```
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=0,Az=0,C=0,Phi=0,phi=0,theta=0,r=0,a=const,b=const,c=const,Lx=0,Ly=0,Lz=0);
Warning, the protected names norm and trace have been redefined and
unprotected

Warning, the protected name close has been redefined and unprotected

Warning, the names &^, d and wdegree have been redefined
```

Spheres in 3D space and Cartan Connections

R. M. Kiehn
last update 04/20/2000, May 1, 2001, June 25,2001,Ju;y 29,2002
rkiehn2352@aol.com

Introduction:

The Cartan connection coefficients will be computed for both the map from spherical coordinates into cartesian coordinates (part1) and the map from cartesian coordinates into spherical coordinates, (Part2)

Part 1: The map is from an initial "spherical state" {r,theta,phi} to a final "Cartesian state" {x,y,z} .

```
> x:=r*cos(theta)*sin(phi);y:=r*sin(theta)*sin(phi);z:=r*cos(phi);
> `dx`:=d(x);`dy`:=d(y);`dz`:=d(z);
```

$$x := r\cos(\theta)\sin(\phi)$$

$$y := r\sin(\theta)\sin(\phi)$$

$$z := r\cos(\phi)$$

$$dx := \cos(\theta)\sin(\phi)\,\mathrm{d}(r) - r\sin(\phi)\sin(\theta)\,\mathrm{d}(\theta) + r\cos(\theta)\cos(\phi)\,\mathrm{d}(\phi)$$

$$dy := \sin(\theta)\sin(\phi)\,\mathrm{d}(r) + r\sin(\phi)\cos(\theta)\,\mathrm{d}(\theta) + r\sin(\theta)\cos(\phi)\,\mathrm{d}(\phi)$$

$$dz := \cos(\phi)\,\mathrm{d}(r) - r\sin(\phi)\,\mathrm{d}(\phi)$$

The linear map from the differentials of the spherical coordinate domain (initial state) to the differentials of the cartesian coordinate range (final state) is the Jacobian matrix of co-variant row vectors. Each row is a "gradient" of the given mapping functions. Each component of the Jacobian matrix is a function of the cylindrical coordinates, NOT the cartesian coordinates. The columns of the Jacobian matrix can be interpreted as a set of contravariant vector columns. When the Jacobian matrix has an inverse, the matrix of columns can serve as a basis FRAME [FF] for the space, with inverse matrix [GG]. The differentials of the column basis vectors of the Jacobian FRAME matrix are linear combinations of the column basis vectors. The differentials do not produce vectors outside of the basis set. This concept is called differential closure. The linear mapping of differentials is defined as a connection (the right Cartan connection of 1-forms). The elements of the connection are functions of variables and their differentials (spherical coordinates in this case) on the initial state..
d[F] = [F] [C (right)]
NOte that the FRAME is on the target range, but the arguments of the Frame functions are on the initial domain.

**

When the Frame [FF] is constructed from the JAcobian matrix of the mapping to the Euclidean space, then the Cartan right connection coefficients are exactly equal to the Christoffel coefficients computed from the induced (pullback) metric on the initial state.

```
> R:=[x,y,z]:`Position Vector`:=R;FF:=jacobian(R,[r,theta,phi]):`Cartan Frame`:=evalm(FF);GG:=simplify(inverse(FF)):`Inverse Frame`:=evalm(GG);DETFF:=simplify(det(FF));
```

$$\mathit{Position\ Vector} := [r\cos(\theta)\sin(\phi),\ r\sin(\theta)\sin(\phi),\ r\cos(\phi)]$$

$$\mathit{Cartan\ Frame} := \begin{bmatrix} \cos(\theta)\sin(\phi) & -r\sin(\theta)\sin(\phi) & r\cos(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -r\sin(\phi) \end{bmatrix}$$

$$\mathit{Inverse\ Frame} := \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ -\dfrac{\sin(\theta)}{r\sin(\phi)} & \dfrac{\cos(\theta)}{r\sin(\phi)} & 0 \\ \dfrac{\cos(\theta)\cos(\phi)}{r} & \dfrac{\sin(\theta)\cos(\phi)}{r} & -\dfrac{\sin(\phi)}{r} \end{bmatrix}$$

$$\mathit{DETFF} := -\sin(\phi)\, r^2$$

NOte that the Frame has a zero determinant at r= 0, and at the poles.
It is possible to rescale the columns of the Frame matrix to produce a Frame with a constant determinant value (such as one or zero).
IF the scaling is such that det = 0, then the Frame is not a basis for the 3D space.

The next equation checks to see that the specified frame produces the desired differential structures: "a contravariant column of components" = |sigmaF> =[FF]|d(r),d(theta),d(zz)>

```
> sigmaF:=innerprod(FF,[d(r),d(theta),d(phi)]);
>
```

$$\mathit{sigmaF} := [\cos(\theta)\sin(\phi)\,d(r) - r\sin(\phi)\sin(\theta)\,d(\theta) + r\cos(\theta)\cos(\phi)\,d(\phi),$$
$$\sin(\theta)\sin(\phi)\,d(r) + r\sin(\phi)\cos(\theta)\,d(\theta) + r\sin(\theta)\cos(\phi)\,d(\phi),\ \cos(\phi)\,d(r) - r\sin(\phi)\,d(\phi)]$$

The resulting 1-forms on the final state as expressed in terms of the variables of the initial state are not exact.
There are no singularities for sigmaF,
BUT each 1-form is closed:

```
> d(sigmaF);
```

$$[0, 0, 0]$$

It is also true that the "vector" of 2-forms, d|sigmaF> - [C (right)] |sigmaF> =0.

The metric on the target xyz is the unit matrix of constants, by assumption.

```
>
> metricf := matrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]]);
```

$$\mathit{metricf} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The induced pulled back metric on the cylindrical coordinate initial state is:

```
> pullbackmetric:=simplify((innerprod(transpose(FF),metricf,FF)));
```

$$\mathit{pullbackmetric} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -r^2(-1+\cos(\phi)^2) & 0 \\ 0 & 0 & r^2 \end{bmatrix}$$

```
>
```

Now Compute the Right Cartan Connection Matrix [C (right)]

Computation by matrix method:

```
> cartan:=simplify(innerprod(inverse(FF),d(FF)));
```

$$cartan := \begin{bmatrix} 0 & d(\theta)\, r\,(-1+\cos(\phi)^2) & -r\, d(\phi) \\ \frac{d(\theta)}{r} & \frac{\sin(\phi)\, d(r) + r\cos(\phi)\, d(\phi)}{r\sin(\phi)} & \frac{\cos(\phi)\, d(\theta)}{\sin(\phi)} \\ \frac{d(\phi)}{r} & -d(\theta)\sin(\phi)\cos(\phi) & \frac{d(r)}{r} \end{bmatrix}$$

The matrix elements of the Right Cartan connection matrix (indices and values on the initial non-cartesian state) using the matrix methods:

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);
```

$$\Gamma 11 := 0$$

$$\Gamma 12 := d(\theta)\, r\,(-1+\cos(\phi)^2)$$

$$\Gamma 13 := -r\, d(\phi)$$

```
> Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(cartan[2,3]);
```

$$\Gamma 21 := \frac{d(\theta)}{r}$$

$$\Gamma 22 := \frac{d(r)}{r} + \frac{\cos(\phi)\, d(\phi)}{\sin(\phi)}$$

$$\Gamma 23 := \frac{\cos(\phi)\, d(\theta)}{\sin(\phi)}$$

```
> Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(cartan[3,3]);
```

$$\Gamma 31 := \frac{d(\phi)}{r}$$

$$\Gamma 32 := -d(\theta)\sin(\phi)\cos(\phi)$$

$$\Gamma 33 := \frac{d(r)}{r}$$

The 3-index right Cartan connection Coefficients are:

```
> CR(11,1):=getcoeff(cartan[1,1]&^d(theta)&^d(phi));CR(11,2):=getcoeff(cartan[1,1]&^d(phi)&^d(r));CR(11,3):=getcoeff(cartan[1,1]&^d(r)&^d(theta));CR(21,1):=getcoeff(cartan[2,1]&^d(theta)&^d(phi));CR(21,2):=getcoeff(cartan[2,1]&^d(phi)&^d(r));CR(21,3):=getcoeff(cartan[2,1]&^d(r)&^d(theta));CR(31,1):=getcoeff(cartan[3,1]&^d(theta)&^d(phi));CR(31,2):=getcoeff(cartan[3,1]&^d(zz)&^d(r));CR(31,3):=getcoeff(cartan[3,1]&^d(r)&^d(theta));
> CR(12,1):=getcoeff(cartan[1,2]&^d(theta)&^d(phi));CR(12,2):=getcoeff(cartan[1,2]&^d(phi)&^d(r));CR(12,3):=getcoeff(cartan[1,2]&^d(r)&^d(theta));CR(22,1):=getcoeff(cartan[2,2]&^d(theta)&^d(phi));CR(22,2):=getcoeff(cartan[2,2]&^d(phi)&^d(r));CR(22,3):=getcoeff(cartan[2,2]&^d(r)&^d(theta));CR(32,1):=getcoeff(cartan[3,2]&^d(theta)&^d(phi));CR(32,2):=getcoeff(cartan[3,2]&^d(phi)&^d(r));CR(32,3):=getcoeff(cartan[3,2]&^d(r)&^d(theta));
> CR(13,1):=getcoeff(cartan[1,3]&^d(theta)&^d(phi));CR(13,2):=getcoeff(cartan[1,3]&^d(phi)&^d(r));CR(13,3):=getcoeff(cartan[1,3]&^d(r)&^d(theta));CR(23,1):=getcoeff(cartan[2,3]&^d(theta)&^d(phi));CR(23,2):=getcoeff(cartan[2,3]&^d(phi)&^d(r));CR(23,3):=getcoeff(cartan[2,3]&^d(r)&^d(theta));CR(33,1):=getcoeff(cartan[3,3]&^d(theta)&^d(phi));CR(33,2):=getcoeff(cartan[3,3]&^d(phi)&^d(r));CR(33,3):=getcoeff(cartan[3,3]&^d(r)&^d(theta));
```

$$CR(11, 1) := 0$$

$$CR(11, 2) := 0$$

$$CR(11, 3) := 0$$

$$CR(21, 1) := 0$$

$$CR(21, 2) := \frac{1}{r}$$

$$CR(21, 3) := 0$$

$$CR(31, 1) := 0$$

$$CR(31, 2) := \frac{1}{r}$$

$$CR(31, 3) := \frac{1}{r}$$

$$CR(12, 1) := 0$$

$$CR(12, 2) := r\,(-1 + \cos(\phi)^2)$$

$$CR(12, 3) := 0$$

$$CR(22, 1) := \frac{1}{r}$$

$$CR(22, 2) := 0$$

$$CR(22, 3) := \frac{\cos(\phi)}{\sin(\phi)}$$

$$CR(32, 1) := 0$$

$$CR(32, 2) := -\cos(\phi)\sin(\phi)$$

$$CR(32, 3) := 0$$

$$CR(13, 1) := 0$$

$$CR(13, 2) := 0$$

$$CR(13, 3) := -r$$

$$CR(23, 1) := 0$$

$$CR(23, 2) := \frac{\cos(\phi)}{\sin(\phi)}$$

$$CR(23, 3) := 0$$

$$CR(33, 1) := \frac{1}{r}$$

$$CR(33, 2) := 0$$

$$CR(33, 3) := 0$$

Compute the Cartan matrix of curvature 2-forms based on the Frame FF and the right Cartan Connection.

