```
> restart: with (linalg):with(liesymm):with(difforms):with(plots):
```

topdynamicsexamples.mws

Examples of a topological thermodynamic System

See Vol. 1, "Non-Equilibrium Thermodynamics" http://www.lulu.com/kiehn chapter 3.8.

R. M. Kiehn

updated Sept/05/2005 --Feb 6 2008

A Maple program that uses the Jacobian Matrix (dyadic) of a 3D velocity field of a dynamical system to compute the similarity invariants, Xm, Yg, Za, and the Brand invariants

```
> setup(x,y,z,t);defform(x=0,y=0,z=0,u=0,f=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,
Ax=0,Ay=0,Az=0,Phi=0,a=const,b=const,c=const,q=const,Lx=0,Ly=0,Lz=0,A=const,B=
const,C=const,D=const,Omega=const,r=const);
```

$$[x, y, z, t] \tag{1}$$

STF dynamical system

```
> AA:=[C*z*x-Omega*y,(C-R)*z*y+Omega*x,A-B*z^2-S*x^2-T*y^2];
```

$$AA := \left[C\,z\,x - \Omega\,y,\ (C-R)\,z\,y + \Omega\,x,\ A - B\,z^2 - S\,x^2 - T\,y^2\right] \tag{2}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=evalm([AA[1],AA[2],AA[3],phi]);
```

$$\phi := C^2 z^2 x^2 + \Omega^2 y^2 + z^2 y^2 C^2 - 2\,z^2 y^2 C\,R + z^2 y^2 R^2 - 2\,z\,y\,R\,\Omega\,x + \Omega^2 x^2 + A^2 - 2\,A\,B\,z^2 - 2\,A\,S\,x^2 - 2\,A\,T\,y^2 + B^2 z^4 + 2\,B\,z^2 S\,x^2 + 2\,B\,z^2\,T\,y^2 + S^2 x^4 + 2\,S\,x^2\,T\,y^2 + T^2 y^4$$

$$VV := \left[C\,z\,x - \Omega\,y,\ (C-R)\,z\,y + \Omega\,x,\ A - B\,z^2 - S\,x^2 - T\,y^2,\ C^2 z^2 x^2 + \Omega^2 y^2 + z^2 y^2 C^2 - 2\,z^2 y^2 C\,R + z^2 y^2 R^2 - 2\,z\,y\,R\,\Omega\,x + \Omega^2 x^2 + A^2 - 2\,A\,B\,z^2 - 2\,A\,S\,x^2 - 2\,A\,T\,y^2 + B^2 z^4 + 2\,B\,z^2 S\,x^2 + 2\,B\,z^2\,T\,y^2 + S^2 x^4 + 2\,S\,x^2\,T\,y^2 + T^2 y^4\right] \tag{3}$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t])):collect(simplify(expand(charpoly(JAC,q))
```

```
),q);
```

$$q^4+(zR+2Bz-2Cz)\,q^3+\left(2Ty^2C+\Omega^2-Cz^2R+C^2z^2+2z^2RB+2Sx^2C-4Cz^2B-2Ty^2R\right)q^2+\left(2CzTy^2R+2Sx\Omega yR+2\Omega^2Bz+2\Omega TyCx-2Sx\Omega yC+2C^2z^3B-2Sx^2C^2z+2Sx^2CzR-2Cz^3RB-2C^2zTy^2\right)q \quad (4)$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
(innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=factor((1/2)*(Xm*S1-S2));Za:=collect(collect(factor((1/3)*(Yg*S1-
Xm*S2+S3)),z),Omega);Tk:=factor(det(JAC));
```

$$Xm:=-z\,(-2C+R+2B)$$

$$Yg:=2Ty^2C+\Omega^2-Cz^2R+C^2z^2+2z^2RB+2Sx^2C-4Cz^2B-2Ty^2R$$

$$Za:=-2\Omega^2Bz+(-2CxTy+2ySxC-2ySxR)\,\Omega+\left(2CRB-2C^2B\right)z^3+\left(-2Ty^2CR+2Ty^2C^2+2Sx^2C^2-2Sx^2CR\right)z$$

$$Tk:=0 \quad (5)$$

```
> ;BBB:=solve(Xm,B);YgB:=factor(subs(B=BBB,Yg));RRR:=solve(YgB,Omega^2);ZaBR:=
factor(subs(B=BBB,Omega^2=RRR,Za));CriticalPoint:=collect(ZaBR-1,Omega);
```

$$BBB:=C-\frac{1}{2}R$$

$$YgB:=2Ty^2C+\Omega^2+3Cz^2R-3C^2z^2-z^2R^2+2Sx^2C-2Ty^2R$$

Warning, solving for expressions other than names or functions is not recommended.

$$RRR:=-2Ty^2C-2Sx^2C-3Cz^2R+3C^2z^2+z^2R^2+2Ty^2R$$

$$ZaBR:=6C^2zTy^2-8CzTy^2R+6Sx^2C^2z-4Sx^2CzR+12C^2z^3R-6Cz^3R^2-8C^3z^3+z^3R^3+2Ty^2zR^2-2\Omega TyCx+2Sx\Omega yC-2Sx\Omega yR$$

$$CriticalPoint:=(-2CxTy+2ySxC-2ySxR)\,\Omega+6C^2zTy^2-8CzTy^2R+6Sx^2C^2z-4Sx^2CzR+12C^2z^3R-6Cz^3R^2-8C^3z^3+z^3R^3+2Ty^2zR^2-1 \quad (6)$$

Thermodynamic crtical point is a surface in {x,y,z} (It is not locally stable)

```
> BBB:=solve(Xm,B);ZaB:=factor(subs(B=BBB,Za));RRR:=solve(ZaB,Omega^2);YYg:=
factor(subs(B=BBB,Omega^2=RRR,Yg));
Hopfcrit:=collect(YYg,Omega);
```

$$BBB:=C-\frac{1}{2}R$$

$$ZaB:=-2Cz\Omega^2+\Omega^2zR-2\Omega TyCx+2Sx\Omega yC-2Sx\Omega yR+3C^2z^3R-Cz^3R^2-2C^3z^3-2CzTy^2R+2C^2zTy^2+2Sx^2C^2z-2Sx^2CzR$$

Warning, solving for expressions other than names or functions is not recommended.

$$RRR := -\frac{1}{z\,(2\,C-R)}\,\big(2\,S\,x\,\Omega\,y\,R-3\,C^2\,z^3\,R+2\,\Omega\,T\,y\,C\,x-2\,S\,x\,\Omega\,y\,C+2\,C\,z\,T\,y^2\,R-2\,C^2\,z\,T\,y^2+C\,z^3\,R^2+2\,C^3\,z^3-2\,S\,x^2\,C^2\,z+2\,S\,x^2\,C\,z\,R\big)$$

$$YYg := -\frac{1}{z\,(2\,C-R)}\,\big(-6\,C^2\,z\,T\,y^2+8\,C\,z\,T\,y^2\,R-6\,S\,x^2\,C^2\,z+4\,S\,x^2\,C\,z\,R-12\,C^2\,z^3\,R+6\,C\,z^3\,R^2+8\,C^3\,z^3-z^3\,R^3-2\,T\,y^2\,z\,R^2+2\,\Omega\,T\,y\,C\,x-2\,S\,x\,\Omega\,y\,C+2\,S\,x\,\Omega\,y\,R\big)$$

$$Hopfcrit := -\frac{(2\,y\,S\,x\,R+2\,C\,x\,T\,y-2\,y\,S\,x\,C)\,\Omega}{z\,(2\,C-R)}-\frac{1}{z\,(2\,C-R)}\,\big(-6\,C^2\,z\,T\,y^2+8\,C\,z\,T\,y^2\,R-6\,S\,x^2\,C^2\,z+4\,S\,x^2\,C\,z\,R-12\,C^2\,z^3\,R+6\,C\,z^3\,R^2+8\,C^3\,z^3-z^3\,R^3-2\,T\,y^2\,z\,R^2\big) \quad \textbf{(7)}$$

```
> TT:=solve(factor(2*S*x*y*R+2*T*y*C*x-2*S*x*y*C),T);ZZ:=subs(T=TT,collect
(collect(collect((4*S*x^2*C*z*R+8*C*z*T*y^2*R-6*C^2*z*T*y^2-6*S*x^2*C^2*z-2*T*
y^2*z*R^2+8*C^3*z^3-z^3*R^3-12*C^2*z^3*R+6*C*z^3*R^2)/z,z),x),y));
```

$$TT := \frac{(C-R)\,S}{C}$$

$$ZZ := \left(-6\,(C-R)\,S\,C-\frac{2\,(C-R)\,S\,R^2}{C}+8\,(C-R)\,S\,R\right)y^2+\big(-6\,S\,C^2+4\,S\,C\,R\big)\,x^2+\big(-12\,C^2\,R+6\,C\,R^2+8\,C^3-R^3\big)\,z^2 \quad \textbf{(8)}$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := -2\,T\,y-C\,y+y\,R$$

$$CURLy := C\,x+2\,S\,x$$

$$CURLz := 2\,\Omega \quad \textbf{(9)}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),Omega);
```

$$HELICITY := \big(C\,y^2-R\,y^2+x^2\,C+2\,A-2\,B\,z^2\big)\,\Omega-2\,C\,z\,x\,T\,y+2\,z\,y\,C\,S\,x-2\,z\,y\,R\,S\,x \quad \textbf{(10)}$$

```
> KKK:=simplify((2*S*x*y*R+2*T*y*C*x-2*S*x*y*C)/(x*y));
```

$$KKK := 2\,S\,R+2\,C\,T-2\,S\,C \quad \textbf{(11)}$$

if KKK is zero then helicity depends upon Omega.

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),Omega);
```

$$enstrophy := 4\,T^2\,y^2+4\,T\,y^2\,C-4\,T\,y^2\,R+C^2\,y^2-2\,C\,y^2\,R+y^2\,R^2+C^2\,x^2+4\,S\,x^2\,C+4\,S^2\,x^2+4\,\Omega^2 \quad \textbf{(12)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \big[-2\,T\,y\,z\,C-z\,y\,C^2+z\,y\,C\,R+\Omega\,C\,x-2\,S\,x\,\Omega,\ -2\,T\,y\,\Omega+\Omega\,y\,C-\Omega\,y\,R+C^2\,z\,x+2\,S\,x\,C\,z-x\,C\,z\,R-2\,z\,R\,S\,x,\ 2\,y\,S\,x\,C-2\,y\,S\,x\,R-2\,C\,x\,T\,y-4\,B\,z\,\Omega\big] \quad \textbf{(13)}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),Omega);ST1:=factor(coeff
  (stretch,Omega^2));st0:=factor(subs(Omega=0,stretch));
```

$$stretch := -8\,\Omega^2 B z + \left(-8\,C x T y + 8\,y S x C - 8\,y S x R\right)\Omega + 4\,C^2 z T y^2 + 4\,S x^2 C^2 z - 4\,C z T y^2 R + z y^2 C R^2 + z y^2 C^3 + 4\,T^2 y^2 z C - 4\,S x^2 C z R + C^3 z x^2 + 4\,S^2 x^2 C z - x^2 C^2 z R - 2\,z y^2 C^2 R - 4\,z R S^2 x^2$$

$$ST1 := -8\,B z$$

$$st0 := z\left(4\,T y^2 C^2 + 4\,S x^2 C^2 - 4\,T y^2 C R + R^2 C y^2 + C^3 y^2 + 4\,C y^2 T^2 - 4\,S x^2 C R + x^2 C^3 + 4\,C x^2 S^2 - R C^2 x^2 - 2\,R C^2 y^2 - 4\,x^2 S^2 R\right) \tag{14}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),Omega);BI2:=factor(coeff(BI,
  Omega^2));BI1:=factor(coeff(BI,Omega));BI0:=collect(factor(subs(Omega=0,BI)),
  z);
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$BI := \left(-4\,S x^2 C - 4\,T y^2 C + 4\,T y^2 R - 2\,C y^2 R + 4\,S^2 x^2 + 16\,B^2 z^2 + C^2 x^2 + C^2 y^2 + 4\,T^2 y^2 + y^2 R^2\right)\Omega^2 + \left(16\,T y C x B z + 8\,C^2 z y S x - 16\,S x y B z C + 16\,S x y B z R + 4\,S x z y R^2 + 4\,z C x T y R - 2\,z y C^2 R x + 2\,y R^2 x C z - 8\,z C^2 x T y + 8\,T y z R S x - 12\,C z y S x R\right)\Omega + 4\,C^3 z^2 x^2 S + 8\,y^2 S x^2 R C T + 4\,S^2 x^2 C^2 z^2 + x^2 C^2 z^2 R^2 + 4\,z^2 R^2 S^2 x^2 + 4\,T^2 y^2 z^2 C^2 + 4\,T y^2 z^2 C^3 - 2\,z^2 y^2 C^3 R + z^2 y^2 C^2 R^2 - 8\,C^2 z^2 x^2 R S - 8\,S^2 x^2 C z^2 R + 4\,x^2 C z^2 R^2 S - 8\,y^2 S^2 x^2 C R - 8\,y^2 S x^2 C^2 T - 2\,C^3 z^2 x^2 R + 4\,y^2 S^2 x^2 R^2 + 4\,C^2 x^2 T^2 y^2 - 4\,T y^2 z^2 C^2 R + 4\,y^2 S^2 x^2 C^2 + z^2 y^2 C^4 + C^4 z^2 x^2$$

$$BI2 := -4\,S x^2 C - 4\,T y^2 C + 4\,T y^2 R - 2\,C y^2 R + 4\,S^2 x^2 + 16\,B^2 z^2 + C^2 x^2 + C^2 y^2 + 4\,T^2 y^2 + y^2 R^2$$

$$BI1 := -2\,y x z\left(8\,S C B - 4\,S R T + 4\,T C^2 + 6\,S C R + C^2 R - 8\,B R S - 8\,C T B - 4\,S C^2 - C R^2 - 2\,T C R - 2\,S R^2\right)$$

$$BI0 := \left(4\,C^2 T^2 y^2 + 4\,T y^2 C^3 - 2\,y^2 C^3 R + y^2 C^2 R^2 + 4\,C^3 x^2 S - 2\,C^3 x^2 R + 4\,S^2 x^2 C^2 + x^2 C^2 R^2 + 4\,S^2 x^2 R^2 + y^2 C^4 - 8\,S^2 x^2 C R + C^4 x^2 - 4\,T y^2 C^2 R - 8\,C^2 x^2 R S + 4\,x^2 C R^2 S\right) z^2 - 8\,y^2 S x^2 C^2 T + 8\,y^2 S x^2 R C T + 4\,y^2 S^2 x^2 R^2 + 4\,C^2 x^2 T^2 y^2 - 8\,y^2 S^2 x^2 C R + 4\,y^2 S^2 x^2 C^2$$

$$\Theta := -16\,\Omega B^2 z^3 + 8\,B z^2 S x C y - 8\,B z^2 S x y R + 16\,z^2 y C R S x - 8\,B z^2 T y C x + 8\,C^2 z^2 x T y + 4\,C^2 z^2 x y R - 8\,z y^2 R \Omega T + 12\,z y^2 R \Omega C - 8\,A S x C y + 8\,A S x y R + 8\,S x T y^3 C - 8\,S x T y^3 R - 8\,z^2 y C^2 S x - 4\,z^2 y R^2 C x - 8\,z^2 y R^2 S x + 4\,z R \Omega x^2 C + 8\,z R \Omega x^2 S + 8\,A T y C x - 8\,S x^3 T y C - 16\,\Omega B z S x^2 - 16\,\Omega B z T y^2 - 4\,z y^2 R^2 \Omega + 8\,\Omega^2 x T y + 4\,\Omega^2 x y R + 8\,S^2 x^3 C y - 8\,S^2 x^3 y R - 8\,\Omega^2 y S x - 8\,T^2 y^3 C x - 8\,\Omega C^2 z x^2 \tag{15}$$

$$-8\,\Omega\,z\,y^2\,C^2+16\,\Omega\,A\,B\,z$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t2:=factor(coeff(Theta,Omega^2));t1:=factor(coeff(Theta,Omega));
> t0:=factor(subs(Omega=0,Theta));
```

$$\Theta := (8\,x\,y\,T+4\,y\,R\,x-8\,y\,S\,x)\,\Omega^2+\big(12\,z\,y^2\,C\,R-8\,z\,T\,y^2\,R+4\,x^2\,C\,z\,R+8\,S\,x^2\,z\,R-4\,z\,y^2\,R^2+16\,A\,B\,z-16\,B\,z\,S\,x^2-16\,B\,z\,T\,y^2-16\,B^2\,z^3-8\,C^2\,z\,x^2-8\,z\,y^2\,C^2\big)\,\Omega-8\,z^2\,y\,C^2\,S\,x+8\,B\,z^2\,S\,x\,C\,y-8\,B\,z^2\,S\,x\,y\,R+16\,z^2\,y\,C\,R\,S\,x-8\,B\,z^2\,T\,y\,C\,x+8\,C^2\,z^2\,x\,T\,y+4\,C^2\,z^2\,x\,y\,R+8\,S\,x\,T\,y^3\,C-8\,S\,x\,T\,y^3\,R-8\,A\,S\,x\,C\,y+8\,A\,S\,x\,y\,R-8\,z^2\,y\,R^2\,S\,x-8\,T^2\,y^3\,C\,x-8\,S^2\,x^3\,y\,R-4\,z^2\,y\,R^2\,C\,x-8\,S\,x^3\,T\,y\,C+8\,S^2\,x^3\,C\,y+8\,A\,T\,y\,C\,x$$

$$t2 := 4\,y\,x\,(2\,T+R-2\,S)$$

$$t1 := 4\,z\,\big(-4\,B^2\,z^2+3\,C\,y^2\,R-2\,T\,y^2\,R+x^2\,C\,R+2\,S\,x^2\,R-y^2\,R^2+4\,A\,B-4\,B\,S\,x^2-4\,B\,T\,y^2-2\,C^2\,x^2-2\,C^2\,y^2\big)$$

$$t0 := -4\,y\,x\,\big(2\,x^2\,S\,T\,C-2\,C\,x^2\,S^2+2\,x^2\,S^2\,R+2\,y^2\,S\,R\,T-2\,y^2\,S\,T\,C+2\,C\,y^2\,T^2+2\,z^2\,C^2\,S-2\,B\,z^2\,S\,C+2\,B\,z^2\,S\,R-4\,z^2\,C\,R\,S+2\,B\,z^2\,T\,C-2\,T\,z^2\,C^2-C^2\,z^2\,R+2\,z^2\,R^2\,S+2\,A\,S\,C-2\,A\,S\,R-2\,A\,T\,C+C\,z^2\,R^2\big) \quad \textbf{(16)}$$

```
> (collect(innerprod(AA,grad(HELICITY,[x,y,z])),Omega));SPHERE:=collect(collect(factor(innerprod(AA,[x,y,z])/z),x^2),y^2);
```

$$-2\,\Omega^2\,x\,y\,R+\big(4\,B\,z\,S\,x^2+4\,B^2\,z^3-4\,z\,y^2\,C\,R+2\,y^2\,z\,R\,S+4\,B\,z\,T\,y^2+2\,S\,x^2\,C\,z-2\,y^2\,z\,C\,S+2\,C^2\,z\,x^2-2\,x^2\,C\,z\,T-2\,S\,x^2\,z\,R+2\,z\,y^2\,R^2+2\,z\,y^2\,C^2-4\,A\,B\,z+2\,T\,y^2\,C\,z\big)\,\Omega+2\,S^2\,x^3\,y\,R-2\,B\,z^2\,S\,x\,C\,y+2\,B\,z^2\,S\,x\,y\,R-6\,z^2\,y\,C\,R\,S\,x+2\,B\,z^2\,T\,y\,C\,x-4\,C^2\,z^2\,x\,T\,y+2\,S\,x\,T\,y^3\,R+2\,A\,S\,x\,C\,y-2\,A\,S\,x\,y\,R-2\,S\,x\,T\,y^3\,C-2\,A\,T\,y\,C\,x+4\,z^2\,y\,C^2\,S\,x+2\,z^2\,y\,R^2\,S\,x-2\,S^2\,x^3\,C\,y+2\,z^2\,y\,R\,C\,x\,T+2\,S\,x^3\,T\,y\,C+2\,T^2\,y^3\,C\,x$$

$$SPHERE := (C-R-T)\,y^2+(C-S)\,x^2-B\,z^2+A \quad \textbf{(17)}$$

```
> C-R-T=-A;C-S=-A;B=A;S=C+A;T=C+A-R;T=S-R;
```

$$C-R-T=-A$$
$$C-S=-A$$
$$B=A$$
$$S=C+A$$
$$T=C+A-R$$
$$T=S-R \quad \textbf{(18)}$$

Lorenz dynamical system

```
> AA:=[A*(y-x),-x*z+R*x-y,x*y-B*z];
```

$$AA := [A\,(y-x),\ -z\,x+R\,x-y,\ y\,x-B\,z] \quad (19)$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=[AA[1],AA[2],AA[3],phi];
```

$$\phi := A^2\,y^2-2\,A^2\,y\,x+A^2\,x^2+z^2\,x^2-2\,x^2\,z\,R+2\,y\,x\,z+x^2\,R^2-2\,y\,R\,x+y^2+y^2\,x^2-2\,y\,x\,B\,z+B^2\,z^2$$

$$VV := \left[A\,(y-x),\ -z\,x+R\,x-y,\ y\,x-B\,z,\ A^2\,y^2-2\,A^2\,y\,x+A^2\,x^2+z^2\,x^2-2\,x^2\,z\,R+2\,y\,x\,z+x^2\,R^2-2\,y\,R\,x+y^2+y^2\,x^2-2\,y\,x\,B\,z+B^2\,z^2\right] \quad (20)$$

```
> JAC:=simplify(jacobian(evalm(VV),[x,y,z,t]));collect(simplify(expand(charpoly
  (JAC,q))),q);
```

$$JAC := \left[\left[-A,\ A,\ 0,\ 0\right],\ \left[-z+R,\ -1,\ -x,\ 0\right],\ \left[y,\ x,\ -B,\ 0\right],\ \left[-2\,A^2\,y+2\,A^2\,x+2\,z^2\,x-4\,x\,z\,R+2\,z\,y+2\,x\,R^2-2\,y\,R+2\,y^2\,x-2\,y\,B\,z,\ 2\,A^2\,y-2\,A^2\,x+2\,z\,x-2\,R\,x+2\,y+2\,y\,x^2-2\,x\,B\,z,\ 2\,z\,x^2-2\,x^2\,R+2\,y\,x-2\,y\,x\,B+2\,B^2\,z,\ 0\right]\right]$$

$$q^4+(A+1+B)\,q^3+\left(A\,B+z\,A+B+x^2+A-A\,R\right)\,q^2+\left(-A\,R\,B+y\,A\,x+A\,B\,z+A\,B+A\,x^2\right)\,q \quad (21)$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=factor((1/2)*(Xm*S1-S2));Za:=factor((1/3)*(Yg*S1-Xm*S2+S3));Tk:=
  factor(det(JAC));
```

$$Xm := -1-A-B$$

$$Yg := A\,B+z\,A+B+x^2+A-A\,R$$

$$Za := A\left(-B\,z-B+R\,B-y\,x-x^2\right)$$

$$Tk := 0 \quad (22)$$

```
> BBB:=solve(Xm-3,B);YgB:=factor(subs(B=BBB,Yg));RRR:=solve(YgB-3,R);ZaBR:=
  factor(subs(B=BBB,R=RRR,Za));CriticalPoint:=collect(ZaBR-1,A);
```

$$BBB := -4-A$$

$$YgB := -4\,A-A^2+z\,A-4+x^2-A\,R$$

$$RRR := -\frac{4\,A+A^2-z\,A+7-x^2}{A}$$

$$ZaBR := 27\,A+9\,A^2+A^3+28-4\,x^2-2\,A\,x^2-y\,A\,x$$

$$CriticalPoint := A^3 + 9\,A^2 + \left(-2\,x^2 - y\,x + 27\right)A - 4\,x^2 + 27 \tag{23}$$

Thermodynamic crtical point is a surface in {x,y,z} (It is not locally stable)

```
> BBB:=solve(Xm,B);ZaB:=factor(subs(B=BBB,Za));RRR:=solve(ZaB,R);YYg:=factor
  (subs(B=BBB,R=RRR,Yg));
  Hopfcrit:=collect(YYg,A);
```

$$BBB := -1 - A$$

$$ZaB := -A\left(-z - z\,A - 1 - A + R + A\,R + y\,x + x^2\right)$$

$$RRR := \frac{z + z\,A + 1 + A - y\,x - x^2}{1 + A}$$

$$YYg := -\frac{3\,A + 3\,A^2 + A^3 + 1 - x^2 - 2\,A\,x^2 - y\,A\,x}{1 + A}$$

$$Hopfcrit := -\frac{A^3 + 3\,A^2 + \left(-2\,x^2 - y\,x + 3\right)A - x^2 + 1}{1 + A} \tag{24}$$

```
> INTER:=collect((numer(CriticalPoint)+numer(Hopfcrit)),A);solve(INTER,A);;
```

$$INTER := 24\,A + 6\,A^2 + 26 - 3\,x^2$$

$$-2 + \frac{1}{6}\sqrt{-12 + 18\,x^2},\ -2 - \frac{1}{6}\sqrt{-12 + 18\,x^2} \tag{25}$$

Hopf conditions

```
> AAA:=solve(Xm,A);ZaAAA:=subs(A=AAA,Za);BBA:=solve(ZaAAA,B);YYg:=factor(subs(A=
  AAA,B=BBA[1],Yg));YYg2:=factor(subs(A=AAA,B=BBA[2],Yg));;
```

$$AAA := -1 - B$$

$$ZaAAA := (-1 - B)\left(-B\,z - B + R\,B - y\,x - x^2\right)$$

$$BBA := -1,\ \frac{x\,(y + x)}{-z - 1 + R}$$

$$YYg := (x - 1)\,(x + 1)$$

$$YYg2 := \frac{1}{(-z - 1 + R)^2}\left(-1 - 3\,z + 2\,z^2\,x^2 + 3\,R - 3\,z^2 + 2\,x^2 + 6\,z\,R - 3\,R^2 - z^3 + 3\,R\,z^2 + R^3 - x^4 - 3\,z\,R^2 + 4\,z\,x^2 - 4\,x^2\,z\,R + y\,x - 4\,x^2\,R - 2\,y\,R\,x - y^2\,x^2 + 2\,y\,x\,z - 2\,z\,y\,R\,x + 2\,x^2\,R^2 + y\,R^2\,x - 2\,y\,x^3 + y\,x\,z^2\right) \tag{26}$$

When YYg>0, then get oscillations. Can solve for the zero set of YYg which will divide the space into regions of oscillations and no oscillations. The parameter of the surface is R, can ask for discriminant to see if there is an envelope.

