A4 SPACES
(Under Construction)

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1 Introduction

1.1 Differentiable Maps and A4 Spaces

A4 spaces are 4 dimensional varieties over a set of variables, say $\{x, y, z, Ct\}$, subject to a set of constraints that limit the topological features of the domain. In particular, the Cartan torsion and the Cartan curvature of such A4 domains are globally zero. Such constraints are features of "flat" spaces, like a Euclidean space, but the subspace features of A4 spaces, unlike a Euclidean space, need not be Euclidean. Different A4 spaces have different subspace structures. It was Shipov's idea to define the vacuum [ref1], not as a Euclidean empty space of four dimensions, with all substructures with euclidean properties, but as an A4 space that would support meaningful and physically different (and in particular, different from Euclidean) subspace structures.

To start the development of what is meant by A4 spaces, consider first those special (uniquely integrable) situations where a map $\phi : x^k \Rightarrow \xi^a$ and its differential $\mathrm{d}\phi : dx^k \Rightarrow \varpi^a = d\xi^a$ exist, and are given by the equations

$$\phi \quad : \quad x^k \Rightarrow \xi^a = \phi^a(x^k) \tag{1}$$

$$d\phi \quad : \quad dx^k \Rightarrow d\xi^a = |\varpi^a\rangle = \left[\partial\phi^a(x^k)/\partial x^j\ \right] \circ \left|dx^j\right\rangle \tag{2}$$

The mapping functions are assumed to be C2, such that the induced 1-forms (Vierbeins) ϖ^a are closed.

$$dd\xi^a = d\,|\varpi^a\rangle = 0. \tag{3}$$

The matrix of partial derivatives is defined as the Jacobian matrix of the map.

$$Jacobian\ matrix = \left[\partial\phi^a(x^k)/\partial x^j\ \right]\ = [J^a_k(x)] \tag{4}$$

The Jacobian matrix forms a linear connection between the differentials $d\xi^a$ and the differentials dx^j. The domain considered includes regions where the determinant of the Jacobian matrix is zero, and other regions where it is not zero. When the inverse Jacobian exists, the system is said to be an element of a group of transformations. When the inverse Jacobian does not exist, the system is said to form a semi-group of transformations. The system of differential forms $|\varpi^a\rangle$ defined by the construction in equation 2 is always closed: $d\,|\varpi^a\rangle = 0$.

If the domain is constrained by eliminating those regions where the determinant of the Jacobian matrix is zero, then over the remainder of the variety $\{x^k\}$ there exists a global inverse matrix (that is, over the variety $\{\mathbf{x}\} = \{x\}$ mod $det[J_k^a(x)] = 0$). By convention, this inverse matrix of functions will be defined as the Repere Mobile or Basis Frame on $\{\mathbf{x}\}$.

$$[\mathbf{F}(\mathbf{x})] = [J_k^a(\mathbf{x})]^{-1} = \left[e_a^k(\mathbf{x})\right] \tag{5}$$

Now consider an alternate approach, where the Basis Frame $[\mathbf{F}(\mathbf{x})]$ is given on a variety, but it is not known that such a basis frame is the inverse of a Jacobian matrix. That is, the integral map $\phi : x^k \Rightarrow \xi^a$ is not known. It will be presumed that inverse of the Basis Frame exists, and is defined as $[\mathbf{G}(\mathbf{x})] = [\mathbf{F}(\mathbf{x})]^{-1}$, such that $[\mathbf{G}(\mathbf{x})] \circ [\mathbf{F}(\mathbf{x})] = [\mathbf{1}]$. Such matrices with non-zero inverse define Projective transformations.

1.1.1 Integrable Vierbeins and Kinematic Vector Fields

The question arises: Can the Vierbein equations,

$$Vierbeins : |\varpi^a\rangle = [\mathbf{G}(\mathbf{x})] \circ \left|dx^j\right\rangle, \tag{6}$$

be uniquely integrated such that the inverse matrix of the Basis Frame is equivalent to a Jacobian matrix of some map? The answer is Yes, if the Vierbiens are exact, or at least satisfy the Frobenius integrability conditions,

$$\varpi^a(\mathbf{x})\hat{}d\varpi^a(\mathbf{x}) = 0. \tag{7}$$

When the Frobenius conditions are satisfied, then there exist integrating factors λ^a (a diagonal matrix $[\lambda^a(\mathbf{x})]$) for each 1-form $\varpi^a(\mathbf{x})$ such that

$$d\,|\lambda^a\varpi^a\rangle = d\{[\lambda^a(\mathbf{x})] \circ [\mathbf{G}(\mathbf{x})] \circ \left|\, dx^j\right\rangle\} \Rightarrow 0. \tag{8}$$

When integrable, the situation is said to be described by an integrable Lie group (for those matrices with determinant >0 which include the identity).