```
> cartanR:=cartan;
```

$$cartanR := cartan$$

```
> CRwedgeCR:=array([[wcollect(factor(cartanR[1,1]&^cartanR[1,1]+cartanR[1,2]&^cartanR[2,1]+cartanR[1,3]&^cartanR[3,1])),(wcollect(factor(cartanR[1,1]&^cartanR[1,2]+cartanR[1,2]&^cartanR[2,2]+cartanR[1,3]&^cartanR[3,2]))),wcollect(factor(cartanR[1,1]&^cartanR[1,3]+cartanR[1,2]&^cartanR[2,3]+cartanR[1,3]&^cartanR[3,3]))],[wcollect(factor(cartanR[2,1]&^cartanR[1,1]+cartanR[2,2]&^cartanR[2,1]+cartanR[2,3]&^cartanR[3,1])),(wcollect(factor(cartanR[2,1]&^cartanR[1,2]+cartanR[2,2]&^cartanR[2,2]+cartanR[2,3]&^cartanR[3,2]))),wcollect(factor(cartanR[2,1]&^cartanR[1,3]+cartanR[2,2]&^cartanR[2,3]+cartanR[2,3]&^cartanR[3,3]))],[wcollect(factor(cartanR[3,1]&^cartanR[1,1]+cartanR[3,2]&^cartanR[2,1]+cartanR[3,3]&^cartanR[3,1])),(wcollect(factor(cartanR[3,1]&^cartanR[2,1]+cartanR[3,2]&^cartanR[2,2]+cartanR[3,3]&^cartanR[3,2]))),wcollect(factor(cartanR[3,1]&^cartanR[1,3]+cartanR[3,2]&^cartanR[2,3]+cartanR[3,3]&^cartanR[3,3]))]]);
```

$CRwedgeCR :=$

$$\left[0, \frac{(\sin(\phi)\cos(\phi)^2 - \sin(\phi))\,(d(\theta)\ \&\wedge\ d(r))}{\sin(\phi)} + r\cos(\phi)\sin(\phi)\,\%2 + \frac{(-r\cos(\phi) + r\cos(\phi)^3)\,\%1}{\sin(\phi)},\right.$$
$$\left. -(d(\phi)\ \&\wedge\ d(r))\right]$$
$$\left[\frac{d(r)\ \&\wedge\ d(\theta)}{r^2} + \frac{\cos(\phi)\,\%2}{\sin(\phi)\,r} + \frac{\cos(\phi)\,\%1}{\sin(\phi)\,r}, 0,\right.$$
$$\left. -\%1 + \frac{\cos(\phi)\,(d(r)\ \&\wedge\ d(\theta))}{\sin(\phi)\,r} + \frac{\cos(\phi)^2\,\%2}{\sin(\phi)^2} + \frac{\cos(\phi)\,(d(\theta)\ \&\wedge\ d(r))}{\sin(\phi)\,r}\right]$$
$$\left[\frac{d(r)\ \&\wedge\ d(\phi)}{r^2}, \frac{\%2}{r^2} - \frac{\cos(\phi)\sin(\phi)\,(d(\theta)\ \&\wedge\ d(r))}{r} - \cos(\phi)^2\,\%1 - \frac{\cos(\phi)\sin(\phi)\,(d(r)\ \&\wedge\ d(\theta))}{r}, 0\right]$$
$$\%1 := d(\theta)\ \&\wedge\ d(\phi)$$
$$\%2 := d(\phi)\ \&\wedge\ d(\theta)$$

```
>
> `CartanCurvature2forms`:=(simplify(evalm(CRwedgeCR+d(cartanR))));CMC:=evalm(CRwedgeCR+d(cartanR)):CMC[1,1];wcollect(simplify((CMC[1,2])));
>
```

CartanCurvature2forms :=

$$[0, -\sin(\phi)\,(r\cos(\phi)\,\%1 + r\cos(\phi)\,\%2 + \sin(\phi)\,(d(\theta)\ \&\wedge\ d(r)) + \sin(\phi)\,(d(r)\ \&\wedge\ d(\theta))),$$
$$-(d(\phi)\ \&\wedge\ d(r)) - (d(r)\ \&\wedge\ d(\phi))]$$
$$\left[\frac{\cos(\phi)\,(\%1 + \%2)}{\sin(\phi)\,r}, 0, \frac{-\%2\sin(\phi)\,r + \cos(\phi)\,(d(r)\ \&\wedge\ d(\theta)) + \cos(\phi)\,(d(\theta)\ \&\wedge\ d(r)) - \%1\,r\sin(\phi)}{\sin(\phi)\,r}\right]$$
$$[0, -(-\%1 + \cos(\phi)\,(d(\theta)\ \&\wedge\ d(r))\sin(\phi)\,r + \cos(\phi)^2\,\%2\,r^2 + \cos(\phi)\,(d(r)\ \&\wedge\ d(\theta))\sin(\phi)\,r - \%1\,r^2$$
$$+ 2\,\%1\,r^2\cos(\phi)^2)\big/r^2, 0]$$
$$\%1 := d(\phi)\ \&\wedge\ d(\theta)$$
$$\%2 := d(\theta)\ \&\wedge\ d(\phi)$$

$$0$$

$$-\sin(\phi)^2\,(d(\theta)\ \&\wedge\ d(r)) - r\cos(\phi)\sin(\phi)\,(d(\phi)\ \&\wedge\ d(\theta)) - \sin(\phi)^2\,(d(r)\ \&\wedge\ d(\theta))$$
$$- \sin(\phi)\,r\cos(\phi)\,(d(\theta)\ \&\wedge\ d(\phi))$$

```
>
```

The components of the matrix of Cartan curvature 2-forms vanish (First structural Equation)

d[C]+[C]^[C]=0

Compute the Cartan vector of torsion 2-forms based on the Frame FF and the right Cartan Connection.

```
> `Cartan_Torsion`:=array([[wcollect(factor(cartanR[1,1]&^d(r)+cartanR[1,2]&^d(theta)+cartanR[1,3]&^d(phi)))],[wcollect(factor(cartanR[2,1]&^d(r)+cartanR[2,2]&^d(theta)+cartanR[2,3]&^d(phi)))],[wcollect(factor(cartanR[3,1]&^d(r)+cartanR[3,2]&^d(theta)+cartanR[3,3]&^d(phi)))]]);
>
```

$$Cartan\_Torsion := \begin{bmatrix} 0 \\ \frac{d(\theta)\ \&\wedge\ d(r)}{r} + \frac{d(r)\ \&\wedge\ d(\theta)}{r} + \frac{\cos(\phi)\,(d(\phi)\ \&\wedge\ d(\theta))}{\sin(\phi)} + \frac{\cos(\phi)\,(d(\theta)\ \&\wedge\ d(\phi))}{\sin(\phi)} \\ \frac{d(\phi)\ \&\wedge\ d(r)}{r} + \frac{d(r)\ \&\wedge\ d(\phi)}{r} \end{bmatrix}$$

The components of the vector of Cartan Torsion 2-forms vanish

(Second structural Equation)

[C]^|(sigmaF)>+[C]^|d(sigmaF)>=0

The Cartan Torsion is NOT THE SAME as the Affine torsion, but BOTH are zero for this example.

```
>
```

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=3;coord:=[r,theta,phi];GG:=simplify(inverse(FF));
```

$$dim := 3$$

$$coord := [r, \theta, \phi]$$

$$GG := \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ -\frac{\sin(\theta)}{r\sin(\phi)} & \frac{\cos(\theta)}{r\sin(\phi)} & 0 \\ \frac{\cos(\theta)\cos(\phi)}{r} & \frac{\sin(\theta)\cos(\phi)}{r} & -\frac{\sin(\phi)}{r} \end{bmatrix}$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1GG[i,j,k] := (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do
  ss:=0;for m from 1 to dim do ss := ss+(d1GG[a,m,k]*FF[m,b]);
  CC[a,b,k]:=simplify(-ss) od od od od ;
>
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if
  CC[a,b,k]=0 then else print(`Cartan_RIGHT`(a,b,k)=factor(CC[a,b,k])) fi od od
  od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients.

CC(abk) index (1,-1,-1)

$$\text{Cartan\_RIGHT}(2, 1, 2) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(3, 1, 3) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(1, 2, 2) = r\,(\cos(\phi) - 1)\,(\cos(\phi) + 1)$$

$$\text{Cartan\_RIGHT}(2, 2, 1) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(2, 2, 3) = \frac{\cos(\phi)}{\sin(\phi)}$$

$$\text{Cartan\_RIGHT}(3, 2, 2) = -\cos(\phi)\sin(\phi)$$

$$\text{Cartan\_RIGHT}(1, 3, 3) = -r$$

$$\text{Cartan\_RIGHT}(2, 3, 2) = \frac{\cos(\phi)}{\sin(\phi)}$$

$$\text{Cartan\_RIGHT}(3, 3, 1) = \frac{1}{r}$$

These results agree with matrix method.

Next check for Affine Torsion using the tensor methods by looking for anti-symmetry on the C(i, j, k) -

C(i, k, j) components of the right Cartan connection matrix.

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (CC[i,j,k]-CC[i,k,j])/2; CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  CCTTS[i,j,k]=0 then else print(`RIGHT_AffineTorsion`(i,k,j)=CCTTS[i,k,j]) fi
  od od od ;
```

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

For the given example, the Cartan Torsion vanishes, and there is no Affine assymetry torsion of the Cartan right connection.

IT is possible to compute the matrix elements of the
CARTAN LEFT CONNECTION

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1GG[a,j,k] := simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrix product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  ss:=0;for m to dim do ss := ss+FF[i,m]*(d1GG[m,j,k]); DD[i,j,k]:=simplify(ss)
  od od od od ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  DD[i,j,k]=0 then else print(`Cartan_LEFT`(i,j,k)=factor(DD[i,j,k])) fi od od
  od ;
```

$$\mathrm{Cartan\_LEFT}(1, 1, 1) = -\frac{1 - \cos(\theta)^2 + \cos(\theta)^2 \cos(\phi)^2}{r}$$

$$\mathrm{Cartan\_LEFT}(1, 1, 3) = -\frac{\sin(\phi) \cos(\phi) (\cos(\theta) - 1) (\cos(\theta) + 1)}{(\cos(\phi) - 1) (\cos(\phi) + 1)}$$

$$\mathrm{Cartan\_LEFT}(1, 2, 1) = -\frac{\cos(\theta) (\cos(\phi) - 1) (\cos(\phi) + 1) \sin(\theta)}{r}$$

$$\mathrm{Cartan\_LEFT}(1, 2, 2) = 1$$

$$\mathrm{Cartan\_LEFT}(1, 2, 3) = -\frac{\sin(\theta) \sin(\phi) \cos(\theta) \cos(\phi)}{(\cos(\phi) - 1) (\cos(\phi) + 1)}$$

$$\mathrm{Cartan\_LEFT}(1, 3, 1) = \frac{\cos(\theta) \cos(\phi) \sin(\phi)}{r}$$