```
> YYgN:=numer(YYg);factor(discrim(YYgN,R));
```

$$YYgN := (x - 1)\,(x + 1)$$

$$0 \tag{27}$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := 2\,x$$

$$CURLy := -y$$

$$CURLz := -z + R - A \tag{28}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),B);
```

$$HELICITY := \left(z^2 - z\,R + z\,A\right)\,B + y\,A\,x - 2\,A\,x^2 + y^2 \tag{29}$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),R);
```

$$enstrophy := R^2 + (-2\,A - 2\,z)\,R + 4\,x^2 + y^2 + z^2 + A^2 + 2\,z\,A \tag{30}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[\ -2\,A\,x - y\,A \quad -z\,x + R\,x + y + A\,x \quad y\,x + B\,z - R\,B + A\,B\ \right] \tag{31}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),B);ST1:=factor(coeff
  (stretch,B));
```

$$stretch := \left(-z^2 + 2\,z\,R - 2\,z\,A - R^2 + 2\,A\,R - A^2\right)\,B - 4\,A\,x^2 - 4\,y\,A\,x - y^2$$

$$ST1 := -(z - R + A)^2 \tag{32}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),B);BI2:=factor(coeff(BI,B^2));BI1:=
  factor(coeff(BI,B));BI0:=collect(factor(subs(B=0,BI)),R);
```

$$BI := \left(z^2 - 2\,z\,R + 2\,z\,A + R^2 - 2\,A\,R + A^2\right)\,B^2 + (2\,y\,x\,z - 2\,y\,R\,x + 2\,y\,A\,x)\,B + 5\,A^2\,x^2 + 4\,A^2\,y\,x + A^2\,y^2 + z^2\,x^2 - 2\,x^2\,z\,R - 2\,y\,x\,z - 2\,z\,x^2\,A + x^2\,R^2 + 2\,y\,R\,x + 2\,R\,x^2\,A + y^2 + 2\,y\,A\,x + y^2\,x^2$$

$$BI2 := (z - R + A)^2$$

$$BI1 := 2\,y\,x\,(z - R + A)$$

$$BI0 := x^2\,R^2 + \left(-2\,z\,x^2 + 2\,A\,x^2 + 2\,y\,x\right)\,R + 5\,A^2\,x^2 + 4\,A^2\,y\,x + A^2\,y^2 + z^2\,x^2 - 2\,y\,x\,z - 2\,z\,x^2\,A + y^2 + 2\,y\,A\,x + y^2\,x^2 \tag{33}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),R);;Theta:=
  collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),B);;Theta:=collect
  (factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),A);;Theta:=collect(factor
  (-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),z);;factor(-8*A^2*x^2+4*A*x^2*z+4*
  y^2+4*y*x*A+4*A^2*y^2-4*x^2*z^2+4*A^2*x*y+4*B^2*z^2-4*x^2*y^2-4*x*y*A*B+4*B^2*
  z*A)/4;
```

$$\Theta := -4x^2R^2 + (-4B^2z - 4Ax^2 + 4yxB + 8zx^2)R - 4z^2x^2 + 4y^2 + 4B^2z^2 + 4A^2y^2 - 4y^2x^2 + 4yAx + 4A^2yx - 4yxAB - 8A^2x^2 + 4zx^2A + 4B^2zA$$

$$\Theta := (-4zR + 4z^2 + 4zA)B^2 + (4yRx - 4yAx)B - 4z^2x^2 + 4y^2 + 4A^2yx + 8x^2zR - 4y^2x^2 - 4x^2R^2 + 4zx^2A + 4A^2y^2 - 8A^2x^2 + 4yAx - 4Rx^2A$$

$$\Theta := (-8x^2 + 4yx + 4y^2)A^2 + (-4yxB - 4x^2R + 4zx^2 + 4B^2z + 4yx)A - 4z^2x^2 + 4y^2 + 4B^2z^2 + 8x^2zR - 4y^2x^2 - 4x^2R^2 + 4yxRB - 4B^2zR$$

$$\Theta := (-4x^2 + 4B^2)z^2 + (4AB^2 + 8x^2R - 4B^2R + 4Ax^2)z - 4y^2x^2 + 4y^2 + 4A^2yx + 4A^2y^2 - 8A^2x^2 - 4x^2R^2 + 4yxRB - 4yxAB - 4Rx^2A + 4yAx$$

$$-z^2x^2 + y^2 + B^2z^2 + A^2y^2 - y^2x^2 + yAx + A^2yx - yxAB - 2A^2x^2 + zx^2A + B^2zA \quad \textbf{(34)}$$

HENON dynamical system

```
> AA:=[A*cos(y)+B*sin(z),B*cos(z)+C*sin(x),C*cos(x)+A*sin(y)];
```

$$AA := [A\cos(y) + B\sin(z), B\cos(z) + C\sin(x), C\cos(x) + A\sin(y)] \quad \textbf{(35)}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=[AA[1],AA[2],AA[3],phi];
```

$$\phi := 2A\cos(y)B\sin(z) + B^2 + 2B\cos(z)C\sin(x) + C^2 + 2C\cos(x)A\sin(y) + A^2$$

$$VV := [A\cos(y) + B\sin(z), B\cos(z) + C\sin(x), C\cos(x) + A\sin(y), 2A\cos(y)B\sin(z) + B^2 + 2B\cos(z)C\sin(x) + C^2 + 2C\cos(x)A\sin(y) + A^2] \quad \textbf{(36)}$$

```
> JAC:=simplify(jacobian(evalm(VV),[x,y,z,t]),trig);
```

$$JAC := [[0, -A\sin(y), B\cos(z), 0], [C\cos(x), 0, -B\sin(z), 0], [-C\sin(x), A\cos(y), 0, 0], [-2C(-B\cos(z)\cos(x) + \sin(x)A\sin(y)), -2A(\sin(y)B\sin(z) - C\cos(x)\cos(y)), 2B(A\cos(y)\cos(z) - \sin(z)C\sin(x)), 0]] \quad \textbf{(37)}$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
(innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=factor((1/2)*(Xm*S1-S2));Za:=factor((1/3)*(Yg*S1-Xm*S2+S3));Tk:=
factor(det(JAC));
```

$$Xm := 0$$

$$Yg := C\cos(x)A\sin(y) + B\cos(z)C\sin(x) + A\cos(y)B\sin(z)$$

$$Za := ABC(\cos(y)\cos(z)\cos(x) - \sin(y)\sin(z)\sin(x))$$

$$Tk := 0 \quad \textbf{(38)}$$

Thermodynamic crtical point is a surface in {x,y,z} (It is not locally stable)

```
> YYg:=factor(subs(A=0,Yg));
```

$$YYg := B\cos(z)\ C\sin(x) \tag{39}$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := A\cos(y) + B\sin(z)$$
$$CURLy := B\cos(z) + C\sin(x)$$
$$CURLz := C\cos(x) + A\sin(y) \tag{40}$$

```
> HELICITY:=collect(simplify(innerprod(AA,VORTICITY),trig),B);
```

$$HELICITY := B^2 + (2\cos(z)\ C\sin(x) + 2\ A\cos(y)\ \sin(z))\ B + 2\ C\cos(x)\ A\sin(y) + A^2 + C^2 \tag{41}$$

```
> enstrophy:=collect(simplify(innerprod(VORTICITY,VORTICITY),trig),R);
```

$$enstrophy := 2\ A\cos(y)\ B\sin(z) + B^2 + 2\ B\cos(z)\ C\sin(x) + C^2 + 2\ C\cos(x)\ A\sin(y) + A^2 \tag{42}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := [-C\sin(x)\ A\sin(y) + B\cos(z)\ C\cos(x),\ C\cos(x)\ A\cos(y) - A\sin(y)\ B\sin(z),\ -B\sin(z)\ C\sin(x) + A\cos(y)\ B\cos(z)] \tag{43}$$

```
> stretch:=simplify(innerprod(BRAND,VORTICITY),trig);
```

$$stretch := -C\sin(x)\ A^2\sin(y)\ \cos(y) - 3\ A\sin(y)\ B\sin(z)\ C\sin(x) + 3\ A\cos(y)\ B\cos(z)\ C\cos(x) + B^2\cos(z)\ C\cos(x)\ \sin(z) + C^2\cos(x)\ A\cos(y)\ \sin(x) - A\sin(y)\ B^2\sin(z)\ \cos(z) - B\sin(z)\ C^2\sin(x)\ \cos(x) + A^2\cos(y)\ B\cos(z)\ \sin(y) \tag{44}$$

```
> BI:=(simplify(innerprod(BRAND,BRAND),trig));
```

$$BI := C^2 A^2 - C^2 A^2\cos(y)^2 - C^2 A^2\cos(x)^2 + 2\ C^2\cos(x)^2 A^2\cos(y)^2 - 2\ C^2\sin(x)\ A\sin(y)\ B\cos(z)\ \cos(x) + 2\ B^2\cos(z)^2 C^2\cos(x)^2 - 2\ C\cos(x)\ A^2\cos(y)\ \sin(y)\ B\sin(z) + A^2 B^2 - A^2 B^2\cos(z)^2 - A^2 B^2\cos(y)^2 + 2\ A^2\cos(y)^2 B^2\cos(z)^2 + B^2 C^2 - B^2 C^2\cos(x)^2 - B^2 C^2\cos(z)^2 - 2\ B^2\sin(z)\ C\sin(x)\ A\cos(y)\ \cos(z) \tag{45}$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta := -12\ A\cos(y)\ B\cos(z)\ C\cos(x) - 4\ B^2\cos(z)\ C\cos(x)\ \sin(z) + 4\ C\sin(x)\ A^2\sin(y)\ \cos(y) + 12\ A\sin(y)\ B\sin(z)\ C\sin(x) + 4\ A\sin(y)\ B^2\sin(z)\ \cos(z) - 4\ C^2\cos(x)\ A\cos(y)\ \sin(x) - 4\ A^2\cos(y)\ B\cos(z)\ \sin(y) \tag{46}$$

$$+4\,B\sin(z)\,C^2\sin(x)\cos(x)$$

Taylor-Green dynamical system

```
> AA:=[A*sin(x)*cos(y)*cos(z),B*cos(x)*sin(y)*cos(z),C*cos(x)*cos(y)*sin(z)];
```

$$AA := [A\sin(x)\cos(y)\cos(z),\ B\cos(x)\sin(y)\cos(z),\ C\cos(x)\cos(y)\sin(z)] \quad (47)$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=[AA[1],AA[2],AA[3],phi];
```

$$\phi := A^2\cos(y)^2\cos(z)^2 - A^2\cos(y)^2\cos(z)^2\cos(x)^2 + B^2\cos(x)^2\cos(z)^2 - B^2\cos(x)^2\cos(z)^2\cos(y)^2 + C^2\cos(x)^2\cos(y)^2 - C^2\cos(x)^2\cos(y)^2\cos(z)^2$$

$$VV := \big[A\sin(x)\cos(y)\cos(z),\ B\cos(x)\sin(y)\cos(z),\ C\cos(x)\cos(y)\sin(z),\ A^2\cos(y)^2\cos(z)^2 - A^2\cos(y)^2\cos(z)^2\cos(x)^2 + B^2\cos(x)^2\cos(z)^2 - B^2\cos(x)^2\cos(z)^2\cos(y)^2 + C^2\cos(x)^2\cos(y)^2 - C^2\cos(x)^2\cos(y)^2\cos(z)^2\big] \quad (48)$$

```
> JAC:=simplify(jacobian(evalm(VV),[x,y,z,t]));
```

$$JAC := \big[\big[A\cos(x)\cos(y)\cos(z),\ -A\sin(x)\sin(y)\cos(z),\ -A\sin(x)\cos(y)\sin(z),\ 0\big],$$
$$\big[-B\sin(x)\sin(y)\cos(z),\ B\cos(x)\cos(y)\cos(z),\ -B\cos(x)\sin(y)\sin(z),\ 0\big],$$
$$\big[-C\sin(x)\cos(y)\sin(z),\ -C\cos(x)\sin(y)\sin(z),\ C\cos(x)\cos(y)\cos(z),\ 0\big],$$
$$\big[2\cos(x)\sin(x)\,\big(A^2\cos(y)^2\cos(z)^2 - B^2\cos(z)^2 + \cos(y)^2B^2\cos(z)^2 - C^2\cos(y)^2 + C^2\cos(y)^2\cos(z)^2\big),\ 2\cos(y)\sin(y)\,\big(-A^2\cos(z)^2 + A^2\cos(z)^2\cos(x)^2 + B^2\cos(x)^2\cos(z)^2 - C^2\cos(x)^2 + C^2\cos(x)^2\cos(z)^2\big),\ 2\cos(z)\sin(z)\,\big(-A^2\cos(y)^2 + A^2\cos(y)^2\cos(x)^2 - B^2\cos(x)^2 + B^2\cos(x)^2\cos(y)^2 + C^2\cos(x)^2\cos(y)^2\big),\ 0\big]\big] \quad (49)$$

```
> S1:=factor(trace(JAC)):S2:=simplify(trace(innerprod(JAC,JAC)),trig):S3:=
  simplify(trace(innerprod(JAC,JAC,JAC)),trig):
> Xm:=S1;Yg:=simplify((1/2)*(Xm*S1-S2),trig);Za:=factor((1/3)*(Yg*S1-Xm*S2+S3));
  Tk:=factor(det(JAC));
```

$$Xm := \cos(y)\cos(z)\cos(x)\,(A+B+C)$$

$$Yg := -A\cos(z)^2B + A\cos(z)^2B\cos(y)^2 + A\cos(z)^2B\cos(x)^2 - A\cos(y)^2C + A\cos(y)^2C\cos(x)^2 + A\cos(y)^2C\cos(z)^2 - B\cos(x)^2C + B\cos(x)^2C\cos(z)^2 + B\cos(x)^2C\cos(y)^2$$

$$Za := A\cos(y)\,B\cos(z)\,C\cos(x)\,\big(-2+\cos(y)^2+\cos(x)^2+\cos(z)^2\big)$$

$$Tk := 0 \quad (50)$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := \cos(x)\,\sin(y)\,\sin(z)\,(-C+B)$$
$$CURLy := -\sin(x)\,\cos(y)\,\sin(z)\,(A-C)$$
$$CURLz := \sin(x)\,\sin(y)\,\cos(z)\,(-B+A) \quad \textbf{(51)}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),B);
```

$$HELICITY := 0 \quad \textbf{(52)}$$

```
> enstrophy:=subs(cos(x)=X,cos(y)=Y,cos(z)=Z,(simplify(innerprod(VORTICITY,
VORTICITY),trig)));E1:=collect(enstrophy,A);
```

$$enstrophy := -2 A Z^2 B X^2 Y^2 - 2 A Y^2 C Z^2 X^2 - 2 B X^2 C Y^2 Z^2 + C^2 X^2 + B^2 X^2 + A^2 Y^2 + C^2 Y^2 + B^2 Z^2 + A^2 Z^2 - 2 B X^2 C - 2 A Y^2 C - 2 C^2 X^2 Y^2 - A^2 Y^2 X^2 - B^2 X^2 Y^2 - 2 A Z^2 B - 2 B^2 X^2 Z^2 - A^2 Z^2 X^2 - C^2 X^2 Z^2 - 2 A^2 Y^2 Z^2 - Y^2 B^2 Z^2 - C^2 Y^2 Z^2 + 2 A Y^2 C X^2 + 2 B X^2 C Y^2 + 2 A Z^2 B X^2 + 2 B X^2 C Z^2 + 2 A Z^2 B Y^2 + 2 A Y^2 C Z^2 + 2 A^2 Y^2 Z^2 X^2 + 2 B^2 X^2 Z^2 Y^2 + 2 C^2 X^2 Y^2 Z^2$$

$$E1 := \left(Y^2 + Z^2 - Y^2 X^2 - 2 Y^2 Z^2 + 2 Y^2 Z^2 X^2 - Z^2 X^2\right) A^2 + \left(-2 Y^2 C Z^2 X^2 - 2 Z^2 B X^2 Y^2 - 2 Y^2 C - 2 Z^2 B + 2 Y^2 C Z^2 + 2 Z^2 B X^2 + 2 Y^2 C X^2 + 2 Z^2 B Y^2\right) A - B^2 X^2 Y^2 - 2 B X^2 C - 2 B X^2 C Y^2 Z^2 + C^2 X^2 + B^2 X^2 - C^2 X^2 Z^2 + C^2 Y^2 + B^2 Z^2 + 2 C^2 X^2 Y^2 Z^2 + 2 B X^2 C Y^2 - 2 C^2 X^2 Y^2 - C^2 Y^2 Z^2 - 2 B^2 X^2 Z^2 + 2 B X^2 C Z^2 - Y^2 B^2 Z^2 + 2 B^2 X^2 Z^2 Y^2 \quad \textbf{(53)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := [-A \cos(x)^2 \cos(y)\,\cos(z)\,C \sin(y)\,\sin(z) + A \cos(x)^2 \cos(y)\,\cos(z)\,B \sin(y)\,\sin(z) - C \sin(x)^2 \cos(y)\,\sin(z)\,A \sin(y)\,\cos(z) + B \sin(x)^2 \sin(y)\,\cos(z)\,A \cos(y)\,\sin(z),\; C \cos(x)\,\sin(y)^2 \sin(z)\,B \sin(x)\,\cos(z) - B \cos(x)\,\cos(y)^2 \cos(z)\,A \sin(x)\,\sin(z) + B \cos(x)\,\cos(y)^2 \cos(z)\,C \sin(x)\,\sin(z) - A \sin(x)\,\sin(y)^2 \cos(z)\,B \cos(x)\,\sin(z),\; -B \cos(x)\,\sin(y)\,\sin(z)^2 C \sin(x)\,\cos(y) + A \sin(x)\,\cos(y)\,\sin(z)^2 C \cos(x)\,\sin(y) - \cos(y)\,\cos(z)^2 C \cos(x)\,B \sin(x)\,\sin(y) + \cos(y)\,\cos(z)^2 C \cos(x)\,A \sin(x)\,\sin(y)] \quad \textbf{(54)}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),B);ST1:=factor(coeff
(stretch,B));
```

$$stretch := \cos(y)\,\cos(z)\,\cos(x)\,\left(\cos(x)^2 A \sin(y)^2 \sin(z)^2 + \sin(x)^2 \sin(y)^2 A \sin(z)^2 + \sin(x)^2 \sin(z)^2 C \sin(y)^2 + \cos(z)^2 C \sin(x)^2 \sin(y)^2\right) B^2 + \cos(y)\,\cos(z)\,\cos(x)\,\big(-2 \cos(x)^2 A\, C \sin(y)^2 \sin(z)^2 + C^2 \sin(x)^2 \cos(y)^2 \sin(z)^2 + C^2 \sin(x)^2 \sin(z)^2 \sin(y)^2 + A^2 \sin(x)^2 \cos(y)^2 \sin(z)^2 - 6 A \sin(x)^2 \sin(z)^2 C \sin(y)^2 + A^2 \sin(x)^2 \sin(y)^2 \sin(z)^2$$

$$- 2\,A\sin(x)^2\cos(y)^2\sin(z)^2\,C - 2\cos(z)^2\,C\sin(x)^2\sin(y)^2\,A\big)\,B$$
$$+ \cos(y)\cos(z)\cos(x)\,\big(\cos(x)^2\,A\,C^2\sin(y)^2\sin(z)^2 + \cos(z)^2\,C\,A^2\sin(x)^2\sin(y)^2$$
$$+ C^2\sin(x)^2\sin(z)^2\,A\sin(y)^2 + A^2\sin(x)^2\sin(z)^2\,C\sin(y)^2\big)$$

$$ST1 := \cos(y)\cos(z)\cos(x)\,\big(-2\cos(x)^2\,A\,C\sin(y)^2\sin(z)^2 + C^2\sin(x)^2\cos(y)^2\sin(z)^2 \tag{55}$$
$$+ C^2\sin(x)^2\sin(z)^2\sin(y)^2 + A^2\sin(x)^2\cos(y)^2\sin(z)^2 - 6\,A\sin(x)^2\sin(z)^2\,C\sin(y)^2$$
$$+ A^2\sin(x)^2\sin(y)^2\sin(z)^2 - 2\,A\sin(x)^2\cos(y)^2\sin(z)^2\,C$$
$$- 2\cos(z)^2\,C\sin(x)^2\sin(y)^2\,A\big)$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),B);BI2:=factor(coeff(BI,B^2));BI1:=
  factor(coeff(BI,B));BI0:=collect(factor(subs(B=0,BI)),R);
```

$$BI := \big(-4\,C\cos(x)^2\sin(y)^2\sin(z)^2\sin(x)^2\cos(z)^2\cos(y)^2\,A$$
$$+ A^2\cos(x)^4\cos(y)^2\cos(z)^2\sin(y)^2\sin(z)^2 + \sin(x)^4\sin(y)^2\cos(z)^2\,A^2\cos(y)^2\sin(z)^2$$
$$+ C^2\cos(x)^2\sin(y)^4\sin(z)^2\sin(x)^2\cos(z)^2 + \cos(x)^2\cos(y)^4\cos(z)^2\,A^2\sin(x)^2\sin(z)^2$$
$$+ \cos(x)^2\cos(y)^4\cos(z)^2\,C^2\sin(x)^2\sin(z)^2 + A^2\sin(x)^2\sin(y)^4\cos(z)^2\cos(x)^2\sin(z)^2$$
$$+ \sin(x)^2\cos(y)^2\sin(z)^4\,C^2\cos(x)^2\sin(y)^2 + \cos(y)^2\cos(z)^4\,C^2\cos(x)^2\sin(x)^2\sin(y)^2$$
$$+ 4\,A^2\cos(x)^2\cos(y)^2\cos(z)^2\sin(y)^2\sin(z)^2\sin(x)^2$$
$$+ 4\cos(x)^2\cos(y)^2\cos(z)^2\,C^2\sin(y)^2\sin(z)^2\sin(x)^2$$
$$- 2\,C\cos(x)^2\sin(y)^4\sin(z)^2\sin(x)^2\cos(z)^2\,A$$
$$- 2\cos(x)^2\cos(y)^4\cos(z)^2\,A\sin(x)^2\sin(z)^2\,C\big)\,B^2 + \big($$
$$-2\,A^2\cos(x)^4\cos(y)^2\cos(z)^2\,C\sin(y)^2\sin(z)^2$$
$$- 4\cos(x)^2\sin(y)^2\sin(z)^2\,C^2\sin(x)^2\cos(y)^2\cos(z)^2\,A$$
$$- 2\cos(y)^2\cos(z)^4\,C^2\cos(x)^2\sin(x)^2\sin(y)^2\,A$$
$$- 2\,C\sin(x)^4\cos(y)^2\sin(z)^2\,A^2\sin(y)^2\cos(z)^2$$
$$- 4\,A^2\cos(x)^2\cos(y)^2\cos(z)^2\,C\sin(y)^2\sin(z)^2\sin(x)^2$$
$$- 2\cos(x)^2\sin(y)^2\sin(z)^4\,C^2\sin(x)^2\cos(y)^2\,A\big)\,B$$
$$+ \cos(y)^2\cos(z)^4\,C^2\cos(x)^2\,A^2\sin(x)^2\sin(y)^2$$
$$+ A^2\cos(x)^4\cos(y)^2\cos(z)^2\,C^2\sin(y)^2\sin(z)^2$$
$$+ A^2\sin(x)^2\cos(y)^2\sin(z)^4\,C^2\cos(x)^2\sin(y)^2$$
$$+ C^2\sin(x)^4\cos(y)^2\sin(z)^2\,A^2\sin(y)^2\cos(z)^2$$
$$+ 4\,A^2\cos(x)^2\cos(y)^2\cos(z)^2\,C^2\sin(y)^2\sin(z)^2\sin(x)^2$$

$$BI2 := -4\,C\cos(x)^2\sin(y)^2\sin(z)^2\sin(x)^2\cos(z)^2\cos(y)^2\,A$$
$$+ A^2\cos(x)^4\cos(y)^2\cos(z)^2\sin(y)^2\sin(z)^2 + \sin(x)^4\sin(y)^2\cos(z)^2\,A^2\cos(y)^2\sin(z)^2$$

$$+C^2\cos(x)^2\sin(y)^4\sin(z)^2\sin(x)^2\cos(z)^2+\cos(x)^2\cos(y)^4\cos(z)^2A^2\sin(x)^2\sin(z)^2$$
$$+\cos(x)^2\cos(y)^4\cos(z)^2C^2\sin(x)^2\sin(z)^2+A^2\sin(x)^2\sin(y)^4\cos(z)^2\cos(x)^2\sin(z)^2$$
$$+\sin(x)^2\cos(y)^2\sin(z)^4C^2\cos(x)^2\sin(y)^2+\cos(y)^2\cos(z)^4C^2\cos(x)^2\sin(x)^2\sin(y)^2$$
$$+4A^2\cos(x)^2\cos(y)^2\cos(z)^2\sin(y)^2\sin(z)^2\sin(x)^2$$
$$+4\cos(x)^2\cos(y)^2\cos(z)^2C^2\sin(y)^2\sin(z)^2\sin(x)^2$$
$$-2C\cos(x)^2\sin(y)^4\sin(z)^2\sin(x)^2\cos(z)^2A$$
$$-2\cos(x)^2\cos(y)^4\cos(z)^2A\sin(x)^2\sin(z)^2C$$
$$BI1:=-2A\cos(y)^2C\sin(y)^2\big(A\cos(x)^4\cos(z)^2\sin(z)^2+2\cos(x)^2\sin(z)^2C\sin(x)^2\cos(z)^2$$
$$+\cos(z)^4C\cos(x)^2\sin(x)^2+\sin(x)^4\sin(z)^2A\cos(z)^2$$
$$+2A\cos(x)^2\cos(z)^2\sin(z)^2\sin(x)^2+\cos(x)^2\sin(z)^4C\sin(x)^2\big)$$
$$BI0:=C^2A^2\cos(y)^2\sin(y)^2\big(\cos(z)^4\cos(x)^2\sin(x)^2+\cos(x)^4\cos(z)^2\sin(z)^2 \quad \textbf{(56)}$$
$$+\sin(x)^2\sin(z)^4\cos(x)^2+\sin(x)^4\sin(z)^2\cos(z)^2+4\cos(x)^2\cos(z)^2\sin(z)^2\sin(x)^2\big)$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta:=-4\sin(y)\sin(z)\sin(x)\big(B^2\cos(x)^2C\cos(z)^2-B\cos(x)^2C^2\cos(y)^2 \quad \textbf{(57)}$$
$$-A^2\cos(y)^2C\cos(z)^2+A\cos(y)^2C^2\cos(x)^2+A^2\cos(z)^2B\cos(y)^2$$
$$-A\cos(z)^2B^2\cos(x)^2\big)$$

Moffat dynamical system

```
> AA:=[A*x+Omega*z,B*y+G*x^2+F*x*z,C*z-Omega*x-F*x*y];
```

$$AA:=\left[Ax+\Omega z,\ By+Gx^2+Fxz,\ Cz-\Omega x-Fxy\right] \quad \textbf{(58)}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=evalm([AA[1],AA[2],AA[3],phi]);
```

$$\phi:=A^2x^2+2Ax\Omega z+z^2\Omega^2+B^2y^2+2ByGx^2+2ByFxz+G^2x^4+2Gx^3Fz+F^2x^2z^2$$
$$+C^2z^2-2Cz\Omega x-2CzFxy+\Omega^2x^2+2\Omega x^2Fy+F^2x^2y^2$$
$$VV:=\big[Ax+\Omega z,\ By+Gx^2+Fxz,\ Cz-\Omega x-Fxy,\ A^2x^2+2Ax\Omega z+z^2\Omega^2+B^2y^2 \quad \textbf{(59)}$$
$$+2ByGx^2+2ByFxz+G^2x^4+2Gx^3Fz+F^2x^2z^2+C^2z^2-2Cz\Omega x-2CzFxy$$
$$+\Omega^2x^2+2\Omega x^2Fy+F^2x^2y^2\big]$$

```
> JAC:=simplify(jacobian(evalm(VV),[x,y,z,t]));collect(simplify(expand(charpoly
  (JAC,q))),q);
```

$$JAC:=\big[\left[A,0,\Omega,0\right],$$
$$\left[2Gx+Fz,B,Fx,0\right],$$

$$\left[-\Omega - F\,y,\ -F\,x,\ C,\ 0\right],$$

$$\left[2\,A^2\,x + 2\,A\,\Omega\,z + 4\,B\,y\,G\,x + 2\,B\,y\,F\,z + 4\,G^2\,x^3 + 6\,G\,x^2\,F\,z + 2\,F^2\,x\,z^2 - 2\,C\,z\,\Omega\right.$$

$$- 2\,C\,z\,F\,y + 2\,\Omega^2\,x + 4\,\Omega\,x\,F\,y + 2\,F^2\,x\,y^2,\ 2\,y\,B^2 + 2\,B\,G\,x^2 + 2\,B\,F\,x\,z - 2\,C\,z\,F\,x$$

$$+ 2\,\Omega\,x^2\,F + 2\,F^2\,x^2\,y,\ 2\,A\,x\,\Omega + 2\,\Omega^2\,z + 2\,B\,y\,F\,x + 2\,G\,x^3\,F + 2\,F^2\,x^2\,z + 2\,C^2\,z$$

$$\left.- 2\,\Omega\,C\,x - 2\,C\,F\,x\,y,\ 0\right]\Big]$$

$$q^4 + (-A - C - B)\,q^3 + \left(\Omega\,F\,y + C\,B + F^2\,x^2 + C\,A + A\,B + \Omega^2\right)q^2 + \left(-\Omega^2\,B - A\,F^2\,x^2 - \Omega\,F\,y\,B + F^2\,x\,\Omega\,z - A\,B\,C + 2\,F\,x^2\,\Omega\,G\right)q \qquad \textbf{(60)}$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
(innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=factor((1/2)*(Xm*S1-S2));Za:=collect(factor((1/3)*(Yg*S1-Xm*S2+S3)
),Omega);;Tk:=factor(det(JAC));
```

$$Xm := A + B + C$$

$$Yg := \Omega\,F\,y + C\,B + F^2\,x^2 + C\,A + A\,B + \Omega^2$$

$$Za := \Omega^2\,B + \left(F\,y\,B - 2\,F\,x^2\,G - F^2\,x\,z\right)\Omega + A\,B\,C + A\,F^2\,x^2$$

$$Tk := 0 \qquad \textbf{(61)}$$

```
> BBB:=solve(Xm-3,B);YgB:=factor(subs(B=BBB,Yg));FFF:=solve(YgB-3,F);ZaBR:=
factor(subs(B=BBB,F=FFF,Za));CriticalPoint:=collect(ZaBR-1,Omega);
```

$$BBB := -A - C + 3$$

$$YgB := \Omega\,F\,y - C\,A - C^2 + 3\,C + F^2\,x^2 - A^2 + 3\,A + \Omega^2$$

$$FFF := \frac{1}{2}\,\frac{-\Omega\,y + \sqrt{\Omega^2\,y^2 + 12\,x^2 + 4\,x^2\,C\,A + 4\,C^2\,x^2 - 12\,x^2\,C + 4\,A^2\,x^2 - 12\,A\,x^2 - 4\,\Omega^2\,x^2}}{x^2},$$

$$-\frac{1}{2}\,\frac{\Omega\,y + \sqrt{\Omega^2\,y^2 + 12\,x^2 + 4\,x^2\,C\,A + 4\,C^2\,x^2 - 12\,x^2\,C + 4\,A^2\,x^2 - 12\,A\,x^2 - 4\,\Omega^2\,x^2}}{x^2}$$

$$ZaBR := -\Omega^2\,A - \Omega^2\,C + 3\,\Omega^2$$

$$-\Omega\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$\left.+ \sqrt{\Omega^2\,y^2 + 12\,x^2 + 4\,x^2\,C\,A + 4\,C^2\,x^2 - 12\,x^2\,C + 4\,A^2\,x^2 - 12\,A\,x^2 - 4\,\Omega^2\,x^2}\right),$$

$$\left.-\frac{1}{2}\,\frac{\Omega\,y + \sqrt{\Omega^2\,y^2 + 12\,x^2 + 4\,x^2\,C\,A + 4\,C^2\,x^2 - 12\,x^2\,C + 4\,A^2\,x^2 - 12\,A\,x^2 - 4\,\Omega^2\,x^2}}{x^2}\right)$$

$$y\,A$$

$$-\Omega\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$\left.+\sqrt{\Omega^2\,y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2\,x^2-12\,x^2\,C+4\,A^2\,x^2-12\,A\,x^2-4\,\Omega^2\,x^2}\right),$$

$$\left.-\frac{1}{2}\,\frac{\Omega\,y+\sqrt{\Omega^2\,y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2\,x^2-12\,x^2\,C+4\,A^2\,x^2-12\,A\,x^2-4\,\Omega^2\,x^2}}{x^2}\right)$$

$$y\,C$$

$$+3\,\Omega\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$\left.+\sqrt{\Omega^2\,y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2\,x^2-12\,x^2\,C+4\,A^2\,x^2-12\,A\,x^2-4\,\Omega^2\,x^2}\right),$$

$$\left.-\frac{1}{2}\,\frac{\Omega\,y+\sqrt{\Omega^2\,y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2\,x^2-12\,x^2\,C+4\,A^2\,x^2-12\,A\,x^2-4\,\Omega^2\,x^2}}{x^2}\right)$$

$$y$$

$$-2\,\Omega\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$\left.+\sqrt{\Omega^2\,y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2\,x^2-12\,x^2\,C+4\,A^2\,x^2-12\,A\,x^2-4\,\Omega^2\,x^2}\right),$$

$$\left.-\frac{1}{2}\,\frac{\Omega\,y+\sqrt{\Omega^2\,y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2\,x^2-12\,x^2\,C+4\,A^2\,x^2-12\,A\,x^2-4\,\Omega^2\,x^2}}{x^2}\right)$$

$$x^2\,G$$

$$-\Omega\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$\left.+\sqrt{\Omega^2\,y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2\,x^2-12\,x^2\,C+4\,A^2\,x^2-12\,A\,x^2-4\,\Omega^2\,x^2}\right),$$

$$\left.-\frac{1}{2}\,\frac{\Omega\,y+\sqrt{\Omega^2\,y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2\,x^2-12\,x^2\,C+4\,A^2\,x^2-12\,A\,x^2-4\,\Omega^2\,x^2}}{x^2}\right)$$

