```
> restart:with(linalg):with(plots):
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the name changecoords has been redefined
```

Fresnel Kummer WAVE Surfaces

CHIRALITY

spie07fresnel2.mws   R.M.Kiehn 11/15/97 updated 2/25/2005 and AUgust 12, 2007

This program will compute and plot the Fresnel WAVE surfaces for a generalized constitutive matrix.
The surfaces are specializations of a Kummer quartic surface, and are related to the Clifford algebra Cl(3,3).
Specialized forms have been selected for easy visualization. The formulas can be modified to handle the general case.

The constitutive tensor is a 6 x 6 complex matrix partitioned into 3x3 matrices.
The on-diagonal upper 3x3 matrix is the epsilon matrix.
The epsilon matrix real part describes electric birefringence.
The lower on-diagonal 3x3 matrix is the reciprocal mu matrix.
The real part of the on diagonal reciprocal mu matrix represents magnetic birefringence
The imaginary part 3x3 on-diagonal matrices represents electric and magnetic Faraday effects.
The real part of the off-diagonal 3x3 matrix represents Fresnel-Fizeau effects,
The imaginary part of the odd diagonal 3x3 matrix represents Optical Activity

The determinant of the constitutive matrix is proportional to the admittance (reciprocal radiation impedance) of "constitutive" free space, cubed.

The fundamental reference for the constitutive matrix is
E.J. Post "The Formal Structure of Electromagnetics" Dover 1997

The sign convention of Post is used below.

The determinant of the constitutive matrix is proportional to the cube of the admittance.
The admittance is the reciprocal of the radiation impedance.
The impedance can be complex.
If not real, the impedance implies a dissipative component,
ultimately related to E dot B not zero.
For no dissipation the constitutive matrix must be Hermitean.

\>

The constitutive matrix X is decomposed into 3x3 blocks as:

```
> X(real):=matrix([[`dielectric bifringence`,`Fresnel-Fizeau`],[`Fresnel-Fizeau`,`magnetic birefringence`]]);
```

$$X(real) := \begin{bmatrix} \textit{dielectric bifringence} & \textit{Fresnel-Fizeau} \\ \textit{Fresnel-Fizeau} & \textit{magnetic birefringence} \end{bmatrix}$$

```
> X(imag):=matrix([[`dielectric Faraday`,`Optical Activity`],[`Optical Activity`,`magnetic Faraday`]]);
```

$$X(imag) := \begin{bmatrix} \textit{dielectric Faraday} & \textit{Optical Activity} \\ \textit{Optical Activity} & \textit{magnetic Faraday} \end{bmatrix}$$

```
> constitutive_tensor:=matrix([[-epsilon,Gamma(D)],[Gamma(H),1/mu]]);D=[-epsilon]*(-E)+[Gamma(D)]*B;
> H=[Gamma(H)]*(-E)+[mu^(-1)]*B;
```

$$constitutive\_tensor := \begin{bmatrix} -\varepsilon & \Gamma(D) \\ \Gamma(H) & \frac{1}{\mu} \end{bmatrix}$$

$$D = -[-\varepsilon]\,E + [\Gamma(D)]\,B$$

$$H = -[\Gamma(H)]\,E + \left[\frac{1}{\mu}\right]B$$

\>

In the language of exterior differential forms, EM theory is based on two axioms. F-dA=0, J-dG = 0.

```
> Maxwell_Faraday_PDEs:=dF=0;Maxwell_Ampere_PDEs:=dG=J;dJ=0;
```

$$Maxwell\_Faraday\_PDEs := dF = 0$$

$$Maxwell\_Ampere\_PDEs := dG = J$$

$$dJ = 0$$

For zero J, the Maxwell Faraday and Maxwell Ampere equations become a

differential ideal.
A wave solution searches for a wave vector 1-form, k =[k,omega], that annihilates the 2-forms F and G;

```
> Faraday:=k&^F=0;
```

$$Faraday := k \&\char`^ F = 0$$

```
> Ampere:=k&^G=0;;
```

$$Ampere := k \&\char`^ G = 0$$

These 6 equations have 12 unknown components of the four 3 vectors (D,E,H,B). The equations, AS SUCH, are not soluble.
But if the vectors D, H are related to E, B by a constitutive equation (which can be complex)
THEN THE 12 EQUATIONS IN 12 UNKNOWNS HAVE A HOMOGENEOUS SOLUTION,
IF CRAMERS RULE IS VALID: THE DETERMINANT OF THE 12 HOMOGENEOUS EQUATIONS MUST VANISH.

From the theory of determinants, a matrix equation of 3x3 matrices can be found, such that the
determinant of the matrix equation is the same as the determinant of the system of 12 homogeneous functions.

The matrix equation (CRAMERS) is to be solved for its eigen values which represent the effective index of refraction in the chosen direction. If
the constitutive matrix is Hermitean, then the eigenvalues are real.

Typical Constitutive Matrix entries are shown below. The elements have been scaled such that the "position vector" to a Fresnel surface point in this space has a value equal to the "effective" index of refraction in that direction. The "phase velocity" in that direction is c/n, where c is defined as 1/sqrt(epsilon*mu). The "phase velocity" should be viewed as at the speed of the "momentum flux" (DxB) in the direction of the position vector.

For the Fresnel Ray surface which defines the propagation speed of the energy flux (ExH) (usually described as the "group" velocity)

See Kiehn, R. M. , Kiehn, G. P., and Roberds, (1991), Parity and time-reversal symmetry

breaking, singular solutions and Fresnel surfaces, Phys. Rev A 43, pp. 5165-5671
or http://www22.pair.com/csdc/pdf/timerev.pdf

Note that when the media is dispersive, the energy flux and the momentum flux do not propagate at the same speed, but the product of the group speed times the phase speed is always 1/(epsilon.mu).