$$\mathrm{Cartan\_LEFT}(1, 3, 3) = -\cos(\theta)$$

$$\mathrm{Cartan\_LEFT}(2, 1, 1) = -\frac{\cos(\theta) (\cos(\phi) - 1) (\cos(\phi) + 1) \sin(\theta)}{r}$$

$$\mathrm{Cartan\_LEFT}(2, 1, 2) = -1$$

$$\mathrm{Cartan\_LEFT}(2, 1, 3) = -\frac{\sin(\theta) \sin(\phi) \cos(\theta) \cos(\phi)}{(\cos(\phi) - 1) (\cos(\phi) + 1)}$$

$$\mathrm{Cartan\_LEFT}(2, 2, 1) = \frac{-\cos(\theta)^2 - \cos(\phi)^2 + \cos(\theta)^2 \cos(\phi)^2}{r}$$

$$\mathrm{Cartan\_LEFT}(2, 2, 3) = \frac{\cos(\theta)^2 \sin(\phi) \cos(\phi)}{(\cos(\phi) - 1) (\cos(\phi) + 1)}$$

$$\mathrm{Cartan\_LEFT}(2, 3, 1) = \frac{\sin(\theta) \cos(\phi) \sin(\phi)}{r}$$

$$\mathrm{Cartan\_LEFT}(2, 3, 3) = -\sin(\theta)$$

$$Cartan\_LEFT(3, 1, 1) = \frac{\cos(\theta)\cos(\phi)\sin(\phi)}{r}$$

$$Cartan\_LEFT(3, 1, 3) = \cos(\theta)$$

$$Cartan\_LEFT(3, 2, 1) = \frac{\sin(\theta)\cos(\phi)\sin(\phi)}{r}$$

$$Cartan\_LEFT(3, 2, 3) = \sin(\theta)$$

$$Cartan\_LEFT(3, 3, 1) = \frac{(\cos(\phi)-1)(\cos(\phi)+1)}{r}$$

The LEFT CARTAN Connection components are not the same as the right Cartan connection components.
Check for assymetry (LEFT Torsion)

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (DD[i,j,k]-DD[i,k,j])/2; TTS[i,j,k]:=simplify(ss) od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  TTS[i,j,k]=0 then else print(`LEFT_Torsion`(i,k,j)=TTS[i,k,j]) fi od od od ;
```

$$LEFT\_Torsion(1, 2, 1) = -\frac{1}{2}\frac{\cos(\theta)(-1+\cos(\phi)^2)\sin(\theta)}{r}$$

$$LEFT\_Torsion(1, 3, 1) = \frac{1}{2}\frac{\cos(\phi)\sin(\phi)(-r+r\cos(\theta)^2-\cos(\theta)+\cos(\phi)^2\cos(\theta))}{(-1+\cos(\phi)^2)\,r}$$

$$LEFT\_Torsion(1, 1, 2) = \frac{1}{2}\frac{\cos(\theta)(-1+\cos(\phi)^2)\sin(\theta)}{r}$$

$$LEFT\_Torsion(1, 3, 2) = \frac{1}{2}\frac{\sin(\theta)\sin(\phi)\cos(\theta)\cos(\phi)}{-1+\cos(\phi)^2}$$

$$LEFT\_Torsion(1, 1, 3) = -\frac{1}{2}\frac{\cos(\phi)\sin(\phi)(-r+r\cos(\theta)^2-\cos(\theta)+\cos(\phi)^2\cos(\theta))}{(-1+\cos(\phi)^2)\,r}$$

$$LEFT\_Torsion(1, 2, 3) = -\frac{1}{2}\frac{\sin(\theta)\sin(\phi)\cos(\theta)\cos(\phi)}{-1+\cos(\phi)^2}$$

$$LEFT\_Torsion(2, 2, 1) = \frac{1}{2}\frac{r-\cos(\theta)^2-\cos(\phi)^2+\cos(\theta)^2\cos(\phi)^2}{r}$$

$$LEFT\_Torsion(2, 3, 1) = \frac{1}{2}\frac{\cos(\phi)\sin(\theta)\sin(\phi)(\cos(\theta)\,r-1+\cos(\phi)^2)}{(-1+\cos(\phi)^2)\,r}$$

$$LEFT\_Torsion(2, 1, 2) = -\frac{1}{2}\frac{r-\cos(\theta)^2-\cos(\phi)^2+\cos(\theta)^2\cos(\phi)^2}{r}$$

$$LEFT\_Torsion(2, 3, 2) = -\frac{1}{2}\frac{\cos(\theta)^2\sin(\phi)\cos(\phi)}{-1+\cos(\phi)^2}$$

$$LEFT\_Torsion(2, 1, 3) = -\frac{1}{2}\frac{\cos(\phi)\sin(\theta)\sin(\phi)(\cos(\theta)\,r-1+\cos(\phi)^2)}{(-1+\cos(\phi)^2)\,r}$$

$$LEFT\_Torsion(2, 2, 3) = \frac{1}{2}\frac{\cos(\theta)^2\sin(\phi)\cos(\phi)}{-1+\cos(\phi)^2}$$

$$LEFT\_Torsion(3, 2, 1) = \frac{1}{2}\frac{\sin(\theta)\cos(\phi)\sin(\phi)}{r}$$

$$LEFT\_Torsion(3, 3, 1) = -\frac{1}{2}\frac{\cos(\theta)\,r+1-\cos(\phi)^2}{r}$$

$$\text{LEFT\_Torsion}(3, 1, 2) = -\frac{1}{2} \frac{\sin(\theta) \cos(\phi) \sin(\phi)}{r}$$

$$\text{LEFT\_Torsion}(3, 3, 2) = -\frac{1}{2} \sin(\theta)$$

$$\text{LEFT\_Torsion}(3, 1, 3) = \frac{1}{2} \frac{\cos(\theta)\, r + 1 - \cos(\phi)^2}{r}$$

$$\text{LEFT\_Torsion}(3, 2, 3) = \frac{1}{2} \sin(\theta)$$

**The Cartan Left connection exhibits asymmetry that could be described as a form of AFFINE torsion,
but not Cartan torsion.**

For this example from cylindrical to cartesian coordinates, there is no assymetry for the [CR], but there is assymetry for [CL]
(The physical implication is not clear to me)

Next the Christoffel symbols will be computed for the metric on the initial state.

Christoffel Connection coefficients from the pullback (induced) metric

```
>
> metric:= evalm(pullbackmetric);
```

$$metric := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -r^2 (-1 + \cos(\phi)^2) & 0 \\ 0 & 0 & r^2 \end{bmatrix}$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do
  C1S[i,j,k] := 0  od od od; for i from 1 to dim do for j from 1 to dim do for
  k from 1 to dim do C1S[i,j,k] :=
  1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss :=
  0; for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then  else
  print(`Christoffel_Gamma2`(i,j,k)=simplify(C2S[i,j,k])) fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind   on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

$$\text{Christoffel\_Gamma2}(1, 2, 2) = -r + r \cos(\phi)^2$$

$$\text{Christoffel\_Gamma2}(1,3,3) = -r$$

$$\text{Christoffel\_Gamma2}(2,1,2) = \frac{1}{r}$$

$$\text{Christoffel\_Gamma2}(2,2,1) = \frac{1}{r}$$

$$\text{Christoffel\_Gamma2}(2,2,3) = -\frac{\cos(\phi)\sin(\phi)}{-1+\cos(\phi)^2}$$

$$\text{Christoffel\_Gamma2}(2,3,2) = -\frac{\cos(\phi)\sin(\phi)}{-1+\cos(\phi)^2}$$

$$\text{Christoffel\_Gamma2}(3,1,3) = \frac{1}{r}$$

$$\text{Christoffel\_Gamma2}(3,2,2) = -\cos(\phi)\sin(\phi)$$

$$\text{Christoffel\_Gamma2}(3,3,1) = \frac{1}{r}$$

The Christoffel metric based connection equals the Cartan right Frame field connection in this example.

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Ricci Rotation coefficients, T(i,j,k)

CartanRight(ijk) = ChristoffelGamma(ijk) + T(ijk)

Compute the T(i,j,k):

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0;
  ss := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(SHIPTR[i,j,k])) fi od od od ;
```

T(ijk) index (1,-1,-1)

$$T(1,2,2) = 0$$

$$T(1,3,3) = 0$$

$$T(2,1,2) = 0$$

$$T(2,2,1) = 0$$

$$T(2,2,3) = 0$$

$$T(2,3,2) = 0$$

$$T(3,1,3) = 0$$

$$T(3,2,2) = 0$$

$$T(3, 3, 1) = 0$$

Right Cartan(ijk) = Gamma(ijk) + T(ijk)

In this example the RICCI rotation coefficients on the domain space of cylindrical coordinates vanish.
The Cartan (right) connection matrix is exactly equal to the the Christoffel symbols on the initial state.

```
>
>
*******************************************************************************
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=
  0,Az=0,C=0,Phi=0,phi=0,theta=0,r=0,a=const,b=const,c=const,Lx=0,Ly=0,Lz=0);
Warning, the protected names norm and trace have been redefined and
unprotected

Warning, the protected name close has been redefined and unprotected

Warning, the names &^, d and wdegree have been redefined
>
```

Part 2 The map is from initial Cartesian state {x,y,z} into the target cylindrical final state, {r,theta,zz}. The constant (Cartesian) metric, g, on the initial state (x,y,z) is pushed forward to the non-constant metric, G, on the final state (r,theta,zz).

```
> r:=(x^2+y^2+z^2)^(1/2);cos(theta):=x/(x^2+y^2)^(1/2);cos(phi):=z/r;sin(theta)
  :=y/(x^2+y^2)^(1/2);sin(phi):=(x^2+y^2)^(1/2)/r;
>
>
```

$$r := \sqrt{x^2 + y^2 + z^2}$$

$$\cos(\theta) := \frac{x}{\sqrt{x^2 + y^2}}$$

$$\cos(\phi) := \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\sin(\theta) := \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sin(\phi) := \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$$

```
>
>
>
```

The map from x,y,z to r, theta, phi is not given explicitly. Yet the differentials can be well defined.

```
> dr:=d(r);`dtheta`:=factor(-d(cos(theta))/(sin(theta)));`dphi`:=factor(-d(cos(phi)/sin(phi)));
```

$$dr := \frac{x\,d(x)}{\sqrt{x^2+y^2+z^2}} + \frac{y\,d(y)}{\sqrt{x^2+y^2+z^2}} + \frac{z\,d(z)}{\sqrt{x^2+y^2+z^2}}$$

$$dtheta := \frac{-y\,d(x)+x\,d(y)}{x^2+y^2}$$

$$dphi := \frac{-d(z)\,x^2 - d(z)\,y^2 + z\,x\,d(x) + z\,y\,d(y)}{(x^2+y^2)^{(3/2)}}$$