2

$$x\,z-C\,A^2-C^2\,A+3\,C\,A$$

$$+A\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$\left.+\sqrt{\Omega^2 y^2+12x^2+4x^2 C A+4C^2x^2-12x^2C+4A^2x^2-12Ax^2-4\Omega^2x^2}\right),$$

$$\left.-\frac{1}{2}\,\frac{\Omega\,y+\sqrt{\Omega^2 y^2+12x^2+4x^2 C A+4C^2x^2-12x^2C+4A^2x^2-12Ax^2-4\Omega^2x^2}}{x^2}\right)$$

$$2$$

$$x^2$$

$$CriticalPoint := \Omega^2\left(-A-C+3\right)+\Bigg(\qquad \textbf{(62)}$$

$$-\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$\left.+\sqrt{\Omega^2 y^2+12x^2+4x^2 C A+4C^2x^2-12x^2C+4A^2x^2-12Ax^2-4\Omega^2x^2}\right),$$

$$\left.-\frac{1}{2}\,\frac{\Omega\,y+\sqrt{\Omega^2 y^2+12x^2+4x^2 C A+4C^2x^2-12x^2C+4A^2x^2-12Ax^2-4\Omega^2x^2}}{x^2}\right)$$

$$y\,A$$

$$-\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$\left.+\sqrt{\Omega^2 y^2+12x^2+4x^2 C A+4C^2x^2-12x^2C+4A^2x^2-12Ax^2-4\Omega^2x^2}\right),$$

$$\left.-\frac{1}{2}\,\frac{\Omega\,y+\sqrt{\Omega^2 y^2+12x^2+4x^2 C A+4C^2x^2-12x^2C+4A^2x^2-12Ax^2-4\Omega^2x^2}}{x^2}\right)$$

$$y\,C$$

$$+3\left(\frac{1}{2}\,\frac{1}{x^2}\left(-\Omega\,y\right.\right.$$

$$+\sqrt{\Omega^2 y^2 + 12\, x^2 + 4\, x^2\, C\, A + 4\, C^2\, x^2 - 12\, x^2\, C + 4\, A^2\, x^2 - 12\, A\, x^2 - 4\, \Omega^2\, x^2}\ \Big),$$

$$-\frac{1}{2}\ \frac{\Omega\, y + \sqrt{\Omega^2 y^2 + 12\, x^2 + 4\, x^2\, C\, A + 4\, C^2\, x^2 - 12\, x^2\, C + 4\, A^2\, x^2 - 12\, A\, x^2 - 4\, \Omega^2\, x^2}}{x^2}\ \Bigg)$$

$$y$$

$$-\, 2\, \Bigg(\frac{1}{2}\ \frac{1}{x^2}\ \Big(-\Omega\, y$$

$$+\sqrt{\Omega^2 y^2 + 12\, x^2 + 4\, x^2\, C\, A + 4\, C^2\, x^2 - 12\, x^2\, C + 4\, A^2\, x^2 - 12\, A\, x^2 - 4\, \Omega^2\, x^2}\ \Big),$$

$$-\frac{1}{2}\ \frac{\Omega\, y + \sqrt{\Omega^2 y^2 + 12\, x^2 + 4\, x^2\, C\, A + 4\, C^2\, x^2 - 12\, x^2\, C + 4\, A^2\, x^2 - 12\, A\, x^2 - 4\, \Omega^2\, x^2}}{x^2}\ \Bigg)$$

$$x^2\, G$$

$$-\, \Bigg(\frac{1}{2}\ \frac{1}{x^2}\ \Big(-\Omega\, y$$

$$+\sqrt{\Omega^2 y^2 + 12\, x^2 + 4\, x^2\, C\, A + 4\, C^2\, x^2 - 12\, x^2\, C + 4\, A^2\, x^2 - 12\, A\, x^2 - 4\, \Omega^2\, x^2}\ \Big),$$

$$-\frac{1}{2}\ \frac{\Omega\, y + \sqrt{\Omega^2 y^2 + 12\, x^2 + 4\, x^2\, C\, A + 4\, C^2\, x^2 - 12\, x^2\, C + 4\, A^2\, x^2 - 12\, A\, x^2 - 4\, \Omega^2\, x^2}}{x^2}\ \Bigg)$$

$$^{2}\ x\, z\, \Bigg)\ \Omega - C\, A^2 - C^2\, A + 3\, C\, A$$

$$+\, A\, \Bigg(\frac{1}{2}\ \frac{1}{x^2}\ \Big(-\Omega\, y$$

$$+\sqrt{\Omega^2 y^2 + 12\, x^2 + 4\, x^2\, C\, A + 4\, C^2\, x^2 - 12\, x^2\, C + 4\, A^2\, x^2 - 12\, A\, x^2 - 4\, \Omega^2\, x^2}\ \Big),$$

$$- \frac{1}{2} \frac{\Omega y + \sqrt{\Omega^2 y^2 + 12 x^2 + 4 x^2 C A + 4 C^2 x^2 - 12 x^2 C + 4 A^2 x^2 - 12 A x^2 - 4 \Omega^2 x^2}}{x^2} \Bigg)$$

^{2}

$$x^2 - 1$$

```
> BBB:=solve(Xm,B);ZaB:=factor(subs(B=BBB,Za));GGG:=solve(ZaB,G);YYg:=factor
  (subs(B=BBB,G=GGG,Yg));
  Hopfcrit:=collect(YYg,A);
>
```

$$BBB := -C - A$$

$$ZaB := -\Omega^2 C - \Omega^2 A - \Omega C F y - \Omega A F y - 2 F x^2 \Omega G - F^2 x \Omega z - C^2 A - C A^2 + A F^2 x^2$$

$$GGG := -\frac{1}{2} \frac{\Omega^2 C + \Omega^2 A + \Omega C F y + \Omega A F y - A F^2 x^2 + F^2 x \Omega z + C^2 A + C A^2}{\Omega x^2 F}$$

$$YYg := \Omega F y - C^2 - C A + F^2 x^2 - A^2 + \Omega^2$$

$$Hopfcrit := \Omega F y - C^2 - C A + F^2 x^2 - A^2 + \Omega^2 \quad \textbf{(63)}$$

```
> INTER:=collect((numer(CriticalPoint)+numer(Hopfcrit)),A);solve(INTER,A);;
```

$$INTER := (-1 - C) A^2 + \Bigg(-C^2$$

$$- \Omega \Bigg(\frac{1}{2} \frac{1}{x^2} \Big(-\Omega y$$

$$+ \sqrt{\Omega^2 y^2 + 12 x^2 + 4 x^2 C A + 4 C^2 x^2 - 12 x^2 C + 4 A^2 x^2 - 12 A x^2 - 4 \Omega^2 x^2} \Big),$$

$$- \frac{1}{2} \frac{\Omega y + \sqrt{\Omega^2 y^2 + 12 x^2 + 4 x^2 C A + 4 C^2 x^2 - 12 x^2 C + 4 A^2 x^2 - 12 A x^2 - 4 \Omega^2 x^2}}{x^2} \Bigg)$$

$$y - \Omega^2 + 2 C$$

$$+ \Bigg(\frac{1}{2} \frac{1}{x^2} \Big(-\Omega y$$

$$+ \sqrt{\Omega^2 y^2 + 12 x^2 + 4 x^2 C A + 4 C^2 x^2 - 12 x^2 C + 4 A^2 x^2 - 12 A x^2 - 4 \Omega^2 x^2} \Big),$$

$$-\frac{1}{2}\;\frac{\Omega\, y+\sqrt{\Omega^2 y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2 x^2-12\,x^2\,C+4\,A^2 x^2-12\,A\,x^2-4\,\Omega^2 x^2}}{x^2}\Bigg)$$

$$^2\;x^2\Bigg)A+\Omega\,F\,y-\Omega^2\,C+4\,\Omega^2$$

$$-\Omega\left(\frac{1}{2}\;\frac{1}{x^2}\Big(-\Omega\,y\right.$$

$$\left.+\sqrt{\Omega^2 y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2 x^2-12\,x^2\,C+4\,A^2 x^2-12\,A\,x^2-4\,\Omega^2 x^2}\Big)\right.,$$

$$-\frac{1}{2}\;\frac{\Omega\, y+\sqrt{\Omega^2 y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2 x^2-12\,x^2\,C+4\,A^2 x^2-12\,A\,x^2-4\,\Omega^2 x^2}}{x^2}\Bigg)$$

$$^2\;x\,z$$

$$-\Omega\left(\frac{1}{2}\;\frac{1}{x^2}\Big(-\Omega\,y\right.$$

$$\left.+\sqrt{\Omega^2 y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2 x^2-12\,x^2\,C+4\,A^2 x^2-12\,A\,x^2-4\,\Omega^2 x^2}\Big)\right.,$$

$$-\frac{1}{2}\;\frac{\Omega\, y+\sqrt{\Omega^2 y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2 x^2-12\,x^2\,C+4\,A^2 x^2-12\,A\,x^2-4\,\Omega^2 x^2}}{x^2}\Bigg)$$

$$y\,C$$

$$+3\,\Omega\left(\frac{1}{2}\;\frac{1}{x^2}\Big(-\Omega\,y\right.$$

$$\left.+\sqrt{\Omega^2 y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2 x^2-12\,x^2\,C+4\,A^2 x^2-12\,A\,x^2-4\,\Omega^2 x^2}\Big)\right.,$$

$$-\frac{1}{2}\;\frac{\Omega\, y+\sqrt{\Omega^2 y^2+12\,x^2+4\,x^2\,C\,A+4\,C^2 x^2-12\,x^2\,C+4\,A^2 x^2-12\,A\,x^2-4\,\Omega^2 x^2}}{x^2}\Bigg)$$

$$y$$

$$-2\,\Omega\left(\frac{1}{2}\;\frac{1}{x^2}\Big(-\Omega\,y\right.$$

$$+\sqrt{\Omega^2 y^2 + 12 x^2 + 4 x^2 C A + 4 C^2 x^2 - 12 x^2 C + 4 A^2 x^2 - 12 A x^2 - 4 \Omega^2 x^2}\Big),$$

$$-\frac{1}{2}\left(\frac{\Omega y + \sqrt{\Omega^2 y^2 + 12 x^2 + 4 x^2 C A + 4 C^2 x^2 - 12 x^2 C + 4 A^2 x^2 - 12 A x^2 - 4 \Omega^2 x^2}}{x^2}\right)$$

$$x^2 G + F^2 x^2 - 1 - C^2$$

Warning, solutions may have been lost

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := -2 F x$$

$$CURLy := 2 \Omega + F y$$

$$CURLz := 2 G x + F z \quad \textbf{(64)}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),Omega);
```

$$HELICITY := (-F x z + 2 B y) \Omega - 2 F x^2 A + 2 C z G x + C z^2 F + B y^2 F - G x^2 F y \quad \textbf{(65)}$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),Omega);
```

$$enstrophy := 4 F^2 x^2 + 4 \Omega^2 + 4 \Omega F y + F^2 y^2 + 4 G^2 x^2 + 4 G x F z + F^2 z^2 \quad \textbf{(66)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \quad \textbf{(67)}$$

$$\left[-2 A F x + 2 \Omega G x + \Omega F z, -2 F x^2 G - F^2 x z + 2 \Omega B + F y B, F^2 x y + 2 C G x + C F z\right]$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),Omega);ST1:=factor(coeff
  (stretch,Omega));st0:=collect(factor(subs(Omega=0,stretch)),F);
```

$$stretch := 4 \Omega^2 B + \left(-8 F x^2 G - 4 F^2 x z + 4 F y B\right) \Omega + 4 A F^2 x^2 + F^2 y^2 B + 4 C G^2 x^2 + 4 C G x F z + C F^2 z^2$$

$$ST1 := 4 F \left(-F x z + B y - 2 G x^2\right)$$

$$st0 := \left(4 A x^2 + y^2 B + C z^2\right) F^2 + 4 C G x F z + 4 C G^2 x^2 \quad \textbf{(68)}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),Omega);BI2:=factor(coeff(BI,
  Omega^2));BI1:=factor(coeff(BI,Omega));BI0:=collect(factor(subs(Omega=0,BI)),
```

```
F);
```

$$BI := \left(4\,G^2 x^2 + 4\,G\,x\,F\,z + F^2 z^2 + 4\,B^2\right)\Omega^2 + \left(-8\,A\,F\,x^2 G - 8\,B\,F\,x^2 G + 4\,B^2 F\,y - 4\,B\,F^2 x\,z - 4\,A\,F^2 x\,z\right)\Omega + 4\,A^2 F^2 x^2 + 2\,F^3 x\,y\,C\,z + F^2 y^2 B^2 + F^4 x^2 y^2 + 4\,F^2 x^2 y\,C\,G - 4\,F^2 x^2 G\,y\,B + 4\,F^2 x^4 G^2 + 4\,F^3 x^3 G\,z + C^2 F^2 z^2 + F^4 x^2 z^2 + 4\,C^2 G^2 x^2 - 2\,F^3 x\,z\,y\,B + 4\,C^2 G\,x\,F\,z$$

$$BI2 := 4\,G^2 x^2 + 4\,G\,x\,F\,z + F^2 z^2 + 4\,B^2$$

$$BI1 := -4\,F\left(B\,F\,x\,z + F\,A\,z\,x + 2\,G\,x^2 A + 2\,B\,G\,x^2 - y\,B^2\right)$$

$$BI0 := \left(z^2 x^2 + y^2 x^2\right)F^4 + \left(4\,z\,G\,x^3 + 2\,x\,y\,C\,z - 2\,y\,x\,B\,z\right)F^3 + \left(-4\,B\,y\,G\,x^2 + 4\,A^2 x^2 + B^2 y^2 + 4\,x^2 y\,C\,G + 4\,G^2 x^4 + C^2 z^2\right)F^2 + 4\,C^2 G\,x\,F\,z + 4\,C^2 G^2 x^2 \tag{69}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t2:=
  factor(coeff(Theta,Omega^2));t1:=collect(factor(-coeff(Theta,Omega)),F);;
```

$$\Theta := \left(-4\,z^2 F - 8\,z\,x\,G\right)\Omega^2 + \left(4\,F^2 x^2 y - 8\,G\,x^2 A + 8\,G\,C\,x^2 - 8\,B\,G\,x^2 - 8\,y\,B^2 + 4\,C\,z\,F\,x - 8\,B\,F\,x\,z + 4\,F\,A\,z\,x\right)\Omega - 8\,z\,x\,G\,C^2 + 4\,F\,x^2 B\,y\,G + 8\,F\,x^4 G^2 + 4\,F^3 x^2 y^2 + 8\,F\,x^2 A^2 - 4\,C^2 z^2 F - 4\,y^2 B^2 F + 12\,F^2 x^3 G\,z + 8\,C\,F\,x^2 y\,G + 4\,F^3 x^2 z^2$$

$$t2 := -4\,z\,(2\,G\,x + F\,z)$$

$$t1 := -4\,F^2 x^2 y + (-4\,C\,z\,x + 8\,x\,B\,z - 4\,A\,z\,x)\,F + 8\,y\,B^2 + 8\,G\,x^2 A - 8\,G\,C\,x^2 + 8\,B\,G\,x^2 \tag{70}$$

```
> t0:=collect(factor(subs(Omega=0,Theta)),F);;
```

$$t0 := \left(4\,z^2 x^2 + 4\,y^2 x^2\right)F^3 + 12\,F^2 x^3 G\,z + \left(4\,B\,y\,G\,x^2 + 8\,G^2 x^4 + 8\,A^2 x^2 - 4\,C^2 z^2 - 4\,B^2 y^2 + 8\,x^2 y\,C\,G\right)F - 8\,z\,x\,G\,C^2 \tag{71}$$

Brusselator dynamical system

```
> AA:=[A-Omega*(x-G*x^2*y)-D*x,Omega*(x-G*x^2*y),0];
```

$$AA := \left[A - \Omega\left(x - G\,x^2 y\right) - \mathrm{D}\,x,\ \Omega\left(x - G\,x^2 y\right),\ 0\right] \tag{72}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=evalm([AA[1],AA[2],AA[3],phi]);
```

$$\phi := A^2 - 2\,A\,x\,\Omega + 2\,A\,\Omega\,G\,x^2 y - 2\,A\,\mathrm{D}\,x + 2\,\Omega^2 x^2 - 4\,\Omega^2 x^3 G\,y + 2\,\Omega\,x^2\,\mathrm{D} + 2\,\Omega^2 G^2 x^4 y^2 - 2\,\Omega\,G\,x^3 y\,\mathrm{D} + \mathrm{D}^2 x^2$$

$$VV := \left[A - \Omega\left(x - G\,x^2 y\right) - \mathrm{D}\,x,\ \Omega\left(x - G\,x^2 y\right),\ 0,\ A^2 - 2\,A\,x\,\Omega + 2\,A\,\Omega\,G\,x^2 y - 2\,A\,\mathrm{D}\,x + 2\,\Omega^2 x^2 - 4\,\Omega^2 x^3 G\,y + 2\,\Omega\,x^2\,\mathrm{D} + 2\,\Omega^2 G^2 x^4 y^2 - 2\,\Omega\,G\,x^3 y\,\mathrm{D} + \mathrm{D}^2 x^2\right] \tag{73}$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t]));collect(simplify(expand(charpoly(JAC,q))
  ),q);
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
```

$$JAC := [[-\Omega + 2\,\Omega\,G\,x\,y - D,\ G\,x^2\,\Omega,\ 0,\ 0],$$
$$[-\Omega\,(-1 + 2\,G\,x\,y),\ -G\,x^2\,\Omega,\ 0,\ 0],$$
$$[0,\ 0,\ 0,\ 0],$$
$$[-2\,\Omega\,A + 4\,A\,\Omega\,G\,x\,y - 2\,A\,D + 4\,\Omega^2\,x - 12\,\Omega^2\,x^2\,G\,y + 4\,\Omega\,x\,D + 8\,\Omega^2\,G^2\,x^3\,y^2$$
$$- 6\,\Omega\,G\,x^2\,y\,D + 2\,D^2\,x,\ 2\,A\,x^2\,\Omega\,G - 4\,\Omega^2\,x^3\,G + 4\,\Omega^2\,G^2\,x^4\,y - 2\,\Omega\,G\,x^3\,D,\ 0,\ 0]]$$
$$q^4 + (D + \Omega - 2\,\Omega\,G\,x\,y + G\,x^2\,\Omega)\,q^3 + q^2\,D\,G\,x^2\,\Omega \tag{74}$$

```
> Xm:=collect(S1,Omega);;Yg:=factor((1/2)*(Xm*S1-S2));Za:=factor((1/3)*(Yg*S1-Xm*S2+S3));Tk:=factor(det(JAC));
```

$$Xm := (-1 - G\,x^2 + 2\,G\,x\,y)\,\Omega - D$$
$$Yg := D\,G\,x^2\,\Omega$$
$$Za := 0$$
$$Tk := 0 \tag{75}$$

Loss of Local Stability

```
> DDD:=solve(Xm,D);YgD:=subs(D=DDD,Yg);solve(YgD,Omega);
```

$$DDD := -(1 + G\,x^2 - 2\,G\,x\,y)\,\Omega$$
$$YgD := -(1 + G\,x^2 - 2\,G\,x\,y)\,\Omega^2\,G\,x^2$$
$$0,\ 0 \tag{76}$$

thermal Critical Point

```
> DDD:=solve(Xm-1,D);YgD:=subs(D=DDD,Yg-1);solve(YgD,Omega);
```

$$DDD := -\Omega + 2\,\Omega\,G\,x\,y - G\,x^2\,\Omega - 1$$
$$YgD := (-\Omega + 2\,\Omega\,G\,x\,y - G\,x^2\,\Omega - 1)\,G\,x^2\,\Omega - 1$$
$$-\frac{1}{2}\,\frac{G\,x - \sqrt{-3\,G^2\,x^2 + 8\,G^2\,y\,x - 4\,G}}{G\,(1 + G\,x^2 - 2\,G\,x\,y)\,x},\ -\frac{1}{2}\,\frac{G\,x + \sqrt{-3\,G^2\,x^2 + 8\,G^2\,y\,x - 4\,G}}{G\,(1 + G\,x^2 - 2\,G\,x\,y)\,x} \tag{77}$$

Hopf conditions

```
> DDD:=solve(Xm,D);YYg:=subs(D=DDD,Yg);EV:=eigenvalues(JAC):EV1:=factor(subs(D=DDD,EV[3]));EV2:=factor(subs(D=DDD,EV[4]));EEVV:=(YYg)^(1/2);
```

$$DDD := -(1 + G\,x^2 - 2\,G\,x\,y)\,\Omega$$
$$YYg := -(1 + G\,x^2 - 2\,G\,x\,y)\,\Omega^2\,G\,x^2$$
$$EV1 := \sqrt{(1 + G\,x^2 - 2\,G\,x\,y)\,\Omega^2\,G\,x^2}$$
$$EV2 := -\sqrt{(1 + G\,x^2 - 2\,G\,x\,y)\,\Omega^2\,G\,x^2}$$
$$EEVV := \sqrt{-(1 + G\,x^2 - 2\,G\,x\,y)\,\Omega^2\,G\,x^2} \tag{78}$$

When YYg>0 get oscillations.

This implies that $(1-2\,G\,x\,y+G\,x^2)\,\Omega^2\,G$ **< 0**

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z]));CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$VORTICITY := \left[\begin{array}{ccc} 0 & 0 & \Omega\,(1-2\,G\,x\,y)-G\,x^2\,\Omega \end{array}\right]$$

$$CURLx := 0$$

$$CURLy := 0$$

$$CURLz := \Omega-2\,\Omega\,G\,x\,y-G\,x^2\,\Omega \tag{79}$$

```
> HELICITY:=(factor(innerprod(AA,VORTICITY)));
```

$$HELICITY := 0 \tag{80}$$

```
> enstrophy:=(factor(innerprod(VORTICITY,VORTICITY)));
```

$$enstrophy := \Omega^2\,\left(-1+2\,G\,x\,y+G\,x^2\right)^2 \tag{81}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right] \tag{82}$$

```
> stretch:=factor(innerprod(BRAND,VORTICITY));
```

$$stretch := 0 \tag{83}$$

```
> BI:=(factor(innerprod(BRAND,BRAND)));
```

$$BI := 0 \tag{84}$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta := 0 \tag{85}$$

```
> E:=grad(phi,[x,y,z]);AA;TC:=evalm((crossprod(E,AA)+evalm((phi*VORTICITY))));
```

$$E := \left[-2\,\Omega\,A+4\,A\,\Omega\,G\,x\,y-2\,A\,\mathrm{D}+4\,\Omega^2\,x-12\,\Omega^2\,x^2\,G\,y+4\,\Omega\,x\,\mathrm{D}+8\,\Omega^2\,G^2\,x^3\,y^2 -6\,\Omega\,G\,x^2\,y\,\mathrm{D}+2\,\mathrm{D}^2\,x,\; 2\,A\,x^2\,\Omega\,G-4\,\Omega^2\,x^3\,G+4\,\Omega^2\,G^2\,x^4\,y-2\,\Omega\,G\,x^3\,\mathrm{D},\; 0\right]$$

$$\left[A-\Omega\,(x-G\,x^2\,y)-\mathrm{D}\,x,\; \Omega\,(x-G\,x^2\,y),\; 0\right]$$

$$TC := \left[0, 0, \left(-2\,\Omega\,A+4\,A\,\Omega\,G\,x\,y-2\,A\,\mathrm{D}+4\,\Omega^2\,x-12\,\Omega^2\,x^2\,G\,y+4\,\Omega\,x\,\mathrm{D}+8\,\Omega^2\,G^2\,x^3\,y^2 -6\,\Omega\,G\,x^2\,y\,\mathrm{D}+2\,\mathrm{D}^2\,x\right)\,\Omega\,(x-G\,x^2\,y)-\left(2\,A\,x^2\,\Omega\,G-4\,\Omega^2\,x^3\,G+4\,\Omega^2\,G^2\,x^4\,y -2\,\Omega\,G\,x^3\,\mathrm{D}\right)\,\left(A-\Omega\,(x-G\,x^2\,y)-\mathrm{D}\,x\right)+\left(A^2-2\,A\,x\,\Omega+2\,A\,\Omega\,G\,x^2\,y-2\,A\,\mathrm{D}\,x\right. \tag{86}$$

$$+2\,\Omega^2 x^2 - 4\,\Omega^2 x^3 G y + 2\,\Omega x^2 D + 2\,\Omega^2 G^2 x^4 y^2 - 2\,\Omega G x^3 y D + D^2 x^2\big)\,\big(\Omega\,(1 - 2\,G x y) - G x^2 \Omega\big)\Big]$$

```
> TC3:=collect(factor(TC[3]),Omega);;
```

$$TC3 := \left(6\,x^2 - 24\,G x^3 y + 12\,x^5 G^2 y - 12\,G^3 x^5 y^3 + 30\,G^2 x^4 y^2 - 6\,G x^4 - 6\,G^3 x^6 y^2\right)\Omega^3 + \left(10\,G^2 x^4 y^2 D + 12\,A G x^2 y + 6\,x^2 D + 8\,A x^3 G + 8\,G^2 x^5 y D - 16\,G x^3 y D - 4\,A x - 8\,x^4 G D - 8\,A G^2 x^3 y^2 - 8\,A x^4 G^2 y\right)\Omega^2 + \left(-4\,A D x + A^2 - 3\,A^2 x^2 G - 3\,G x^4 D^2 + 6\,A x^3 G D + 3\,D^2 x^2 - 2\,A^2 G x y + 6\,A D G x^2 y - 4\,D^2 x^3 G y\right)\Omega \quad \textbf{(87)}$$

```
> a1:=factor(coeff(TC3,Omega^3));a2:=factor(coeff(TC3,Omega^2));a3:=factor(coeff(TC3,Omega));
```

$$a1 := -6\,x^2\left(-1 + 2\,G x y + G x^2\right)(-1 + G x y)^2$$

$$a2 := -2\,(-1 + G x y)\,x\left(-4\,x^3 G D - 5\,D G x^2 y + 4\,G x^2 A + 4\,A G x y + 3\,D x - 2\,A\right)$$

$$a3 := -(-D x + A)\left(-A + 3\,G x^2 A + 2\,A G x y + 3\,D x - 4\,D G x^2 y - 3\,x^3 G D\right) \quad \textbf{(88)}$$

Chiral Separation (KONDEPUDI)

```
> AA:=[(B)*x+2*C*x*y,+D+(B)*y-C*(y^2+x^2)-E*(y^2-x^2),0];
```

$$AA := \left[B x + 2\,y x C,\; D + B y - C\left(y^2 + x^2\right) - E\left(y^2 - x^2\right),\; 0\right] \quad \textbf{(89)}$$

```
> phi:=((AA[1]^2+AA[2]^2+AA[3]^2));VV:=evalm([AA[1],AA[2],AA[3],phi]);
```

$$\phi := (B x + 2\,y x C)^2 + \left(D + B y - C\left(y^2 + x^2\right) - E\left(y^2 - x^2\right)\right)^2$$

$$VV := \left[B x + 2\,y x C,\; D + B y - C\left(y^2 + x^2\right) - E\left(y^2 - x^2\right),\; 0,\; (B x + 2\,y x C)^2 + \left(D + B y - C\left(y^2 + x^2\right) - E\left(y^2 - x^2\right)\right)^2\right] \quad \textbf{(90)}$$

```
> JAC:=(jacobian(VV,[x,y,z,t]));
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace(innerprod(JAC,JAC,JAC))):
```

$$JAC := \Big[\left[B + 2\,C y,\; 2\,C x,\; 0,\; 0\right],\; \left[-2\,C x + 2\,E x,\; B - 2\,C y - 2\,E y,\; 0,\; 0\right],\; \left[0,\; 0,\; 0,\; 0\right],\; \Big[2\,(B x + 2\,y x C)\,(B + 2\,C y) + 2\left(D + B y - C\left(y^2 + x^2\right) - E\left(y^2 - x^2\right)\right)(-2\,C x + 2\,E x),\; 4\,(B x + 2\,y x C)\,C x + 2\left(D + B y - C\left(y^2 + x^2\right) - E\left(y^2 - x^2\right)\right)(B - 2\,C y - 2\,E y),\; 0,\; 0\Big]\Big] \quad \textbf{(91)}$$

```
> Xm:=collect(S1,Omega);;Yg:=factor((1/2)*(Xm*S1-S2));Za:=factor((1/3)*(Yg*S1-Xm*S2+S3));Tk:=factor(det(JAC));
```

$$Xm := 2\,B - 2\,E y$$

$$Yg := B^2 - 2\,B\,E\,y - 4\,C^2\,y^2 + 4\,C^2\,x^2 - 4\,C\,x^2\,E - 4\,C\,y^2\,E$$
$$Za := 0$$
$$Tk := 0 \qquad \textbf{(92)}$$

HOpf conditions

```
> BBB:=solve(Xm,B);
```

$$BBB := E\,y \qquad \textbf{(93)}$$

```
> EV:=eigenvalues(JAC);EV3:=subs(B=BBB,EV[3]);EV4:=subs(B=BBB,EV[4]);
```

$$EV := 0,\, 0,\, B - E\,y + \sqrt{E^2\,y^2 + 4\,C^2\,y^2 - 4\,C^2\,x^2 + 4\,C\,x^2\,E + 4\,C\,y^2\,E},\, B - E\,y - \sqrt{E^2\,y^2 + 4\,C^2\,y^2 - 4\,C^2\,x^2 + 4\,C\,x^2\,E + 4\,C\,y^2\,E}$$
$$EV3 := \sqrt{E^2\,y^2 + 4\,C^2\,y^2 - 4\,C^2\,x^2 + 4\,C\,x^2\,E + 4\,C\,y^2\,E}$$
$$EV4 := -\sqrt{E^2\,y^2 + 4\,C^2\,y^2 - 4\,C^2\,x^2 + 4\,C\,x^2\,E + 4\,C\,y^2\,E} \qquad \textbf{(94)}$$

```
> YYg:=factor(subs(B=BBB,Yg));CCC:=solve(YYg,C);CCC1:=factor(CCC[1]);CCC2:=
  factor(CCC[2]);
```

$$YYg := -E^2\,y^2 - 4\,C^2\,y^2 + 4\,C^2\,x^2 - 4\,C\,x^2\,E - 4\,C\,y^2\,E$$
$$CCC := \frac{1}{2}\,\frac{\left(y^2 + x^2 + \sqrt{3\,y^2\,x^2 + x^4}\right)E}{x^2 - y^2},\, -\frac{1}{2}\,\frac{\left(-y^2 - x^2 + \sqrt{3\,y^2\,x^2 + x^4}\right)E}{x^2 - y^2}$$
$$CCC1 := \frac{1}{2}\,\frac{\left(y^2 + x^2 + \sqrt{x^2\,(3\,y^2 + x^2)}\right)E}{(x - y)\,(y + x)}$$
$$CCC2 := -\frac{1}{2}\,\frac{\left(-y^2 - x^2 + \sqrt{x^2\,(3\,y^2 + x^2)}\right)E}{(x - y)\,(y + x)} \qquad \textbf{(95)}$$

```
> XX0:=solve(AA[1],x);YY0:=solve(subs(x=XX,AA[2]),y);YY00:=solve(subs(AA[1]),y);
  XX00:=solve(subs(y=YY00,AA[2]),x);
```

$$XX0 := 0$$
$$YY0 := \frac{1}{2}\,\frac{B + \sqrt{B^2 + 4\,C\,\mathrm{D} - 4\,C^2\,XX^2 + 4\,E\,\mathrm{D} + 4\,E^2\,XX^2}}{C + E},\, \frac{1}{2}\,\frac{B - \sqrt{B^2 + 4\,C\,\mathrm{D} - 4\,C^2\,XX^2 + 4\,E\,\mathrm{D} + 4\,E^2\,XX^2}}{C + E}$$
$$YY00 := -\frac{1}{2}\,\frac{B}{C}$$
$$XX00 := \frac{1}{2}\,\frac{\sqrt{-(C - E)\,\left(-4\,\mathrm{D}\,C^2 + 3\,B^2\,C + E\,B^2\right)}}{(C - E)\,C}, \qquad \textbf{(96)}$$

$$-\frac{1}{2}\frac{\sqrt{-(C-E)\left(-4\,\mathrm{D}\,C^2+3\,B^2\,C+E\,B^2\right)}}{(C-E)\,C}$$

Fixed Points and Hopf.

