\text{Cartan\_LEFT}(3, 1, 1) = \frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$\text{Cartan\_LEFT}(3, 1, 2) = \cos(\phi)$$

$$Cartan\_LEFT(3, 2, 1) = \frac{\cos(\theta)\sin(\phi)\sin(\theta)}{r}$$

$$Cartan\_LEFT(3, 2, 2) = \sin(\phi)$$

$$Cartan\_LEFT(3, 3, 1) = \frac{-1 + \cos(\theta)^2}{r}$$

The anti-symmetric part of the LEFT CARTAN Connection appear above.
Check for assymetry (LEFT Torsion)

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (DD[i,j,k]-DD[i,k,j])/2; TTS[i,j,k]:=simplify(ss) od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  TTS[i,j,k]=0 then else print(`LEFT_Torsion`(i,k,j)=TTS[i,k,j]) fi od od od ;
```

$$LEFT\_Torsion(1, 2, 1) = \frac{1}{2}\,\frac{-\sin(\theta)\cos(\theta)\,r + \sin(\theta)\cos(\theta)\,r\cos(\phi)^2 - \cos(\phi)\sin(\phi) + 2\cos(\phi)\sin(\phi)\cos(\theta)^2 - \cos(\phi)\sin(\phi)\cos(\theta)^4}{r\,(-1 + \cos(\theta)^2)}$$

$$LEFT\_Torsion(1, 3, 1) = \frac{1}{2}\,\frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$LEFT\_Torsion(1, 1, 2) = -\frac{1}{2}\,\frac{-\sin(\theta)\cos(\theta)\,r + \sin(\theta)\cos(\theta)\,r\cos(\phi)^2 - \cos(\phi)\sin(\phi) + 2\cos(\phi)\sin(\phi)\cos(\theta)^2 - \cos(\phi)\sin(\phi)\cos(\theta)^4}{r\,(-1 + \cos(\theta)^2)}$$

$$LEFT\_Torsion(1, 3, 2) = -\frac{1}{2}\cos(\phi) - \frac{1}{2}$$

$$LEFT\_Torsion(1, 1, 3) = -\frac{1}{2}\,\frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$LEFT\_Torsion(1, 2, 3) = \frac{1}{2} + \frac{1}{2}\cos(\phi)$$

$$LEFT\_Torsion(2, 2, 1) = \frac{1}{2}\,\frac{\sin(\theta)\cos(\phi)\sin(\phi)\cos(\theta)\,r + \cos(\theta)^2 - \cos(\theta)^4 - 2\cos(\theta)^2\cos(\phi)^2 + \cos(\theta)^4\cos(\phi)^2 + \cos(\phi)^2}{r\,(-1 + \cos(\theta)^2)}$$

$$LEFT\_Torsion(2, 3, 1) = \frac{1}{2}\,\frac{r + \sin(\theta)\sin(\phi)\cos(\theta)}{r}$$

$$LEFT\_Torsion(2, 1, 2) = -\frac{1}{2}\,\frac{\sin(\theta)\cos(\phi)\sin(\phi)\cos(\theta)\,r + \cos(\theta)^2 - \cos(\theta)^4 - 2\cos(\theta)^2\cos(\phi)^2 + \cos(\theta)^4\cos(\phi)^2 + \cos(\phi)^2}{r\,(-1 + \cos(\theta)^2)}$$

$$LEFT\_Torsion(2, 3, 2) = -\frac{1}{2}\sin(\phi)$$

$$LEFT\_Torsion(2, 1, 3) = -\frac{1}{2}\,\frac{r + \sin(\theta)\sin(\phi)\cos(\theta)}{r}$$

$$LEFT\_Torsion(2, 2, 3) = \frac{1}{2}\sin(\phi)$$

$$\mathrm{LEFT\_Torsion}(3,2,1) = -\frac{1}{2}\,\frac{\cos(\phi)\,r - \sin(\theta)\sin(\phi)\cos(\theta)}{r}$$

$$\mathrm{LEFT\_Torsion}(3,3,1) = \frac{1}{2}\,\frac{-1+\cos(\theta)^2}{r}$$

$$\mathrm{LEFT\_Torsion}(3,1,2) = \frac{1}{2}\,\frac{\cos(\phi)\,r - \sin(\theta)\sin(\phi)\cos(\theta)}{r}$$

$$\mathrm{LEFT\_Torsion}(3,1,3) = -\frac{1}{2}\,\frac{-1+\cos(\theta)^2}{r}$$

For this example from cartesian to spherical coordinates, there is no assymetry for the [CR], but there is assymetry for [CL]
(The physical implication is not clear to me)

Next the Christoffel symbols will be computed for the metric on the initial state.
As the metric on {x,y,z} is presumed to be the unit matrix, all the Christoffel symbols should be zero

Christoffel Connection coefficients from the induced metric

```
>
> metric:= evalm(inducedmetric);
```

$$metric := \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 - r^2\cos(\theta)^2 \end{bmatrix}$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k]
  := 0  od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to
  dim do C1S[i,j,k] := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od
  od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0;
  for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then  else print(`Christoffel_Gamma2`(i-1,j-1,k-1)=C2S[i,j,k]) fi
  od od od;
```

The non zero Christoffel Connection coefficients 2nd kind   on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

$$\mathrm{Christoffel\_Gamma2}(0,1,1) = -r$$

$$\text{Christoffel\_Gamma2}(0,2,2) = -r + r\cos(\theta)^2$$

$$\text{Christoffel\_Gamma2}(1,0,1) = \frac{1}{r}$$

$$\text{Christoffel\_Gamma2}(1,1,0) = \frac{1}{r}$$

$$\text{Christoffel\_Gamma2}(1,2,2) = -\cos(\theta)\sin(\theta)$$

$$\text{Christoffel\_Gamma2}(2,0,2) = \frac{1}{r}$$

$$\text{Christoffel\_Gamma2}(2,1,2) = -\frac{\cos(\theta)\sin(\theta)}{-1+\cos(\theta)^2}$$

$$\text{Christoffel\_Gamma2}(2,2,0) = \frac{1}{r}$$

$$\text{Christoffel\_Gamma2}(2,2,1) = -\frac{\cos(\theta)\sin(\theta)}{-1+\cos(\theta)^2}$$

If no entries appear above the Christoffel symbols on the domain space vanish

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) = ChristoffelGamma(ijk) + T(ijk)

Compute the T(i,j,k):

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss
  := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else print(`T`(i,j,k)=simplify(SHIPTR[i,j,k]))
  fi od od od ;
```

T(ijk) index (1,-1,-1)

$$T(1,2,2) = 0$$
$$T(1,3,3) = 0$$
$$T(2,1,2) = 0$$
$$T(2,2,1) = 0$$
$$T(2,3,3) = 0$$
$$T(3,1,3) = 0$$
$$T(3,2,3) = 0$$
$$T(3,3,1) = 0$$
$$T(3,3,2) = 0$$

Right Cartan(ijk) = Gamma(ijk) + T(ijk)

In this example the rotation coefficients on the domain space vanish for the Cartan right matrix is exactly equal to the the Christoffel symbols.

Note the differences between the map from cartesian space and the map to cartesian space.

>

>