The Frame matrix of functions on the initial state (x,y,z) that causes the differential structures dr,dtheta,dphi to be created linearly in terms of dx,dy,dz is deduced from the linearmapping above. An explicit form for the mapping is not used. The Frame matrix [FF] so deduced is given below:

```
> FF:=(array([[x/r,y/r,z/r],[-y/(x^2+y^2),x/(x^2+y^2),0],[z*x/(x^2+y^2)^(3/2),z*y/(x^2+y^2)^(3/2),-1/(x^2+y^2)^(1/2)]]));GG:=evalm(inverse(FF));DETFF:=factor(det(FF));
```

$$FF := \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ \frac{z\,x}{(x^2+y^2)^{(3/2)}} & \frac{z\,y}{(x^2+y^2)^{(3/2)}} & -\frac{1}{\sqrt{x^2+y^2}} \end{bmatrix}$$

$$GG := \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & -y & \frac{z\,x\,(x^2+y^2)^{(3/2)}}{\%1} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & x & \frac{z\,y\,(x^2+y^2)^{(3/2)}}{\%1} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & 0 & -\frac{(x^2+y^2)^{(5/2)}}{\%1} \end{bmatrix}$$

$$\%1 := x^4 + 2\,x^2\,y^2 + y^4 + y^2\,z^2 + z^2\,x^2$$

$$DETFF := -\frac{\sqrt{x^2+y^2+z^2}}{(x^2+y^2)^{(3/2)}}$$

TO check to see if FF does indeed map the Cartesian variables to the desired 1-forms on the final state, construct sigmaF. The Push forward of the initial state differentials to produce 1-forms on the final state (in terms cylindrical coordinates) is given by:

```
> sigmaF:=innerprod(FF,[d(x),d(y),d(z)]);`dsigmaF`:=factor(d(sigmaF));
```

$$sigmaF := \left[\frac{x\,d(x)+y\,d(y)+z\,d(z)}{\sqrt{x^2+y^2+z^2}}, \frac{-y\,d(x)+x\,d(y)}{x^2+y^2}, \frac{-d(z)\,x^2 - d(z)\,y^2 + z\,x\,d(x) + z\,y\,d(y)}{(x^2+y^2)^{(3/2)}}\right]$$

$$dsigmaF := [0, 0, 0]$$

Note that each component of the sigmaF is closed, but not necessarily exact. In particular note that there is a singularity at x^2+y^2 = 0 for d(theta) and d(phi) and a singularity at the origin for dr.

```
> SCALE:=array([[(x^2+y^2)^(1/2)/r,0,0],[0,(x^2+y^2)^(1/2),0],[0,0,(x^2+y^2)^(1/2)]]);
```

$$SCALE := \begin{bmatrix} \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 & 0 \\ 0 & \sqrt{x^2+y^2} & 0 \\ 0 & 0 & \sqrt{x^2+y^2} \end{bmatrix}$$

```
> FFN:=innerprod(FF,SCALE);DETFFN:=det(FFN);GGN:=inverse(FFN);
```

$$FFN := \begin{bmatrix} \frac{x\sqrt{x^2+y^2}}{\%1} & \frac{y\sqrt{x^2+y^2}}{\sqrt{\%1}} & \frac{z\sqrt{x^2+y^2}}{\sqrt{\%1}} \\ -\frac{y}{\sqrt{x^2+y^2}\sqrt{\%1}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ \frac{z\,x}{\sqrt{\%1}\,(x^2+y^2)} & \frac{z\,y}{x^2+y^2} & -1 \end{bmatrix}$$

$$\%1 := x^2+y^2+z^2$$

$$DETFFN := -1$$

$$GGN := \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y\sqrt{\%1}}{\sqrt{x^2+y^2}} & \frac{z\,x}{\sqrt{\%1}} \\ \frac{y}{\sqrt{x^2+y^2}\sqrt{\%1}} & \frac{x}{\sqrt{x^2+y^2}} & \frac{z\,y}{\%1} \\ \frac{z}{\sqrt{x^2+y^2}\sqrt{\%1}} & 0 & -\frac{x^2+y^2}{\%1} \end{bmatrix}$$

$$\%1 := x^2+y^2+z^2$$

NOte that it is possible to scale each vector such that the determinant has a fixed value. Such cases will be examined later.

****

```
>
>
```

The metric on the domain xyz is the unit matrix of constants, by assumption. As such the Christoffel symbols will be zero. The pushed forward metric on the spherical coordinate range is (but with arguments on the domain x,y,z) is

```
> pushedmetric:=simplify(innerprod(transpose(GG),GG));inducedmetricinverse:=inn
  erprod(FF,transpose(FF));
```

$$pushedmetric := \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2+y^2 & 0 \\ 0 & 0 & \frac{(x^2+y^2)^2}{x^2+y^2+z^2} \end{bmatrix}$$

$$inducedmetricinverse := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{x^2+y^2} & 0 \\ 0 & 0 & \frac{x^2+y^2+z^2}{(x^2+y^2)^2} \end{bmatrix}$$

Now Compute the Right Cartan Matrix [CR]

```
> evalm(GG):factor(d(FF)):
>
>

> cartan:=(factor(innerprod(GG,factor(d(FF)))));
```

$cartan :=$

$$\Big[-(z^4 y^3 d(y)+2 z^4 x^2 y\, d(y)+z^4 d(x) y^2 x+2 z^4 d(x) x^3-x^4 z^3 d(z)-z^3 x^2 d(z) y^2+z^2 d(x) x^5+2 z^2 y^5 d(y)$$
$$+3 x^2 y^3 z^2 d(y)+x^4 z^2 y\, d(y)+3 x^3 z^2 y^2 d(x)+2 x z^2 y^4 d(x)+d(x) x y^6+2 y^5 d(y) x^2+y^7 d(y)+y^3 d(y) x^4$$
$$+2 x^3 y^4 d(x)+x^5 y^2 d(x)) \Big/ (\%1^2 (x^2+y^2)^2), (-2 z^4 x^2 y\, d(x)-z^4 y^3 d(x)+z^4 x^3 d(y)+z^3 y\, d(z) x^3$$
$$+z^3 y^3 d(z) x+3 x z^2 d(y) y^4+5 x^3 y^2 z^2 d(y)-3 x^2 z^2 y^3 d(x)+2 x^5 z^2 d(y)-x^4 z^2 y\, d(x)-2 y^5 d(x) z^2$$
$$+2 x\, d(y) y^6+5 x^3 d(y) y^4-y^7 d(x)+4 x^5 d(y) y^2-2 x^2 y^5 d(x)-x^4 y^3 d(x)+x^7 d(y)) \Big/ (\%1^2 (x^2+y^2)^2),$$
$$\frac{(z^3 d(x) x+z^3 y\, d(y)+d(z) y^4+2 x^2 d(z) y^2+x^4 d(z))\, x}{\%2\ \%1}\Big]$$

$$\Big[-(2 z^4 x y^2 d(y)-z^4 y^3 d(x)+z^4 x^3 d(y)-z^3 y\, d(z) x^3-z^3 y^3 d(z) x+x z^2 d(y) y^4+3 x^3 y^2 z^2 d(y)$$
$$-5 x^2 z^2 y^3 d(x)+2 x^5 z^2 d(y)-3 x^4 z^2 y\, d(x)-2 y^5 d(x) z^2+2 x^5 d(y) y^2+x^3 d(y) y^4-4 x^2 y^5 d(x)+x^7 d(y)$$
$$-y^7 d(x)-2 x^6 y\, d(x)-5 x^4 y^3 d(x)) \Big/ (\%1^2 (x^2+y^2)^2), -(2 z^4 y^3 d(y)+z^4 d(x) x^3+z^4 x^2 y\, d(y)$$
$$+2 z^4 d(x) y^2 x-z^3 y^4 d(z)-z^3 x^2 d(z) y^2+z^2 y^5 d(y)+3 x^3 z^2 y^2 d(x)+2 x^4 z^2 y\, d(y)+x z^2 y^4 d(x)$$
$$+3 x^2 y^3 z^2 d(y)+2 z^2 d(x) x^5+x^7 d(x)+x^3 y^4 d(x)+2 x^5 y^2 d(x)+y\, d(y) x^6+y^5 d(y) x^2+2 y^3 d(y) x^4) \Big/ ($$
$$\%1^2 (x^2+y^2)^2), \frac{(z^3 d(x) x+z^3 y\, d(y)+d(z) y^4+2 x^2 d(z) y^2+x^4 d(z))\, y}{\%2\ \%1}\Big]$$

$$\Big[(3 z^3 d(x) x+3 z^3 y\, d(y)-2 z^2 d(z) y^2-2 z^2 x^2 d(z)+2 z\, d(x) x^3+2 z\, d(x) y^2 x+2 z x^2 y\, d(y)+2 z y^3 d(y)$$
$$-x^4 d(z)-d(z) y^4-2 x^2 d(z) y^2)\, x / (\%2\ \%1), (3 z^3 d(x) x+3 z^3 y\, d(y)-2 z^2 d(z) y^2-2 z^2 x^2 d(z)$$
$$+2 z\, d(x) x^3+2 z\, d(x) y^2 x+2 z x^2 y\, d(y)+2 z y^3 d(y)-x^4 d(z)-d(z) y^4-2 x^2 d(z) y^2)\, y / (\%2\ \%1),$$
$$-\frac{d(x) x^3-x^2 z\, d(z)+x^2 y\, d(y)+2 x\, d(x) z^2+d(x) y^2 x+2 y\, d(y) z^2-z y^2 d(z)+y^3 d(y)}{\%1^2}\Big]$$

$$\%1 := x^2+y^2+z^2$$
$$\%2 := x^4+2 x^2 y^2+y^4+y^2 z^2+z^2 x^2$$

The matrix elements of the Right Cartan connection matrix (indices and values on the initial Cartesian state) using the matrix methods: positive = upper index, negative = lower index.

```
> Gamma(1,-1):=wcollect(cartan[1,1]);Gamma(1,-2):=wcollect(cartan[1,2]);Gamma(1
  ,-3):=wcollect(cartan[1,3]);
```

$$\Gamma(1,-1) := -\frac{(2 z^4 x^3+x^5 z^2+2 x z^2 y^4+x y^6+3 x^3 y^2 z^2+z^4 x y^2+2 x^3 y^4+x^5 y^2)\, d(x)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}$$
$$-\frac{(x^4 y^3+2 y^5 z^2+z^4 y^3+2 z^4 x^2 y+y^7+3 x^2 z^2 y^3+x^4 z^2 y+2 x^2 y^5)\, d(y)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}-\frac{(-x^4 z^3-z^3 x^2 y^2)\, d(z)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}$$

$$\Gamma(1,-2) := \frac{(-2 y^5 z^2-3 x^2 z^2 y^3-2 z^4 x^2 y-x^4 z^2 y-z^4 y^3-2 x^2 y^5-x^4 y^3-y^7)\, d(x)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}$$
$$+\frac{(4 x^5 y^2+z^4 x^3+2 x^5 z^2+3 x z^2 y^4+5 x^3 y^2 z^2+2 x y^6+5 x^3 y^4+x^7)\, d(y)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}+\frac{(z^3 y x^3+z^3 y^3 x)\, d(z)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}$$

$$\Gamma(1,-3) := \frac{z^3 x^2 d(x)}{\%1\,(x^2+y^2+z^2)}+\frac{z^3 y x\, d(y)}{\%1\,(x^2+y^2+z^2)}+\frac{(x^4+2 x^2 y^2+y^4)\, x\, d(z)}{\%1\,(x^2+y^2+z^2)}$$
$$\%1 := x^4+2 x^2 y^2+y^4+y^2 z^2+z^2 x^2$$

```
> Gamma(2,-1):=wcollect(cartan[2,1]);Gamma(2,-2):=wcollect(cartan[2,2]);Gamma(2
  ,-3):=wcollect(cartan[2,3]);
```