```
> BBB:=solve(Xm-1,B);YgB:=factor(subs(B=BBB,Yg));EEE:=solve(YgB-1,E);
```

$$BBB := E\,y+\frac{1}{2}$$

$$YgB := -E^2\,y^2+\frac{1}{4}-4\,C^2\,y^2+4\,C^2\,x^2-4\,C\,x^2\,E-4\,C\,y^2\,E$$

$$EEE := -\frac{1}{2}\frac{4\,C\,y^2+4\,x^2\,C-\sqrt{48\,C^2\,x^2\,y^2+16\,x^4\,C^2-3\,y^2}}{y^2},\;-\frac{1}{2}\frac{4\,C\,y^2+4\,x^2\,C+\sqrt{48\,C^2\,x^2\,y^2+16\,x^4\,C^2-3\,y^2}}{y^2} \qquad \textbf{(97)}$$

There are two thermo critcal points given by the two values of E

Thermodynamic crtical point is a surface in {x,y,z} (It is not locally stable)

```
> BBB:=solve(Xm,B);YYg:=factor(subs(B=BBB,Yg));
  Hopfcrit:=simplify(YYg);
> EEE[1];
```

$$BBB := E\,y$$

$$YYg := -E^2\,y^2-4\,C^2\,y^2+4\,C^2\,x^2-4\,C\,x^2\,E-4\,C\,y^2\,E$$

$$Hopfcrit := -E^2\,y^2-4\,C^2\,y^2+4\,C^2\,x^2-4\,C\,x^2\,E-4\,C\,y^2\,E$$

$$-\frac{1}{2}\frac{4\,C\,y^2+4\,x^2\,C-\sqrt{48\,C^2\,x^2\,y^2+16\,x^4\,C^2-3\,y^2}}{y^2} \qquad \textbf{(98)}$$

```
> INTER:=factor(subs(E=EEE[2],Hopfcrit));
```

$$INTER := \frac{3}{4} \qquad \textbf{(99)}$$

```
> YYYG1:=factor(subs(x=XX0,y=YY0[1],YYg));YYYG2:=factor(subs(x=XX0,y=YY0[2],YYg)
  );YYYYG1:=factor(subs(x=XX00[1],y=YY00,(YYg)));YYYYG2:=simplify(subs(x=XX00
  [2],y=YY00,(YYg)));
```

$$YYYG1 := -\frac{1}{4}\frac{\left(B+\sqrt{B^2+4\,C\,\mathrm{D}-4\,C^2\,XX^2+4\,E\,\mathrm{D}+4\,E^2\,XX^2}\right)^2(2\,C+E)^2}{(C+E)^2}$$

$$YYYG2 := -\frac{1}{4}\frac{\left(B-\sqrt{B^2+4\,C\,\mathrm{D}-4\,C^2\,XX^2+4\,E\,\mathrm{D}+4\,E^2\,XX^2}\right)^2(2\,C+E)^2}{(C+E)^2}$$

$$YYYYG1 := -\frac{1}{4}\,\frac{-16\,D\,C^3+16\,B^2\,C^2+8\,C\,E\,B^2+E^2\,B^2}{C^2}$$

$$YYYYG2 := -\frac{1}{4}\,\frac{-16\,D\,C^3+16\,B^2\,C^2+8\,C\,E\,B^2+E^2\,B^2}{C^2} \qquad \textbf{(100)}$$

If B goes to zero, then Yg is always negative implying that it is a minimal surface of the negative gauss curvature type (soap film), not a Hopf bifurcation for the Racemic alpha = 0 fixed point. BUT if B goes to zero then it is true that Yg is positive depending on CD....Yg=4CD

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z]));CURLx:=(VORTICITY[1]);CURLy:=(VORTICITY[2])
;CURLz:=(VORTICITY[3]);
```

$$VORTICITY := \begin{bmatrix} 0 & 0 & -4\,C\,x+2\,E\,x \end{bmatrix}$$

$$CURLx := 0$$

$$CURLy := 0$$

$$CURLz := -4\,C\,x+2\,E\,x \qquad \textbf{(101)}$$

```
> HELICITY:=(factor(innerprod(AA,VORTICITY)));
```

$$HELICITY := 0 \qquad \textbf{(102)}$$

```
> enstrophy:=(factor(innerprod(VORTICITY,VORTICITY)));
```

$$enstrophy := 4\,x^2\,(2\,C-E)^2 \qquad \textbf{(103)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \textbf{(104)}$$

```
> stretch:=factor(innerprod(BRAND,VORTICITY));
```

$$stretch := 0 \qquad \textbf{(105)}$$

```
> BI:=(factor(innerprod(BRAND,BRAND)));
```

$$BI := 0 \qquad \textbf{(106)}$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta := 0 \qquad \textbf{(107)}$$

```
> EE:=grad(phi,[x,y,z]);AA:=AA;Bphi:=[phi*VORTICITY[1],Phi*VORTICITY[2],phi*
VORTICITY[3]];TC:=evalm((crossprod(EE,AA)+Bphi));CP:=crossprod(EE,AA);EA3:=
factor(CP[3]);
```

$$EE := \big[2\,(B\,x+2\,y\,x\,C)\,(B+2\,C\,y)+2\,\big(D+B\,y-C\,(y^2+x^2)-E\,(y^2-x^2)\big)\,(-2\,C\,x+2\,E\,x),\;4\,(B\,x+2\,y\,x\,C)\,C\,x+2\,\big(D+B\,y-C\,(y^2+x^2)-E\,(y^2-x^2)\big)\,(B-2\,C\,y-2\,E\,y),\;0\big]$$

$$AA := \big[B\,x+2\,y\,x\,C,\;D+B\,y-C\,(y^2+x^2)-E\,(y^2-x^2),\;0\big]$$

$$Bphi := \left[0, 0, \left((B x + 2 y x C)^2 + \left(\mathrm{D} + B y - C \left(y^2 + x^2\right) - E \left(y^2 - x^2\right)\right)^2\right) (-4 C x + 2 E x)\right]$$

$$TC := \left[0, 0, \left(2 (B x + 2 y x C) (B + 2 C y) + 2 \left(\mathrm{D} + B y - C \left(y^2 + x^2\right) - E \left(y^2 - x^2\right)\right) (-2 C x + 2 E x)\right) \left(\mathrm{D} + B y - C \left(y^2 + x^2\right) - E \left(y^2 - x^2\right)\right) - \left(4 (B x + 2 y x C) C x + 2 \left(\mathrm{D} + B y - C \left(y^2 + x^2\right) - E \left(y^2 - x^2\right)\right) (B - 2 C y - 2 E y)\right) (B x + 2 y x C) + \left((B x + 2 y x C)^2 + \left(\mathrm{D} + B y - C \left(y^2 + x^2\right) - E \left(y^2 - x^2\right)\right)^2\right) (-4 C x + 2 E x)\right]$$

$$CP := \left[0, 0, \left(2 (B x + 2 y x C) (B + 2 C y) + 2 \left(\mathrm{D} + B y - C \left(y^2 + x^2\right) - E \left(y^2 - x^2\right)\right) (-2 C x + 2 E x)\right) \left(\mathrm{D} + B y - C \left(y^2 + x^2\right) - E \left(y^2 - x^2\right)\right) - \left(4 (B x + 2 y x C) C x + 2 \left(\mathrm{D} + B y - C \left(y^2 + x^2\right) - E \left(y^2 - x^2\right)\right) (B - 2 C y - 2 E y)\right) (B x + 2 y x C)\right]$$

$$EA3 := -4 x \left(3 C x^2 B y E + 4 C^2 x^2 B y + 4 C x^2 \mathrm{D} E - 4 C x^2 E^2 y^2 - 4 C^2 x^2 y^2 E - 3 \mathrm{D} B E y - 2 \mathrm{D} C y^2 E + B y^3 C E - 3 E^2 x^2 B y + C x^2 B^2 + 10 C^3 x^2 y^2 - 2 C^2 x^2 \mathrm{D} - 3 C^2 x^4 E + 3 C x^4 E^2 - 6 \mathrm{D} C^2 y^2 + 2 \mathrm{D} E^2 y^2 - y^2 B^2 C - 2 y^2 B^2 E - 4 B y^3 C^2 + 3 B y^3 E^2 + 7 C^2 y^4 E + C y^4 E^2 + 2 E^3 x^2 y^2 + C^3 x^4 + 5 C^3 y^4 - E^3 y^4 - 2 x^2 \mathrm{D} E^2 + \mathrm{D}^2 C - \mathrm{D}^2 E - x^4 E^3\right) \quad \textbf{(108)}$$

```
> TC3:=collect(factor(TC[3]),B);;
```

$$TC3 := -2 x \left(4 x^2 C - 5 E y^2 - E x^2\right) B^2 - 2 x \left(4 \mathrm{D} C y + 8 E^2 y^3 - 12 C^2 y^3 + 8 x^2 C E y - 8 \mathrm{D} E y + 12 C^2 x^2 y - 8 E^2 x^2 y\right) B - 2 x \left(10 C x^4 E^2 - 16 \mathrm{D} C^2 y^2 + 14 C x^2 \mathrm{D} E - 12 C x^2 E^2 y^2 - 14 C^2 x^2 y^2 E + 6 E^3 x^2 y^2 + 17 C^2 y^4 E - 6 \mathrm{D} C y^2 E - 11 C^2 x^4 E - 6 x^2 \mathrm{D} E^2 + 32 C^3 x^2 y^2 - 8 C^2 x^2 \mathrm{D} - 3 x^4 E^3 - 3 E^3 y^4 + 4 \mathrm{D}^2 C + 6 \mathrm{D} E^2 y^2 + 2 C y^4 E^2 + 4 C^3 x^4 + 12 C^3 y^4 - 3 \mathrm{D}^2 E\right) \quad \textbf{(109)}$$

```
> a2:=factor(coeff(TC3,B^2));a1:=factor(coeff(TC3,B));a0:=collect(factor(subs(B=
  0,TC3)),C);;
```

$$a2 := -2 x \left(4 x^2 C - 5 E y^2 - E x^2\right)$$

$$a1 := -8 x y \left(2 E^2 y^2 - 3 C^2 y^2 + C \mathrm{D} + 3 C^2 x^2 - 2 x^2 E^2 + 2 C x^2 E - 2 E \mathrm{D}\right)$$

$$a0 := -2 x \left(32 y^2 x^2 + 4 x^4 + 12 y^4\right) C^3 - 2 x \left(-16 y^2 \mathrm{D} - 14 x^2 y^2 E - 8 x^2 \mathrm{D} - 11 x^4 E + 17 y^4 E\right) C^2 - 2 x \left(-12 E^2 y^2 x^2 - 6 \mathrm{D} E y^2 + 2 E^2 y^4 + 4 \mathrm{D}^2 + 14 \mathrm{D} E x^2 + 10 E^2 x^4\right) C - 2 x \left(-3 E^3 y^4 + 6 E^3 x^2 y^2 + 6 \mathrm{D} E^2 y^2 - 3 x^4 E^3 - 6 x^2 \mathrm{D} E^2 - 3 \mathrm{D}^2 E\right) \quad \textbf{(110)}$$

```
> factor(subs(B=BBB,C=CCC1,TC3)):
```

Dynamo dynamical system

```
> AA:=[-B*x+C*z*y+Omega*y,-B*y+C*z*x-Omega*x,1-x*y+D*z^2];
```

$$AA := \left[-B x + C z y + \Omega y, -B y + C z x - \Omega x, 1 - y x + \mathrm{D} z^2\right] \quad \textbf{(111)}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=evalm([AA[1],AA[2],AA[3],phi]);
```

$$\phi := B^2 x^2 - 4\,B\,x\,C\,z\,y + C^2 z^2 y^2 + 2\,C\,z\,y^2\,\Omega + \Omega^2 y^2 + B^2 y^2 + C^2 z^2 x^2 - 2\,C\,z\,x^2\,\Omega + \Omega^2 x^2 + 1 - 2\,y\,x + 2\,\mathrm{D}\,z^2 + y^2 x^2 - 2\,y\,x\,\mathrm{D}\,z^2 + \mathrm{D}^2 z^4$$

$$VV := \left[-B\,x + C\,z\,y + \Omega\,y,\ -B\,y + C\,z\,x - \Omega\,x,\ 1 - y\,x + \mathrm{D}\,z^2,\ B^2 x^2 - 4\,B\,x\,C\,z\,y + C^2 z^2 y^2 + 2\,C\,z\,y^2\,\Omega + \Omega^2 y^2 + B^2 y^2 + C^2 z^2 x^2 - 2\,C\,z\,x^2\,\Omega + \Omega^2 x^2 + 1 - 2\,y\,x + 2\,\mathrm{D}\,z^2 + y^2 x^2 - 2\,y\,x\,\mathrm{D}\,z^2 + \mathrm{D}^2 z^4 \right] \quad (112)$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t]));collect(simplify(expand(charpoly(JAC,q))
  ),q);
```

$$JAC := \left[\left[-B,\ C\,z + \Omega,\ C\,y,\ 0 \right], \left[C\,z - \Omega,\ -B,\ C\,x,\ 0 \right], \left[-y,\ -x,\ 2\,\mathrm{D}\,z,\ 0 \right], \left[2\,B^2 x - 4\,B\,C\,z\,y + 2\,C^2 z^2 x - 4\,C\,z\,x\,\Omega + 2\,\Omega^2 x - 2\,y + 2\,y^2 x - 2\,y\,\mathrm{D}\,z^2,\ -4\,B\,x\,C\,z + 2\,C^2 z^2 y + 4\,C\,z\,y\,\Omega + 2\,\Omega^2 y + 2\,y\,B^2 - 2\,x + 2\,x^2 y - 2\,x\,\mathrm{D}\,z^2,\ -4\,B\,x\,C\,y + 2\,C^2 z\,y^2 + 2\,C\,y^2\,\Omega + 2\,C^2 z\,x^2 - 2\,C\,x^2\,\Omega + 4\,\mathrm{D}\,z - 4\,y\,x\,\mathrm{D}\,z + 4\,\mathrm{D}^2 z^3,\ 0 \right] \right]$$

$$q^4 + (2\,B - 2\,\mathrm{D}\,z)\,q^3 + \left(C\,y^2 - 4\,B\,\mathrm{D}\,z + x^2 C + B^2 + \Omega^2 - C^2 z^2 \right) q^2 + \left(-2\,\Omega^2\,\mathrm{D}\,z + C\,y^2 B - 2\,B^2\,\mathrm{D}\,z + B\,x^2 C + 2\,C^2 z^3\,\mathrm{D} + 2\,C^2 z\,x\,y \right) q \quad (113)$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=factor((1/2)*(Xm*S1-S2));Za:=factor((1/3)*(Yg*S1-Xm*S2+S3));Tk:=
  factor(det(JAC));
```

$$Xm := -2\,B + 2\,\mathrm{D}\,z$$

$$Yg := C\,y^2 - 4\,B\,\mathrm{D}\,z + x^2 C + B^2 + \Omega^2 - C^2 z^2$$

$$Za := -2\,C^2 z\,x\,y - C\,y^2 B + 2\,B^2\,\mathrm{D}\,z - B\,x^2 C - 2\,C^2 z^3\,\mathrm{D} + 2\,\Omega^2\,\mathrm{D}\,z$$

$$Tk := 0 \quad (114)$$

```
> BBB:=solve(Xm-3,B);YgB:=factor(subs(B=BBB,Yg));RRR:=solve(YgB-3,Omega^2);
  ZaBR:=factor(subs(B=BBB,Omega^2=RRR,Za));CriticalPoint:=collect(ZaBR-1,C);
```

$$BBB := \mathrm{D}\,z - \frac{3}{2}$$

$$YgB := C\,y^2 - 3\,\mathrm{D}^2 z^2 + 3\,\mathrm{D}\,z + x^2 C + \frac{9}{4} + \Omega^2 - C^2 z^2$$

Warning, solving for expressions other than names or functions is not recommended.

$$RRR := -C\,y^2 + 3\,\mathrm{D}^2 z^2 - 3\,\mathrm{D}\,z - x^2 C + \frac{3}{4} + C^2 z^2$$

$$ZaBR := -2\,C^2\,z\,x\,y - 3\,C\,y^2\,\mathrm{D}\,z + \frac{3}{2}\,C\,y^2 + 8\,\mathrm{D}^3\,z^3 - 12\,\mathrm{D}^2\,z^2 + 6\,\mathrm{D}\,z - 3\,C\,x^2\,\mathrm{D}\,z + \frac{3}{2}\,x^2\,C$$

$$CriticalPoint := -2\,C^2\,z\,x\,y + \left(\frac{3}{2}\,y^2 - 3\,y^2\,\mathrm{D}\,z - 3\,x^2\,\mathrm{D}\,z + \frac{3}{2}\,x^2\right)C - 12\,\mathrm{D}^2\,z^2 + 6\,\mathrm{D}\,z + 8\,\mathrm{D}^3\,z^3 - 1 \quad (115)$$

Thermodynamic crtical point is a surface in {x,y,z} (It is not locally stable)

```
> BBB:=solve(Xm,B);ZaB:=factor(subs(B=BBB,Za));RRR:=solve(ZaB,Omega^2);YYg:=
  factor(subs(B=BBB,Omega^2=RRR,Yg));
  Hopfcrit:=collect(YYg,A);
```

$$BBB := \mathrm{D}\,z$$

$$ZaB := -z\left(2\,C^2\,x\,y + C\,y^2\,\mathrm{D} - 2\,z^2\,\mathrm{D}^3 + C\,x^2\,\mathrm{D} + 2\,z^2\,C^2\,\mathrm{D} - 2\,\Omega^2\,\mathrm{D}\right)$$

Warning, solving for expressions other than names or functions is not recommended.

$$RRR := \frac{1}{2}\,\frac{2\,C^2\,x\,y + C\,y^2\,\mathrm{D} - 2\,z^2\,\mathrm{D}^3 + C\,x^2\,\mathrm{D} + 2\,z^2\,C^2\,\mathrm{D}}{\mathrm{D}}$$

$$YYg := \frac{1}{2}\,\frac{3\,C\,y^2\,\mathrm{D} - 8\,z^2\,\mathrm{D}^3 + 3\,C\,x^2\,\mathrm{D} + 2\,C^2\,x\,y}{\mathrm{D}}$$

$$Hopfcrit := \frac{1}{2}\,\frac{3\,C\,y^2\,\mathrm{D} - 8\,z^2\,\mathrm{D}^3 + 3\,C\,x^2\,\mathrm{D} + 2\,C^2\,x\,y}{\mathrm{D}} \quad (116)$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := -x - C\,x$$

$$CURLy := C\,y + y$$

$$CURLz := -2\,\Omega \quad (117)$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),Omega);
```

$$HELICITY := \left(-2\,y\,x\,C - 2 - 2\,\mathrm{D}\,z^2\right)\Omega + B\,x^2 + B\,x^2\,C - C\,y^2\,B - y^2\,B \quad (118)$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),Omega);
```

$$enstrophy := x^2 + 2\,x^2\,C + C^2\,x^2 + C^2\,y^2 + 2\,C\,y^2 + y^2 + 4\,\Omega^2 \quad (119)$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[B\,x + B\,C\,x + C^2\,y\,z + C\,z\,y - C\,y\,\Omega + \Omega\,y,\ -C\,z\,x - C^2\,x\,z + \Omega\,x - C\,x\,\Omega - B\,C\,y - B\,y,\ -4\,\Omega\,\mathrm{D}\,z\right] \quad (120)$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),Omega);ST1:=factor(coeff
  (stretch,Omega^2));st0:=factor(subs(Omega=0,stretch));
```

$$stretch := -B\,x^2 - 2\,B\,x^2\,C - B\,C^2\,x^2 - 4\,C^2\,z\,x\,y - 2\,C^3\,y\,z\,x - 2\,x\,y\,C\,z - B\,C^2\,y^2 - 2\,C\,y^2\,B - y^2\,B + 8\,\Omega^2\,\mathrm{D}\,z$$

$$ST1 := 8\,\mathrm{D}\,z$$

$$st0 := -(1+C)^2\,(2\,x\,y\,C\,z + B\,x^2 + y^2\,B) \quad \textbf{(121)}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),Omega);BI2:=factor(coeff(BI,Omega^2));BI1:=factor(coeff(BI,Omega));BI0:=collect(factor(subs(Omega=0,BI)),z);
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$BI := (y^2 + x^2 + C^2\,y^2 + 16\,\mathrm{D}^2\,z^2 + C^2\,x^2 - 2\,x^2\,C - 2\,C\,y^2)\,\Omega^2 + (2\,C^3\,x^2\,z - 2\,C^3\,y^2\,z - 2\,C\,z\,x^2 + 2\,C\,z\,y^2)\,\Omega + 4\,B\,x\,C\,z\,y + B^2\,y^2 + 2\,C^3\,y^2\,z^2 + C^2\,z^2\,x^2 + C^4\,y^2\,z^2 + C^2\,z^2\,y^2 + 2\,C\,x^2\,B^2 + C^4\,x^2\,z^2 + B^2\,C^2\,y^2 + B^2\,x^2 + B^2\,C^2\,x^2 + 2\,C^3\,z^2\,x^2 + 2\,y^2\,B^2\,C + 8\,B\,x\,C^2\,y\,z + 4\,B\,C^3\,x\,y\,z$$

$$BI2 := y^2 + x^2 + C^2\,y^2 + 16\,\mathrm{D}^2\,z^2 + C^2\,x^2 - 2\,x^2\,C - 2\,C\,y^2$$

$$BI1 := 2\,C\,z\,(x-y)\,(y+x)\,(-1+C)\,(1+C)$$

$$BI0 := (1+C)^2\,(C^2\,y^2 + C^2\,x^2)\,z^2 + 4\,(1+C)^2\,B\,x\,C\,y\,z + (1+C)^2\,(B^2\,y^2 + B^2\,x^2)$$

$$\Theta := -4\,\Omega^2\,y^2 + 4\,\Omega^2\,x^2 - 8\,C\,z\,x^2\,\Omega - 8\,C\,z\,y^2\,\Omega + 16\,\Omega\,\mathrm{D}\,z + 16\,\Omega\,\mathrm{D}^2\,z^3 + 4\,C^3\,z^2\,x^2 - 4\,\Omega^2\,x^2\,C + 4\,C\,y^2\,\Omega^2 - 4\,C^3\,y^2\,z^2 + 4\,C^2\,z^2\,x^2 - 4\,C^2\,z^2\,y^2 - 4\,B^2\,y^2 + 4\,C\,x^2\,B^2 - 4\,y^2\,B^2\,C - 16\,\Omega\,y\,x\,\mathrm{D}\,z + 4\,B^2\,x^2 - 16\,B\,x\,C\,y\,\Omega \quad \textbf{(122)}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t2:=factor(coeff(Theta,Omega^2));t1:=factor(coeff(Theta,Omega));
> t0:=factor(subs(Omega=0,Theta));
```

$$\Theta := (4\,x^2 - 4\,x^2\,C + 4\,C\,y^2 - 4\,y^2)\,\Omega^2 + (16\,\mathrm{D}\,z + 16\,\mathrm{D}^2\,z^3 - 8\,C\,z\,x^2 - 16\,y\,x\,\mathrm{D}\,z - 8\,C\,z\,y^2 - 16\,B\,x\,C\,y)\,\Omega + 4\,C\,x^2\,B^2 - 4\,y^2\,B^2\,C + 4\,C^3\,z^2\,x^2 + 4\,B^2\,x^2 - 4\,B^2\,y^2 - 4\,C^3\,y^2\,z^2 + 4\,C^2\,z^2\,x^2 - 4\,C^2\,z^2\,y^2$$

$$t2 := -4\,(x-y)\,(y+x)\,(-1+C)$$

$$t1 := 16\,\mathrm{D}\,z + 16\,\mathrm{D}^2\,z^3 - 8\,C\,z\,x^2 - 16\,y\,x\,\mathrm{D}\,z - 8\,C\,z\,y^2 - 16\,B\,x\,C\,y$$

$$t0 := 4\,(x-y)\,(y+x)\,(1+C)\,(C^2\,z^2 + B^2) \quad \textbf{(123)}$$

SPINNING TOP-Eulerian Rotator dynamical system

```
> AA:=[A*y*z,B*z*x,C*x*y];
```

$$AA := [A\,y\,z,\, x\,B\,z,\, y\,x\,C] \quad \textbf{(124)}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=evalm([AA[1],AA[2],AA[3],phi]);
```

$$\phi := A^2 y^2 z^2 + x^2 B^2 z^2 + C^2 x^2 y^2$$

$$VV := \left[\; A\,y\,z \;\; x\,B\,z \;\; y\,x\,C \;\; A^2 y^2 z^2 + x^2 B^2 z^2 + C^2 x^2 y^2 \; \right] \tag{125}$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t]));collect(simplify(expand(charpoly(JAC,q))
),q);
```

$$JAC := \begin{bmatrix} 0 & z\,A & A\,y & 0 \\ B\,z & 0 & B\,x & 0 \\ C\,y & C\,x & 0 & 0 \\ 2\,x\,B^2 z^2 + 2\,C^2 y^2 x & 2\,A^2 y\,z^2 + 2\,C^2 x^2 y & 2\,A^2 y^2 z + 2\,x^2 B^2 z & 0 \end{bmatrix}$$

$$q^4 + \left(-C\,y^2 A - B\,x^2 C - B\,z^2 A\right) q^2 - 2\,B\,z\,C\,x\,A\,y\,q \tag{126}$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
(innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=factor((1/2)*(Xm*S1-S2));Za:=factor((1/3)*(Yg*S1-Xm*S2+S3));Tk:=
factor(det(JAC));
```

$$Xm := 0$$

$$Yg := -C\,y^2 A - B\,x^2 C - B\,z^2 A$$

$$Za := 2\,B\,z\,C\,x\,A\,y$$

$$Tk := 0 \tag{127}$$

```
> Hopfcrit:=factor(subs(z=0,Yg));
```

$$Hopfcrit := -C\left(y^2 A + B\,x^2\right) \tag{128}$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := C\,x - B\,x$$

$$CURLy := A\,y - C\,y$$

$$CURLz := B\,z - z\,A \tag{129}$$

```
> HELICITY:=(factor(innerprod(AA,VORTICITY)));
```

$$HELICITY := 0 \tag{130}$$

```
> enstrophy:=factor(innerprod(VORTICITY,VORTICITY));
```

$$enstrophy := C^2 x^2 - 2\,B\,x^2 C + B^2 x^2 + A^2 y^2 - 2\,C\,y^2 A + C^2 y^2 + B^2 z^2 - 2\,B\,z^2 A + z^2 A^2 \tag{131}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[\; -C\,y\,z\,A + B\,z\,A\,y \;\; B\,x\,C\,z - z\,A\,B\,x \;\; -B\,x\,C\,y + A\,y\,C\,x \; \right] \tag{132}$$

```
> stretch:=(factor(innerprod(BRAND,VORTICITY)/(1)));factor(C^2*A+B^2*A+C^2*B+
A^2*B+B^2*C+A^2*C);
```

$$stretch := -y\,x\,z\left(C^2 A - 6\,B\,C\,A + B^2 A + C^2 B + A^2 B + B^2 C + C\,A^2\right)$$

$$C^2 A + B^2 A + C^2 B + A^2 B + B^2 C + C A^2 \quad (133)$$

```
> BI:=simplify(factor(innerprod(BRAND,BRAND)));zz:=(C^2*y^2*A^2*z^2-2*C*y^2*A^2*
  z^2*B+B^2*z^2*A^2*y^2)*Omega^2+(C^2*x^2*B^2*z^2-2*C*x^2*B^2*z^2*A+A^2*z^2*B^2*
  x^2)*Omega+B^2*x^2*C^2*y^2-2*B*x^2*C^2*y^2*A+A^2*y^2*C^2*x^2;
```

$$BI := C^2 y^2 z^2 A^2 - 2 C y^2 z^2 A^2 B + B^2 z^2 A^2 y^2 + B^2 x^2 C^2 z^2 - 2 B^2 x^2 C z^2 A + z^2 A^2 B^2 x^2 + B^2 x^2 C^2 y^2 - 2 B x^2 C^2 y^2 A + A^2 y^2 C^2 x^2$$

$$zz := \left(C^2 y^2 z^2 A^2 - 2 C y^2 z^2 A^2 B + B^2 z^2 A^2 y^2\right) \Omega^2 + \left(B^2 x^2 C^2 z^2 - 2 B^2 x^2 C z^2 A + z^2 A^2 B^2 x^2\right) \Omega + B^2 x^2 C^2 y^2 - 2 B x^2 C^2 y^2 A + A^2 y^2 C^2 x^2 \quad (134)$$

```
> factor(coeff(zz,Omega^2));factor(coeff(zz,Omega));
```

$$A^2 y^2 z^2 (B - C)^2$$

$$x^2 B^2 z^2 (A - C)^2 \quad (135)$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta := -4 B^2 x^2 C z^2 + 4 B x^2 C^2 y^2 + 4 C y^2 z^2 A^2 - 4 x^2 C^2 y^2 A - 4 y^2 z^2 A^2 B + 4 B^2 x^2 z^2 A \quad (136)$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t2:=
  factor(coeff(Theta,Omega^2));t1:=factor(coeff(Theta,Omega));
> t0:=factor(subs(Omega=0,Theta));SYMM:=factor(subs(C=0,B=-A,Theta));factor(subs
  (I1=1,I2=1,I3=1/2,SYMM));
```

$$\Theta := -4 B^2 x^2 C z^2 + 4 B x^2 C^2 y^2 + 4 C y^2 z^2 A^2 - 4 x^2 C^2 y^2 A - 4 y^2 z^2 A^2 B + 4 B^2 x^2 z^2 A$$

$$t2 := 0$$

$$t1 := 0$$

$$t0 := -4 B^2 x^2 C z^2 + 4 B x^2 C^2 y^2 + 4 C y^2 z^2 A^2 - 4 x^2 C^2 y^2 A - 4 y^2 z^2 A^2 B + 4 B^2 x^2 z^2 A$$

$$SYMM := 4 A^3 z^2 \left(y^2 + x^2\right)$$

$$4 A^3 z^2 \left(y^2 + x^2\right) \quad (137)$$

ROSSLER dynamical system