**In that which follows typical entries have been encoded into the submatrices.
With Maple you can change the entries to anything you want.**

The complex epsilon (permittivity) matrix

Dielectric birefringence (real) and dielectric Faraday (imag) effects

```
>
> eps := matrix([[A*epsilon+I*fdd, I*fdr, 0], [-I*fdr,
  B*epsilon+I*fdd, 0], [0, 0, C*epsilon+I*fdd]]);
```

$$eps := \begin{bmatrix} A\,\varepsilon + I\,fdd & I\,fdr & 0 \\ -I\,fdr & B\,\varepsilon + I\,fdd & 0 \\ 0 & 0 & C\,\varepsilon + I\,fdd \end{bmatrix}$$

**The real (symmetric) part of the epsilon matrix (above) is the permittivity matrix and leads to bifrefringence;
the complex antisymmetric part represents dielectric Faraday effects.**

The Gamma submatrix (Real and Imaginary Parts)

Fresnel-Fizeau (real) and Optical Activity (imag) effects

```
> Gamma(real):=evalm(matrix([[gr,s,0],[-s,gr,0],[0,0,gr]]));
```

$$\Gamma(real) := \begin{bmatrix} gr & s & 0 \\ -s & gr & 0 \\ 0 & 0 & gr \end{bmatrix}$$

**The real part of the off-diagonal gamma matrix is given above.
The Fresnel-Fizeau and Sagnac effect is contained the real part of the off diagonal**

gamma matrix
and optical activity in the imaginary part.

```
> Gamma(imag):=evalm(matrix([[I*gi,I*p,0],[-I*p,I*gi,0],[0,0,I*gi]]));
```

$$\Gamma(imag) := \begin{bmatrix} I\,gi & I\,p & 0 \\ -I\,p & I\,gi & 0 \\ 0 & 0 & I\,gi \end{bmatrix}$$

```
>
```

The optical activity is due to the complex part of the off diagonal submatrix above. Realize that the coefficients used above are representative and can be adjusted to suit.

```
> Gamma(D):=evalm(Gamma(real)+Gamma(imag));
```

$$\Gamma(\mathrm{D}) := \begin{bmatrix} gr + I\,gi & s + I\,p & 0 \\ -s - I\,p & gr + I\,gi & 0 \\ 0 & 0 & gr + I\,gi \end{bmatrix}$$

The Hermitean conjugate (complex conjugate transpose) of the Gamma Matrix

```
> Gamma(Conj):=evalm(alpha*transpose(Gamma(real))-evalm(beta*transpose(Gamma(imag))));Gamma(H):=subs(alpha=1,beta=1,Gamma(Conj));
```

$$\Gamma(Conj) := \begin{bmatrix} \alpha\,gr - I\,\beta\,gi & -\alpha\,s + I\,\beta\,p & 0 \\ \alpha\,s - I\,\beta\,p & \alpha\,gr - I\,\beta\,gi & 0 \\ 0 & 0 & \alpha\,gr - I\,\beta\,gi \end{bmatrix}$$

$$\Gamma(H) := \begin{bmatrix} gr - I\,gi & -s + I\,p & 0 \\ s - I\,p & gr - I\,gi & 0 \\ 0 & 0 & gr - I\,gi \end{bmatrix}$$

The inverse mu matrix (reciprocal permeability matrix)

Magnetic birefringence (real) and magnetic Faraday effects (imag)

```
> mu(permeability):=evalm(matrix([[a/(mu),I*fmr,0],[-I*fmr,b/(mu),0],[0,0,c/(mu)+I*fmd]]));
```

$$\mu(permeability) := \begin{bmatrix} \frac{a}{\mu} & I\, fmr & 0 \\ -I\, fmr & \frac{b}{\mu} & 0 \\ 0 & 0 & \frac{c}{\mu} + I\, fmd \end{bmatrix}$$

The real (symetric) part of the mu matrix is the magnetic birefringence part; the imaginary antisymmetric part represents magnetic Faraday effects.

```
>
> XX:=matrix([[A*epsilon+I*fdd,I*fdr,0,gr+I*gi,s+I*p,0],[-I*fdr,B*ep
silon+I*fdd,0,-s-I*p,gr+I*gi,0],[0,0,C*epsilon+I*fdd,0,0,gr+I*gi],
[gr-I*gi,-s+I*p,0,a/mu,I*fmr,0],[s-I*p,gr-I*gi,0,-I*fmr,b/mu,0],[0
,0,gr-I*gi,0,0,c/mu+I*fmd]]);
```

$$XX := \begin{bmatrix} A\,\varepsilon + I\, fdd & I\, fdr & 0 & gr + I\, gi & s + I\, p & 0 \\ -I\, fdr & B\,\varepsilon + I\, fdd & 0 & -s - I\, p & gr + I\, gi & 0 \\ 0 & 0 & C\,\varepsilon + I\, fdd & 0 & 0 & gr + I\, gi \\ gr - I\, gi & -s + I\, p & 0 & \frac{a}{\mu} & I\, fmr & 0 \\ s - I\, p & gr - I\, gi & 0 & -I\, fmr & \frac{b}{\mu} & 0 \\ 0 & 0 & gr - I\, gi & 0 & 0 & \frac{c}{\mu} + I\, fmd \end{bmatrix}$$

```
>
>
> mu(inverse):=evalm(inverse(mu(permeability)));
```

$$\mu(inverse) := \begin{bmatrix} \frac{\mu\, b}{a\, b - fmr^2\, \mu^2} & \frac{-I\, \mu^2\, fmr}{a\, b - fmr^2\, \mu^2} & 0 \\ \frac{I\, \mu^2\, fmr}{a\, b - fmr^2\, \mu^2} & \frac{\mu\, a}{a\, b - fmr^2\, \mu^2} & 0 \\ 0 & 0 & \frac{\mu}{c + I\, fmd\, \mu} \end{bmatrix}$$

The matrices are now scaled for ease of computation and plotting.
Values of n = [x.y.z] > 1 imply a phase velocity less than c =1/sqrt(epsilon*mu).
Values of n < 1 imply phase speeds greater than c.

The N matrix (Index of Refraction Operator)

```
> NV:=evalm(matrix([[0,z,-y],[-z,0,x],[y,-x,0]]));N:=evalm(NV*((epsilon)^(1/2)*(mu)^(1/2)));
```

$$NV := \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}$$

$$N := \begin{bmatrix} 0 & \sqrt{\varepsilon}\,\sqrt{\mu}\,z & -\sqrt{\varepsilon}\,\sqrt{\mu}\,y \\ -\sqrt{\varepsilon}\,\sqrt{\mu}\,z & 0 & \sqrt{\varepsilon}\,\sqrt{\mu}\,x \\ \sqrt{\varepsilon}\,\sqrt{\mu}\,y & -\sqrt{\varepsilon}\,\sqrt{\mu}\,x & 0 \end{bmatrix}$$

The N matrix above is the scaled "index of refraction matrix" which acts like a cross product operator. The N matrix has three components which form the position vector to the Kummer surface. The magnitude of N is the "index" of refraction, n, in the direction of the vector N. The phase speed is then related to 1/n.