$$\Gamma(2, -1) := -\frac{(-z^4 y^3 - 5 x^2 z^2 y^3 - 3 x^4 z^2 y - 2 y^5 z^2 - 4 x^2 y^5 - y^7 - 2 x^6 y - 5 x^4 y^3)\, \mathrm{d}(x)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}$$
$$- \frac{(2 z^4 x y^2 + z^4 x^3 + x z^2 y^4 + 3 x^3 y^2 z^2 + 2 x^5 z^2 + 2 x^5 y^2 + x^3 y^4 + x^7)\, \mathrm{d}(y)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2} - \frac{(-z^3 y x^3 - z^3 y^3 x)\, \mathrm{d}(z)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}$$
$$\Gamma(2, -2) := -\frac{(2 z^4 x y^2 + z^4 x^3 + x z^2 y^4 + 3 x^3 y^2 z^2 + 2 x^5 z^2 + 2 x^5 y^2 + x^3 y^4 + x^7)\, \mathrm{d}(x)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}$$
$$- \frac{(z^4 x^2 y + x^2 y^5 + 2 z^4 y^3 + 2 x^4 y^3 + y^5 z^2 + x^6 y + 2 x^4 z^2 y + 3 x^2 z^2 y^3)\, \mathrm{d}(y)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2} - \frac{(-z^3 y^4 - z^3 x^2 y^2)\, \mathrm{d}(z)}{(x^2+y^2+z^2)^2 (x^2+y^2)^2}$$
$$\Gamma(2, -3) := \frac{z^3 y x\, \mathrm{d}(x)}{\%1\,(x^2+y^2+z^2)} + \frac{z^3 y^2\, \mathrm{d}(y)}{\%1\,(x^2+y^2+z^2)} + \frac{(x^4 + 2 x^2 y^2 + y^4)\, y\, \mathrm{d}(z)}{\%1\,(x^2+y^2+z^2)}$$
$$\%1 := x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2$$

```
> Gamma(3,-1):=wcollect(cartan[3,1]);Gamma(3,-2):=wcollect(cartan[3,2]);Gamma(3
,-3):=wcollect(cartan[3,3]);
```

$$\Gamma(3, -1) := \frac{(3 z^3 x + 2 z x^3 + 2 z y^2 x)\, x\, \mathrm{d}(x)}{\%1\,(x^2+y^2+z^2)} + \frac{(3 z^3 y + 2 z x^2 y + 2 z y^3)\, x\, \mathrm{d}(y)}{\%1\,(x^2+y^2+z^2)}$$
$$+ \frac{(-x^4 - 2 y^2 z^2 - 2 z^2 x^2 - y^4 - 2 x^2 y^2)\, x\, \mathrm{d}(z)}{\%1\,(x^2+y^2+z^2)}$$
$$\%1 := x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2$$
$$\Gamma(3, -2) := \frac{(3 z^3 x + 2 z x^3 + 2 z y^2 x)\, y\, \mathrm{d}(x)}{\%1\,(x^2+y^2+z^2)} + \frac{(3 z^3 y + 2 z x^2 y + 2 z y^3)\, y\, \mathrm{d}(y)}{\%1\,(x^2+y^2+z^2)}$$
$$+ \frac{(-x^4 - 2 y^2 z^2 - 2 z^2 x^2 - y^4 - 2 x^2 y^2)\, y\, \mathrm{d}(z)}{\%1\,(x^2+y^2+z^2)}$$
$$\%1 := x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2$$
$$\Gamma(3, -3) := -\frac{(x^3 + y^2 x + 2 x z^2)\, \mathrm{d}(x)}{(x^2+y^2+z^2)^2} - \frac{(x^2 y + y^3 + 2 y z^2)\, \mathrm{d}(y)}{(x^2+y^2+z^2)^2} - \frac{(-x^2 z - z y^2)\, \mathrm{d}(z)}{(x^2+y^2+z^2)^2}$$

```
>
> CR(11,1):=factor(getcoeff(cartan[1,1]&^d(y)&^d(z)));CR(11,2):=factor(getcoeff
(cartan[1,1]&^d(z)&^d(x)));CR(11,3):=factor(getcoeff(cartan[1,1]&^d(x)&^d(y))
);CR(21,1):=factor(getcoeff(cartan[2,1]&^d(y)&^d(z)));CR(21,2):=factor(getcoe
ff(cartan[2,1]&^d(z)&^d(x)));CR(21,3):=factor(getcoeff(cartan[2,1]&^d(x)&^d(y
)));CR(31,1):=factor(getcoeff(cartan[3,1]&^d(y)&^d(z)));CR(31,2):=factor(getc
oeff(cartan[3,1]&^d(z)&^d(x)));CR(31,3):=factor(getcoeff(cartan[3,1]&^d(x)&^d
(y)));
> CR(12,1):=factor(getcoeff(cartan[1,2]&^d(y)&^d(z)));CR(12,2):=factor(getcoeff
(cartan[1,2]&^d(z)&^d(x)));CR(12,3):=factor(getcoeff(cartan[1,2]&^d(x)&^d(y))
);CR(22,1):=factor(getcoeff(cartan[2,2]&^d(y)&^d(z)));CR(22,2):=factor(getcoe
ff(cartan[2,2]&^d(z)&^d(x)));CR(22,3):=factor(getcoeff(cartan[2,2]&^d(x)&^d(y
)));CR(32,1):=factor(getcoeff(cartan[3,2]&^d(y)&^d(z)));CR(32,2):=factor(getc
oeff(cartan[3,2]&^d(z)&^d(x)));CR(32,3):=factor(getcoeff(cartan[3,2]&^d(x)&^d
(y)));
> CR(13,1):=factor(getcoeff(cartan[1,3]&^d(y)&^d(z)));CR(13,2):=factor(getcoeff
(cartan[1,3]&^d(z)&^d(x)));CR(13,3):=factor(getcoeff(cartan[1,3]&^d(x)&^d(y))
);CR(23,1):=factor(getcoeff(cartan[2,3]&^d(y)&^d(z)));CR(23,2):=factor(getcoe
ff(cartan[2,3]&^d(z)&^d(x)));CR(23,3):=factor(getcoeff(cartan[2,3]&^d(x)&^d(y
)));CR(33,1):=factor(getcoeff(cartan[3,3]&^d(y)&^d(z)));CR(33,2):=factor(getc
oeff(cartan[3,3]&^d(z)&^d(x)));CR(33,3):=factor(getcoeff(cartan[3,3]&^d(x)&^d
(y)));
```

$$CR(11, 1) := -\frac{x\,(2\,z^4\,x^2 + z^2\,x^4 + 2\,y^4\,z^2 + y^6 + 3\,z^2\,x^2\,y^2 + z^4\,y^2 + 2\,x^2\,y^4 + x^4\,y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$CR(11, 2) := -\frac{y\,(2\,z^4\,x^2 + z^2\,x^4 + 2\,y^4\,z^2 + y^6 + 3\,z^2\,x^2\,y^2 + z^4\,y^2 + 2\,x^2\,y^4 + x^4\,y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$CR(11, 3) := \frac{z^3\,x^2}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(21, 1) := \frac{y\,(z^4\,y^2 + 5\,z^2\,x^2\,y^2 + 3\,z^2\,x^4 + 2\,y^4\,z^2 + 4\,x^2\,y^4 + y^6 + 2\,x^6 + 5\,x^4\,y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$CR(21, 2) := -\frac{x\,(2\,z^4\,y^2 + z^4\,x^2 + y^4\,z^2 + 3\,z^2\,x^2\,y^2 + 2\,z^2\,x^4 + 2\,x^4\,y^2 + x^2\,y^4 + x^6)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$CR(21, 3) := \frac{z^3\,y\,x}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(31, 1) := \frac{x^2\,z\,(3\,z^2 + 2\,x^2 + 2\,y^2)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(31, 2) := \frac{x\,z\,y\,(3\,z^2 + 2\,x^2 + 2\,y^2)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(31, 3) := -\frac{(x^2+y^2+2\,z^2)\,x}{(x^2+y^2+z^2)^2}$$

$$CR(12, 1) := -\frac{y\,(2\,z^4\,x^2 + z^2\,x^4 + 2\,y^4\,z^2 + y^6 + 3\,z^2\,x^2\,y^2 + z^4\,y^2 + 2\,x^2\,y^4 + x^4\,y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$CR(12, 2) := \frac{x\,(4\,x^4\,y^2 + z^4\,x^2 + 2\,z^2\,x^4 + 3\,y^4\,z^2 + 5\,z^2\,x^2\,y^2 + 2\,y^6 + 5\,x^2\,y^4 + x^6)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$CR(12, 3) := \frac{z^3\,y\,x}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(22, 1) := -\frac{x\,(2\,z^4\,y^2 + z^4\,x^2 + y^4\,z^2 + 3\,z^2\,x^2\,y^2 + 2\,z^2\,x^4 + 2\,x^4\,y^2 + x^2\,y^4 + x^6)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$CR(22, 2) := -\frac{y\,(2\,z^4\,y^2 + z^4\,x^2 + y^4\,z^2 + 3\,z^2\,x^2\,y^2 + 2\,z^2\,x^4 + 2\,x^4\,y^2 + x^2\,y^4 + x^6)}{(x^2+y^2)^2\,(x^2+y^2+z^2)^2}$$

$$CR(22, 3) := \frac{z^3\,y^2}{(x^2+y^2+z^2)^2\,(x^2+y^2)}$$

$$CR(32, 1) := \frac{x\,z\,y\,(3\,z^2 + 2\,x^2 + 2\,y^2)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(32, 2) := \frac{y^2\,z\,(3\,z^2 + 2\,x^2 + 2\,y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)}$$

$$CR(32, 3) := -\frac{(x^2+y^2+2\,z^2)\,y}{(x^2+y^2+z^2)^2}$$

$$CR(13, 1) := \frac{z^3\,x^2}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(13,2) := \frac{z^3\, y\, x}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(13,3) := \frac{(x^2+y^2)\, x}{(x^2+y^2+z^2)^2}$$

$$CR(23,1) := \frac{z^3\, y\, x}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$CR(23,2) := \frac{z^3\, y^2}{(x^2+y^2+z^2)^2\,(x^2+y^2)}$$

$$CR(23,3) := \frac{(x^2+y^2)\, y}{(x^2+y^2+z^2)^2}$$

$$CR(33,1) := -\frac{(x^2+y^2+2\,z^2)\, x}{(x^2+y^2+z^2)^2}$$

$$CR(33,2) := -\frac{(x^2+y^2+2\,z^2)\, y}{(x^2+y^2+z^2)^2}$$

$$CR(33,3) := \frac{z\,(x^2+y^2)}{(x^2+y^2+z^2)^2}$$

Note that the Affine Torsion of right Cartan matrix is ZERO.