```
> AA:= [W*y-z,W*x-A*y,B+x*z-C*y+T*z];
```

$$AA := [W y - z, W x - A y, B + z x - C y + T z] \quad (138)$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=evalm([AA[1],AA[2],AA[3],phi]);
```

$$\phi := W^2 y^2 - 2 W y z + z^2 + W^2 x^2 - 2 W x A y + A^2 y^2 + B^2 + 2 x B z - 2 B C y + 2 B T z + z^2 x^2 - 2 x y C z + 2 z^2 x T + C^2 y^2 - 2 C y T z + T^2 z^2$$

$$VV := \left[W y - z, W x - A y, B + z x - C y + T z, W^2 y^2 - 2 W y z + z^2 + W^2 x^2 - 2 W x A y + A^2 y^2 + B^2 + 2 x B z - 2 B C y + 2 B T z + z^2 x^2 - 2 x y C z + 2 z^2 x T + C^2 y^2 - 2 C y T z + T^2 z^2\right] \quad (139)$$

```
> JAC:=(jacobian(VV,[x,y,z,t]));
```

$$JAC := [[0, W, -1, 0], [W, -A, 0, 0], [z, -C, x+T, 0], [2W^2x - 2WAy + 2Bz + 2z^2x - 2Czy + 2z^2T, 2W^2y - 2Wz - 2WxA + 2A^2y - 2BC - 2Czx + 2C^2y - 2CTz, -2Wy + 2z + 2Bx + 2BT + 2zx^2 - 2yxC + 4zxT - 2CyT + 2T^2z, 0]] \quad (140)$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=factor((1/2)*(Xm*S1-S2));Za:=factor((1/3)*(Yg*S1-Xm*S2+S3));Tk:=
  factor(det(JAC));
```

$$Xm := -A + x + T$$

$$Yg := -Ax - AT - W^2 + z$$

$$Za := -zA - W^2T - W^2x + CW$$

$$Tk := 0 \quad (141)$$

```
> TTT:=solve(Xm-3,T);ZaB:=factor(subs(T=TTT,Za));AAA:=solve(ZaB-3,A);YgBR:=
  factor(subs(T=TTT,A=AAA,Yg));CriticalPoint:=collect(factor(expand(YgBR-1)),W);
```

$$TTT := A - x + 3$$

$$ZaB := -zA - W^2A - 3W^2 + CW$$

$$AAA := \frac{-3 - 3W^2 + CW}{z + W^2}$$

$$YgBR := -\frac{9 - 9z - z^3 - 3CW^3 + C^2W^2 - W^2z^2 + W^4z + W^6 + 9W^2 - 6CW - 9W^2z + 3WCz}{(z + W^2)^2}$$

$$CriticalPoint := -\frac{1}{(z + W^2)^2}\left(W^6 + (z+1)W^4 - 3CW^3 + (-7z + C^2 - z^2 + 9)W^2 + (3Cz - 6C)W + 9 - 9z - z^3 + z^2\right) \quad (142)$$

Thermodynamic crtical point is a surface in {x,y,z} (It is not locally stable)

```
> TTT:=solve(Xm,T);ZaB:=factor(subs(T=TTT,Za));CCC:=solve(ZaB,C);YYg:=factor
  (subs(T=TTT,C=CCC,Yg));
  Hopfcrit:=collect(YYg,A);
```

$$TTT := A - x$$

$$ZaB := -zA - W^2A + CW$$

$$CCC := \frac{(z + W^2)\, A}{W}$$

$$YYg := -A^2 - W^2 + z$$

$$Hopfcrit := -A^2 - W^2 + z \quad \textbf{(143)}$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := -C$$

$$CURLy := -1 - z$$

$$CURLz := 0 \quad \textbf{(144)}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),W);
```

$$HELICITY := (-x - z\,x - C\,y)\, W + A\,y + C\,z + A\,y\,z \quad \textbf{(145)}$$

```
> enstrophy:=(factor(innerprod(VORTICITY,VORTICITY)));
```

$$enstrophy := C^2 + 1 + 2\,z + z^2 \quad \textbf{(146)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[\; -W\,(z+1) \quad -C\,W + A + z\,A \quad C \;\right] \quad \textbf{(147)}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),W);
```

$$stretch := 2\,W\,(z+1)\,C - (z+1)\,(z\,A + A) \quad \textbf{(148)}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),W);BI2:=factor(coeff(BI,W^2));BI1:=
  factor(coeff(BI,W));BI0:=collect(factor(subs(W=0,BI)),z);
```

$$BI := (C^2 + 1 + 2\,z + z^2)\, W^2 + (-2\,C\,A - 2\,A\,C\,z)\, W + z^2 A^2 + C^2 + A^2 + 2\,z\,A^2$$

$$BI2 := C^2 + 1 + 2\,z + z^2$$

$$BI1 := -2\,C\,A\,(z+1)$$

$$BI0 := z^2 A^2 + C^2 + A^2 + 2\,z\,A^2 \quad \textbf{(149)}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),W);Theta:=
  collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),A);
```

$$\Theta := (4\,y + 4\,y\,z + 4\,C\,x)\, W^2 + (-4\,C\,y\,A - 4\,A\,x - 4\,A\,z\,x - 4\,z^2 - 4\,z)\, W - 4\,C\,T\,z + 4\,z\,A^2\,y - 4\,B\,C - 4\,C\,z\,x + 4\,A^2\,y + 4\,C^2\,y$$

$$\Theta := (4\,y + 4\,y\,z)\, A^2 + (-4\,C\,W\,y - 4\,W\,x - 4\,W\,z\,x)\, A + 4\,C\,W^2\,x - 4\,W\,z^2 + 4\,W^2\,y + 4\,W^2\,y\,z - 4\,W\,z - 4\,C\,T\,z - 4\,B\,C - 4\,C\,z\,x + 4\,C^2\,y \quad \textbf{(150)}$$

Zhabotinsky dynamical system

```
> AA:=[0,0,0];VV:=[0,0,0,0];phi:=0;
```

$$AA := [0, 0, 0]$$
$$VV := [0, 0, 0, 0]$$
$$\phi := 0 \tag{151}$$

```
> AA:=evalm([A*y+B*x-C*x*y-2*D*x*x,-A*y-C*x*y+F*E*z/2,2*B*x-E*z]);
```

$$AA := \left[\, A\,y + B\,x - y\,x\,C - 2\,x^2\,\mathrm{D} \quad -A\,y - y\,x\,C + \frac{1}{2}\,F\,E\,z \quad 2\,B\,x - E\,z \,\right] \tag{152}$$

```
> phi:=factor((AA[1]^2+AA[2]^2+AA[3]^2));VV:=evalm([AA[1],AA[2],AA[3],phi]);
```

$$\phi := 2\,A^2\,y^2 + 2\,A\,y\,B\,x - 4\,A\,y\,x^2\,\mathrm{D} + 5\,B^2\,x^2 - 2\,B\,x^2\,C\,y - 4\,B\,x^3\,\mathrm{D} + 2\,C^2\,x^2\,y^2 + 4\,y\,x^3\,C\,\mathrm{D} + 4\,x^4\,\mathrm{D}^2 - A\,y\,F\,E\,z - y\,x\,C\,F\,E\,z + \frac{1}{4}\,F^2\,E^2\,z^2 - 4\,B\,x\,E\,z + E^2\,z^2$$

$$VV := \Big[A\,y + B\,x - y\,x\,C - 2\,x^2\,\mathrm{D},\ -A\,y - y\,x\,C + \frac{1}{2}\,F\,E\,z,\ 2\,B\,x - E\,z,\ 2\,A^2\,y^2 + 2\,A\,y\,B\,x - 4\,A\,y\,x^2\,\mathrm{D} + 5\,B^2\,x^2 - 2\,B\,x^2\,C\,y - 4\,B\,x^3\,\mathrm{D} + 2\,C^2\,x^2\,y^2 + 4\,y\,x^3\,C\,\mathrm{D} + 4\,x^4\,\mathrm{D}^2 - A\,y\,F\,E\,z - y\,x\,C\,F\,E\,z + \frac{1}{4}\,F^2\,E^2\,z^2 - 4\,B\,x\,E\,z + E^2\,z^2\Big] \tag{153}$$

```
> JAC:=(jacobian(VV,[x,y,z,t]));
```

$$JAC := \Big[\big[B - C\,y - 4\,\mathrm{D}\,x,\ A - C\,x,\ 0,\ 0\big],\ \big[-C\,y,\ -A - C\,x,\ \frac{1}{2}\,F\,E,\ 0\big],\ \big[2\,B,\ 0,\ -E,\ 0\big],\ \big[2\,A\,y\,B - 8\,A\,y\,x\,\mathrm{D} + 10\,B^2\,x - 4\,B\,x\,C\,y - 12\,B\,x^2\,\mathrm{D} + 4\,C^2\,y^2\,x + 12\,y\,x^2\,C\,\mathrm{D} + 16\,x^3\,\mathrm{D}^2 - y\,C\,F\,E\,z - 4\,z\,E\,B,\ 4\,A^2\,y + 2\,A\,B\,x - 4\,A\,x^2\,\mathrm{D} - 2\,B\,x^2\,C + 4\,C^2\,x^2\,y + 4\,x^3\,C\,\mathrm{D} - A\,F\,E\,z - x\,C\,F\,E\,z,\ -A\,y\,F\,E - y\,x\,C\,F\,E + \frac{1}{2}\,F^2\,E^2\,z - 4\,E\,x\,B + 2\,E^2\,z,\ 0\big]\Big] \tag{154}$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=collect(factor((1/2)*(Xm*S1-S2)),E);;Za:=factor((1/3)*(Yg*S1-Xm*S2+
  S3));Tk:=factor(det(JAC));
```

$$Xm := B - C\,y - 4\,\mathrm{D}\,x - A - C\,x - E$$
$$Yg := (A - B + 4\,\mathrm{D}\,x + C\,y + C\,x)\,E - B\,C\,x + 4\,\mathrm{D}\,x\,A + 2\,C\,y\,A - A\,B + 4\,C\,x^2\,\mathrm{D}$$
$$Za := E\,\left(F\,B\,A - 2\,C\,y\,A - 4\,\mathrm{D}\,x\,A + A\,B + B\,C\,x - 4\,C\,x^2\,\mathrm{D} - F\,B\,C\,x\right)$$
$$Tk := 0 \tag{155}$$

Hopf conditions

```
> AAA:=factor(solve(Xm,A));ZaEEE:=factor(subs(A=AAA,Za));FFF:=solve(ZaEEE,F);
  YgAF:=factor(subs(A=AAA,F=FFF,Yg));
```

$$AAA := B - C\,y - 4\,\mathrm{D}\,x - C\,x - E$$

$$ZaEEE := E\,\big(F\,B^2 - F\,B\,C\,y - 4\,F\,B\,\mathrm{D}\,x - 2\,F\,B\,C\,x - F\,E\,B - 3\,B\,C\,y + 2\,C^2\,y^2 + 12\,C\,y\,\mathrm{D}\,x + 2\,C^2\,x\,y + 2\,C\,y\,E - 8\,B\,\mathrm{D}\,x + 16\,\mathrm{D}^2\,x^2 + 4\,\mathrm{D}\,x\,E + B^2 - E\,B\big)$$

$$FFF := -\frac{1}{B\,(B - C\,y - 4\,\mathrm{D}\,x - 2\,C\,x - E)}\,\big(B^2 - E\,B + 2\,C^2\,y^2 + 12\,C\,y\,\mathrm{D}\,x + 2\,C^2\,x\,y - 3\,B\,C\,y - 8\,B\,\mathrm{D}\,x + 16\,\mathrm{D}^2\,x^2 + 4\,\mathrm{D}\,x\,E + 2\,C\,y\,E\big)$$

$$YgAF := -E^2 + 8\,B\,\mathrm{D}\,x - 12\,C\,y\,\mathrm{D}\,x - 16\,\mathrm{D}^2\,x^2 - 4\,\mathrm{D}\,x\,E + 3\,B\,C\,y - 2\,C^2\,y^2 - 2\,C^2\,x\,y - 2\,C\,y\,E - B^2 + E\,B \qquad \textbf{(156)}$$

Chiral - Brand invariants

```
> VORTICITY:=(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=simplify
  (VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := -\frac{1}{2}\,F\,E$$

$$CURLy := -2\,B$$

$$CURLz := -C\,y - A + C\,x \qquad \textbf{(157)}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),F);
```

$$HELICITY := \left(-\frac{1}{2}\,E\,x\,B + \mathrm{D}\,E\,x^2 - \frac{1}{2}\,E\,y\,A - z\,E\,B + \frac{1}{2}\,E\,C\,x\,y\right) F + 2\,A\,y\,B + E\,z\,A - 2\,A\,B\,x + 2\,B\,x^2\,C + E\,z\,C\,y - E\,z\,C\,x \qquad \textbf{(158)}$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),F);
```

$$enstrophy := \frac{1}{4}\,F^2\,E^2 + 4\,B^2 + C^2\,y^2 + 2\,C\,y\,A - 2\,C^2\,x\,y + A^2 - 2\,A\,C\,x + C^2\,x^2 \qquad \textbf{(159)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[-\frac{1}{2}\,F\,E\,B + \frac{1}{2}\,F\,E\,C\,y + 2\,F\,E\,\mathrm{D}\,x - 2\,A\,B + 2\,B\,C\,x,\ 2\,A\,B + 2\,B\,C\,x - \frac{1}{2}\,F\,E\,A + \frac{1}{2}\,F\,E\,C\,x,\ -F\,E\,B + C\,y\,E + A\,E - C\,x\,E\right] \qquad \textbf{(160)}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),F);ST2:=factor(coeff
  (stretch,F^2));ST1:=factor(coeff(stretch,F));st0:=(factor(subs(F=0,stretch)));
  st0e:=collect(st0,E);E0:=factor(coeff(st0e,E));
```

$$stretch := \left(-\frac{1}{4}\,C\,y\,E^2 - \mathrm{D}\,x\,E^2 + \frac{1}{4}\,E^2\,B\right) F^2 + (3\,E\,B\,A + E\,B\,C\,y - 3\,E\,B\,C\,x)\,F - E\,C^2\,x^2$$

$$-4B^2A-4CxB^2-A^2E-C^2y^2E-2AECy+2C^2yEx+2AECx$$

$$ST2 := \frac{1}{4}E^2(B-Cy-4\mathrm{D}x)$$

$$ST1 := EB(3A+Cy-3Cx)$$

$$st0 := -EC^2x^2-4B^2A-4CxB^2-A^2E-C^2y^2E-2AECy+2C^2yEx+2AECx$$

$$st0e := (-C^2x^2-A^2-C^2y^2-2CyA+2C^2xy+2ACx)E-4B^2A-4CxB^2$$

$$E0 := -(Cy+A-Cx)^2 \quad \textbf{(161)}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),F);BI2:=factor(coeff(BI,F^2));BI1:=
  factor(coeff(BI,F));BI0:=collect(factor(subs(F=0,BI)),E);
```

$$BI := \left(2E^2Cy\mathrm{D}x+\frac{1}{4}C^2y^2E^2-2E^2B\mathrm{D}x+\frac{1}{4}C^2x^2E^2+\frac{1}{4}A^2E^2+4E^2\mathrm{D}^2x^2-\frac{1}{2}AE^2Cx+\frac{5}{4}E^2B^2-\frac{1}{2}E^2BCy\right)F^2+(8E\mathrm{D}x^2BC+2EB^2A-2EB^2Cx-2ECyAB-2E^2BCy-8E\mathrm{D}xAB+2E^2BCx-2E^2BA+2EC^2yBx-2A^2BE+2BC^2x^2E)F+8B^2C^2x^2+A^2E^2-2AE^2Cx+8A^2B^2+C^2y^2E^2-2C^2yE^2x+C^2x^2E^2+2CyE^2A$$

$$BI2 := \frac{1}{4}E^2(-8B\mathrm{D}x-2BCy+8Cy\mathrm{D}x+A^2+5B^2+C^2y^2+16\mathrm{D}^2x^2+C^2x^2-2ACx)$$

$$BI1 := -2BE(CyE+AE-CxE-AB+BCx+CyA-C^2xy+4\mathrm{D}xA-4Cx^2\mathrm{D}+A^2-C^2x^2)$$

$$BI0 := (C^2y^2+2CyA-2C^2xy+A^2-2ACx+C^2x^2)E^2+8B^2C^2x^2+8A^2B^2 \quad \textbf{(162)}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),F);T1:=factor
  (coeff(Theta,F));
```

$$\Theta := (-E^2zCx+E^2zA)F^2+(2EC^2x^2y+2BAyE+16Ex^3\mathrm{D}^2-4BE^2z-4zEBCx-2EA^2y-12EBx^2\mathrm{D}-8EAyx\mathrm{D}+12Eyx^2C\mathrm{D}-2Ay^2EC-4ExBCy+2EC^2y^2x-4zEBA+10ExB^2)F+4E^2zCy-4E^2zCx+16A^2yB+8B^2xA+8Ex^2BC-8ExBCy+16C^2x^2By-8Cx^2B^2+16Bx^3\mathrm{D}C+4E^2zA-8ExBA-16Bx^2\mathrm{D}A$$

$$T1 := -2E(-C^2x^2y-AyB-8x^3\mathrm{D}^2+2zEB+2BxCz+A^2y+6Bx^2\mathrm{D}+4Ayx\mathrm{D}-6yx^2C\mathrm{D}+Cy^2A+2BxCy-C^2y^2x+2BzA-5B^2x) \quad \textbf{(163)}$$

```
> factor(subs(F=0,Theta));
```

$$4E^2zCy-4E^2zCx+16A^2yB+8B^2xA+8Ex^2BC-8ExBCy+16C^2x^2By \quad \textbf{(164)}$$

$$-8\,C\,x^2\,B^2+16\,B\,x^3\,\mathrm{D}\,C+4\,E^2\,z\,A-8\,E\,x\,B\,A-16\,B\,x^2\,\mathrm{D}\,A$$

Pitchfork Hopf OK

```
> AA:= [e*A-G+C*z^2+T*(x^2+y^2)+Omega*y,e*A-G+C*z^2+T*(x^2+y^2)-Omega*x,z*(A+B*
  z^2+D*(x^2+y^2))];
```

$$AA:=\left[e\,A-G+C\,z^2+T\left(y^2+x^2\right)+\Omega\,y,\,e\,A-G+C\,z^2+T\left(y^2+x^2\right)-\Omega\,x,\,z\left(A+B\,z^2+\mathrm{D}\left(y^2+x^2\right)\right)\right] \tag{165}$$

```
> phi:=factor((AA[1]^2+AA[2]^2+AA[3]^2));VV:=[AA[1],AA[2],AA[3],phi];
```

$$\phi:=\Omega^2\,y^2+\Omega^2\,x^2-4\,G\,C\,z^2+2\,T\,y^3\,\Omega-4\,G\,T\,y^2+2\,G^2+z^2\,A^2+2\,e^2\,A^2+2\,C^2\,z^4+2\,T^2\,y^4+2\,x^4\,T^2+z^6\,B^2-4\,e\,A\,G+4\,T^2\,y^2\,x^2-4\,G\,x^2\,T-2\,G\,\Omega\,y-2\,x^3\,T\,\Omega+2\,G\,\Omega\,x+2\,z^4\,B\,A+z^2\,\mathrm{D}^2\,y^4+z^2\,x^4\,\mathrm{D}^2+4\,e\,A\,C\,z^2+4\,e\,A\,T\,y^2+4\,e\,A\,x^2\,T+2\,e\,A\,\Omega\,y+4\,C\,z^2\,T\,y^2+4\,C\,z^2\,x^2\,T+2\,C\,z^2\,\Omega\,y+2\,x^2\,T\,\Omega\,y-2\,e\,A\,\Omega\,x-2\,C\,z^2\,\Omega\,x-2\,T\,y^2\,\Omega\,x+2\,z^2\,A\,y^2\,\mathrm{D}+2\,z^2\,A\,x^2\,\mathrm{D}+2\,z^4\,B\,y^2\,\mathrm{D}+2\,z^4\,B\,x^2\,\mathrm{D}+2\,z^2\,y^2\,\mathrm{D}^2\,x^2$$

$$VV:=\left[e\,A-G+C\,z^2+T\left(y^2+x^2\right)+\Omega\,y,\,e\,A-G+C\,z^2+T\left(y^2+x^2\right)-\Omega\,x,\,z\left(A+B\,z^2+\mathrm{D}\left(y^2+x^2\right)\right),\,\Omega^2\,y^2+\Omega^2\,x^2-4\,G\,C\,z^2+2\,T\,y^3\,\Omega-4\,G\,T\,y^2+2\,G^2+z^2\,A^2+2\,e^2\,A^2+2\,C^2\,z^4+2\,T^2\,y^4+2\,x^4\,T^2+z^6\,B^2-4\,e\,A\,G+4\,T^2\,y^2\,x^2-4\,G\,x^2\,T-2\,G\,\Omega\,y-2\,x^3\,T\,\Omega+2\,G\,\Omega\,x+2\,z^4\,B\,A+z^2\,\mathrm{D}^2\,y^4+z^2\,x^4\,\mathrm{D}^2+4\,e\,A\,C\,z^2+4\,e\,A\,T\,y^2+4\,e\,A\,x^2\,T+2\,e\,A\,\Omega\,y+4\,C\,z^2\,T\,y^2+4\,C\,z^2\,x^2\,T+2\,C\,z^2\,\Omega\,y+2\,x^2\,T\,\Omega\,y-2\,e\,A\,\Omega\,x-2\,C\,z^2\,\Omega\,x-2\,T\,y^2\,\Omega\,x+2\,z^2\,A\,y^2\,\mathrm{D}+2\,z^2\,A\,x^2\,\mathrm{D}+2\,z^4\,B\,y^2\,\mathrm{D}+2\,z^4\,B\,x^2\,\mathrm{D}+2\,z^2\,y^2\,\mathrm{D}^2\,x^2\right] \tag{166}$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t]));collect(simplify(expand(charpoly(JAC,q))
  ),q):
```

$$JAC:=\left[\left[2\,x\,T,\,2\,T\,y+\Omega,\,2\,C\,z,\,0\right],\right. \tag{167}$$

$$\left[2\,x\,T-\Omega,\,2\,T\,y,\,2\,C\,z,\,0\right],$$

$$\left[2\,\mathrm{D}\,z\,x,\,2\,\mathrm{D}\,z\,y,\,A+3\,B\,z^2+y^2\,\mathrm{D}+x^2\,\mathrm{D},\,0\right],$$

$$\left[2\,\Omega^2\,x+8\,x^3\,T^2+8\,T^2\,y^2\,x-8\,G\,x\,T-6\,x^2\,T\,\Omega+2\,\Omega\,G+4\,z^2\,x^3\,\mathrm{D}^2+8\,e\,A\,x\,T+8\,C\,z^2\,x\,T+4\,x\,T\,\Omega\,y-2\,\Omega\,A\,e-2\,\Omega\,z^2\,C-2\,T\,y^2\,\Omega+4\,z^2\,A\,x\,\mathrm{D}+4\,z^4\,B\,x\,\mathrm{D}\right.$$

$$+ 4\,\mathrm{D}^2 z^2 y^2 x,\ 2\,\Omega^2 y + 6\,T y^2 \Omega - 8\,G\,T y + 8\,T^2 y^3 + 8\,T^2 y\,x^2 - 2\,\Omega\,G + 4\,z^2\,\mathrm{D}^2 y^3$$

$$+ 8\,e\,A\,T y + 2\,\Omega\,A\,e + 8\,C\,z^2\,T y + 2\,\Omega\,z^2\,C + 2\,x^2\,T\,\Omega - 4\,x\,T\,\Omega\,y + 4\,z^2\,A\,y\,\mathrm{D}$$

$$+ 4\,z^4\,B\,y\,\mathrm{D} + 4\,\mathrm{D}^2 z^2 y\,x^2,\ -8\,G\,C\,z + 2\,z\,A^2 + 8\,C^2 z^3 + 6\,z^5 B^2 + 8\,z^3\,B\,A + 2\,z\,\mathrm{D}^2 y^4$$

$$+ 2\,z\,x^4\,\mathrm{D}^2 + 8\,e\,A\,C\,z + 8\,C\,z\,T y^2 + 8\,C\,z\,x^2\,T + 4\,C\,z\,y\,\Omega - 4\,C\,z\,x\,\Omega + 4\,z\,A\,y^2\,\mathrm{D}$$

$$+ 4\,z\,A\,x^2\,\mathrm{D} + 8\,z^3\,B\,y^2\,\mathrm{D} + 8\,z^3\,B\,x^2\,\mathrm{D} + 4\,z\,y^2\,\mathrm{D}^2 x^2,\ 0]]$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=collect(factor((1/2)*(Xm*S1-S2)),Omega);Za:=collect(factor((1/3)*
  (Yg*S1-Xm*S2+S3)),Omega);Tk:=factor(det(JAC));Yg0:=factor(subs(Omega=0,Yg));
  Za1:=factor(coeff(Za,Omega));
```

$$Xm := 2\,x\,T + 2\,T y + A + 3\,B\,z^2 + y^2\,\mathrm{D} + x^2\,\mathrm{D}$$

$$Yg := \Omega^2 + (2\,T y - 2\,x\,T)\,\Omega + 2\,T y\,A + 2\,T y^3\,\mathrm{D} + 6\,T y\,B\,z^2 + 2\,T y\,x^2\,\mathrm{D} - 4\,\mathrm{D}\,z^2\,y\,C + 2\,A\,x\,T - 4\,\mathrm{D}\,z^2\,x\,C + 2\,x^3\,T\mathrm{D} + 6\,x\,T\,B\,z^2 + 2\,x\,T y^2\,\mathrm{D}$$

$$Za := \left(A + 3\,B\,z^2 + y^2\,\mathrm{D} + x^2\,\mathrm{D}\right)\Omega^2 + \left(-2\,A\,x\,T - 2\,x^3\,T\mathrm{D} + 2\,T y\,A + 2\,T y^3\,\mathrm{D} - 2\,x\,T y^2\,\mathrm{D} + 6\,T y\,B\,z^2 + 2\,T y\,x^2\,\mathrm{D} - 4\,\mathrm{D}\,z^2\,y\,C + 4\,\mathrm{D}\,z^2\,x\,C - 6\,x\,T\,B\,z^2\right)\Omega$$

$$Tk := 0$$

$$Yg0 := 2\,(y + x)\,\left(T\mathrm{D}\,x^2 + A\,T - 2\,C\,z^2\,\mathrm{D} + 3\,T\,B\,z^2 + T y^2\,\mathrm{D}\right)$$

$$Za1 := -2\,(x - y)\,\left(T\mathrm{D}\,x^2 + A\,T - 2\,C\,z^2\,\mathrm{D} + 3\,T\,B\,z^2 + T y^2\,\mathrm{D}\right) \qquad \textbf{(168)}$$

Hopf conditions

```
> AAA:=factor(solve(Xm,A));ZaAAA:=subs(A=AAA,Za);DDD:=(solve(ZaAAA,D));YYg:=
  factor(subs(A=AAA,D=DDD,Yg));
```

$$AAA := -2\,x\,T - 2\,T y - 3\,B\,z^2 - y^2\,\mathrm{D} - x^2\,\mathrm{D}$$

$$ZaAAA := (-2\,x\,T - 2\,T y)\,\Omega^2 + \left(-2\,\left(-2\,x\,T - 2\,T y - 3\,B\,z^2 - y^2\,\mathrm{D} - x^2\,\mathrm{D}\right) x\,T - 2\,x^3\,T\mathrm{D} + 2\,T y\,\left(-2\,x\,T - 2\,T y - 3\,B\,z^2 - y^2\,\mathrm{D} - x^2\,\mathrm{D}\right) + 2\,T y^3\,\mathrm{D} - 2\,x\,T y^2\,\mathrm{D} + 6\,T y\,B\,z^2 + 2\,T y\,x^2\,\mathrm{D} - 4\,\mathrm{D}\,z^2\,y\,C + 4\,\mathrm{D}\,z^2\,x\,C - 6\,x\,T\,B\,z^2\right)\Omega$$

$$DDD := -\frac{1}{2}\,\frac{T\left(-\Omega\,x - \Omega\,y + 2\,x^2\,T - 2\,T y^2\right)}{C\,z^2\,(x - y)}$$

$$YYg := -\frac{\Omega\left(-\Omega\,x + \Omega\,y + 4\,x^2\,T + 4\,T y^2\right)}{x - y} \qquad \textbf{(169)}$$