The elements of the constitutive matrix have been scaled (below) for algebraic reduction purposes.

```
> M:=evalm(innerprod(N,mu(permeability)));MM:=evalm(innerprod(M,N));
```

$$M := \begin{bmatrix} -I\sqrt{\varepsilon}\,\sqrt{\mu}\,z\,fmr & \dfrac{\sqrt{\varepsilon}\,z\,b}{\sqrt{\mu}} & -\dfrac{\sqrt{\varepsilon}\,y\,(c + I\,fmd\,\mu)}{\sqrt{\mu}} \\ -\dfrac{\sqrt{\varepsilon}\,z\,a}{\sqrt{\mu}} & -I\sqrt{\varepsilon}\,\sqrt{\mu}\,z\,fmr & \dfrac{\sqrt{\varepsilon}\,x\,(c + I\,fmd\,\mu)}{\sqrt{\mu}} \\ \dfrac{\sqrt{\varepsilon}\,(y\,a + I\,\mu\,x\,fmr)}{\sqrt{\mu}} & \dfrac{\sqrt{\varepsilon}\,(I\,\mu\,y\,fmr - x\,b)}{\sqrt{\mu}} & 0 \end{bmatrix}$$

$MM :=$

$$\begin{bmatrix} -\varepsilon z^2 b - \varepsilon y^2 c - I\varepsilon y^2 fmd\,\mu, & -I\varepsilon\mu z^2 fmr + \varepsilon y x c + I\varepsilon y x\,fmd\,\mu, & I\varepsilon\mu z\,fmr\,y + \varepsilon z b x \\ I\varepsilon\mu z^2 fmr + \varepsilon y x c + I\varepsilon y x\,fmd\,\mu, & -\varepsilon z^2 a - \varepsilon x^2 c - I\varepsilon x^2 fmd\,\mu, & \varepsilon z a y - I\varepsilon\mu z\,fmr\,x \\ -\varepsilon(I\mu y\,fmr - x b)\,z, & \varepsilon(y a + I\mu x\,fmr)\,z, & -\varepsilon y^2 a - \varepsilon x^2 b \end{bmatrix}$$

```
> CRAMERS:=evalm(eps)+simplify(evalm(innerprod(Gamma(D),N)-innerprod(N,Gamma(H))))+evalm(MM);
```

$$CRAMERS := \begin{bmatrix} A\varepsilon + I\,fdd & I\,fdr & 0 \\ -I\,fdr & B\varepsilon + I\,fdd & 0 \\ 0 & 0 & C\varepsilon + I\,fdd \end{bmatrix}$$

$$+ \begin{bmatrix} -2\sqrt{\varepsilon}\sqrt{\mu}\,z\,s, & 2I\sqrt{\varepsilon}\sqrt{\mu}\,z\,gi, & -2I\sqrt{\varepsilon}\sqrt{\mu}\,y\,gi + \sqrt{\varepsilon}\sqrt{\mu}\,x\,s + I\sqrt{\varepsilon}\sqrt{\mu}\,x\,p \\ -2I\sqrt{\varepsilon}\sqrt{\mu}\,z\,gi, & -2\sqrt{\varepsilon}\sqrt{\mu}\,z\,s, & \sqrt{\varepsilon}\sqrt{\mu}\,y\,s + I\sqrt{\varepsilon}\sqrt{\mu}\,y\,p + 2I\sqrt{\varepsilon}\sqrt{\mu}\,x\,gi \\ 2I\sqrt{\varepsilon}\sqrt{\mu}\,y\,gi + \sqrt{\varepsilon}\sqrt{\mu}\,x\,s - I\sqrt{\varepsilon}\sqrt{\mu}\,x\,p, & -2I\sqrt{\varepsilon}\sqrt{\mu}\,x\,gi + \sqrt{\varepsilon}\sqrt{\mu}\,y\,s - I\sqrt{\varepsilon}\sqrt{\mu}\,y\,p, & 0 \end{bmatrix}$$

$$+\begin{bmatrix} -\varepsilon z^2 b - \varepsilon y^2 c - I\,\varepsilon\,y^2\,fmd\,\mu\,, & -I\,\varepsilon\,\mu\,z^2\,fmr + \varepsilon\,y\,x\,c + I\,\varepsilon\,y\,x\,fmd\,\mu\,, & I\,\varepsilon\,\mu\,z\,fmr\,y + \varepsilon\,z\,b\,x \\ I\,\varepsilon\,\mu\,z^2\,fmr + \varepsilon\,y\,x\,c + I\,\varepsilon\,y\,x\,fmd\,\mu\,, & -\varepsilon z^2 a - \varepsilon x^2 c - I\,\varepsilon\,x^2\,fmd\,\mu\,, & \varepsilon\,z\,a\,y - I\,\varepsilon\,\mu\,z\,fmr\,x \\ -\varepsilon\,(I\,\mu\,y\,fmr - x\,b)\,z\,, & \varepsilon\,(y\,a + I\,\mu\,x\,fmr)\,z\,, & -\varepsilon\,y^2\,a - \varepsilon\,x^2\,b \end{bmatrix}$$

The CRAMERS matrix is a matrix decomposition of the determinant of the 12 by 12 homogeneous system.
Note that the CRAMERS matrix ewuation does not depend upon the diagonal real part, gr, of the gamma matrix.
This is FRESNEL FIZEAU isotropic expansion,
The Hermitean conjugate of the upper right Gamma matrix is used for the lower left Gamma matrix,
The CRAMERS matrix still depends upon the real anti-symmetric part sa (rotations - Sagnac)
The determinant of the Crammer matrix is equal to the Kummer polynomial function, whose zero set defines the eigenvalues for the index of refraction, n, and therefor the phase velocity of propagation.

```
> SUBSa:=fdd=fdda*epsilon,fdr=fdra*epsilon,p=pa*epsilon^(1/2)/(mu)^(1/2),gr=gra*epsilon^(1/2)/(mu)^(1/2),r=ra*epsilon^(1/2)/(mu)^(1/2),fmr=fmra/mu,fmd=fmda/mu,s=sa*epsilon^(1/2)/(mu)^(1/2),gi=gia*epsilon^(1/2)/(mu)^(1/2);
```

$$SUBSa := fdd = fdda\,\varepsilon,\ fdr = fdra\,\varepsilon,\ p = \frac{pa\sqrt{\varepsilon}}{\sqrt{\mu}},\ gr = \frac{gra\sqrt{\varepsilon}}{\sqrt{\mu}},\ r = \frac{ra\sqrt{\varepsilon}}{\sqrt{\mu}},\ fmr = \frac{fmra}{\mu},\ fmd = \frac{fmda}{\mu},\ s = \frac{sa\sqrt{\varepsilon}}{\sqrt{\mu}},\ gi = \frac{gia\sqrt{\varepsilon}}{\sqrt{\mu}}$$