Next compute the Cartan matrix of curvature 2-forms, based on the Frame and the Cartan right connection.

```
> cartanR:=cartan;
```

$$cartanR := cartan$$

```
> CRwedgeCR:=array([[wcollect(factor(cartanR[1,1]&^cartanR[1,1]+cartanR[1,2]&^cartanR[2,1]+cartanR[1,3]&^cartanR[3,1])),(wcollect(factor(cartanR[1,1]&^cartanR[1,2]+cartanR[1,2]&^cartanR[2,2]+cartanR[1,3]&^cartanR[3,2]))),wcollect(factor(cartanR[1,1]&^cartanR[1,3]+cartanR[1,2]&^cartanR[2,3]+cartanR[1,3]&^cartanR[3,3]))],[wcollect(factor(cartanR[2,1]&^cartanR[1,1]+cartanR[2,2]&^cartanR[2,1]+cartanR[2,3]&^cartanR[3,1])),(wcollect(factor(cartanR[2,1]&^cartanR[1,2]+cartanR[2,2]&^cartanR[2,2]+cartanR[2,3]&^cartanR[3,2]))),wcollect(factor(cartanR[2,1]&^cartanR[1,3]+cartanR[2,2]&^cartanR[2,3]+cartanR[2,3]&^cartanR[3,3]))],[wcollect(factor(cartanR[3,1]&^cartanR[1,1]+cartanR[3,2]&^cartanR[2,1]+cartanR[3,3]&^cartanR[3,1])),(wcollect(factor(cartanR[3,1]&^cartanR[2,1]+cartanR[3,2]&^cartanR[2,2]+cartanR[3,3]&^cartanR[3,2]))),wcollect(factor(cartanR[3,1]&^cartanR[1,3]+cartanR[3,2]&^cartanR[2,3]+cartanR[3,3]&^cartanR[3,3]))]]):
>
> `CartanCurvature2forms`:=wcollect(factor(evalm(CRwedgeCR+d(cartanR)))):
> C2F:=evalm(`CartanCurvature2forms`);simplify(C2F[1,1]);
```

$C2F :=$

$$\Bigg[2\,\frac{z^3\, y^2\, x\,(d(z)\ \&\hat{}\ d(x))}{(x^2+y^2)^2\ \%1^2} - \frac{2\,x\,(y\,z^4 - x^4\, y - z^2\, x^2\, y - y^3\, z^2 - 2\,x^2\, y^3 - y^5)\,(d(y)\ \&\hat{}\ d(x))}{(x^2+y^2)^2\ \%1^2}$$

$$- \frac{2\,x\,(-z^3\, x\, y + z\, x^3\, y + z\, x\, y^3)\,(d(y)\ \&\hat{}\ d(z))}{(x^2+y^2)^2\ \%1^2} - \frac{2\,x\,(z\, x^4 + z\, x^2\, y^2)\,(d(x)\ \&\hat{}\ d(z))}{(x^2+y^2)^2\ \%1^2}$$

$$+ \frac{2\,(z^2 - x^2 - y^2)\, z\, x^2\, y\,(d(z)\ \&\hat{}\ d(y))}{\%1^2\,(x^2+y^2)^2} - \frac{2\,(y^2\, z^2 + x^4 + x^2\, y^2)\, z\, x\,(d(z)\ \&\hat{}\ d(x))}{\%1^2\,(x^2+y^2)^2}$$

$$+ \frac{2\,x\, y\,(-x^4 - z^2\, x^2 - 2\,x^2\, y^2 - y^4 - y^2\, z^2 + z^4)\,(d(y)\ \&\hat{}\ d(x))}{\%1^2\,(x^2+y^2)^2}, - \frac{(-z^3\, x^3 + z^3\, y^2\, x)\,(d(z)\ \&\hat{}\ d(y))}{(x^2+y^2)^2\ \%1^2}$$

$$-\frac{\%3\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)^2\ \%1^2}-\frac{(2\,z\,x\,y^4+2\,z\,x^3\,y^2)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)^2\ \%1^2}$$

$$-\frac{(-z^3\,x^2\,y+z^3\,y^3+2\,z\,x^2\,y^3+2\,z\,x^4\,y)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)^2\ \%1^2}-\frac{(z^2\,x^2-y^2\,z^2+2\,x^2\,y^2+2\,y^4)\,z\,x\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{\%1^2\,(x^2+y^2)^2}$$

$$+\frac{(z^2\,x^2-y^2\,z^2-2\,x^4-2\,x^2\,y^2)\,z\,y\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(x))}{\%1^2\,(x^2+y^2)^2}-\frac{\%3\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x))}{\%1^2\,(x^2+y^2)^2},-\frac{z^3\,y\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)\ \%1^2}$$

$$-\frac{(z^2\,x\,y-2\,x^3\,y-2\,x\,y^3)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)\ \%1^2}-\frac{(z^2\,x^2+y^4-x^4)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)\ \%1^2}$$

$$-\frac{y\,x\,(z^2-2\,x^2-2\,y^2)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)\ \%1^2}-\frac{(z^2\,x^2+y^4-x^4)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(x))}{(x^2+y^2)\ \%1^2}-\frac{z^3\,y\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x))}{\%1\ \%2}\Bigg]$$

$$\Bigg[-\frac{\%3\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)^2\ \%1^2}-\frac{(z^3\,x^3-z^3\,y^2\,x+2\,z\,x^3\,y^2+2\,z\,x\,y^4)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)^2\ \%1^2}$$

$$-\frac{(-z^3\,x^2\,y+z^3\,y^3+2\,z\,x^2\,y^3+2\,z\,x^4\,y)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)^2\ \%1^2}-\frac{(z^2\,x^2-y^2\,z^2+2\,x^2\,y^2+2\,y^4)\,z\,x\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{\%1^2\,(x^2+y^2)^2}$$

$$+\frac{(z^2\,x^2-y^2\,z^2-2\,x^4-2\,x^2\,y^2)\,z\,y\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(x))}{\%1^2\,(x^2+y^2)^2}-\frac{\%3\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x))}{\%1^2\,(x^2+y^2)^2},$$

$$2\,\frac{y\,(x\,z^4-z^2\,x^3-z^2\,x\,y^2-x^5-x\,y^4-2\,x^3\,y^2)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x))}{(x^2+y^2)^2\ \%1^2}+\frac{2\,y\,(-z^3\,x^2-y^4\,z-z\,x^2\,y^2)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)^2\ \%1^2}$$

$$+\frac{2\,y\,(-z\,x\,y^3+z^3\,x\,y-z\,x^3\,y)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)^2\ \%1^2}-\frac{2\,(z^2\,x^2+x^2\,y^2+y^4)\,z\,y\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{\%1^2\,(x^2+y^2)^2}$$

$$+\frac{2\,(z^2-x^2-y^2)\,z\,y^2\,x\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(x))}{\%1^2\,(x^2+y^2)^2}-\frac{2\,x\,y\,(-x^4-z^2\,x^2-2\,x^2\,y^2-y^4-y^2\,z^2+z^4)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x))}{\%1^2\,(x^2+y^2)^2},$$

$$-\frac{(-z^2\,x\,y^3-x^5\,y-x^3\,y^3)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(x))}{(x^2+y^2)^2\ \%1^2}-\frac{(z^3\,y^2\,x+z^3\,x^3)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x))}{(x^2+y^2)^2\ \%1^2}$$

$$-\frac{(z^2\,x^2\,y^2+y^4\,z^2+x^4\,y^2-x^2\,y^4+x^6-y^6)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)^2\ \%1^2}-\frac{(z^2\,x^3\,y-2\,x\,y^5-5\,x^3\,y^3-3\,x^5\,y)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)^2\ \%1^2}$$

$$-\frac{(y^2\,z^2-y^4+x^4)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)\ \%1^2}-\frac{y\,x\,(z^2-2\,x^2-2\,y^2)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(x))}{(x^2+y^2)\ \%1^2}+\frac{x\,z^3\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x))}{\%1\ \%2}\Bigg]$$

$$\Bigg[-\frac{(x\,y^3+x^3\,y)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)\ \%1^2}-\frac{(3\,z^3\,y+2\,z\,x^2\,y+2\,z\,y^3)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)\ \%1^2}$$

$$-\frac{(3\,z^2\,x\,y+x^3\,y+x\,y^3)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)\ \%1^2}-\frac{(z^2\,x^2-2\,y^2\,z^2-y^4-2\,x^2\,y^2-x^4)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)\ \%1^2}$$

$$-\frac{3\,z^2\,x\,y\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{\%2\ \%1}-\frac{(z^2\,x^2-2\,y^2\,z^2-y^4-2\,x^2\,y^2-x^4)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(x))}{\%2\ \%1}$$

$$-\frac{z\,y\,(3\,z^2+2\,x^2+2\,y^2)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(x))}{\%2\ \%1},-\frac{(x^2\,y^2+y^4)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)\ \%1^2}$$

$$-\frac{(3\,z^3\,x+2\,z\,x^3+2\,z\,y^2\,x)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(y))}{(x^2+y^2)\ \%1^2}-\frac{(2\,z^2\,x^2+y^2\,z^2+x^4+x^2\,y^2)\,(\mathrm{d}(y)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)\ \%1^2}$$

$$-\frac{(-z^2\,x\,y-2\,x^3\,y-2\,x\,y^3)\,(\mathrm{d}(x)\ \&\wedge\ \mathrm{d}(z))}{(x^2+y^2)\ \%1^2}+\frac{(2\,z^2\,x^2-y^2\,z^2+x^4+2\,x^2\,y^2+y^4)\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(y))}{\%1\ \%2}$$

$$-\frac{3 z^2 x y (d(z) \&\wedge d(x))}{\%2\ \%1}+\frac{z x (3 z^2+2 x^2+2 y^2)(d(y) \&\wedge d(x))}{\%2\ \%1},$$

$$2 \frac{z y (d(y) \&\wedge d(z))}{\%1^2}+\frac{2 z (d(x) \&\wedge d(z)) x}{\%1^2}+\frac{2 y z (d(z) \&\wedge d(y))}{\%1^2}+\frac{2 x z (d(z) \&\wedge d(x))}{\%1^2}\Bigg]$$

$\%1 := x^2+y^2+z^2$

$\%2 := x^4+2 x^2 y^2+y^4+y^2 z^2+z^2 x^2$

$\%3 := z^4 x^2-z^4 y^2-z^2 x^4+y^4 z^2+x^2 y^4-x^6+y^6-x^4 y^2$

$$2 z x^2 (y z^2 (d(y) \&\wedge d(z))-x^2 y (d(y) \&\wedge d(z))-y^3 (d(y) \&\wedge d(z))-x^3 (d(x) \&\wedge d(z)) - y^2 x (d(x) \&\wedge d(z))+y z^2 (d(z) \&\wedge d(y))-x^2 y (d(z) \&\wedge d(y))-y^3 (d(z) \&\wedge d(y))-x^3 (d(z) \&\wedge d(x)) - y^2 x (d(z) \&\wedge d(x)))\Big/ ((x^2+y^2)^2 (x^2+y^2+z^2)^2)$$

```
> (C2F[3,1]&^d(x)+C2F[3,1]&^d(y)+C2F[3,1]&^d(z));
```

0

```
>
```

The components of the Cartan curvature 2-forms vanish.

```
> CTOR:=array([[wcollect(factor(cartanR[1,1]&^d(x)+cartanR[1,2]&^d(y)+cartanR[1,3]&^d(z)))],[wcollect(factor(cartanR[2,1]&^d(x)+cartanR[2,2]&^d(y)+cartanR[2,3]&^d(z)))],[wcollect(factor(cartanR[3,1]&^d(x)+cartanR[3,2]&^d(y)+cartanR[3,3]&^d(z)))]]):
> `Cartan_Torsion2forms`:=evalm(CTOR);
```