Oscillations occur when YYg>0
Fixed points

```
> RRR:=solve(ZaAAA,Omega);HopfCrit:=factor(subs(A=AAA,Omega=RRR[1],Yg));
  HopfCrit:=factor(subs(A=AAA,Omega=RRR[2],Yg));
```

$$RRR := 0, \frac{2\left(x^2 T^2 - T^2 y^2 - \mathrm{D} z^2 y C + \mathrm{D} z^2 x C\right)}{T\,(y+x)}$$

$$HopfCrit := -4\,(y+x)\left(x T^2 + T^2 y + C z^2 \mathrm{D}\right)$$

$$HopfCrit := \frac{1}{T^2 (y+x)^2}\Big(4\left(x T^2 + T^2 y + C z^2 \mathrm{D}\right)\left(C x^2 \mathrm{D} z^2 - 2 y \mathrm{D} z^2 C x + C y^2 \mathrm{D} z^2 - 3 T^2 y x^2 - 3 T^2 y^2 x - T^2 y^3 - x^3 T^2\right)\Big) \quad (170)$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);EE1:=-factor(diff(phi,x))
  ;EE2:=-factor(diff(phi,y));EE3:=-factor(diff(phi,z));
```

$$CURLx := 2\,\mathrm{D}\,z\,y - 2\,C\,z$$

$$CURLy := 2\,C\,z - 2\,\mathrm{D}\,z\,x$$

$$CURLz := 2\,x\,T - 2\,\Omega - 2\,T\,y$$

$$EE1 := -2\,\Omega^2 x - 8 x^3 T^2 - 8 T^2 y^2 x + 8 G x T + 6 x^2 T \Omega - 2 \Omega G - 4 z^2 x^3 \mathrm{D}^2 - 8 e A x T - 8 C z^2 x T - 4 x T \Omega y + 2 \Omega A e + 2 \Omega z^2 C + 2 T y^2 \Omega - 4 z^2 A x \mathrm{D} - 4 z^4 B x \mathrm{D} - 4 \mathrm{D}^2 z^2 y^2 x$$

$$EE2 := -2\,\Omega^2 y - 6 T y^2 \Omega + 8 G T y - 8 T^2 y^3 - 8 T^2 y x^2 + 2 \Omega G - 4 z^2 \mathrm{D}^2 y^3 - 8 e A T y - 2 \Omega A e - 8 C z^2 T y - 2 \Omega z^2 C - 2 x^2 T \Omega + 4 x T \Omega y - 4 z^2 A y \mathrm{D} - 4 z^4 B y \mathrm{D} - 4 \mathrm{D}^2 z^2 y x^2$$

$$EE3 := -2\,z\left(3 B^2 z^4 + 4 C^2 z^2 + 4 B z^2 A + 4 B z^2 y^2 \mathrm{D} + 4 B z^2 x^2 \mathrm{D} - 4 C G + A^2 + x^4 \mathrm{D}^2 + 2 y^2 \mathrm{D}^2 x^2 + 4 T y^2 C + \mathrm{D}^2 y^4 + 2 C y \Omega + 4 C A e + 2 A y^2 \mathrm{D} + 4 T x^2 C - 2 C x \Omega + 2 A x^2 \mathrm{D}\right) \quad (171)$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),Omega);H1:=factor(coeff
  (HELICITY,Omega));H0:=factor(subs(Omega=0,HELICITY));
```

$$HELICITY := -2\,z\left(C x + C y + B z^2 + A\right)\Omega - 2\,z\left(-\mathrm{D} z^2 y C - x T B z^2 + T y B z^2 + G \mathrm{D} y + \mathrm{D} z^2 x C - A x T + T y A - e A \mathrm{D} y - G \mathrm{D} x + e A \mathrm{D} x\right)$$

$$H1 := -2\,z\left(C x + C y + B z^2 + A\right)$$

$$H0 := -2\,z\,(x-y)\left(-T B z^2 + C z^2 \mathrm{D} - A T - G \mathrm{D} + e A \mathrm{D}\right) \quad (172)$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),Omega);E0:=factor
  (subs(Omega=0,enstrophy));
```

$$enstrophy := 4\,\Omega^2 + (8\,T\,y - 8\,x\,T)\,\Omega + 4\,D^2 z^2 y^2 - 8\,D\,z^2 y\,C + 8\,C^2 z^2 - 8\,D\,z^2 x\,C + 4\,D^2 z^2 x^2 + 4\,x^2 T^2 + 4\,T^2 y^2 - 8\,T^2 y\,x$$

$$E0 := 4\,D^2 z^2 y^2 - 8\,D\,z^2 y\,C + 8\,C^2 z^2 - 8\,D\,z^2 x\,C + 4\,D^2 z^2 x^2 + 4\,x^2 T^2 + 4\,T^2 y^2 - 8\,T^2 y\,x \qquad \textbf{(173)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := [-2\,C\,z\,\Omega - 2\,D\,z\,x\,\Omega,\ -2\,D\,z\,y\,\Omega - 2\,C\,z\,\Omega,\ -4\,D\,z^2 x\,C + 4\,D\,z^2 y\,C + 2\,A\,x\,T - 2\,\Omega\,A - 2\,T\,y\,A + 6\,x\,T\,B\,z^2 - 6\,\Omega\,B\,z^2 - 6\,T\,y\,B\,z^2 + 2\,x\,T\,y^2 D - 2\,\Omega\,y^2 D - 2\,T\,y^3 D + 2\,x^3\,T\,D - 2\,\Omega\,x^2 D - 2\,T\,y\,x^2 D] \qquad \textbf{(174)}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),Omega);ST2:=factor(coeff(stretch,Omega^2));ST1:=factor(coeff(stretch,Omega));st0:=factor(subs(Omega=0,stretch));
```

$$stretch := (12\,B\,z^2 + 4\,y^2 D + 4\,x^2 D + 4\,A)\,\Omega^2 + (-8\,x^3\,T\,D + 8\,T\,y\,A + 8\,T\,y^3 D - 8\,A\,x\,T - 24\,x\,T\,B\,z^2 + 24\,T\,y\,B\,z^2 + 8\,T\,y\,x^2 D - 16\,D\,z^2 y\,C - 8\,x\,T\,y^2 D + 16\,D\,z^2 x\,C)\,\Omega - 8\,T^2 y\,A\,x + 12\,T^2 y^2 B\,z^2 - 8\,C\,z^2 D\,x^2 T - 8\,C\,z^2 D\,y^2 T - 8\,T^2 y^3 D\,x + 4\,T^2 y^4 D - 8\,T^2 y\,x^3 D + 8\,T^2 y^2 x^2 D + 12\,x^2 T^2 B\,z^2 + 16\,C\,z^2 D\,x\,T\,y + 4\,x^4 T^2 D + 4\,T^2 y^2 A + 4\,A\,x^2 T^2 - 24\,T^2 y\,B\,z^2 x$$

$$ST2 := 12\,B\,z^2 + 4\,y^2 D + 4\,x^2 D + 4\,A$$

$$ST1 := -8\,(x-y)\,(T\,D\,x^2 + A\,T - 2\,C\,z^2 D + 3\,T\,B\,z^2 + T\,y^2 D)$$

$$st0 := 4\,T\,(x-y)^2\,(T\,D\,x^2 + A\,T - 2\,C\,z^2 D + 3\,T\,B\,z^2 + T\,y^2 D) \qquad \textbf{(175)}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),Omega);BI2:=factor(coeff(BI,Omega^2));BI1:=factor(coeff(BI,Omega));BI0:=(factor(subs(Omega=0,BI)));
```

$$BI := -32\,D^2 z^4 y\,C^2 x + 4\,A^2 x^2 T^2 - 16\,D\,z^2 y^2 C\,T\,A + 4\,T^2 y^2 A^2 + 4\,T^2 y^6 D^2 + 4\,x^6 T^2 D^2 - 16\,D\,z^2 x^2 C\,A\,T - 48\,D\,z^4 x^2 C\,T\,B - 32\,D^2 z^2 x^2 C\,T\,y^2 + 32\,D^2 z^2 x\,C\,T\,y^3 + 32\,D^2 z^2 x^3 C\,T\,y + (48\,T\,y\,B\,z^2 x^2 D + 16\,T\,y^3 A\,D + 16\,T\,y^3 D^2 x^2 + 72\,T\,y\,B^2 z^4 + 8\,T\,y\,x^4 D^2 - 16\,A\,x^3\,T\,D - 16\,x^3\,T\,D^2 y^2 - 8\,x\,T\,y^4 D^2 + 8\,T\,y\,A^2 + 8\,T\,y^5 D^2 - 8\,A^2 x\,T - 8\,x^5\,T\,D^2 - 72\,x\,T\,B^2 z^4 + 16\,C\,z^2 D\,x\,A + 48\,C\,z^4 D\,x\,B + 16\,C\,z^2 D^2 x\,y^2 - 16\,C\,z^2 D\,y\,A - 48\,C\,z^4 D\,y\,B - 16\,C\,z^2 D^2 y\,x^2 - 48\,x\,T\,y^2 D\,B\,z^2 + 48\,T\,y\,A\,B\,z^2 + 16\,T\,y\,A\,x^2 D + 48\,T\,y^3 D\,B\,z^2 - 48\,A\,x\,T\,B\,z^2 - 16\,A\,x\,T\,y^2 D - 48\,x^3\,T\,D\,B\,z^2 + 16\,C\,z^2 D^2 x^3 - 16\,C\,z^2 D^2 y^3)\,\Omega + 32\,D\,z^2 x\,C\,T\,y\,A - 16\,D^2 z^2 x^4 C\,T - 16\,D^2 z^2 y^4 C\,T + 24\,A\,x^2 T^2 B\,z^2 + 16\,A\,x^2 T^2 y^2 D - 16\,A\,x\,T^2 y^3 D - 16\,A\,x^3 T^2 y\,D + 24\,T^2 y^2 A\,B\,z^2 - 72\,x\,T^2 B^2 z^4 y + 24\,x^4 T^2 B\,z^2 D + 24\,T^2 y^4 B\,z^2 D + 16\,D^2 z^4 y^2 C^2 + 16\,D^2 z^4 x^2 C^2 + (4\,D^2 z^2 x^2 + 8\,y^2 D^2 x^2 + 4\,D^2 z^2 y^2 + 8\,C^2 z^2 + 36\,B^2 z^4 + 8\,A\,x^2 D + 8\,D\,z^2 x\,C + 8\,D\,z^2 y\,C + 4\,D^2 y^4$$

$$+4A^2+24Bz^2y^2\mathrm{D}+24Bz^2x^2\mathrm{D}+24Bz^2A+8Ay^2\mathrm{D}+4x^4\mathrm{D}^2)\,\Omega^2-8A^2xT^2y$$
$$+8Ax^4T^2\mathrm{D}+8T^2y^4A\mathrm{D}+36x^2T^2B^2z^4+36T^2y^2B^2z^4+12x^2T^2y^4\mathrm{D}^2-8xT^2y^5\mathrm{D}^2$$
$$+12x^4T^2y^2\mathrm{D}^2-16x^3T^2y^3\mathrm{D}^2-8x^5T^2\mathrm{D}^2y+96\mathrm{D}z^4xCTyB-48\mathrm{D}z^4y^2CTB$$
$$-48AxT^2yBz^2+48x^2T^2Bz^2y^2\mathrm{D}-48xT^2Bz^2y^3\mathrm{D}-48x^3T^2Bz^2y\mathrm{D}$$

$$BI2:=4\mathrm{D}^2z^2x^2+8y^2\mathrm{D}^2x^2+4\mathrm{D}^2z^2y^2+8C^2z^2+36B^2z^4+8Ax^2\mathrm{D}+8\mathrm{D}z^2xC$$
$$+8\mathrm{D}z^2yC+4\mathrm{D}^2y^4+4A^2+24Bz^2y^2\mathrm{D}+24Bz^2x^2\mathrm{D}+24Bz^2A+8Ay^2\mathrm{D}+4x^4\mathrm{D}^2$$

$$BI1:=-8\,(x-y)\,\left(A+3Bz^2+y^2\mathrm{D}+x^2\mathrm{D}\right)\left(T\mathrm{D}x^2+AT-2Cz^2\mathrm{D}+3TBz^2+Ty^2\mathrm{D}\right)$$

$$BI0:=4\,(x-y)^2\left(T\mathrm{D}x^2+AT-2Cz^2\mathrm{D}+3TBz^2+Ty^2\mathrm{D}\right)^2 \quad \textbf{(176)}$$

```
> Theta:=collect(factor(-2*innerprod([EE1,EE2,EE3],VORTICITY)),Omega);
```

$$\Theta:=8z\,(-Cy+Cx)\,\Omega^2+8z\left(-2C^2z^2-A^2-2Ty^2C-\mathrm{D}^2y^4-eA\mathrm{D}y-eA\mathrm{D}x-Tyx^2\mathrm{D}\right. \quad \textbf{(177)}$$
$$-xTy^2\mathrm{D}-4Bz^2A-2CAe-x^4\mathrm{D}^2-2Ax^2\mathrm{D}-3B^2z^4-Ty^3\mathrm{D}-x^3T\mathrm{D}-2y^2\mathrm{D}^2x^2$$
$$-2Ay^2\mathrm{D}-\mathrm{D}z^2yC-\mathrm{D}z^2xC-2Tx^2C-4Bz^2y^2\mathrm{D}-4Bz^2x^2\mathrm{D}+2CG+G\mathrm{D}y$$
$$\left.+G\mathrm{D}x\right)\Omega+8z\left(-4TyBz^2x^2\mathrm{D}-2Ty^3A\mathrm{D}-2Ty^3\mathrm{D}^2x^2-3TyB^2z^4-Tyx^4\mathrm{D}^2\right.$$
$$+2Ax^3T\mathrm{D}+2x^3T\mathrm{D}^2y^2+xTy^4\mathrm{D}^2-TyA^2-Ty^5\mathrm{D}^2+A^2xT+x^5T\mathrm{D}^2+3xTB^2z^4$$
$$-2Cz^2\mathrm{D}xA-2Cz^4\mathrm{D}xB-2Cz^2\mathrm{D}^2xy^2+2Cz^2\mathrm{D}yA+2Cz^4\mathrm{D}yB+2Cz^2\mathrm{D}^2yx^2$$
$$+4xTy^2\mathrm{D}Bz^2-4TyABz^2-2TyAx^2\mathrm{D}-4Ty^3\mathrm{D}Bz^2+4AxTBz^2+2AxTy^2\mathrm{D}$$
$$\left.+4x^3T\mathrm{D}Bz^2-2Cz^2\mathrm{D}^2x^3+2Cz^2\mathrm{D}^2y^3\right)$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t2:=
  factor(coeff(Theta,Omega^2));t1:=factor(coeff(Theta,Omega));
> t0:=factor(subs(Omega=0,Theta));
```

$$\Theta:=-8z\,(-Cy+Cx)\,\Omega^2-8z\left(-2C^2z^2-A^2-2Ty^2C-\mathrm{D}^2y^4-eA\mathrm{D}y-eA\mathrm{D}x\right.$$
$$-Tyx^2\mathrm{D}-xTy^2\mathrm{D}-4Bz^2A-2CAe-x^4\mathrm{D}^2-2Ax^2\mathrm{D}-3B^2z^4-Ty^3\mathrm{D}-x^3T\mathrm{D}$$
$$-2y^2\mathrm{D}^2x^2-2Ay^2\mathrm{D}-\mathrm{D}z^2yC-\mathrm{D}z^2xC-2Tx^2C-4Bz^2y^2\mathrm{D}-4Bz^2x^2\mathrm{D}+2CG$$
$$\left.+G\mathrm{D}y+G\mathrm{D}x\right)\Omega-8z\left(-4TyBz^2x^2\mathrm{D}-2Ty^3A\mathrm{D}-2Ty^3\mathrm{D}^2x^2-3TyB^2z^4\right.$$
$$-Tyx^4\mathrm{D}^2+2Ax^3T\mathrm{D}+2x^3T\mathrm{D}^2y^2+xTy^4\mathrm{D}^2-TyA^2-Ty^5\mathrm{D}^2+A^2xT+x^5T\mathrm{D}^2$$
$$+3xTB^2z^4-2Cz^2\mathrm{D}xA-2Cz^4\mathrm{D}xB-2Cz^2\mathrm{D}^2xy^2+2Cz^2\mathrm{D}yA+2Cz^4\mathrm{D}yB$$
$$+2Cz^2\mathrm{D}^2yx^2+4xTy^2\mathrm{D}Bz^2-4TyABz^2-2TyAx^2\mathrm{D}-4Ty^3\mathrm{D}Bz^2+4AxTBz^2$$
$$\left.+2AxTy^2\mathrm{D}+4x^3T\mathrm{D}Bz^2-2Cz^2\mathrm{D}^2x^3+2Cz^2\mathrm{D}^2y^3\right)$$

$$t2:=-8zC\,(x-y)$$

$$t1:=8z\left(2C^2z^2+A^2+2Ty^2C+\mathrm{D}^2y^4+eA\mathrm{D}y+eA\mathrm{D}x+Tyx^2\mathrm{D}+xTy^2\mathrm{D}+4Bz^2A\right.$$
$$+2CAe+x^4\mathrm{D}^2+2Ax^2\mathrm{D}+3B^2z^4+Ty^3\mathrm{D}+x^3T\mathrm{D}+2y^2\mathrm{D}^2x^2+2Ay^2\mathrm{D}+\mathrm{D}z^2yC$$
$$\left.+\mathrm{D}z^2xC+2Tx^2C+4Bz^2y^2\mathrm{D}+4Bz^2x^2\mathrm{D}-2CG-G\mathrm{D}y-G\mathrm{D}x\right)$$

$$t0 := -8\, z\, (x-y)\, \left(A + B z^2 + y^2 \mathrm{D} + x^2 \mathrm{D}\right) \left(T \mathrm{D} x^2 + A T - 2 C z^2 \mathrm{D} + 3 T B z^2 + T y^2 \mathrm{D}\right) \quad \textbf{(178)}$$

```
> t11:=(t1/(-8*z));collect(t11,D);t10:=factor(subs(D=0,t11));
> beta:=D*x^2+A+B*z^2+D*y^2;betasq:=expand(beta^2);
```

$$t11 := -2 C^2 z^2 - A^2 - 2 T y^2 C - \mathrm{D}^2 y^4 - e A \mathrm{D} y - e A \mathrm{D} x - T y x^2 \mathrm{D} - x T y^2 \mathrm{D} - 4 B z^2 A - 2 C A e - x^4 \mathrm{D}^2 - 2 A x^2 \mathrm{D} - 3 B^2 z^4 - T y^3 \mathrm{D} - x^3 T \mathrm{D} - 2 y^2 \mathrm{D}^2 x^2 - 2 A y^2 \mathrm{D} - \mathrm{D} z^2 y C - \mathrm{D} z^2 x C - 2 T x^2 C - 4 B z^2 y^2 \mathrm{D} - 4 B z^2 x^2 \mathrm{D} + 2 C G + G \mathrm{D} y + G \mathrm{D} x$$

$$\left(-x^4 - y^4 - 2 y^2 x^2\right) \mathrm{D}^2 + \left(-2 y^2 A - x T y^2 - e A x - 2 A x^2 - e A y - T y^3 + G y - T y x^2 - 4 y^2 z^2 B - C z^2 y - x^3 T + G x - 4 x^2 B z^2 - C z^2 x\right) \mathrm{D} - 2 C^2 z^2 - A^2 - 2 T y^2 C - 4 B z^2 A - 2 C A e - 2 T x^2 C - 3 B^2 z^4 + 2 C G$$

$$t10 := -2 C^2 z^2 - A^2 - 2 T y^2 C - 4 B z^2 A - 2 C A e - 2 T x^2 C - 3 B^2 z^4 + 2 C G$$

$$\beta := A + B z^2 + y^2 \mathrm{D} + x^2 \mathrm{D}$$

$$betasq := A^2 + 2 B z^2 A + 2 A y^2 \mathrm{D} + 2 A x^2 \mathrm{D} + B^2 z^4 + 2 B z^2 y^2 \mathrm{D} + 2 B z^2 x^2 \mathrm{D} + \mathrm{D}^2 y^4 + 2 y^2 \mathrm{D}^2 x^2 + x^4 \mathrm{D}^2 \quad \textbf{(179)}$$

```
> simplify(t11-expand(-(D*x^2+D*y^2+A+B*z^2)^2));
```

$$-2 C^2 z^2 - 2 T y^2 C - e A \mathrm{D} y - e A \mathrm{D} x - T y x^2 \mathrm{D} - x T y^2 \mathrm{D} - 2 B z^2 A - 2 C A e - 2 B^2 z^4 - T y^3 \mathrm{D} - x^3 T \mathrm{D} - \mathrm{D} z^2 y C - \mathrm{D} z^2 x C - 2 T x^2 C - 2 B z^2 y^2 \mathrm{D} - 2 B z^2 x^2 \mathrm{D} + 2 C G + G \mathrm{D} y + G \mathrm{D} x \quad \textbf{(180)}$$

TRANSCRITICAL HOPF OK

```
> AA:= [x*(A-G+C*z)+Omega*y,y*(A-G+C*z)-Omega*x,A*z+B*z^2+D*(x^2+y^2)];
```

$$AA := \left[x\, (A - G + C z) + \Omega y,\ y\, (A - G + C z) - \Omega x,\ z A + B z^2 + \mathrm{D}\left(y^2 + x^2\right)\right] \quad \textbf{(181)}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=[AA[1],AA[2],AA[3],phi];
```

$$\phi := \Omega^2 y^2 + \Omega^2 x^2 + \mathrm{D}^2 y^4 + 2 z A y^2 \mathrm{D} + 2 z A x^2 \mathrm{D} + A^2 x^2 + G^2 y^2 + C^2 z^2 x^2 + C^2 z^2 y^2 + 2 z^3 B A - 2 G x^2 C z + 2 A y^2 C z - 2 G y^2 C z + x^4 \mathrm{D}^2 - 2 G x^2 A + A^2 y^2 + z^2 A^2 + G^2 x^2 + B^2 z^4 - 2 A y^2 G + 2 A C z x^2 + 2 y^2 \mathrm{D}^2 x^2 + 2 B z^2 y^2 \mathrm{D} + 2 B z^2 x^2 \mathrm{D}$$

$$VV := \left[x\, (A - G + C z) + \Omega y,\ y\, (A - G + C z) - \Omega x,\ z A + B z^2 + \mathrm{D}\left(y^2 + x^2\right),\ \Omega^2 y^2 + \Omega^2 x^2 + \mathrm{D}^2 y^4 + 2 z A y^2 \mathrm{D} + 2 z A x^2 \mathrm{D} + A^2 x^2 + G^2 y^2 + C^2 z^2 x^2 + C^2 z^2 y^2 + 2 z^3 B A - 2 G x^2 C z + 2 A y^2 C z - 2 G y^2 C z + x^4 \mathrm{D}^2 - 2 G x^2 A + A^2 y^2 + z^2 A^2 + G^2 x^2 + B^2 z^4 - 2 A y^2 G + 2 A C z x^2 + 2 y^2 \mathrm{D}^2 x^2 + 2 B z^2 y^2 \mathrm{D} + 2 B z^2 x^2 \mathrm{D}\right] \quad \textbf{(182)}$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t]));collect(simplify(expand(charpoly(JAC,q))
  ),q);
```

$$JAC := \left[\left[A - G + C z, \Omega, C x, 0\right],\right.$$

$$\left[-\Omega, A-G+C z, C y, 0\right],$$

$$\left[2 \mathrm{D} x, 2 \mathrm{D} y, A+2 B z, 0\right],$$

$$\left[2 \Omega^2 x+4 \mathrm{D} z x A+2 A^2 x+2 C^2 z^2 x-4 G x C z+4 x^3 \mathrm{D}^2-4 G x A+2 G^2 x+4 A C z x\right.$$
$$+4 \mathrm{D}^2 y^2 x+4 B x \mathrm{D} z^2, 2 \Omega^2 y+4 \mathrm{D}^2 y^3+4 \mathrm{D} z y A+2 G^2 y+2 C^2 z^2 y+4 C y z A$$
$$-4 G y C z+2 A^2 y-4 A y G+4 \mathrm{D}^2 y x^2+4 B y \mathrm{D} z^2, 2 A y^2 \mathrm{D}+2 A x^2 \mathrm{D}+2 C^2 z x^2$$
$$+2 C^2 z y^2+6 B z^2 A-2 G C x^2+2 C y^2 A-2 G y^2 C+2 z A^2+4 B^2 z^3+2 x^2 C A$$
$$\left.+4 B z y^2 \mathrm{D}+4 B z x^2 \mathrm{D}, 0\right]\Big]$$

$$q^4+(-2 C z-3 A+2 G-2 B z) q^3+\left(4 B z A-2 C x^2 \mathrm{D}-2 C y^2 \mathrm{D}-4 G B z+C^2 z^2\right.$$
$$\left.+4 C z^2 B-2 G C z+4 A C z+3 A^2-4 G A+G^2+\Omega^2\right) q^2+\left(2 A \mathrm{D} y^2 C-A^3\right.$$
$$+4 A G B z-4 A C z^2 B-2 G^2 B z+2 G C z A-2 z A^2 B-2 A^2 C z-2 \Omega^2 B z$$
$$+2 \mathrm{D} x^2 A C-C^2 z^2 A-2 C^2 z^3 B-2 G \mathrm{D} y^2 C+2 C^2 z \mathrm{D} y^2-G^2 A+4 G C z^2 B$$
$$\left.+2 \mathrm{D} x^2 C^2 z+2 G A^2-2 \mathrm{D} x^2 C G-\Omega^2 A\right) q \quad \textbf{(183)}$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=collect(factor((1/2)*(Xm*S1-S2)),Omega);Za:=collect(factor((1/3)*
  (Yg*S1-Xm*S2+S3)),Omega);Tk:=factor(det(JAC));Yg0:=factor(subs(Omega=0,Yg));
  Za1:=factor(coeff(Za,Omega));Za0:=factor(subs(Omega=0,Za));
```

$$Xm := 3 A-2 G+2 C z+2 B z$$

$$Yg := 4 B z A-2 C x^2 \mathrm{D}-2 C y^2 \mathrm{D}-4 G B z+C^2 z^2+4 C z^2 B-2 G C z+4 A C z+3 A^2$$
$$-4 G A+G^2+\Omega^2$$

$$Za := (A+2 B z) \Omega^2+A^3-2 \mathrm{D} x^2 A C-4 A G B z+4 A C z^2 B-2 A \mathrm{D} y^2 C-4 G C z^2 B$$
$$+2 G \mathrm{D} y^2 C-2 C^2 z \mathrm{D} y^2+2 \mathrm{D} x^2 C G-2 \mathrm{D} x^2 C^2 z-2 G C z A+2 G^2 B z+2 C^2 z^3 B$$
$$+C^2 z^2 A-2 G A^2+G^2 A+2 z A^2 B+2 A^2 C z$$

$$Tk := 0$$

$$Yg0 := 4 B z A-2 C x^2 \mathrm{D}-2 C y^2 \mathrm{D}-4 G B z+C^2 z^2+4 C z^2 B-2 G C z+4 A C z+3 A^2$$
$$-4 G A+G^2$$

$$Za1 := 0$$

$$Za0 := (A-G+C z)\left(A^2+A C z-G A+2 B z A-2 G B z-2 C x^2 \mathrm{D}+2 C z^2 B-2 C y^2 \mathrm{D}\right) \quad \textbf{(184)}$$

Hopf conditions

```
> GGG:=factor(solve(Xm,G));ZaGGG:=factor(subs(G=GGG,Za));RRR:=solve(ZaGGG,
  Omega^2);YYg:=factor(subs(G=GGG,Omega^2=RRR,Yg));
```

$$GGG := \frac{3}{2} A+C z+B z$$

$$ZaGGG := \frac{1}{4}\,(A+2\,B\,z)\,\left(A^2+4\,B\,z\,A+4\,C\,x^2\,\mathrm{D}+4\,C\,y^2\,\mathrm{D}+4\,B^2\,z^2+4\,\Omega^2\right)$$

Warning, solving for expressions other than names or functions is not recommended.

$$RRR := -\frac{1}{4}\,A^2-B\,z\,A-C\,x^2\,\mathrm{D}-C\,y^2\,\mathrm{D}-B^2\,z^2$$

$$YYg := -4\,B\,z\,A-3\,C\,x^2\,\mathrm{D}-3\,C\,y^2\,\mathrm{D}-4\,B^2\,z^2-A^2 \quad \textbf{(185)}$$

Oscillations occur when YYg>0