A useful case for physical applications is when the Vierbiens form a vector of functions times a single differential parameter, ds,

$$|\varpi^a\rangle = |W^\sigma ds\rangle = |W^\sigma\rangle\, ds, \tag{9}$$

such that

$$\left|\, dx^j\right\rangle = [F] \circ |W^\sigma\rangle\, ds = |V^\sigma(x,y,z,t)\rangle\, ds. \tag{10}$$

In otherwords, the differentials $\left|\, dx^j\right\rangle$ can be written in terms of a kinematic representation of a single parameter group, thereby permitting a kinematic Velocity field to be defined as a single parameter groujp. It follows that the existence of a kinematic Velocity field requires that the Vierbeins need not be exact, but all of them must be of Pfaff dimenension 2, at most. That is

$$|d\varpi^a\rangle = |dV^\sigma\hat{}ds\rangle \neq 0 \;\; but \;\; |\varpi^a\hat{}d\varpi^a\rangle = 0. \tag{11}$$

If the Vierbeins are not integrable, then the concept of a kinematic Velocity field does not exist.

1.1.2 Cartan Structural Equations

Independent from the questions of integrability of the Vierbeins, and solely from the existence of the differentiable inverse matrix, it follows that

$$d[\mathbf{F}] = [\mathbf{F}] \circ [\mathbf{C}] \qquad where \qquad [\mathbf{C}] = -d[\mathbf{G}] \circ [\mathbf{F}]. \tag{12}$$

The matrix $[\mathbf{C}]$ is defined as the Cartan connection (a matrix of 1-forms). Forming the exterior derivative preceeding equation yields the equation,

$$[\mathbf{F}] \circ \{[\mathbf{C}]\,\hat{}\,[\mathbf{C}] + d[\mathbf{C}]\} = 0. \tag{13}$$

By linearity, the bracket factor, which is a matrix of 2-forms, must vanish.

Multiplying the Vierbein definition by $[\mathbf{F}(\mathbf{x})]$ yields the equations

$$[\mathbf{F}(\mathbf{x})] \circ |\varpi^a\rangle = \left|dx^j\right\rangle. \tag{14}$$

Taking the exterior derivative on both sides (which must be zero as it is assumed that $ddx^j = 0$) yields

$$[\mathbf{F}] \circ \{[\mathbf{C}]\,\hat{}\,|\varpi^a\rangle + |d\varpi^a\rangle\} = 0 \tag{15}$$

By linearity, the bracket factor (which is a column vector of 2-forms) must vanish. Observe that this result does not depend upon the integrability of the Vierbeins.

The zero values of the two bracket equations (Cartan's equations of structure)

$$\text{Cartan Torsion 2-forms} \;\; \{[\mathbf{C}]\,\hat{}\,|\varpi^a\rangle + |d\varpi^a\rangle\} = 0 \tag{16}$$

$$\text{Cartan Curvature 2-forms} \;\; \{[\mathbf{C}]\,\hat{}\,[\mathbf{C}] + d[\mathbf{C}]\} = 0. \tag{17}$$

define an A4 space on the variety {x,y,z,t}.

Note that an integrable mapping defines a A4 space, but the conditions for an A4 space do not require that the system of Vierbeins is integrable. Hence,

Theorem 1 *An A4 space on a 4D variety is necessary but not sufficient to produce an integrable mapping. There exist A4 spaces which DO NOT support a kinematic Velocity Field. (The unique limit of $dx^k/ds = V^k(x,y,z,Ct)$ does not exist.)*

The first structural equation on an A4 space does lead to a relationship between Frobenius integrability of the Vierbeins and the Cartan connection (no sum on a):

$$\langle \varpi^a | \hat{} \, | d\varpi^a \rangle = - \langle \varpi^a | \hat{} \, [\mathbf{C}^a_b] \hat{} \, | \varpi^b \rangle \tag{18}$$

There exist manifolds for which the Cartan structural equations are not equal to zero. These manifolds are best studied as subspaces of higher dimensional euclidean spaces, for which the Cartan equations of structure are indeed zero, but for which the Cartan equations of structure for the subspaces are not necessarily zero. Such an approach will be applied herein to A4 spaces, by partitioning the A4 domain into a "spatial" 3D region and a "timelike" 1D domain. The Cartan curvature and the Cartan torsion of the subspaces need not be zero. For any Euclidean subspace, the Cartan curvature and the Cartan torsion 2-forms vanish.