The Constitutive matrix CHIH

```
> CHIH:=subs(SUBSa,evalm(XX));
```

$$CHIH := \begin{bmatrix} A\,\varepsilon + I\,fdda\,\varepsilon, & I\,fdra\,\varepsilon, & 0, & \frac{gra\sqrt{\varepsilon}}{\sqrt{\mu}} + \frac{I\,gia\sqrt{\varepsilon}}{\sqrt{\mu}}, & \frac{sa\sqrt{\varepsilon}}{\sqrt{\mu}} + \frac{I\,pa\sqrt{\varepsilon}}{\sqrt{\mu}}, & 0 \\ -I\,fdra\,\varepsilon, & B\,\varepsilon + I\,fdda\,\varepsilon, & 0, & -\frac{sa\sqrt{\varepsilon}}{\sqrt{\mu}} - \frac{I\,pa\sqrt{\varepsilon}}{\sqrt{\mu}}, & \frac{gra\sqrt{\varepsilon}}{\sqrt{\mu}} + \frac{I\,gia\sqrt{\varepsilon}}{\sqrt{\mu}}, & 0 \\ 0, & 0, & C\,\varepsilon + I\,fdda\,\varepsilon, & 0, & 0, & \frac{gra\sqrt{\varepsilon}}{\sqrt{\mu}} + \frac{I\,gia\sqrt{\varepsilon}}{\sqrt{\mu}} \\ \frac{gra\sqrt{\varepsilon}}{\sqrt{\mu}} - \frac{I\,gia\sqrt{\varepsilon}}{\sqrt{\mu}}, & -\frac{sa\sqrt{\varepsilon}}{\sqrt{\mu}} + \frac{I\,pa\sqrt{\varepsilon}}{\sqrt{\mu}}, & 0, & \frac{a}{\mu}, & \frac{I\,fmra}{\mu}, & 0 \\ \frac{sa\sqrt{\varepsilon}}{\sqrt{\mu}} - \frac{I\,pa\sqrt{\varepsilon}}{\sqrt{\mu}}, & \frac{gra\sqrt{\varepsilon}}{\sqrt{\mu}} - \frac{I\,gia\sqrt{\varepsilon}}{\sqrt{\mu}}, & 0, & \frac{-I\,fmra}{\mu}, & \frac{b}{\mu}, & 0 \\ 0, & 0, & \frac{gra\sqrt{\varepsilon}}{\sqrt{\mu}} - \frac{I\,gia\sqrt{\varepsilon}}{\sqrt{\mu}}, & 0, & 0, & \frac{c}{\mu} + \frac{I\,fmda}{\mu} \end{bmatrix}$$

>

The Fresnel Hamiltonian Matrix HHH

```
> HHH:=subs(SUBSa,evalm(CRAMERS));
```

$HHH :=$

$[A\,\varepsilon + I\,fdda\,\varepsilon - 2\,\varepsilon\,z\,sa - \varepsilon\,z^2\,b - \varepsilon\,y^2\,c - I\,\varepsilon\,y^2\,fmda\,,$

$I\,fdra\,\varepsilon + 2\,I\,\varepsilon\,z\,gia - I\,\varepsilon\,z^2\,fmra + \varepsilon\,y\,x\,c + I\,\varepsilon\,y\,x\,fmda\,,$

$-2\,I\,\varepsilon\,y\,gia + \varepsilon\,x\,sa + I\,\varepsilon\,x\,pa + I\,\varepsilon\,z\,fmra\,y + \varepsilon\,z\,b\,x]$

$[-I\,fdra\,\varepsilon - 2\,I\,\varepsilon\,z\,gia + I\,\varepsilon\,z^2\,fmra + \varepsilon\,y\,x\,c + I\,\varepsilon\,y\,x\,fmda\,,$

$B\,\varepsilon + I\,fdda\,\varepsilon - 2\,\varepsilon\,z\,sa - \varepsilon\,z^2\,a - \varepsilon\,x^2\,c - I\,\varepsilon\,x^2\,fmda\,,$

$\varepsilon\,y\,sa + I\,\varepsilon\,y\,pa + 2\,I\,\varepsilon\,x\,gia + \varepsilon\,z\,a\,y - I\,\varepsilon\,z\,fmra\,x]$

$[2\,I\,\varepsilon\,y\,gia + \varepsilon\,x\,sa - I\,\varepsilon\,x\,pa - \varepsilon\,(I\,y\,fmra - x\,b)\,z\,,$

$-2\,I\,\varepsilon\,x\,gia + \varepsilon\,y\,sa - I\,\varepsilon\,y\,pa + \varepsilon\,(y\,a + I\,x\,fmra)\,z\,, C\,\varepsilon + I\,fdda\,\varepsilon - \varepsilon\,y^2\,a - \varepsilon\,x^2\,b]$

The notation is such that :

fdda = dielectric Faraday expansion

fdra = dielectric Faraday rotation

fmda = magnetic Faraday expansion

fmra = magnetic Faraday rotation

gra = Fresnel-Fizeau expansion

gia = OA expansion

pa = OA rotation

sa = Fresnel-Fizeau rotation

The Cramers matrix equation does not depend upon gra.

Now compute the various matrices whose determinant create the Kummer equation.

```
> ADZ:=(mu^3/epsilon^3)*(det(CHIH));
```

$$ADZ := -(-C\,c - I\,C\,fmda - I\,fdda\,c + fdda\,fmda + gra^2 + gia^2)\,(-2\,fdra\,fmra\,sa^2 + gra^4 - 2\,fdra\,fmra\,gra^2 - A\,gia^2\,a + 2\,gra^2\,gia^2 + 2\,gra^2\,sa^2 - 2\,gra^2\,pa^2 - 2\,sa^2\,gia^2 + 2\,sa^2\,pa^2 - A\,pa^2\,b - A\,a\,gra^2 - B\,b\,gra^2 - B\,a\,sa^2 + A\,B\,a\,b - 2\,A\,sa\,fmra\,gia + 2\,A\,gra\,fmra\,pa + 2\,gra\,fdra\,b\,pa + 2\,gra\,B\,fmra\,pa + 8\,gra\,gia\,sa\,pa - 2\,sa\,fdra\,a\,gia - 2\,sa\,B\,fmra\,gia + fdda^2\,fmra^2 + fdra^2\,fmra^2 + 2\,gia^2\,pa^2 - I\,fdda\,B\,fmra^2 - A\,B\,fmra^2 - A\,b\,sa^2 + gia^4 + pa^4 + sa^4 + I\,A\,fdda\,a\,b - 4\,I\,fdda\,sa\,fmra\,gia + 4\,I\,fdda\,gra\,fmra\,pa - I\,fdda\,b\,sa^2 - I\,fdda\,pa^2\,b - I\,fdda\,a\,gra^2 - I\,fdda\,gia^2\,a - I\,A\,fdda\,fmra^2 - I\,fdda\,b\,gra^2 - I\,gia^2\,fdda\,b - I\,fdda\,a\,sa^2 - I\,pa^2\,fdda\,a - fdda^2\,a\,b - 2\,fdra\,pa^2\,fmra - 2\,fdra\,gia^2\,fmra - fdra^2\,a\,b - gia^2\,B\,b - pa^2\,B\,a + 2\,fdra\,gra\,a\,pa - 2\,fdra\,sa\,b\,gia + I\,fdda\,B\,a\,b)$$

The Impedance factor is complex if fdda, and fmda, are not zero.