```
> RRR:=solve(ZaGGG,Omega^2);HopfCrit:=factor(subs(G=GGG,Omega^2=RRR,Yg));
```

Warning, solving for expressions other than names or functions is not recommended.

$$RRR := -\frac{1}{4}\,A^2-B\,z\,A-C\,x^2\,\mathrm{D}-C\,y^2\,\mathrm{D}-B^2\,z^2$$

$$HopfCrit := -4\,B\,z\,A-3\,C\,x^2\,\mathrm{D}-3\,C\,y^2\,\mathrm{D}-4\,B^2\,z^2-A^2 \quad \textbf{(186)}$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := 2\,\mathrm{D}\,y-C\,y$$

$$CURLy := C\,x-2\,\mathrm{D}\,x$$

$$CURLz := -2\,\Omega \quad \textbf{(187)}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),Omega);
```

$$HELICITY := \Omega\,\left(-C\,y^2-x^2\,C-2\,z\,A-2\,B\,z^2\right) \quad \textbf{(188)}$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),Omega);E0:=factor
  (subs(Omega=0,enstrophy));
```

$$enstrophy := 4\,\mathrm{D}^2\,y^2-4\,C\,y^2\,\mathrm{D}+C^2\,y^2+C^2\,x^2-4\,C\,x^2\,\mathrm{D}+4\,\mathrm{D}^2\,x^2+4\,\Omega^2$$

$$E0 := \left(y^2+x^2\right)\,(C-2\,\mathrm{D})^2 \quad \textbf{(189)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[2\,A\,y\,\mathrm{D}-C\,y\,A-2\,G\,\mathrm{D}\,y+C\,y\,G+2\,C\,y\,\mathrm{D}\,z-C^2\,y\,z-C\,x\,\Omega-2\,\Omega\,x\,\mathrm{D},\ -2\,\mathrm{D}\,y\,\Omega-C\,y\,\Omega+A\,C\,x-2\,\mathrm{D}\,x\,A-C\,x\,G+2\,G\,\mathrm{D}\,x+C^2\,x\,z-2\,C\,x\,\mathrm{D}\,z,\ -2\,\Omega\,A-4\,\Omega\,B\,z\right] \quad \textbf{(190)}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),Omega);ST2:=factor(coeff
  (stretch,Omega^2));st0:=factor(subs(Omega=0,stretch));
```

$$stretch := (8\,B\,z+4\,A)\,\Omega^2+4\,A\,y^2\,\mathrm{D}^2-4\,A\,\mathrm{D}\,y^2\,C+C^2\,y^2\,A-4\,G\,\mathrm{D}^2\,y^2+4\,G\,\mathrm{D}\,y^2\,C-C^2\,y^2\,G+4\,C\,y^2\,\mathrm{D}^2\,z-4\,C^2\,z\,\mathrm{D}\,y^2+C^3\,y^2\,z+x^2\,C^2\,A-4\,\mathrm{D}\,x^2\,A\,C+4\,\mathrm{D}^2\,x^2\,A-C^2\,x^2\,G+4\,\mathrm{D}\,x^2\,C\,G-4\,G\,\mathrm{D}^2\,x^2+C^3\,x^2\,z-4\,\mathrm{D}\,x^2\,C^2\,z+4\,C\,x^2\,\mathrm{D}^2\,z$$

$$ST2 := 8\,B\,z + 4\,A$$

$$st0 := \left(y^2 + x^2\right)\,(C - 2\,\mathrm{D})^2\,(A - G + C\,z) \qquad \textbf{(191)}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),Omega);BI2:=factor(coeff(BI,
  Omega^2));BI1:=factor(coeff(BI,Omega));BI0:=(factor(subs(Omega=0,BI)));
```

$$BI := A^2 C^2 x^2 + C^2 A^2 y^2 + C^4 x^2 z^2 + C^4 y^2 z^2 + C^2 G^2 x^2 + 8\,A\,C\,x^2\,G\,\mathrm{D} - 8\,A\,C^2 x^2\,\mathrm{D}\,z + 8\,\mathrm{D}^2 x^2\,A\,C\,z + 8\,C^2 x^2\,G\,\mathrm{D}\,z - 8\,G\,\mathrm{D}^2 x^2\,C\,z - 4\,A^2 y^2\,\mathrm{D}\,C - 8\,A\,y^2\,\mathrm{D}^2\,G - 2\,C^2 y^2\,A\,G + 2\,C^3 y^2\,A\,z - 4\,G^2\,\mathrm{D}\,y^2\,C - 2\,C^3 y^2\,G\,z + 4\,C^2 y^2\,\mathrm{D}^2 z^2 - 4\,C^3 y^2\,\mathrm{D}\,z^2 - 4\,A^2\,C\,x^2\,\mathrm{D} - 2\,A\,C^2 x^2\,G + 2\,A\,C^3 x^2\,z - 8\,\mathrm{D}^2 x^2\,A\,G - 4\,C\,x^2\,G^2\,\mathrm{D} - 2\,C^3 x^2\,G\,z - 4\,C^3 x^2 z^2\,\mathrm{D} + 4\,C^2 x^2\,\mathrm{D}^2 z^2 + 8\,A\,y^2\,\mathrm{D}\,C\,G + 8\,A\,y^2\,\mathrm{D}^2\,C\,z - 8\,A\,y^2\,\mathrm{D}\,C^2 z - 8\,G\,\mathrm{D}^2 y^2\,C\,z + 8\,G\,\mathrm{D}\,y^2\,C^2 z + 4\,G^2\,\mathrm{D}^2 x^2 + 4\,A^2 y^2\,\mathrm{D}^2 + 4\,\mathrm{D}^2 x^2 A^2 + 4\,G^2\,\mathrm{D}^2 y^2 + C^2 y^2 G^2 + \left(4\,\mathrm{D}^2 x^2 + 4\,\mathrm{D}^2 y^2 + C^2 y^2 + C^2 x^2 + 4\,A^2 + 4\,C\,x^2\,\mathrm{D} + 4\,C\,y^2\,\mathrm{D} + 16\,B\,z\,A + 16\,B^2 z^2\right)\Omega^2$$

$$BI2 := 4\,\mathrm{D}^2 x^2 + 4\,\mathrm{D}^2 y^2 + C^2 y^2 + C^2 x^2 + 4\,A^2 + 4\,C\,x^2\,\mathrm{D} + 4\,C\,y^2\,\mathrm{D} + 16\,B\,z\,A + 16\,B^2 z^2$$

$$BI1 := 0$$

$$BI0 := \left(y^2 + x^2\right)\,(C - 2\,\mathrm{D})^2\,(A - G + C\,z)^2 \qquad \textbf{(192)}$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta := 8\,\Omega\,\left(3\,B\,z^2 A + A\,y^2\,\mathrm{D} + A\,x^2\,\mathrm{D} + 2\,B^2 z^3 + z\,A^2 + C^2 z\,x^2 + C^2 z\,y^2 + 2\,B\,z\,y^2\,\mathrm{D} + 2\,B\,z\,x^2\,\mathrm{D} - G\,C\,x^2 + C\,y^2 A - G\,y^2\,C + x^2\,C\,A\right) \qquad \textbf{(193)}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t2:=
  factor(coeff(Theta,Omega^2));t1:=factor(coeff(Theta,Omega));
> t0:=factor(subs(Omega=1,Theta)/(-8));
```

$$\Theta := \left(24\,B\,z^2 A + 8\,A\,y^2\,\mathrm{D} + 8\,A\,x^2\,\mathrm{D} + 16\,B^2 z^3 + 8\,z\,A^2 + 8\,C^2 z\,x^2 + 8\,C^2 z\,y^2 + 16\,B\,z\,y^2\,\mathrm{D} + 16\,B\,z\,x^2\,\mathrm{D} - 8\,G\,C\,x^2 + 8\,C\,y^2 A - 8\,G\,y^2\,C + 8\,x^2\,C\,A\right)\Omega$$

$$t2 := 0$$

$$t1 := 24\,B\,z^2 A + 8\,A\,y^2\,\mathrm{D} + 8\,A\,x^2\,\mathrm{D} + 16\,B^2 z^3 + 8\,z\,A^2 + 8\,C^2 z\,x^2 + 8\,C^2 z\,y^2 + 16\,B\,z\,y^2\,\mathrm{D} + 16\,B\,z\,x^2\,\mathrm{D} - 8\,G\,C\,x^2 + 8\,C\,y^2 A - 8\,G\,y^2\,C + 8\,x^2\,C\,A$$

$$t0 := -3\,B\,z^2 A - A\,y^2\,\mathrm{D} - A\,x^2\,\mathrm{D} - 2\,B^2 z^3 - z\,A^2 - C^2 z\,x^2 - C^2 z\,y^2 - 2\,B\,z\,y^2\,\mathrm{D} - 2\,B\,z\,x^2\,\mathrm{D} + G\,C\,x^2 - C\,y^2 A + G\,y^2\,C - x^2\,C\,A \qquad \textbf{(194)}$$

SADDLE NODE HOPF OK

```
> AA:= [x*(G+C*z)+Omega*y,y*(G+C*z)-Omega*x,A+B*z^2+D*(x^2+y^2)];
```

$$AA := \left[x\,(G + C\,z) + \Omega\,y,\, y\,(G + C\,z) - \Omega\,x,\, A + B\,z^2 + \mathrm{D}\left(y^2 + x^2\right)\right] \quad \textbf{(195)}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=[AA[1],AA[2],AA[3],phi];
```

$$\phi := G^2 x^2 + 2\,G\,x^2\,C\,z + C^2 z^2 x^2 + \Omega^2 y^2 + G^2 y^2 + 2\,G\,y^2\,C\,z + C^2 z^2 y^2 + \Omega^2 x^2 + A^2 + 2\,B\,z^2 A + 2\,A\,y^2\,\mathrm{D} + 2\,A\,x^2\,\mathrm{D} + B^2 z^4 + 2\,B\,z^2 y^2\,\mathrm{D} + 2\,B\,z^2 x^2\,\mathrm{D} + \mathrm{D}^2 y^4 + 2\,y^2\,\mathrm{D}^2 x^2 + x^4\,\mathrm{D}^2$$

$$VV := \left[x\,(G + C\,z) + \Omega\,y,\, y\,(G + C\,z) - \Omega\,x,\, A + B\,z^2 + \mathrm{D}\left(y^2 + x^2\right),\, G^2 x^2 + 2\,G\,x^2\,C\,z + C^2 z^2 x^2 + \Omega^2 y^2 + G^2 y^2 + 2\,G\,y^2\,C\,z + C^2 z^2 y^2 + \Omega^2 x^2 + A^2 + 2\,B\,z^2 A + 2\,A\,y^2\,\mathrm{D} + 2\,A\,x^2\,\mathrm{D} + B^2 z^4 + 2\,B\,z^2 y^2\,\mathrm{D} + 2\,B\,z^2 x^2\,\mathrm{D} + \mathrm{D}^2 y^4 + 2\,y^2\,\mathrm{D}^2 x^2 + x^4\,\mathrm{D}^2\right] \quad \textbf{(196)}$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t]));collect(simplify(expand(charpoly(JAC,q))
  ),q);
```

$$JAC := \left[\left[G + C\,z,\, \Omega,\, C\,x,\, 0\right],\; \left[-\Omega,\, G + C\,z,\, C\,y,\, 0\right],\; \left[2\,\mathrm{D}\,x,\, 2\,\mathrm{D}\,y,\, 2\,B\,z,\, 0\right],\; \left[2\,G^2 x + 4\,G\,x\,C\,z + 2\,C^2 z^2 x + 2\,\Omega^2 x + 4\,\mathrm{D}\,x\,A + 4\,B\,x\,\mathrm{D}\,z^2 + 4\,\mathrm{D}^2 y^2 x + 4\,x^3\,\mathrm{D}^2,\, 2\,\Omega^2 y + 2\,G^2 y + 4\,G\,y\,C\,z + 2\,C^2 z^2 y + 4\,A\,y\,\mathrm{D} + 4\,B\,y\,\mathrm{D}\,z^2 + 4\,\mathrm{D}^2 y^3 + 4\,\mathrm{D}^2 y\,x^2,\, 2\,G\,C\,x^2 + 2\,C^2 z\,x^2 + 2\,G\,y^2 C + 2\,C^2 z\,y^2 + 4\,B\,z\,A + 4\,B^2 z^3 + 4\,B\,z\,y^2\,\mathrm{D} + 4\,B\,z\,x^2\,\mathrm{D},\, 0\right]\right]$$

$$q^4 + (-2\,C\,z - 2\,B\,z - 2\,G)\,q^3 + \left(-2\,C\,x^2\,\mathrm{D} + 2\,G\,C\,z + \Omega^2 + G^2 + C^2 z^2 + 4\,C\,z^2 B - 2\,C\,y^2\,\mathrm{D} + 4\,G\,B\,z\right) q^2 + \left(-2\,C^2 z^3 B + 2\,C^2 z\,\mathrm{D}\,y^2 + 2\,\mathrm{D}\,x^2\,C^2 z - 2\,\Omega^2 B\,z - 2\,G^2 B\,z + 2\,\mathrm{D}\,x^2\,C\,G - 4\,G\,C\,z^2 B + 2\,G\,\mathrm{D}\,y^2\,C\right) q \quad \textbf{(197)}$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=collect(factor((1/2)*(Xm*S1-S2)),Omega);Za:=collect(factor((1/3)*
  (Yg*S1-Xm*S2+S3)),Omega);Tk:=factor(det(JAC));Yg0:=factor(subs(Omega=0,Yg));
  Za1:=factor(coeff(Za,Omega));Za0:=factor(subs(Omega=0,Za));
```

$$Xm := 2\,G + 2\,C\,z + 2\,B\,z$$

$$Yg := -2\,C\,x^2\,\mathrm{D} + 2\,G\,C\,z + \Omega^2 + G^2 + C^2 z^2 + 4\,C\,z^2 B - 2\,C\,y^2\,\mathrm{D} + 4\,G\,B\,z$$

$$Za := 4\,G\,C\,z^2 B - 2\,G\,\mathrm{D}\,y^2\,C - 2\,C^2 z\,\mathrm{D}\,y^2 - 2\,\mathrm{D}\,x^2\,C\,G - 2\,\mathrm{D}\,x^2\,C^2 z + 2\,G^2 B\,z + 2\,C^2 z^3 B + 2\,\Omega^2 B\,z$$

$$Tk := 0$$

$$Yg0 := -2\,C\,x^2\,\mathrm{D} + 2\,G\,C\,z - 2\,C\,y^2\,\mathrm{D} + G^2 + C^2 z^2 + 4\,C\,z^2 B + 4\,G\,B\,z$$

$$Za1 := 0$$

$$Za0 := 2\,(G + C\,z)\left(C\,z^2 B - C\,y^2\,\mathrm{D} - C\,x^2\,\mathrm{D} + G\,B\,z\right) \quad \textbf{(198)}$$

Hopf conditions

```
> GGG:=factor(solve(Xm,G));ZaGGG:=subs(G=GGG,Za);DDD:=solve(ZaGGG,D);YYg:=factor
  (subs(G=GGG,D=DDD,Yg));EV:=eigenvalues(JAC):EV3:=factor(subs(G=GGG,D=DDD,EV[3]
  ));EV2:=simplify(subs(G=GGG,D=DDD,EV[2]));EV4:=factor(subs(G=GGG,D=DDD,EV[4]))
  ;factor(subs(G=GGG,D=DDD,EV[1]));factor(EV3*EV4*EV2);
```

$$GGG := -z\,(B+C)$$

$$ZaGGG := -4\,z^3\,(B+C)\,C\,B + 2\,z\,(B+C)\,\mathrm{D}\,y^2\,C - 2\,C^2\,z\,\mathrm{D}\,y^2 + 2\,\mathrm{D}\,x^2\,C\,z\,(B+C) - 2\,\mathrm{D}\,x^2\,C^2\,z + 2\,z^3\,(B+C)^2\,B + 2\,C^2\,z^3\,B + 2\,\Omega^2\,B\,z$$

$$DDD := -\frac{B^2\,z^2+\Omega^2}{C\,(y^2+x^2)}$$

$$YYg := -B^2\,z^2 + 3\,\Omega^2$$

$$EV3 := \frac{1}{\left(-3\,(B^2 z^2 - 3\,\Omega^2)^3\right)^{1/6}}\left(\frac{1}{6}\,\mathrm{I}\left(\mathrm{I}\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3} - 3\,\mathrm{I}\,3^{1/3}\,\Omega^2 - 3^{5/6}\,B^2 z^2 + 3\,3^{5/6}\,\Omega^2 + \mathrm{I}\,3^{1/3}\,B^2 z^2 + \sqrt{3}\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3}\right)3^{1/3}\right)$$

$$EV2 := \frac{1}{3}\,\frac{\sqrt{3}\left(\left(-(B^2 z^2-3\,\Omega^2)^3\right)^{1/3} + B^2 z^2 - 3\,\Omega^2\right)}{\left(-(B^2 z^2-3\,\Omega^2)^3\right)^{1/6}}$$

$$EV4 := \frac{1}{\left(-3\,(B^2 z^2 - 3\,\Omega^2)^3\right)^{1/6}}\left(\frac{1}{6}\,\mathrm{I}\left(\mathrm{I}\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3} - 3\,\mathrm{I}\,3^{1/3}\,\Omega^2 + 3^{5/6}\,B^2 z^2 - 3\,3^{5/6}\,\Omega^2 + \mathrm{I}\,3^{1/3}\,B^2 z^2 - \sqrt{3}\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3}\right)3^{1/3}\right)$$

$$0$$

$$-\frac{1}{36}\,\frac{1}{\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3}\left(-(B^2 z^2-3\,\Omega^2)^3\right)^{1/6}}\Bigg(\left(\mathrm{I}\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3} - 3\,\mathrm{I}\,3^{1/3}\,\Omega^2 - 3^{5/6}\,B^2 z^2 + 3\,3^{5/6}\,\Omega^2 + \mathrm{I}\,3^{1/3}\,B^2 z^2 + \sqrt{3}\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3}\right) 3^{1/6}\left(\mathrm{I}\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3} - 3\,\mathrm{I}\,3^{1/3}\,\Omega^2 + 3^{5/6}\,B^2 z^2 - 3\,3^{5/6}\,\Omega^2 + \mathrm{I}\,3^{1/3}\,B^2 z^2 - \sqrt{3}\left(-3\,(B^2 z^2-3\,\Omega^2)^3\right)^{1/3}\right)\left(\left(-(B^2 z^2-3\,\Omega^2)^3\right)^{1/3} + B^2 z^2 - 3\,\Omega^2\right)\Bigg) \qquad \textbf{(199)}$$

Oscillations occur when YYg>0 Hence if 3(Omega)^2 exceeds (Bz)^2, oscillations occur.
Fixed points

```
> RRR:=solve(ZaGGG,Omega^2);HopfCrit:=factor(subs(G=GGG,Omega^2=RRR,Yg));
```

Warning, solving for expressions other than names or functions is not recommended.

$$RRR := -C\,y^2\,\mathrm{D} - C\,x^2\,\mathrm{D} - B^2\,z^2$$
$$HopfCrit := -3\,C\,x^2\,\mathrm{D} - 3\,C\,y^2\,\mathrm{D} - 4\,B^2\,z^2 \qquad (200)$$

```
> Xm;
```

$$2\,G + 2\,C\,z + 2\,B\,z \qquad (201)$$

```
> ZaCB:=(factor(subs(G=0,C=-B,Za)));DDD:=solve(ZaCB,D);(factor(subs(C=-B,G=0,Yg)
  ));
```

$$ZaCB := 2\,B\,z\,\left(B^2\,z^2 - \mathrm{D}\,y^2\,B - B\,x^2\,\mathrm{D} + \Omega^2\right)$$
$$DDD := \frac{B^2\,z^2 + \Omega^2}{B\,(y^2 + x^2)}$$
$$2\,B\,x^2\,\mathrm{D} + \Omega^2 - 3\,B^2\,z^2 + 2\,\mathrm{D}\,y^2\,B \qquad (202)$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := 2\,\mathrm{D}\,y - C\,y$$
$$CURLy := C\,x - 2\,\mathrm{D}\,x$$
$$CURLz := -2\,\Omega \qquad (203)$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),Omega);
```

$$HELICITY := \Omega\,\left(-C\,y^2 - x^2\,C - 2\,A - 2\,B\,z^2\right) \qquad (204)$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),Omega);E0:=factor
  (subs(Omega=0,enstrophy));
```

$$enstrophy := 4\,\mathrm{D}^2\,y^2 - 4\,C\,y^2\,\mathrm{D} + C^2\,y^2 + C^2\,x^2 - 4\,C\,x^2\,\mathrm{D} + 4\,\mathrm{D}^2\,x^2 + 4\,\Omega^2$$
$$E0 := \left(y^2 + x^2\right)\,(C - 2\,\mathrm{D})^2 \qquad (205)$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[2\,G\,\mathrm{D}\,y - C\,y\,G + 2\,C\,y\,\mathrm{D}\,z - C^2\,y\,z - C\,x\,\Omega - 2\,\Omega\,x\,\mathrm{D},\ -2\,\mathrm{D}\,y\,\Omega - C\,y\,\Omega + C\,x\,G - 2\,G\,\mathrm{D}\,x + C^2\,x\,z - 2\,C\,x\,\mathrm{D}\,z,\ -4\,\Omega\,B\,z\right] \qquad (206)$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),Omega);ST2:=factor(coeff
  (stretch,Omega^2));st0:=factor(subs(Omega=0,stretch));
```

$$stretch := 4\,G\,\mathrm{D}^2\,y^2 - 4\,G\,\mathrm{D}\,y^2\,C + C^2\,y^2\,G + 4\,C\,y^2\,\mathrm{D}^2\,z - 4\,C^2\,z\,\mathrm{D}\,y^2 + C^3\,y^2\,z + C^2\,x^2\,G - 4\,\mathrm{D}\,x^2\,C\,G + 4\,G\,\mathrm{D}^2\,x^2 + C^3\,x^2\,z - 4\,\mathrm{D}\,x^2\,C^2\,z + 4\,C\,x^2\,\mathrm{D}^2\,z + 8\,\Omega^2\,B\,z$$
$$ST2 := 8\,B\,z$$
$$st0 := \left(y^2 + x^2\right)\,(C - 2\,\mathrm{D})^2\,(G + C\,z) \qquad (207)$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),Omega);BI2:=factor(coeff(BI,
```

```
Omega^2));BI1:=factor(coeff(BI,Omega));BI0:=(factor(subs(Omega=0,BI)));
```

$$BI := \left(4\,D^2 x^2 + 4\,D^2 y^2 + C^2 y^2 + C^2 x^2 + 4\,C\,x^2\,D + 4\,C\,y^2\,D + 16\,B^2 z^2\right)\Omega^2 + C^4 y^2 z^2 + 4\,C^2 x^2 D^2 z^2 - 4\,C\,x^2 G^2 D + C^4 x^2 z^2 + 8\,G\,D^2 x^2 C\,z - 4\,G^2 D\,y^2 C + C^2 G^2 x^2 - 8\,C^2 x^2 G\,D\,z - 4\,C^3 y^2 D\,z^2 - 4\,C^3 x^2 z^2 D + 2\,C^3 y^2 G\,z + 4\,C^2 y^2 D^2 z^2 + 2\,C^3 x^2 G\,z + 4\,G^2 D^2 x^2 + 8\,G\,D^2 y^2 C\,z - 8\,G\,D\,y^2 C^2 z + 4\,G^2 D^2 y^2 + C^2 y^2 G^2$$

$$BI2 := 4\,D^2 x^2 + 4\,D^2 y^2 + C^2 y^2 + C^2 x^2 + 4\,C\,x^2\,D + 4\,C\,y^2\,D + 16\,B^2 z^2$$

$$BI1 := 0$$

$$BI0 := \left(y^2 + x^2\right)\left(G + C\,z\right)^2\left(C - 2\,D\right)^2 \qquad \textbf{(208)}$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta := 8\,\Omega\left(2\,B^2 z^3 + 2\,B\,z\,A + C^2 z\,x^2 + C^2 z\,y^2 + 2\,B\,z\,y^2 D + 2\,B\,z\,x^2 D + G\,C\,x^2 + G\,y^2 C\right) \qquad \textbf{(209)}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t2:=
  factor(coeff(Theta,Omega^2));t1:=factor(coeff(Theta,Omega));
> t0:=factor(subs(Omega=1,Theta)/8);ThetaHopf:=(factor(subs(G=GGG,D=DDD,Theta)))
  ;;FalacoCRIT:=YYg;boundinghopf:=factor(x^2*C^2-2*A*C+2*Omega^2+y^2*C^2+2*B^2*
  z^2-2*C*z^2*B);
```

$$\Theta := \left(16\,B^2 z^3 + 16\,B\,z\,A + 8\,C^2 z\,x^2 + 8\,C^2 z\,y^2 + 16\,B\,z\,y^2 D + 16\,B\,z\,x^2 D + 8\,G\,C\,x^2 + 8\,G\,y^2 C\right)\Omega$$

$$t2 := 0$$

$$t1 := 16\,B^2 z^3 + 16\,B\,z\,A + 8\,C^2 z\,x^2 + 8\,C^2 z\,y^2 + 16\,B\,z\,y^2 D + 16\,B\,z\,x^2 D + 8\,G\,C\,x^2 + 8\,G\,y^2 C$$

$$t0 := 2\,B^2 z^3 + 2\,B\,z\,A + C^2 z\,x^2 + C^2 z\,y^2 + 2\,B\,z\,y^2 D + 2\,B\,z\,x^2 D + G\,C\,x^2 + G\,y^2 C$$

$$ThetaHopf := 8\,\Omega\left(-B\,x^2 C + 2\,A\,B + 2\,\Omega^2 - C\,y^2 B + 4\,B^2 z^2\right) z$$

$$FalacoCRIT := -B^2 z^2 + 3\,\Omega^2$$

$$boundinghopf := C^2 x^2 - 2\,C\,A + 2\,\Omega^2 + C^2 y^2 + 2\,B^2 z^2 - 2\,C\,z^2 B \qquad \textbf{(210)}$$

```
> bbb:=(4/3)*solve(YYg,Omega^2)/(z^2);H:=HELICITY;BBB:=solve(H,B);Tension:=subs
  (B=BBB,bbb);Rotat:=(Tension)^(1/2);
```

Warning, solving for expressions other than names or functions is not recommended.

$$bbb := \frac{4}{9}\,B^2$$

$$H := \Omega\left(-C\,y^2 - x^2 C - 2\,A - 2\,B\,z^2\right)$$

$$BBB := -\frac{1}{2}\,\frac{C\,y^2 + x^2 C + 2\,A}{z^2}$$

$$Tension := \frac{1}{9}\,\frac{\left(C\,y^2 + x^2 C + 2\,A\right)^2}{z^4}$$

$$Rotat := \frac{1}{9}\sqrt{9}\sqrt{\frac{\left(C y^2 + x^2 C + 2 A\right)^2}{z^4}} \tag{211}$$

****

HYSTERESIS HOPF OK

```
> AA:= [x*(-G+C*z)+Omega*y,y*(-G+C*z)-Omega*x,A+B*z+E*z^3+D*(x^2+y^2)];
```

$$AA := \left[x\,(-G + C z) + \Omega y,\ y\,(-G + C z) - \Omega x,\ A + B z + E z^3 + \mathrm{D}\left(y^2 + x^2\right)\right] \tag{212}$$

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=[AA[1],AA[2],AA[3],phi];
```

$$\phi := \Omega^2 y^2 + \Omega^2 x^2 + A^2 + \mathrm{D}^2 y^4 + G^2 y^2 + 2 B z A + C^2 z^2 x^2 + C^2 z^2 y^2 + E^2 z^6 - 2 G x^2 C z - 2 G y^2 C z + 2 B z y^2 \mathrm{D} + 2 B z x^2 \mathrm{D} + x^4 \mathrm{D}^2 + B^2 z^2 + G^2 x^2 + 2 A x^2 \mathrm{D} + 2 E z^3 y^2 \mathrm{D} + 2 E z^3 x^2 \mathrm{D} + 2 A E z^3 + 2 B z^4 E + 2 y^2 \mathrm{D}^2 x^2 + 2 A y^2 \mathrm{D}$$

$$VV := \left[x\,(-G + C z) + \Omega y,\ y\,(-G + C z) - \Omega x,\ A + B z + E z^3 + \mathrm{D}\left(y^2 + x^2\right),\ \Omega^2 y^2 + \Omega^2 x^2 + A^2 + \mathrm{D}^2 y^4 + G^2 y^2 + 2 B z A + C^2 z^2 x^2 + C^2 z^2 y^2 + E^2 z^6 - 2 G x^2 C z - 2 G y^2 C z + 2 B z y^2 \mathrm{D} + 2 B z x^2 \mathrm{D} + x^4 \mathrm{D}^2 + B^2 z^2 + G^2 x^2 + 2 A x^2 \mathrm{D} + 2 E z^3 y^2 \mathrm{D} + 2 E z^3 x^2 \mathrm{D} + 2 A E z^3 + 2 B z^4 E + 2 y^2 \mathrm{D}^2 x^2 + 2 A y^2 \mathrm{D}\right] \tag{213}$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t]));collect(simplify(expand(charpoly(JAC,q))
  ),q);
```

$$JAC := \left[\left[-G + C z, \Omega, C x, 0\right], \left[-\Omega, -G + C z, C y, 0\right], \left[2 \mathrm{D} x, 2 \mathrm{D} y, B + 3 E z^2, 0\right], \left[2 \Omega^2 x + 2 C^2 z^2 x - 4 G x C z + 4 B \mathrm{D} z x + 4 x^3 \mathrm{D}^2 + 2 G^2 x + 4 \mathrm{D} x A + 4 E z^3 x \mathrm{D} + 4 \mathrm{D}^2 y^2 x,\ 2 \Omega^2 y + 4 \mathrm{D}^2 y^3 + 2 G^2 y + 2 C^2 z^2 y - 4 G y C z + 4 B \mathrm{D} z y + 4 E z^3 y \mathrm{D} + 4 \mathrm{D}^2 y x^2 + 4 A y \mathrm{D},\ 2 A B + 2 C^2 z x^2 + 2 C^2 z y^2 + 6 E^2 z^5 - 2 G C x^2 - 2 G y^2 C + 2 \mathrm{D} y^2 B + 2 B x^2 \mathrm{D} + 2 B^2 z + 6 E z^2 y^2 \mathrm{D} + 6 E z^2 x^2 \mathrm{D} + 6 A E z^2 + 8 B z^3 E, 0\right]\right]$$

$$q^4 + \left(-B - 2 C z - 3 E z^2 + 2 G\right) q^3 + \left(-2 C y^2 \mathrm{D} + G^2 + C^2 z^2 - 2 G C z - 2 C x^2 \mathrm{D} + \Omega^2 + 2 B C z + 6 C z^3 E - 2 G B - 6 G E z^2\right) q^2 + \left(-G^2 B - 2 G \mathrm{D} y^2 C + 2 C^2 z \mathrm{D} y^2\right. \tag{214}$$

$$-3\,G^2 E z^2 - B C^2 z^2 + 2\,G B C z + 6\,G C z^3 E - 2\,\mathrm{D} x^2 C G + 2\,\mathrm{D} x^2 C^2 z - 3\,C^2 z^4 E - B\,\Omega^2 - 3\,\Omega^2 E z^2\big)\,q$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
(innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=collect(factor((1/2)*(Xm*S1-S2)),Omega);Za:=collect(factor((1/3)*
(Yg*S1-Xm*S2+S3)),Omega);Tk:=factor(det(JAC));Yg0:=factor(subs(Omega=0,Yg));
Za1:=factor(coeff(Za,Omega));Za0:=factor(subs(Omega=0,Za));
```

$$Xm := -2\,G + 2\,C z + B + 3\,E z^2$$

$$Yg := -2\,C y^2\,\mathrm{D} + G^2 + C^2 z^2 - 2\,G C z - 2\,C x^2\,\mathrm{D} + \Omega^2 + 2\,B C z + 6\,C z^3 E - 2\,G B - 6\,G E z^2$$

$$Za := \left(B + 3\,E z^2\right)\Omega^2 + B C^2 z^2 - 2\,\mathrm{D} x^2 C^2 z + 2\,G\,\mathrm{D} y^2 C - 2\,C^2 z\,\mathrm{D} y^2 + 2\,\mathrm{D} x^2 C G + 3\,G^2 E z^2 + G^2 B - 2\,G B C z - 6\,G C z^3 E + 3\,C^2 z^4 E$$

$$Tk := 0$$

$$Yg0 := -2\,C y^2\,\mathrm{D} + G^2 + C^2 z^2 - 2\,G C z - 2\,C x^2\,\mathrm{D} - 6\,G E z^2 + 2\,B C z + 6\,C z^3 E - 2\,G B$$

$$Za1 := 0$$

$$Za0 := (-G + C z)\left(3\,C z^3 E + B C z - 2\,C y^2\,\mathrm{D} - 2\,C x^2\,\mathrm{D} - 3\,G E z^2 - G B\right) \qquad \textbf{(215)}$$

Hopf conditions