1.2 Basis Frames that are NOT integrable

Consider an algebraic variety $\{x^i\} = \{x, y, z, Ct...\}$over a simply connected domain, with perhaps a boundary. As a starting point assume that there exists a matrix $[\mathbf{G}(\mathbf{x})]$ on the domain such that a set of 1-forms can be defined as:

$$|\varpi^a\rangle = [\mathbf{G}(\mathbf{x})] \circ |dx^j\rangle \tag{19}$$

The inverse of $[\mathbf{G}(\mathbf{x})]$ may not exist over the whole domain but it is still possible that $[\mathbf{G}(\mathbf{x})]$ is a Jacobian matrix, such that the system has an integral representation in terms of a set of functions. The criteria for integrability is that

$$\varpi^a \hat{} d\varpi^a = \langle \varpi^a | \hat{} \, | d\varpi^a \rangle = \langle dx^i | \hat{} \left[\widetilde{\mathbf{G}}(\mathbf{x}) \right] \hat{} d \left[\mathbf{G}(\mathbf{x}) \right] \hat{} \, | dx^j \rangle \Rightarrow 0 \tag{20}$$

for each value of the index a. Note that the 3-forms $\varpi^a \hat{} d\varpi^a$ can be constructed from the assumed matrix $[\mathbf{G}(\mathbf{x})]$, with no constraint on the determinant of $[\mathbf{G}(\mathbf{x})]$, (and no sum over a). The integrability requirement in index form for each index a is given by the equation

$$G^a_i \{\partial G^a_j / \partial x^k - \partial G^a_k / \partial x^j\} \equiv \{\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \bullet \mathbf{B}\}^a \Rightarrow 0 \tag{21}$$

The engineering "electromagnetic" notation on the RHS arises from the represention of *each* Vierbein as a 1-form of action having a Vector potential $\mathbf{A}$ for the spatial parts, and a scalar potential ϕ for the timelike part. Such a notation has helped the present author in the understanding of integrable vierbeins. For individual cases, the 4 component vector field, given by the engineering notation, defines what has been called the Topological Torsion vector. For a specific case, $G^a \Rightarrow A^a$ and $F^a = dA^a$ such that $\mathbf{B} = curl\mathbf{A}$, and $\mathbf{E} = -grad\phi - \partial\mathbf{A}/\partial t$. Then $A^a \hat{} dA^a = A^a \hat{} F^a \equiv \{\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \bullet \mathbf{B}\}^a \Rightarrow 0$ for each integrable Vierbein 1-form. It follows that the divergence of the Topological Torsion vector for integrable Vierbeins must be zero, $\mathbf{E} \bullet \mathbf{B} = 0$. Methods of constructing the matrix $[\mathbf{G}(\mathbf{x})]$ will be described below.

Next consider what is known as basis frame of functions $\left[\mathbf{F}(x^i)\right]$ at an arbitrary point p of the domain . The Basis Frame is the assumed inverse of the matrrix $[\mathbf{G}(\mathbf{x})]$.

Depending on the choice of functions that are used to construct the $[\mathbf{G}(\mathbf{x})]$, the determinant of $[\mathbf{G}(\mathbf{x})]$ may or may not be zero at the point p. However, now refine the domain of definition to be limited to those sets for which the determinant of the matrix of functions $[\mathbf{G}(\mathbf{x})]$ is NOT zero, and therefor an inverse matrix of functions, $[\mathbf{F}(\mathbf{x})]$, exists. The restricted domain is where $\mathbf{x} = \{x^i : \det\left[\mathbf{F}(x^i)\right] \neq 0 \}$. The original, simply connected, domain $\{x^i\}$ now can become $\{\mathbf{x}\}$ composed of disconnected components, each of which may or may not be simply connected.