This implies dissipation for the complex term.
for Gamma (H) = hermitean conjugate Gamma (D) !!!
The HHH Fresnel HAmiltonian does not depend upon gra.
Hence, could SET DIAGONAL ELEMENTS OF THE FARADAY MATRICES TO ZERO

```
>
```

The Kummer Fresnel WAVE Quartic Polynomial

```
> HAMILTONIAN:=factor(subs(fdda=0,fmda=0,det(HHH)/epsilon^3));ADM:=factor(subs(fdda=0,fmda=0,ADZ));
```

$$HAMILTONIAN := -A\,y^2\,pa^2 + 2\,fdra\,z^2\,fmra\,C - 4\,A\,x^2\,gia^2 + y^4\,c\,B\,a - y^2\,c\,B\,C + z^4\,b\,a\,C + 2\,z^3\,b\,sa\,C - z^2\,b\,B\,C + z^2\,b\,y^2\,sa^2 + z^2\,b\,y^2\,pa^2 + 2\,z^3\,sa\,a\,C - 2\,z\,sa\,B\,C + A\,B\,C - A\,B\,y^2\,a + A\,x^4\,c\,b - A\,x^2\,c\,C - A\,y^2\,sa^2 - 4\,fdra\,y^2\,gia\,sa - A\,z^2\,fmra^2\,x^2 + 2\,fdra\,x\,pa\,z\,a\,y - A\,z^2\,a\,C - 2\,A\,z\,sa\,C - A\,B\,x^2\,b + 2\,A\,y\,pa\,z\,fmra\,x - fdra^2\,C - 2\,z\,b\,x\,fdra\,y\,pa - 2\,x\,pa\,z\,fmra\,y\,B + 2\,y^2\,x^2\,c\,pa^2 - z^2\,fmra^2\,y^2\,B + x^2\,pa^2\,z^2\,a + 2\,y^2\,x^2\,c\,sa^2 + 2\,x^2\,pa^2\,z\,sa - 4\,x^2\,sa\,fdra\,gia + x^2\,sa^2\,z^2\,a + 2\,z\,sa\,B\,y^2\,a - x^2\,pa^2\,B + x^4\,pa^2\,c + fdra^2\,x^2\,b + z^2\,b\,x^2\,c\,C + 2\,z\,sa\,y^2\,pa^2 - 4\,z^2\,gia^2\,C - z^4\,fmra^2\,C - 4\,y^2\,gia^2\,B + y^2\,c\,z^2\,a\,C + 2\,y^2\,c\,z\,sa\,C + 2\,fdra\,z\,fmra\,y^2\,sa + 2\,z\,sa\,x^2\,c\,C + fdra^2\,y^2\,a + y^2\,c\,B\,x^2\,b + z^2\,b\,B\,y^2\,a + 2\,x^2\,sa\,fdra\,z\,fmra + 4\,y^2\,gia\,z\,fmra\,B$$

$+ 4\, y\, gia\, x\, pa\, B + A\, x^2\, c\, y^2\, a + A\, z^2\, a\, x^2\, b + 2\, A\, z\, sa\, x^2\, b + 4\, A\, x^2\, gia\, z\, fmra - 4\, A\, y\, pa\, x\, gia$
$- 4\, fdra\, z\, gia\, C + 4\, z^3\, gia\, fmra\, C + 4\, z^2\, sa^2\, C + 2\, z\, sa^3\, y^2 + y^4\, c\, pa^2 + y^4\, c\, sa^2 + 2\, x^2\, sa^3\, z$
$- x^2\, sa^2\, B + x^4\, sa^2\, c$

$ADM := -(-C\, c + gra^2 + gia^2)\,(-2\, fdra\, fmra\, sa^2 + gra^4 - 2\, fdra\, fmra\, gra^2 - A\, gia^2\, a$
$+ 2\, gra^2\, gia^2 + 2\, gra^2\, sa^2 - 2\, gra^2\, pa^2 - 2\, sa^2\, gia^2 + 2\, sa^2\, pa^2 - A\, pa^2\, b - A\, a\, gra^2 - B\, b\, gra^2$
$- B\, a\, sa^2 + A\, B\, a\, b - 2\, A\, sa\, fmra\, gia + 2\, A\, gra\, fmra\, pa + 2\, gra\, fdra\, b\, pa + 2\, gra\, B\, fmra\, pa$
$+ 8\, gra\, gia\, sa\, pa - 2\, sa\, fdra\, a\, gia - 2\, sa\, B\, fmra\, gia + fdra^2\, fmra^2 + 2\, gia^2\, pa^2 - A\, B\, fmra^2$
$- A\, b\, sa^2 + gia^4 + pa^4 + sa^4 - 2\, fdra\, pa^2\, fmra - 2\, fdra\, gia^2\, fmra - fdra^2\, a\, b - gia^2\, B\, b - pa^2\, B\, a$
$+ 2\, fdra\, gra\, a\, pa - 2\, fdra\, sa\, b\, gia)$

For specific examples choose values for the matrix elements. A,B,C are numeric factors times epsilon. a,b,c are numeric factors times mu. The Optical Activity part is scaled by the impedance of "free space" or sqrt(epsilon/mu). The algebraic method presented permits symbolic factorization. The dielectric Faraday fd is scaled by epsilon. The magnetic faraday is scaled by mu. The example below is for a chiral Optical Activity coefficient gamma of sqrt(2)/2. Note that the phase velocity of one of the polarization states can be faster than the speed of light and the other is slower.!!!
Select the effects to be studied algebraically by elimating all effects but one or two of the scalars,
fma,fda,ra,pa,ga
Example 1 studies optical activity algebraically (chirality term?), by setting all factors to zero, except ga:
Example 2 studies optical activity algebraically, by setting all factors to zero, except pa:
Example 3 studies optical activity combined with Faraday rotation algebraically, by setting all factors to zero, except ga and pa:

EXAMPLES

AO = imaginary part of Gamma matrix, Fresnel Fizeau = real part of Gamma matrix

Example 1 CHIRALITY

AO diagonal only gia =1/3

>

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMILTONIAN:
>
```

Set anisotropic coefficients:

```
>
> SUBS1:=A=1,B=1,C=1,a=1,b=1,c=1;
```

$$SUBS1 := A = 1, B = 1, C = 1, a = 1, b = 1, c = 1$$

```
> HAMA:=subs(SUBS1,HAMRED):ADMa:=evalm(subs(SUBS1,evalm(ADM))):
```

Set the numeric values for the coefficients

Zero non used components

```
> SUBS:=fmra=0,fdra=0,pa=0,gra=0;Admittance:=factor(subs(SUBS,evalm(
  ADMa)));
```

$$SUBS := fmra = 0, fdra = 0, pa = 0, gra = 0$$

$$Admittance := -(gia - 1)(gia + 1)(1 + sa - gia)(sa + 1 + gia)(-1 - gia + sa)(sa - 1 + gia)$$

Note that the admittance can be ZERO for gia = +1 or -1.

Set numeric values

```
> HAM:=factor(subs(gia=1/3,sa=1/2,(SUBS,HAMA))):
>
```

Kummer polynomial = determinant factor = 0 (fourth order)

```
> KUMM:=factor(HAM);
```

$$KUMM := 1 + \frac{9}{4} z^2 x^2 + \frac{9}{4} z x^2 + \frac{5}{2} y^2 x^2 + \frac{9}{4} z^2 y^2 - 2 z - \frac{13}{9} z^2 - \frac{97}{36} x^2 - \frac{97}{36} y^2 + \frac{5}{4} x^4 + 2 z^3 + \frac{9}{4} z y^2 + \frac{5}{4} y^4 + z^4$$

>

```
> KUMMX:=factor(subs(x=0,KUMM));solve(subs(y=0,KUMMX),z);
```

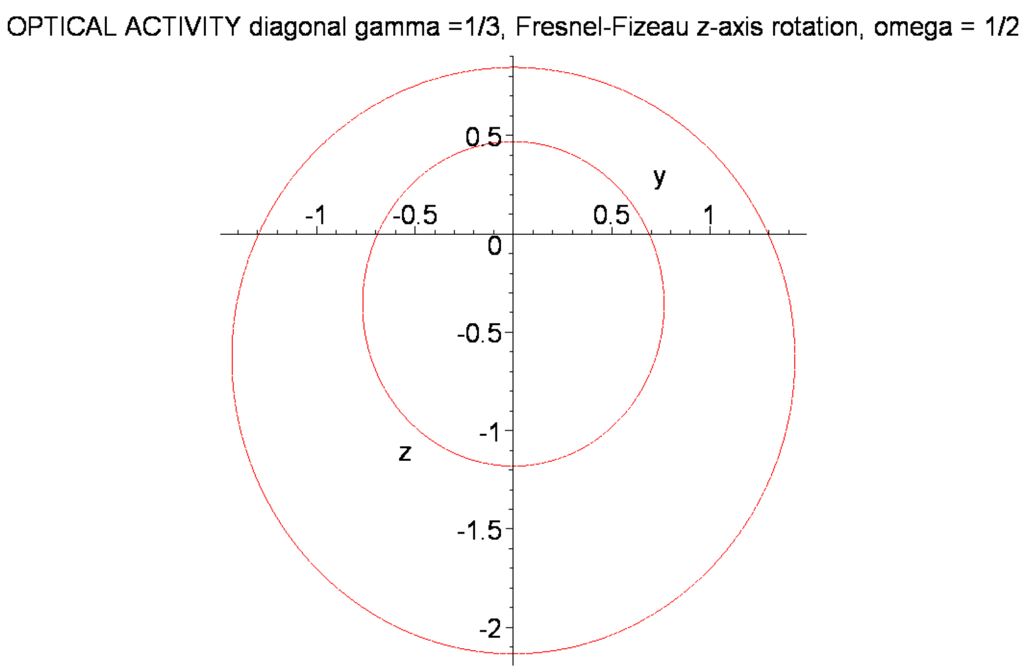
$$KUMMX := 1+\frac{9}{4}z^2y^2-2z-\frac{13}{9}z^2-\frac{97}{36}y^2+2z^3+\frac{9}{4}zy^2+\frac{5}{4}y^4+z^4$$

$$-\frac{1}{6}+\frac{1}{6}\sqrt{37},\ -\frac{1}{6}-\frac{1}{6}\sqrt{37},\ -\frac{5}{6}+\frac{1}{6}\sqrt{61},\ -\frac{5}{6}-\frac{1}{6}\sqrt{61}$$

```
>
```

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-3..2,numpoints =
15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY diagonal gamma
=1/3, Fresnel-Fizeau z-axis rotation, omega = 1/2`);
```

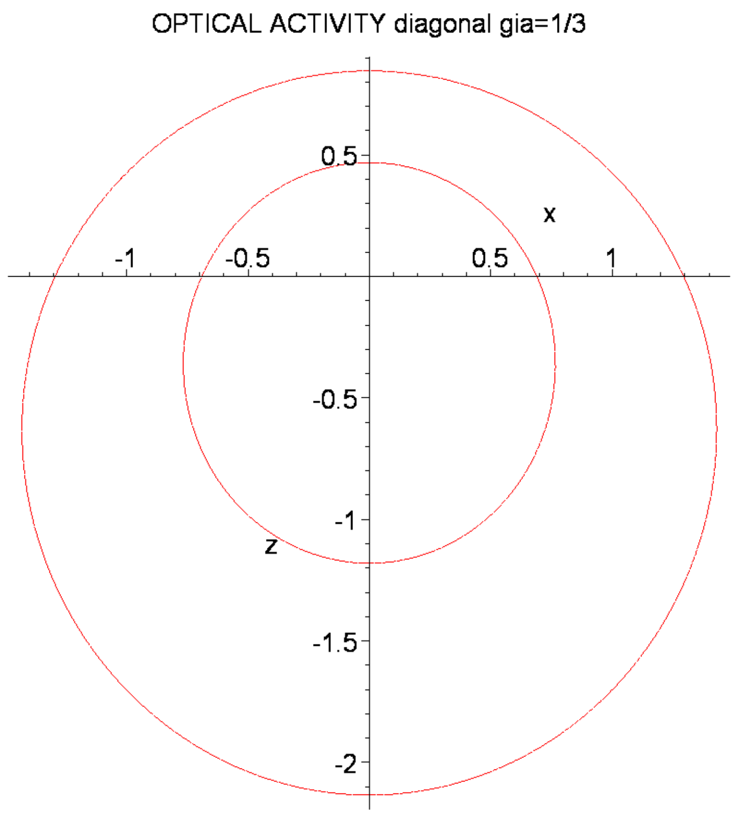


```
> KUMMY:=subs(y=0,KUMM);
```

$$KUMMY := 1+\frac{9}{4}z^2x^2+\frac{9}{4}z\,x^2-2\,z-\frac{13}{9}z^2-\frac{97}{36}x^2+\frac{5}{4}x^4+2\,z^3+z^4$$