```
> GGG:=factor(solve(Xm,G));ZaGGG:=subs(G=GGG,Za);DDD:=factor(solve(ZaGGG,D));
YYg:=factor(subs(G=GGG,D=DDD,Yg));
```

$$GGG := C z + \frac{1}{2}\,B + \frac{3}{2}\,E z^2$$

$$ZaGGG := \left(B + 3\,E z^2\right)\Omega^2 + B C^2 z^2 - 2\,\mathrm{D} x^2 C^2 z + 2\left(C z + \frac{1}{2}\,B + \frac{3}{2}\,E z^2\right)\mathrm{D} y^2 C - 2\,C^2 z\,\mathrm{D} y^2 + 2\,\mathrm{D} x^2 C\left(C z + \frac{1}{2}\,B + \frac{3}{2}\,E z^2\right) + 3\left(C z + \frac{1}{2}\,B + \frac{3}{2}\,E z^2\right)^2 E z^2 + \left(C z + \frac{1}{2}\,B + \frac{3}{2}\,E z^2\right)^2 B - 2\left(C z + \frac{1}{2}\,B + \frac{3}{2}\,E z^2\right) B C z - 6\left(C z + \frac{1}{2}\,B + \frac{3}{2}\,E z^2\right) C z^3 E + 3\,C^2 z^4 E$$

$$DDD := -\frac{1}{4}\,\frac{B^2 + 6\,B E z^2 + 9\,E^2 z^4 + 4\,\Omega^2}{C\left(y^2 + x^2\right)}$$

$$YYg := 3\,\Omega^2 - \frac{3}{2}\,B E z^2 - \frac{9}{4}\,E^2 z^4 - \frac{1}{4}\,B^2 \qquad \textbf{(216)}$$

Oscillations occur when YYg>0
The Hopf critical surface is

```
> RRR:=solve(ZaGGG,Omega^2);HopfCrit:=factor(subs(G=GGG,Omega^2=RRR,Yg));
```

Warning, solving for expressions other than names or functions is not recommended.

$$RRR := -\frac{1}{4}B^2 - \frac{3}{2}B\,E\,z^2 - C\,x^2\,\mathrm{D} - C\,y^2\,\mathrm{D} - \frac{9}{4}E^2\,z^4$$

$$HopfCrit := -3\,C\,y^2\,\mathrm{D} - B^2 - 6\,B\,E\,z^2 - 9\,E^2\,z^4 - 3\,C\,x^2\,\mathrm{D} \quad \textbf{(217)}$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := 2\,\mathrm{D}\,y - C\,y$$

$$CURLy := C\,x - 2\,\mathrm{D}\,x$$

$$CURLz := -2\,\Omega \quad \textbf{(218)}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),Omega);
```

$$HELICITY := \Omega\left(-C\,y^2 - x^2\,C - 2\,A - 2\,B\,z - 2\,E\,z^3\right) \quad \textbf{(219)}$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),Omega);E0:=factor
  (subs(Omega=0,enstrophy));
```

$$enstrophy := 4\,\mathrm{D}^2\,y^2 - 4\,C\,y^2\,\mathrm{D} + C^2\,y^2 + C^2\,x^2 - 4\,C\,x^2\,\mathrm{D} + 4\,\mathrm{D}^2\,x^2 + 4\,\Omega^2$$

$$E0 := \left(y^2 + x^2\right)\left(C - 2\,\mathrm{D}\right)^2 \quad \textbf{(220)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[-2\,G\,\mathrm{D}\,y + C\,y\,G + 2\,C\,y\,\mathrm{D}\,z - C^2\,y\,z - C\,x\,\Omega - 2\,\Omega\,x\,\mathrm{D},\ -2\,\mathrm{D}\,y\,\Omega - C\,y\,\Omega - C\,x\,G + 2\,G\,\mathrm{D}\,x + C^2\,x\,z - 2\,C\,x\,\mathrm{D}\,z,\ -2\,B\,\Omega - 6\,\Omega\,E\,z^2\right] \quad \textbf{(221)}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),Omega);ST2:=factor(coeff
  (stretch,Omega^2));st0:=factor(subs(Omega=0,stretch));
```

$$stretch := \left(4\,B + 12\,E\,z^2\right)\Omega^2 - 4\,G\,\mathrm{D}^2\,y^2 + 4\,G\,\mathrm{D}\,y^2\,C - C^2\,y^2\,G + 4\,C\,y^2\,\mathrm{D}^2\,z - 4\,C^2\,z\,\mathrm{D}\,y^2 + C^3\,y^2\,z - C^2\,x^2\,G + 4\,\mathrm{D}\,x^2\,C\,G - 4\,G\,\mathrm{D}^2\,x^2 + C^3\,x^2\,z - 4\,\mathrm{D}\,x^2\,C^2\,z + 4\,C\,x^2\,\mathrm{D}^2\,z$$

$$ST2 := 4\,B + 12\,E\,z^2$$

$$st0 := \left(y^2 + x^2\right)\left(C - 2\,\mathrm{D}\right)^2\left(-G + C\,z\right) \quad \textbf{(222)}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),Omega);BI2:=factor(coeff(BI,
  Omega^2));BI1:=factor(coeff(BI,Omega));BI0:=(factor(subs(Omega=0,BI)));
```

$$BI := \left(4\,\mathrm{D}^2\,x^2 + 4\,\mathrm{D}^2\,y^2 + 4\,B^2 + 36\,E^2\,z^4 + C^2\,y^2 + C^2\,x^2 + 24\,B\,E\,z^2 + 4\,C\,x^2\,\mathrm{D} + 4\,C\,y^2\,\mathrm{D}\right)\Omega^2 - 2\,C^3\,y^2\,G\,z + 4\,C^2\,y^2\,\mathrm{D}^2\,z^2 - 4\,C^3\,y^2\,\mathrm{D}\,z^2 - 4\,C^3\,x^2\,z^2\,\mathrm{D} + 4\,C^2\,x^2\,\mathrm{D}^2\,z^2 + C^4\,x^2\,z^2$$

$$+ C^4 y^2 z^2 - 8 G D^2 x^2 C z - 4 G^2 D y^2 C + C^2 G^2 x^2 + 8 C^2 x^2 G D z - 2 C^3 x^2 G z$$
$$+ 4 G^2 D^2 x^2 + C^2 y^2 G^2 - 4 C x^2 G^2 D + 8 G D y^2 C^2 z + 4 G^2 D^2 y^2 - 8 G D^2 y^2 C z$$

$$BI2 := 4 D^2 x^2 + 4 D^2 y^2 + 4 B^2 + 36 E^2 z^4 + C^2 y^2 + C^2 x^2 + 24 B E z^2 + 4 C x^2 D + 4 C y^2 D$$

$$BI1 := 0$$

$$BI0 := (y^2 + x^2) (-G + C z)^2 (C - 2 D)^2 \tag{223}$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta := 8 \Omega (3 E^2 z^5 + B^2 z + 3 A E z^2 + 4 B z^3 E + D y^2 B + B x^2 D + 3 E z^2 y^2 D + 3 E z^2 x^2 D + A B + C^2 z x^2 + C^2 z y^2 - G C x^2 - G y^2 C) \tag{224}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t1:=
  factor(coeff(Theta,Omega));
> t0:=factor(subs(Omega=1,Theta)/8);
```

$$\Theta := (24 E^2 z^5 + 8 B^2 z + 24 A E z^2 + 32 B z^3 E + 8 D y^2 B + 8 B x^2 D + 24 E z^2 y^2 D + 24 E z^2 x^2 D + 8 A B + 8 C^2 z x^2 + 8 C^2 z y^2 - 8 G C x^2 - 8 G y^2 C) \Omega$$

$$t1 := 24 E^2 z^5 + 8 B^2 z + 24 A E z^2 + 32 B z^3 E + 8 D y^2 B + 8 B x^2 D + 24 E z^2 y^2 D + 24 E z^2 x^2 D + 8 A B + 8 C^2 z x^2 + 8 C^2 z y^2 - 8 G C x^2 - 8 G y^2 C$$

$$t0 := 3 E^2 z^5 + B^2 z + 3 A E z^2 + 4 B z^3 E + D y^2 B + B x^2 D + 3 E z^2 y^2 D + 3 E z^2 x^2 D + A B + C^2 z x^2 + C^2 z y^2 - G C x^2 - G y^2 C \tag{225}$$

```
> ThetaHopf:=factor(subs(G=GGG,D=DDD,Theta));boundingHopf:=factor(9*E^2*z^4-4*C*
  z^3*E+6*B*E*z^2-4*C*z*B-4*A*C+2*y^2*C^2+4*Omega^2+2*x^2*C^2+B^2);
```

$$ThetaHopf := \frac{1}{C} (2 \Omega (B + 3 E z^2) (-B^2 - 6 B E z^2 + 4 B C z - 2 C^2 y^2 - 2 C^2 x^2 - 4 \Omega^2 - 9 E^2 z^4 + 4 C A + 4 C z^3 E))$$

$$boundingHopf := 9 E^2 z^4 - 4 C z^3 E + 6 B E z^2 - 4 B C z - 4 C A + 2 C^2 y^2 + 4 \Omega^2 + 2 C^2 x^2 + B^2 \tag{226}$$

```
> H:=HELICITY;BBB:=z*solve(H,B);BBB*BBB;ZZZ:=Omega^2-3/4*(BBB*BBB);
```

$$H := \Omega (-C y^2 - x^2 C - 2 A - 2 B z - 2 E z^3)$$

$$BBB := -\frac{1}{2} C y^2 - \frac{1}{2} x^2 C - A - E z^3$$

$$\left(-\frac{1}{2} C y^2 - \frac{1}{2} x^2 C - A - E z^3\right)^2$$

$$ZZZ := \Omega^2 - \frac{3}{4} \left(-\frac{1}{2} C y^2 - \frac{1}{2} x^2 C - A - E z^3\right)^2 \tag{227}$$

```
> YYg;
```

$$3 \Omega^2 - \frac{3}{2} B E z^2 - \frac{9}{4} E^2 z^4 - \frac{1}{4} B^2 \tag{228}$$

Generalized Langford

```
> AA:= [x*(G+C*z)+Omega*y,y*(G+C*z)-Omega*x,P*sinh(alpha*z)+A-0*B*z+0*F*z^2+0*E*
  z^3+D*(x^2+y^2)];
```

$$AA := \left[x\,(G+C\,z)+\Omega\,y,\ y\,(G+C\,z)-\Omega\,x,\ P\sinh(\alpha\,z)+A+\mathrm{D}\,(y^2+x^2)\right] \tag{229}$$

```
> #AA:= [x*(G+C*z)+Omega*y,y*(G+C*z)-Omega*x,P*(sinh(z))+D*(x^2+y^2)];
```

```
> phi:=simplify((AA[1]^2+AA[2]^2+AA[3]^2));VV:=[AA[1],AA[2],AA[3],phi];
```

$$\phi := G^2x^2+2\,G\,x^2\,C\,z+C^2z^2x^2+\Omega^2y^2+G^2y^2+2\,G\,y^2\,C\,z+C^2z^2y^2+\Omega^2x^2+P^2\sinh(\alpha\,z)^2 +2\,P\sinh(\alpha\,z)\,A+2\,P\sinh(\alpha\,z)\,y^2\,\mathrm{D}+2\,P\sinh(\alpha\,z)\,x^2\,\mathrm{D}+A^2+2\,A\,y^2\,\mathrm{D}+2\,A\,x^2\,\mathrm{D} +\mathrm{D}^2y^4+2\,y^2\,\mathrm{D}^2x^2+x^4\,\mathrm{D}^2$$

$$VV := \left[x\,(G+C\,z)+\Omega\,y,\ y\,(G+C\,z)-\Omega\,x,\ P\sinh(\alpha\,z)+A+\mathrm{D}\,(y^2+x^2),\ G^2x^2 +2\,G\,x^2\,C\,z+C^2z^2x^2+\Omega^2y^2+G^2y^2+2\,G\,y^2\,C\,z+C^2z^2y^2+\Omega^2x^2+P^2\sinh(\alpha\,z)^2 +2\,P\sinh(\alpha\,z)\,A+2\,P\sinh(\alpha\,z)\,y^2\,\mathrm{D}+2\,P\sinh(\alpha\,z)\,x^2\,\mathrm{D}+A^2+2\,A\,y^2\,\mathrm{D}+2\,A\,x^2\,\mathrm{D} +\mathrm{D}^2y^4+2\,y^2\,\mathrm{D}^2x^2+x^4\,\mathrm{D}^2\right] \tag{230}$$

```
> JAC:=simplify(jacobian(VV,[x,y,z,t]));collect(simplify(expand(charpoly(JAC,q))
  ),q);
```

$$JAC := \left[\left[G+C\,z,\ \Omega,\ C\,x,\ 0\right],\ \left[-\Omega,\ G+C\,z,\ C\,y,\ 0\right],\ \left[2\,\mathrm{D}\,x,\ 2\,\mathrm{D}\,y,\ P\cosh(\alpha\,z)\,\alpha,\ 0\right],\ \left[2\,x\,\left(G^2+2\,G\,C\,z+C^2z^2+\Omega^2+2\,P\sinh(\alpha\,z)\,\mathrm{D}+2\,A\,\mathrm{D}+2\,\mathrm{D}^2y^2+2\,\mathrm{D}^2x^2\right),\ 2\,y\,\left(G^2 +2\,G\,C\,z+C^2z^2+\Omega^2+2\,P\sinh(\alpha\,z)\,\mathrm{D}+2\,A\,\mathrm{D}+2\,\mathrm{D}^2y^2+2\,\mathrm{D}^2x^2\right),\ 2\,G\,C\,x^2 +2\,C^2z\,x^2+2\,G\,y^2\,C+2\,C^2z\,y^2+2\,P^2\sinh(\alpha\,z)\cosh(\alpha\,z)\,\alpha+2\,P\cosh(\alpha\,z)\,\alpha\,A +2\,P\cosh(\alpha\,z)\,\alpha\,y^2\,\mathrm{D}+2\,P\cosh(\alpha\,z)\,\alpha\,x^2\,\mathrm{D},\ 0\right]\right]$$

$$q^4+\left(-2\,C\,z-P\cosh(\alpha\,z)\,\alpha-2\,G\right)q^3+\left(-2\,C\,x^2\,\mathrm{D}+2\,G\,C\,z+\Omega^2+G^2+C^2z^2 +2\,C\,z\,P\cosh(\alpha\,z)\,\alpha-2\,C\,y^2\,\mathrm{D}+2\,G\,P\cosh(\alpha\,z)\,\alpha\right)q^2+\left(-C^2z^2\,P\cosh(\alpha\,z)\,\alpha +2\,C^2z\,\mathrm{D}\,y^2+2\,\mathrm{D}\,x^2\,C^2z-\Omega^2\,P\cosh(\alpha\,z)\,\alpha-G^2\,P\cosh(\alpha\,z)\,\alpha+2\,\mathrm{D}\,x^2\,C\,G\right. \tag{231}$$

$$-2\,G\,C\,z\,P\cosh(\alpha z)\,\alpha+2\,G\,\mathrm{D}\,y^2\,C\big)\,q$$

```
> S1:=factor(trace(JAC)):S2:=factor(trace(innerprod(JAC,JAC))):S3:=factor(trace
  (innerprod(JAC,JAC,JAC))):
> Xm:=S1;Yg:=collect(factor((1/2)*(Xm*S1-S2)),Omega);Za:=collect(factor((1/3)*
  (Yg*S1-Xm*S2+S3)),Omega);Tk:=factor(det(JAC));Yg0:=factor(subs(Omega=0,Yg));
  Za1:=factor(coeff(Za,Omega));Za0:=factor(subs(Omega=0,Za));
```

$$Xm := 2\,G+2\,C\,z+P\cosh(\alpha z)\,\alpha$$

$$Yg := -2\,C\,x^2\,\mathrm{D}+2\,G\,C\,z+\Omega^2+G^2+C^2\,z^2+2\,C\,z\,P\cosh(\alpha z)\,\alpha-2\,C\,y^2\,\mathrm{D}+2\,G\,P\cosh(\alpha z)\,\alpha$$

$$Za := -2\,G\,\mathrm{D}\,y^2\,C-2\,C^2\,z\,\mathrm{D}\,y^2-2\,\mathrm{D}\,x^2\,C\,G-2\,\mathrm{D}\,x^2\,C^2\,z+C^2\,z^2\,P\cosh(\alpha z)\,\alpha+G^2\,P\cosh(\alpha z)\,\alpha+\Omega^2\,P\cosh(\alpha z)\,\alpha+2\,G\,C\,z\,P\cosh(\alpha z)\,\alpha$$

$$Tk := 0$$

$$Yg0 := -2\,C\,x^2\,\mathrm{D}+2\,G\,C\,z-2\,C\,y^2\,\mathrm{D}+G^2+C^2\,z^2+2\,C\,z\,P\cosh(\alpha z)\,\alpha+2\,G\,P\cosh(\alpha z)\,\alpha$$

$$Za1 := 0$$

$$Za0 := -(G+C\,z)\,\big(-C\,z\,P\cosh(\alpha z)\,\alpha+2\,C\,y^2\,\mathrm{D}+2\,C\,x^2\,\mathrm{D}-G\,P\cosh(\alpha z)\,\alpha\big) \tag{232}$$

Hopf conditions

```
> GGG:=factor(solve(Xm,G));ZaGGG:=subs(G=GGG,Za);DDD:=factor(solve(ZaGGG,D));
  YYg:=factor(subs(G=GGG,D=DDD,Yg));
```

$$GGG := -C\,z-\frac{1}{2}\,P\cosh(\alpha z)\,\alpha$$

$$ZaGGG := -2\left(-C\,z-\frac{1}{2}\,P\cosh(\alpha z)\,\alpha\right)\mathrm{D}\,y^2\,C-2\,C^2\,z\,\mathrm{D}\,y^2-2\,\mathrm{D}\,x^2\,C\left(-C\,z-\frac{1}{2}\,P\cosh(\alpha z)\,\alpha\right)-2\,\mathrm{D}\,x^2\,C^2\,z+C^2\,z^2\,P\cosh(\alpha z)\,\alpha+\left(-C\,z-\frac{1}{2}\,P\cosh(\alpha z)\,\alpha\right)^2 P\cosh(\alpha z)\,\alpha+\Omega^2\,P\cosh(\alpha z)\,\alpha+2\left(-C\,z-\frac{1}{2}\,P\cosh(\alpha z)\,\alpha\right)C\,z\,P\cosh(\alpha z)\,\alpha$$

$$DDD := -\frac{1}{4}\,\frac{P^2\cosh(\alpha z)^2\,\alpha^2+4\,\Omega^2}{C\,(y^2+x^2)}$$

$$YYg := -\frac{1}{4}\,P^2\cosh(\alpha z)^2\,\alpha^2+3\,\Omega^2 \tag{233}$$

Oscillations occur when YYg>0
The Hopf critical surface is

```
> RRR:=solve(ZaGGG,Omega^2);HopfCrit:=factor(subs(G=GGG,Omega^2=RRR,Yg));Pcrit:=
  solve(HopfCrit,P);
```

Warning, solving for expressions other than names or functions is not recommended.

$$RRR := -C\,y^2\,\mathrm{D} - C\,x^2\,\mathrm{D} - \frac{1}{4}\,P^2\cosh(\alpha z)^2\,\alpha^2$$

$$HopfCrit := -3\,C\,x^2\,\mathrm{D} - 3\,C\,y^2\,\mathrm{D} - P^2\cosh(\alpha z)^2\,\alpha^2$$

$$Pcrit := \frac{2\sqrt{-3\,C\,y^2\,\mathrm{D} - 3\,C\,x^2\,\mathrm{D}}\;\mathrm{e}^{\alpha z}}{\left(\left(\mathrm{e}^{\alpha z}\right)^2+1\right)\alpha},\ -\frac{2\sqrt{-3\,C\,y^2\,\mathrm{D} - 3\,C\,x^2\,\mathrm{D}}\;\mathrm{e}^{\alpha z}}{\left(\left(\mathrm{e}^{\alpha z}\right)^2+1\right)\alpha} \quad \textbf{(234)}$$

Chiral - Brand invariants

```
> VORTICITY:=evalm(curl(AA,[x,y,z])):CURLx:=simplify(VORTICITY[1]);CURLy:=
  simplify(VORTICITY[2]);CURLz:=simplify(VORTICITY[3]);
```

$$CURLx := 2\,\mathrm{D}\,y - C\,y$$

$$CURLy := C\,x - 2\,\mathrm{D}\,x$$

$$CURLz := -2\,\Omega \quad \textbf{(235)}$$

```
> HELICITY:=collect(factor(innerprod(AA,VORTICITY)),Omega);
```

$$HELICITY := \left(-C\,y^2 - x^2\,C - 2\,P\sinh(\alpha z) - 2\,A\right)\Omega \quad \textbf{(236)}$$

```
> enstrophy:=collect(factor(innerprod(VORTICITY,VORTICITY)),Omega);E0:=factor
  (subs(Omega=0,enstrophy));
```

$$enstrophy := 4\,\mathrm{D}^2\,y^2 - 4\,C\,y^2\,\mathrm{D} + C^2\,y^2 + C^2\,x^2 - 4\,C\,x^2\,\mathrm{D} + 4\,\mathrm{D}^2\,x^2 + 4\,\Omega^2$$

$$E0 := \left(y^2 + x^2\right)\,(C - 2\,\mathrm{D})^2 \quad \textbf{(237)}$$

```
> BRAND:=innerprod(jacobian(AA,[x,y,z]),VORTICITY);
```

$$BRAND := \left[2\,G\,\mathrm{D}\,y - C\,y\,G + 2\,C\,y\,\mathrm{D}\,z - C^2\,y\,z - C\,x\,\Omega - 2\,\Omega\,x\,\mathrm{D},\ -2\,\mathrm{D}\,y\,\Omega - C\,y\,\Omega + C\,x\,G - 2\,G\,\mathrm{D}\,x + C^2\,x\,z - 2\,C\,x\,\mathrm{D}\,z,\ -2\,P\cosh(\alpha z)\,\alpha\,\Omega\right] \quad \textbf{(238)}$$

```
> stretch:=collect(factor(innerprod(BRAND,VORTICITY)),Omega);ST2:=factor(coeff
  (stretch,Omega^2));st0:=factor(subs(Omega=0,stretch));
```

$$stretch := 4\,G\,\mathrm{D}^2\,y^2 - 4\,G\,\mathrm{D}\,y^2\,C + C^2\,y^2\,G + 4\,C\,y^2\,\mathrm{D}^2\,z - 4\,C^2\,z\,\mathrm{D}\,y^2 + C^3\,y^2\,z + C^2\,x^2\,G - 4\,\mathrm{D}\,x^2\,C\,G + 4\,G\,\mathrm{D}^2\,x^2 + C^3\,x^2\,z - 4\,\mathrm{D}\,x^2\,C^2\,z + 4\,C\,x^2\,\mathrm{D}^2\,z + 4\,\Omega^2\,P\cosh(\alpha z)\,\alpha$$

$$ST2 := 4\,P\cosh(\alpha z)\,\alpha$$

$$st0 := \left(y^2 + x^2\right)\,(C - 2\,\mathrm{D})^2\,(G + C\,z) \quad \textbf{(239)}$$

```
> BI:=collect(factor(innerprod(BRAND,BRAND)),Omega);BI2:=factor(coeff(BI,
  Omega^2));BI1:=factor(coeff(BI,Omega));BI0:=(factor(subs(Omega=0,BI)));
```

$$BI := \left(4 D^2 x^2 + 4 D^2 y^2 + C^2 y^2 + C^2 x^2 + 4 P^2 \cosh(\alpha z)^2 \alpha^2 + 4 C x^2 D + 4 C y^2 D\right) \Omega^2 + C^4 y^2 z^2 + 4 C^2 x^2 D^2 z^2 - 4 C x^2 G^2 D + C^4 x^2 z^2 + 8 G D^2 x^2 C z - 4 G^2 D y^2 C + C^2 G^2 x^2 - 8 C^2 x^2 G D z - 4 C^3 y^2 D z^2 - 4 C^3 x^2 z^2 D + 2 C^3 y^2 G z + 4 C^2 y^2 D^2 z^2 + 2 C^3 x^2 G z + 4 G^2 D^2 x^2 + 8 G D^2 y^2 C z - 8 G D y^2 C^2 z + 4 G^2 D^2 y^2 + C^2 y^2 G^2$$

$$BI2 := 4 D^2 x^2 + 4 D^2 y^2 + C^2 y^2 + C^2 x^2 + 4 P^2 \cosh(\alpha z)^2 \alpha^2 + 4 C x^2 D + 4 C y^2 D$$

$$BI1 := 0$$

$$BI0 := \left(y^2 + x^2\right) (G + C z)^2 (C - 2 D)^2 \quad \textbf{(240)}$$

```
> Theta:=factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY));
```

$$\Theta := 8 \Omega \left(G C x^2 + C^2 z x^2 + G y^2 C + C^2 z y^2 + P^2 \sinh(\alpha z) \cosh(\alpha z) \alpha + P \cosh(\alpha z) \alpha A + P \cosh(\alpha z) \alpha y^2 D + P \cosh(\alpha z) \alpha x^2 D\right) \quad \textbf{(241)}$$

```
> Theta:=collect(factor(-2*innerprod(grad(phi,[x,y,z]),VORTICITY)),Omega);t1:=
  factor(coeff(Theta,Omega));
> t0:=factor(subs(Omega=1,Theta)/8);
```

$$\Theta := \left(8 G C x^2 + 8 C^2 z x^2 + 8 G y^2 C + 8 C^2 z y^2 + 8 P^2 \sinh(\alpha z) \cosh(\alpha z) \alpha + 8 P \cosh(\alpha z) \alpha A + 8 P \cosh(\alpha z) \alpha y^2 D + 8 P \cosh(\alpha z) \alpha x^2 D\right) \Omega$$

$$t1 := 8 G C x^2 + 8 C^2 z x^2 + 8 G y^2 C + 8 C^2 z y^2 + 8 P^2 \sinh(\alpha z) \cosh(\alpha z) \alpha + 8 P \cosh(\alpha z) \alpha A + 8 P \cosh(\alpha z) \alpha y^2 D + 8 P \cosh(\alpha z) \alpha x^2 D$$

$$t0 := G C x^2 + C^2 z x^2 + G y^2 C + C^2 z y^2 + P^2 \sinh(\alpha z) \cosh(\alpha z) \alpha + P \cosh(\alpha z) \alpha A + P \cosh(\alpha z) \alpha y^2 D + P \cosh(\alpha z) \alpha x^2 D \quad \textbf{(242)}$$

```
> ThetaHopf:=factor(subs(G=GGG,D=DDD,Theta));boundingHopf:=factor(9*E^2*z^4-4*C*
  z^3*E+6*B*E*z^2-4*C*z*B-4*A*C+2*y^2*C^2+4*Omega^2+2*x^2*C^2+B^2);
```

$$ThetaHopf := \frac{1}{C} \left(2 \Omega \left(-2 C^2 x^2 - 2 C^2 y^2 - P^2 \cosh(\alpha z)^2 \alpha^2 + 4 C A + 4 P \sinh(\alpha z) C - 4 \Omega^2\right) P \cosh(\alpha z) \alpha\right)$$

$$boundingHopf := 9 E^2 z^4 - 4 C z^3 E + 6 B E z^2 - 4 B C z - 4 C A + 2 C^2 y^2 + 4 \Omega^2 + 2 C^2 x^2 + B^2 \quad \textbf{(243)}$$

```
> H:=HELICITY;BBB:=z*solve(H,P);BBB*BBB;ZZZ:=Omega^2-3/4*(BBB*BBB);
```

$$H := \left(-C y^2 - x^2 C - 2 P \sinh(\alpha z) - 2 A\right) \Omega$$

$$BBB := -\frac{1}{2} \frac{z \left(C y^2 + x^2 C + 2 A\right)}{\sinh(\alpha z)}$$

$$\frac{1}{4} \frac{z^2 \left(C y^2 + x^2 C + 2 A\right)^2}{\sinh(\alpha z)^2}$$

$$ZZZ := \Omega^2 - \frac{3}{16} \frac{z^2 \left(C y^2 + x^2 C + 2 A\right)^2}{\sinh(\alpha z)^2} \quad \textbf{(244)}$$

```
> YYg;
```

$$-\frac{1}{4} P^2 \cosh(\alpha z)^2 \alpha^2 + 3 \Omega^2 \tag{245}$$