Cartan_Torsion2forms :=

$$\Bigg[-\frac{\%2 (d(z) \&\wedge d(y))}{(x^2+y^2)^2\ \%1^2}-\frac{(x^4 y^3+2 y^5 z^2+z^4 y^3+2 z^4 x^2 y+y^7+3 x^2 z^2 y^3+x^4 z^2 y+2 x^2 y^5)(d(x) \&\wedge d(y))}{(x^2+y^2)^2\ \%1^2}$$
$$-\frac{(-x^4 z^3-z^3 x^2 y^2)(d(z) \&\wedge d(x))}{(x^2+y^2)^2\ \%1^2}$$
$$-\frac{(x^4 y^3+2 y^5 z^2+z^4 y^3+2 z^4 x^2 y+y^7+3 x^2 z^2 y^3+x^4 z^2 y+2 x^2 y^5)(d(y) \&\wedge d(x))}{(x^2+y^2)^2\ \%1^2}-\frac{\%2 (d(y) \&\wedge d(z))}{(x^2+y^2)^2\ \%1^2}$$
$$-\frac{(-x^4 z^3-z^3 x^2 y^2)(d(x) \&\wedge d(z))}{\%1^2 (x^2+y^2)^2}\Bigg]$$

$$\Bigg[-\frac{(-z^3 y^4-z^3 x^2 y^2)(d(z) \&\wedge d(y))}{(x^2+y^2)^2\ \%1^2}$$
$$-\frac{(2 z^4 x y^2+z^4 x^3+x z^2 y^4+3 x^3 y^2 z^2+2 x^5 z^2+2 x^5 y^2+x^3 y^4+x^7)(d(x) \&\wedge d(y))}{(x^2+y^2)^2\ \%1^2}-\frac{\%2 (d(z) \&\wedge d(x))}{(x^2+y^2)^2\ \%1^2}$$
$$-\frac{(2 z^4 x y^2+z^4 x^3+x z^2 y^4+3 x^3 y^2 z^2+2 x^5 z^2+2 x^5 y^2+x^3 y^4+x^7)(d(y) \&\wedge d(x))}{(x^2+y^2)^2\ \%1^2}$$
$$-\frac{(-z^3 y^4-z^3 x^2 y^2)(d(y) \&\wedge d(z))}{(x^2+y^2)^2\ \%1^2}-\frac{\%2 (d(x) \&\wedge d(z))}{(x^2+y^2)^2\ \%1^2}\Bigg]$$

$$\Bigg[\frac{(-y^5-x^4 y-2 x^2 y^3-2 y^3 z^2-2 z^2 x^2 y)(d(z) \&\wedge d(y))}{\%1^2 (x^2+y^2)}+\frac{(3 z^3 x y+2 z x y^3+2 z x^3 y)(d(x) \&\wedge d(y))}{\%1^2 (x^2+y^2)}$$
$$+\frac{(-2 x^3 y^2-x y^4-x^5-2 z^2 x^3-2 z^2 x y^2)(d(z) \&\wedge d(x))}{\%1^2 (x^2+y^2)}+\frac{(3 z^3 x y+2 z x y^3+2 z x^3 y)(d(y) \&\wedge d(x))}{\%1^2 (x^2+y^2)}$$
$$+\frac{(-y^5-x^4 y-2 x^2 y^3-2 y^3 z^2-2 z^2 x^2 y)(d(y) \&\wedge d(z))}{\%1^2 (x^2+y^2)}$$

$$+\frac{(-2x^3y^2-xy^4-x^5-2z^2x^3-2z^2xy^2)\,(d(x)\,\&\wedge\,d(z))}{\%1^2\,(x^2+y^2)}\Bigg]$$

$$\%1 := x^2+y^2+z^2$$

$$\%2 := -z^3\,y\,x^3-z^3\,y^3\,x$$

```
> CTOR[3,1]&^dz;
```

$$0$$

THE CARTAN TORSION 2-FORMS VANISH

```
>
>
```

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=3;coord:=[x,y,z];GG:=inverse(FF);
```

$$dim := 3$$

$$coord := [x, y, z]$$

$$GG := \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & -y & \frac{z\,x\,(x^2+y^2)^{(3/2)}}{\%1} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & x & \frac{z\,y\,(x^2+y^2)^{(3/2)}}{\%1} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & 0 & -\frac{(x^2+y^2)^{(5/2)}}{\%1} \end{bmatrix}$$

$$\%1 := x^4+2x^2y^2+y^4+y^2z^2+z^2x^2$$

```
>
>
```

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1GG[i,j,k] := (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do
  ss:=0;for m from 1 to dim do ss := ss+(d1GG[a,m,k]*FF[m,b]);
  CC[a,b,k]:=simplify(-ss) od od od od ;
>
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if
  CC[a,b,k]=0 then else print(`Cartan_RIGHT`(a,b,k)=factor(CC[a,b,k])) fi od od
  od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients.

CC(abk) index (1,-1,-1)

$$\text{Cartan\_RIGHT}(1,1,1) = -\frac{x\,(2z^4x^2+z^2x^4+2y^4z^2+y^6+3z^2x^2y^2+z^4y^2+2x^2y^4+x^4y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$\text{Cartan\_RIGHT}(1,1,2) = -\frac{y\,(2z^4x^2+z^2x^4+2y^4z^2+y^6+3z^2x^2y^2+z^4y^2+2x^2y^4+x^4y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$\text{Cartan\_RIGHT}(1,1,3) = \frac{z^3\,x^2}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\text{Cartan\_RIGHT}(2,1,1) = \frac{y\,(z^4y^2+5z^2x^2y^2+3z^2x^4+2y^4z^2+4x^2y^4+y^6+2x^6+5x^4y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(2,1,2) = -\frac{x\,(2 z^4 y^2 + z^4 x^2 + y^4 z^2 + 3 z^2 x^2 y^2 + 2 z^2 x^4 + 2 x^4 y^2 + x^2 y^4 + x^6)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(2,1,3) = \frac{z^3\,y\,x}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(3,1,1) = \frac{x^2\,z\,(3 z^2 + 2 x^2 + 2 y^2)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(3,1,2) = \frac{x\,z\,y\,(3 z^2 + 2 x^2 + 2 y^2)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(3,1,3) = -\frac{(x^2+y^2+2 z^2)\,x}{(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(1,2,1) = -\frac{y\,(2 z^4 x^2 + z^2 x^4 + 2 y^4 z^2 + y^6 + 3 z^2 x^2 y^2 + z^4 y^2 + 2 x^2 y^4 + x^4 y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(1,2,2) = \frac{x\,(4 x^4 y^2 + z^4 x^2 + 2 z^2 x^4 + 3 y^4 z^2 + 5 z^2 x^2 y^2 + 2 y^6 + 5 x^2 y^4 + x^6)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(1,2,3) = \frac{z^3\,y\,x}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(2,2,1) = -\frac{x\,(2 z^4 y^2 + z^4 x^2 + y^4 z^2 + 3 z^2 x^2 y^2 + 2 z^2 x^4 + 2 x^4 y^2 + x^2 y^4 + x^6)}{(x^2+y^2+z^2)^2\,(x^2+y^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(2,2,2) = -\frac{y\,(2 z^4 y^2 + z^4 x^2 + y^4 z^2 + 3 z^2 x^2 y^2 + 2 z^2 x^4 + 2 x^4 y^2 + x^2 y^4 + x^6)}{(x^2+y^2)^2\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(2,2,3) = \frac{z^3\,y^2}{(x^2+y^2+z^2)^2\,(x^2+y^2)}$$

$$\mathrm{Cartan\_RIGHT}(3,2,1) = \frac{x\,z\,y\,(3 z^2 + 2 x^2 + 2 y^2)}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(3,2,2) = \frac{y^2\,z\,(3 z^2 + 2 x^2 + 2 y^2)}{(x^2+y^2+z^2)^2\,(x^2+y^2)}$$

$$\mathrm{Cartan\_RIGHT}(3,2,3) = -\frac{(x^2+y^2+2 z^2)\,y}{(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(1,3,1) = \frac{z^3\,x^2}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(1,3,2) = \frac{z^3\,y\,x}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(1,3,3) = \frac{(x^2+y^2)\,x}{(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(2,3,1) = \frac{z^3\,y\,x}{(x^2+y^2)\,(x^2+y^2+z^2)^2}$$

$$\mathrm{Cartan\_RIGHT}(2,3,2) = \frac{z^3\,y^2}{(x^2+y^2+z^2)^2\,(x^2+y^2)}$$

$$\text{Cartan\_RIGHT}(2,3,3)=\frac{(x^2+y^2)\,y}{(x^2+y^2+z^2)^2}$$

$$\text{Cartan\_RIGHT}(3,3,1)=-\frac{(x^2+y^2+2\,z^2)\,x}{(x^2+y^2+z^2)^2}$$

$$\text{Cartan\_RIGHT}(3,3,2)=-\frac{(x^2+y^2+2\,z^2)\,y}{(x^2+y^2+z^2)^2}$$

$$\text{Cartan\_RIGHT}(3,3,3)=\frac{z\,(x^2+y^2)}{(x^2+y^2+z^2)^2}$$

These results agree with matrix method.

Next check for Affine Torsion using the tensor methods:

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss := (CC[i,j,k]-CC[i,k,j])/2; CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if CCTTS[i,j,k]=0 then else print(`RIGHT_AffineTorsion`(i,k,j)=factor(CCTTS[i,k,j])) fi od od od ;
```

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

THE AFFINE TORSION OF THE RIGHT CARTAN MATRIX IS ZERO

Now compute the CARTAN LEFT CONNECTION

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[a,j,k] := simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrix product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0;for m to dim do ss := ss+FF[i,m]*(d1GG[m,j,k]); DD[i,j,k]:=simplify(ss) od od od od ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if DD[i,j,k]=0 then else print(`Cartan_LEFT`(i,j,k)=DD[i,j,k]) fi od od od ;
```

$$\text{Cartan\_LEFT}(1,2,1)=\frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\text{Cartan\_LEFT}(1,2,2)=-\frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\text{Cartan\_LEFT}(1,3,1)=-\frac{x\,z\,\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(1,3,2)=-\frac{y\,\sqrt{x^2+y^2}\,z}{(x^2+y^2+z^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(1,3,3)=\frac{(x^2+y^2)^{(3/2)}}{(x^2+y^2+z^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(2,1,1)=-\frac{y}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)}$$

$$\text{Cartan\_LEFT}(2,1,2)=\frac{x}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)}$$

$$\text{Cartan\_LEFT}(2,2,1)=\frac{x}{x^2+y^2}$$

$$\text{Cartan\_LEFT}(2,2,2)=\frac{y}{x^2+y^2}$$

$$\text{Cartan\_LEFT}(2,3,1)=-\frac{y\,z}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$\text{Cartan\_LEFT}(2,3,2)=\frac{z\,x}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$\text{Cartan\_LEFT}(3,1,1)=\frac{z\,x}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(3,1,2)=\frac{z\,y}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(3,1,3)=-\frac{1}{\sqrt{x^2+y^2+z^2}\,\sqrt{x^2+y^2}}$$

$$\text{Cartan\_LEFT}(3,2,1)=\frac{z\,y}{(x^2+y^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(3,2,2)=-\frac{z\,x}{(x^2+y^2)^{(3/2)}}$$

$$\text{Cartan\_LEFT}(3,3,1)=\frac{x\,(x^2+y^2+2\,z^2)}{x^4+2\,x^2\,y^2+y^4+y^2\,z^2+z^2\,x^2}$$

$$\text{Cartan\_LEFT}(3,3,2)=\frac{y\,(x^2+y^2+2\,z^2)}{x^4+2\,x^2\,y^2+y^4+y^2\,z^2+z^2\,x^2}$$

$$\text{Cartan\_LEFT}(3,3,3)=-\frac{z}{x^2+y^2+z^2}$$

The anti-symmetric part of the LEFT CARTAN Connection appear above.
Check for assymetry (LEFT Torsion)

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (DD[i,j,k]-DD[i,k,j])/2; TTS[i,j,k]:=simplify(ss) od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  TTS[i,j,k]=0 then else print(`LEFT_Torsion`(i,k,j)=TTS[i,k,j]) fi od od od ;
```