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-3..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY diagonal
  gia=1/3`);
```

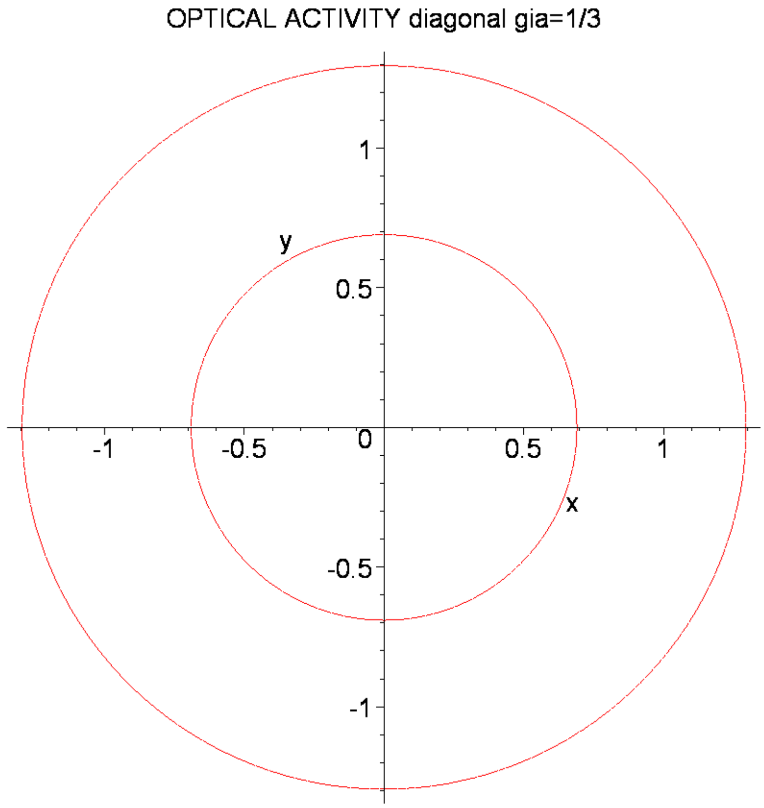


```
> KUMMZ:=subs(z=0,KUMM);
```

$$KUMMZ := 1 + \frac{5}{2} y^2 x^2 - \frac{97}{36} x^2 - \frac{97}{36} y^2 + \frac{5}{4} x^4 + \frac{5}{4} y^4$$

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ,x=-2.0..2.0,y=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY diagonal
  gia=1/3`);
```

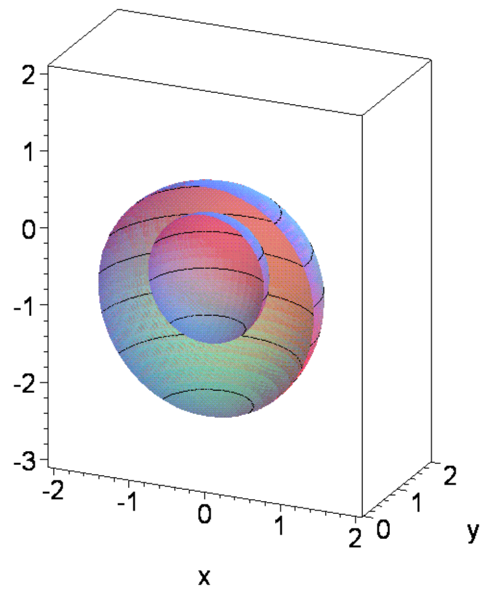


Optical Activity gives Two concentric spheres

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM,x=-2..2,y=-0..2,z=-3..2,shading=XYZ,lightmodel
=light3,axes=BOXED,style=PATCHCONTOUR,scaling=CONSTRAINED,title=`O
PTICAL ACTIVITY diagonal gamma=1/3, Fresnel-Fizeau z-axis
rotation, omega = 1/2`,numpoints=50000,orientation=[-66,69]);
```

OPTICAL ACTIVITY diagonal gamma=1/3, Fresnel-Fizeau z-axis rotation, omega = 1/2



Example 2 CHIRALITY
AO off diagonal only p =2

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMILTONIAN:
>
```

Set anisotropic coefficients:

```
>
> SUBS1:=A=1,B=1,C=1,a=1,b=1,c=1;
```

```
> HAMA:=subs(SUBS1,HAMRED);ADMa:=evalm(subs(SUBS1,evalm(ADM)));
```

Set the numeric values for the coefficients

Zero non used components

```
> SUBS:=fmra=0,fdra=0,pa=pa,sa=0,gra=0,gia=0;Admittance:=factor(subs
  (SUBS,evalm(ADM)));
```

Note that the admittance can be ZERO for various values of the isotropy coefficients and pa.

Set numeric values

```
> HAM:=factor(subs(pa=2,(SUBS,HAMA))):
>
```

Kummer polynomial = determinant factor = 0 (fourth order)

```
> KUMM:=factor(HAM);
>
>
```

```
> KUMMX:=factor(subs(x=0,KUMM));
>
```

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY off diagonal pa
  = 2`);
```

```
> KUMMY:=subs(y=0,KUMM);
```

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY off diagonal
  pa=2`);
```

```
> KUMMZ:=subs(z=0,KUMM);
```

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ,x=-2.0..3.0,y=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY off diagonal pa
  = 2`);
```

Optical Activity gives Two concentric spheres

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM,x=-2..2,y=-.8..1.5,z=-2..2,shading=XYZ,lightmodel=light3,axes=NORMAL,style=PATCHCONTOUR,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY off diagonal pa = 2`
,numpoints=50000,orientation=[-79,83]);
```

Example 3 CHIRALITY - AO both on and off diagonal only,

```
>
```

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMILTONIAN:
>
```

Set anisotropic coefficients:

```
>
> SUBS1:=A=1,B=1,C=1,a=1,b=1,c=1;
> HAMA:=subs(SUBS1,HAMRED):ADMa:=evalm(subs(SUBS1,evalm(ADM))):
```

Set the numeric values for the coefficients

Zero non used components

```
> SUBS:=fmra=0,fdra=0,sa=0,gra=0;Admittance:=factor(subs(SUBS,evalm(ADMa)));
```

Note that the admittance can be ZERO for gia = +1 or -1 or on the circle of values determined by the 2 parameters gia and pa

Set numeric values

```
> HAM:=factor(subs(gia=1/3,pa=1/3,(SUBS,HAMA))):
>
```

Kummer polynomial = determinant factor = 0 (fourth order)

```
> KUMM:=factor(HAM);
>
```

```
> KUMMX:=subs(x=0,KUMM);
```

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY pa=gia=1/3`);
```

```
> KUMMY:=subs(y=.0,KUMM);
```

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY pa=gia=1/3`);

>
```

```
> KUMMZ:=subs(z=.0,KUMM);
```

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ=0,x=-2.0..2.0,y=-2..2,numpoints =
  50000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY pa=gia=1/3`);
```

```
> implicitplot3d(KUMM,x=-2..2,y=-0..2,z=-2..2,shading=XYZ,lightmodel
  =light3,axes=NORMAL,style=PATCHCONTOUR,scaling=CONSTRAINED,title=`
  OPTICAL ACTIVITY
  pa=gia=1/3`,numpoints=50000,orientation=[-74,53]);
```

Example 4 ---- 4 Mode
Fresnel Fizeau (rotation) -sa and diagonal OA-gi,

```
>
```

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMILTONIAN:
>
```

Set anisotropic coefficients:

```
>
> SUBS1:=A=1,B=1,C=1,a=1,b=1,c=1;
> HAMA:=subs(SUBS1,HAMRED):ADMa:=evalm(subs(SUBS1,evalm(ADM))):
```

Set the numeric values for the coefficients

Zero non used components

```
> SUBS:=fmra=0,fdra=0,pa=0,gra=0;Admittance:=factor(subs(SUBS,evalm(
  ADMa)));
```

Note that the admittance can be ZERO for gia = +1 or -1, and for various combinations of sa and gia.

Set numeric values

```
> HAM:=factor(subs(gia=1/6,sa=1/3,(SUBS,HAMA))):
>
```

Kummer polynomial = determinant factor = 0 (fourth order)

```
> KUMM:=factor(HAM);
>
```

[>
[>

```
> KUMMX:=factor(subs(x=0,KUMM));
>
```

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Fresnel rotation sa = 1/3 OA
  gi=1/6`);
```

```
> KUMMY:=subs(y=0,KUMM);
```

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Fresnel rotation sa = 1/3 OA
  gi=1/6`);
```

```
> KUMMZ:=subs(z=0,KUMM);
```

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ,x=-2.0..2.0,y=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Fresnel rotation sa = 1/3 OA
  gi=1/6`);
```

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM,x=-2..2,y=-0..2,z=-2..2,shading=XYZ,lightmodel
=light3,axes=NORMAL,style=PATCHCONTOUR,scaling=CONSTRAINED,title=`
Fresnel rotation sa = 1/3 OA gi=1/6`
,numpoints=100000,orientation=[-60,77]);
```

4 propagation speeds along the z axis.

SAGNAC DUAL POLARIZED RING LASER 4 BEAT SIGNALS

due to Frensel Fizeau rotation and Optical Activity.

Example 5 ---- 4 Mode Faraday (rotation) - fm = 1/6 and diagonal OA - gi = 1/6,

```
>
```

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMILTONIAN:
```

Set anisotropic coefficients:

```
> SUBS1:=A=1,B=1,C=1,a=1,b=1,c=1;
> HAMA:=subs(SUBS1,HAMRED):ADMa:=evalm(subs(SUBS1,evalm(ADM))):
```

Set the numeric values for the coefficients

Zero non used components

```
> SUBS:=fdra=0,sa=0,pa=0,gra=0;Admittance:=factor(subs(SUBS,evalm(AD
  Ma)));
```

Note that the admittance can be ZERO for gia = +1 or -1 and various values of gia

and fmra

Set numeric values

```
> HAM:=factor(subs(gia=1/6,fmra=1/6,(SUBS,HAMA))):
>
```

Kummer polynomial = determinant factor = 0 (fourth order)

```
> KUMM:=factor(HAM);
>
>
>
```

```
> KUMMX:=factor(subs(x=0,KUMM));
>
```

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Faraday rotation fm = 1/6 OA
  gi=1/6`);
```

```
> KUMMY:=subs(y=0,KUMM);
```

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Faraday rotation fm = 1/6 OA
  gi=1/6`);
```

```
> KUMMZ:=subs(z=0,KUMM);
```

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ,x=-2.0..2.0,y=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Faraday rotation fm = 1/6 OA
  gi=1/6`);
```

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM,x=-2..2,y=-0..2,z=-2..2,shading=XYZ,lightmodel
  =light3,axes=NORMAL,style=PATCHCONTOUR,scaling=CONSTRAINED,title=`
  Faraday (rotation) - fm = 1/6  and diagonal OA - gi = 1/6`
  ,numpoints=100000,orientation=[-60,77]);
```

Note 4 propagation speeds along the z axis. Due to Faraday rotation and Optical Activity.

Example 6 4 Mode
FRESNEL - FIZEAU EXPANSION gr = 1/2

,

```
>
```

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMILTONIAN:
```

Set anisotropic coefficients:

```
> SUBS1:=A=1,B=1,C=1,a=1,b=1,c=1;
> HAMA:=subs(SUBS1,HAMRED):ADMa:=evalm(subs(SUBS1,evalm(ADM))):
```

Set the numeric values for the coefficients

Zero non used components

```
> SUBS:=fdra=0,fmra=0,pa=0,gia=0,sa=0;Admittance:=factor(subs(SUBS,e
  valm(ADMa)));
```

Set numeric values

```
> HAM:=factor(subs(gra=1/2,(SUBS,HAMA))):
>
```

Kummer polynomial = determinant factor = 0 (fourth order)

```
> KUMM:=factor(HAM);grad(KUMM,[x,y,z]);
```

The Value of the Fresnel-Fizeau Expansion Factor (gra) does not change the Wave Vector surface.
It looks like the classic Lorentz Kummer surface.

$KUMM := (y^2 + x^2 + z^2 - 1)^2 = 0$

It consists of two spheres of the same radius sandwiched together,
like a minimal surface soap film.
The gradient vanishes on the surface.
However, the Free-Space Impedance does depend upon the Fresnel Fizeau Expansion.

```
> KUMM:=y^2+x^2+z^2-1
>
```

```
> KUMMX:=factor(subs(x=0,KUMM));
>
```

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-3..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Faraday rotation fm = 1/6 OA
  gi=1/6`);
```

```
> KUMMY:=subs(y=0,KUMM);
```

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Faraday rotation fm = 1/6 OA
  gi=1/6`);
```

```
> KUMMZ:=subs(z=0,KUMM);
```

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ,x=-2.0..2.0,y=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`Faraday rotation fm = 1/6 OA
  gi=1/6`);
```

>
>
>