$$\text{LEFT\_Torsion}(1,2,1)=\frac{1}{2}\frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\text{LEFT\_Torsion}(1,3,1)=-\frac{1}{2}\frac{x\,z\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^{(3/2)}}$$

$$\text{LEFT\_Torsion}(1,1,2)=-\frac{1}{2}\frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\text{LEFT\_Torsion}(1,3,2)=-\frac{1}{2}\frac{y\sqrt{x^2+y^2}\,z}{(x^2+y^2+z^2)^{(3/2)}}$$

$$\text{LEFT\_Torsion}(1,1,3)=\frac{1}{2}\frac{x\,z\sqrt{x^2+y^2}}{(x^2+y^2+z^2)^{(3/2)}}$$

$$\text{LEFT\_Torsion}(1,2,3)=\frac{1}{2}\frac{y\sqrt{x^2+y^2}\,z}{(x^2+y^2+z^2)^{(3/2)}}$$

$$\mathrm{LEFT\_Torsion}(2,2,1)=\frac{1}{2}\frac{x\,(-1+\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)}$$

$$\mathrm{LEFT\_Torsion}(2,3,1)=-\frac{1}{2}\frac{y\,z}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$\mathrm{LEFT\_Torsion}(2,1,2)=-\frac{1}{2}\frac{x\,(-1+\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)}$$

$$\mathrm{LEFT\_Torsion}(2,3,2)=\frac{1}{2}\frac{z\,x}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$\mathrm{LEFT\_Torsion}(2,1,3)=\frac{1}{2}\frac{y\,z}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$\mathrm{LEFT\_Torsion}(2,2,3)=-\frac{1}{2}\frac{z\,x}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$\mathrm{LEFT\_Torsion}(3,2,1)=\frac{1}{2}\frac{y\,z\,(-1+\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)^{(3/2)}}$$

$$\mathrm{LEFT\_Torsion}(3,3,1)=$$

$$\frac{1}{2}\frac{x^4+2\,x^2\,y^2+y^4+y^2\,z^2+z^2\,x^2+x^3\sqrt{\%1}\sqrt{x^2+y^2}+x\sqrt{\%1}\sqrt{x^2+y^2}\,y^2+2\,x\sqrt{\%1}\sqrt{x^2+y^2}\,z^2}{\sqrt{\%1}\sqrt{x^2+y^2}\,(x^4+2\,x^2\,y^2+y^4+y^2\,z^2+z^2\,x^2)}$$

$$\%1:=x^2+y^2+z^2$$

$$\mathrm{LEFT\_Torsion}(3,1,2)=-\frac{1}{2}\frac{y\,z\,(-1+\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)^{(3/2)}}$$

$$\mathrm{LEFT\_Torsion}(3,3,2)=\frac{1}{2}\frac{y\,(x^2+y^2+2\,z^2)}{x^4+2\,x^2\,y^2+y^4+y^2\,z^2+z^2\,x^2}$$

$$\mathrm{LEFT\_Torsion}(3,1,3)=$$

$$-\frac{1}{2}\frac{x^4+2\,x^2\,y^2+y^4+y^2\,z^2+z^2\,x^2+x^3\sqrt{\%1}\sqrt{x^2+y^2}+x\sqrt{\%1}\sqrt{x^2+y^2}\,y^2+2\,x\sqrt{\%1}\sqrt{x^2+y^2}\,z^2}{\sqrt{\%1}\sqrt{x^2+y^2}\,(x^4+2\,x^2\,y^2+y^4+y^2\,z^2+z^2\,x^2)}$$

$$\%1:=x^2+y^2+z^2$$

$$\mathrm{LEFT\_Torsion}(3,2,3)=-\frac{1}{2}\frac{y\,(x^2+y^2+2\,z^2)}{x^4+2\,x^2\,y^2+y^4+y^2\,z^2+z^2\,x^2}$$

For this example from cartesian to spherical coordinates, there is no assymetry for the [CR], but there is assymetry for [CL]
(The physical implication is not clear to me)

Next the Christoffel symbols will be computed for the metric on the initial state.
As the metric on {x,y,z} is presumed to be the unit matrix, all the Christoffel symbols sould be zero

Christoffel Connection coefficients from the induced metric

```
>
> metric:= array([[1,0,0],[0,1,0],[0,0,1]]);
```

$$metric:=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

```
> metricinverse:=inverse(metric):
```

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do
  C1S[i,j,k] := 0  od od od; for i from 1 to dim do for j from 1 to dim do for
  k from 1 to dim do C1S[i,j,k] :=
  1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss :=
  0; for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then  else print(`Christoffel_Gamma2`(i-1,j-1,k-1)=C2S[i,j,k])
  fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind   on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

If no entries appear above the Christoffel symbols on the domain space vanish

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) = ChristoffelGamma(ijk) + T(ijk)

Compute the T(i,j,k):

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0;
  ss := (CC[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 and CC[i,j,k]=0 then else
  print(`T`(i,j,k)=simplify(SHIPTR[i,j,k])) fi od od od ;
```

T(ijk) index (1,-1,-1)

$$\mathrm{T}(1,1,1) = -\frac{x\,(2\,z^4\,x^2 + z^2\,x^4 + 2\,y^4\,z^2 + y^6 + 3\,z^2\,x^2\,y^2 + z^4\,y^2 + 2\,x^2\,y^4 + x^4\,y^2)}{(x^4 + 2\,x^2\,y^2 + y^4 + y^2\,z^2 + z^2\,x^2)^2}$$

$$\mathrm{T}(1,1,2) = -\frac{(2\,z^4\,x^2 + z^2\,x^4 + 2\,y^4\,z^2 + y^6 + 3\,z^2\,x^2\,y^2 + z^4\,y^2 + 2\,x^2\,y^4 + x^4\,y^2)\,y}{(x^4 + 2\,x^2\,y^2 + y^4 + y^2\,z^2 + z^2\,x^2)^2}$$

$$\mathrm{T}(1,1,3) = \frac{x^2\,z^3}{(x^2 + y^2 + z^2)\,(x^4 + 2\,x^2\,y^2 + y^4 + y^2\,z^2 + z^2\,x^2)}$$

$$T(1,2,1) = -\frac{(2 z^4 x^2 + z^2 x^4 + 2 y^4 z^2 + y^6 + 3 z^2 x^2 y^2 + z^4 y^2 + 2 x^2 y^4 + x^4 y^2)\, y}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)^2}$$

$$T(1,2,2) = \frac{(4 x^4 y^2 + z^4 x^2 + 2 z^2 x^4 + 3 y^4 z^2 + 5 z^2 x^2 y^2 + 2 y^6 + 5 x^2 y^4 + x^6)\, x}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)^2}$$

$$T(1,2,3) = \frac{x z^3 y}{(x^2 + y^2 + z^2)(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)}$$

$$T(1,3,1) = \frac{x^2 z^3}{(x^2 + y^2 + z^2)(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)}$$

$$T(1,3,2) = \frac{x z^3 y}{(x^2 + y^2 + z^2)(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)}$$

$$T(1,3,3) = \frac{(x^2 + y^2)\, x}{(x^2 + y^2 + z^2)^2}$$

$$T(2,1,1) = \frac{(z^4 y^2 + 5 z^2 x^2 y^2 + 3 z^2 x^4 + 2 y^4 z^2 + 4 x^2 y^4 + y^6 + 2 x^6 + 5 x^4 y^2)\, y}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)^2}$$

$$T(2,1,2) = -\frac{x\,(2 z^4 y^2 + z^4 x^2 + y^4 z^2 + 3 z^2 x^2 y^2 + 2 z^2 x^4 + 2 x^4 y^2 + x^2 y^4 + x^6)}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)^2}$$

$$T(2,1,3) = \frac{x z^3 y}{(x^2 + y^2 + z^2)(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)}$$

$$T(2,2,1) = -\frac{x\,(2 z^4 y^2 + z^4 x^2 + y^4 z^2 + 3 z^2 x^2 y^2 + 2 z^2 x^4 + 2 x^4 y^2 + x^2 y^4 + x^6)}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)^2}$$

$$T(2,2,2) = -\frac{y\,(2 z^4 y^2 + z^4 x^2 + y^4 z^2 + 3 z^2 x^2 y^2 + 2 z^2 x^4 + 2 x^4 y^2 + x^2 y^4 + x^6)}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)^2}$$

$$T(2,2,3) = \frac{y^2 z^3}{(x^2 + y^2 + z^2)(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)}$$

$$T(2,3,1) = \frac{x z^3 y}{(x^2 + y^2 + z^2)(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)}$$

$$T(2,3,2) = \frac{y^2 z^3}{(x^2 + y^2 + z^2)(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)}$$

$$T(2,3,3) = \frac{(x^2 + y^2)\, y}{(x^2 + y^2 + z^2)^2}$$

$$T(3,1,1) = \frac{x^2 z\,(3 z^2 + 2 x^2 + 2 y^2)}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)(x^2 + y^2 + z^2)}$$

$$T(3,1,2) = \frac{z x y\,(3 z^2 + 2 x^2 + 2 y^2)}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)(x^2 + y^2 + z^2)}$$

$$T(3,1,3) = -\frac{(x^2 + y^2 + 2 z^2)\, x}{(x^2 + y^2 + z^2)^2}$$

$$T(3,2,1) = \frac{z x y\,(3 z^2 + 2 x^2 + 2 y^2)}{(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)(x^2 + y^2 + z^2)}$$

$$T(3,2,2) = \frac{z y^2\,(3 z^2 + 2 x^2 + 2 y^2)}{(x^2 + y^2 + z^2)(x^4 + 2 x^2 y^2 + y^4 + y^2 z^2 + z^2 x^2)}$$

$$T(3,2,3) = -\frac{(x^2+y^2+2z^2)\,y}{(x^2+y^2+z^2)^2}$$

$$T(3,3,1) = -\frac{(x^2+y^2+2z^2)\,x}{(x^2+y^2+z^2)^2}$$

$$T(3,3,2) = -\frac{(x^2+y^2+2z^2)\,y}{(x^2+y^2+z^2)^2}$$

$$T(3,3,3) = \frac{z\,(x^2+y^2)}{(x^2+y^2+z^2)^2}$$

Right Cartan(ijk) = Christoffel Gamma(ijk) + T(ijk)

These computations agree with Shipov on page 217, except for the T(3,3,2), T(3,3,1) and the T(2,2,1) terms given above.

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