The choice of a non-zero determinant changes the topology of the original domain, but now the basis frame $[\mathbf{F}(\mathbf{x})]$ is globally defined over the new topology. Further assume that an origin can be defined such that the differentials of a position vector make sense.

$$|d\mathbf{R}\rangle = [\mathbf{I}] \circ \left| \begin{matrix} dx \\ dy \\ dz \\ d(Ct) \end{matrix} \right\rangle = [\mathbf{F}] \circ [\mathbf{G}] \left| \begin{matrix} dx \\ dy \\ dz \\ d(Ct) \end{matrix} \right\rangle = [\mathbf{F}] \circ |\varpi^a\rangle \tag{22}$$

1.3 Types of Basis Frames

Accept the existence of a basis frame $[\mathbf{F}]$ of functions defined at a points p on a 4 dimensional variety, where the determinant does not vanish. Then the existence of the inverse matrix permits the basis frame matrix to be considered as a projective transformation, or an element of the general linear group. Constraints placed upon the functions that make up the basis frame can be used to identify equivalences classes of Basis Frames which are subgroups of the general linear group. For a four dimensional variety, the basis frame $[\mathbf{F}]$ consists of 16 independent functions constrained by the condition that the determinant at any point p is not zero. Hence the set of points, det $[\mathbf{F}] = 0$, define a hypersurface of points on a 16 dimensional space which are excluded.

Next consider those invertible matrices of the form

$$[p\mathbf{A}] = \begin{bmatrix} F_1^1 & F_2^1 & F_3^1 & U \\ F_1^2 & F_2^2 & F_3^2 & V \\ F_1^3 & F_2^3 & F_3^3 & W \\ 0 & 0 & 0 & E \end{bmatrix} \tag{23}$$

where the three elements on the bottom row have been excluded (set to zero). Matrices of this format are elements of an equivalence class of matrices defined as the Affine subgroup of the general linear group. The Affine matrices have

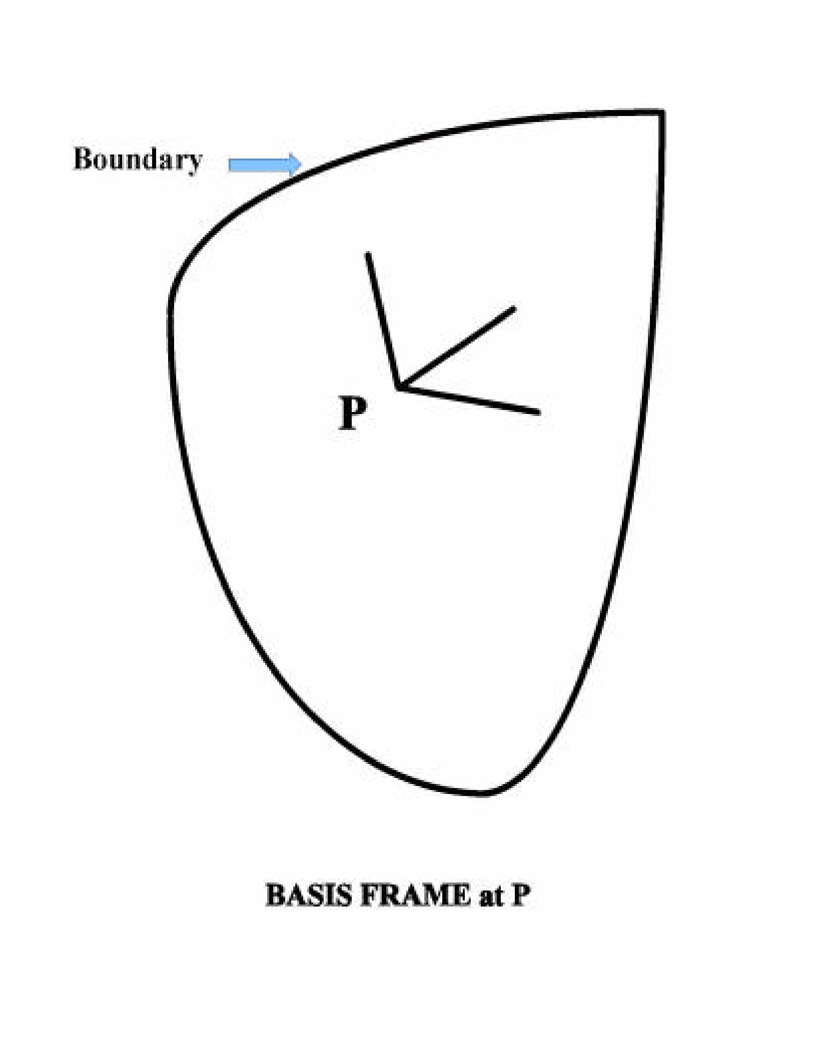

Figure 1:

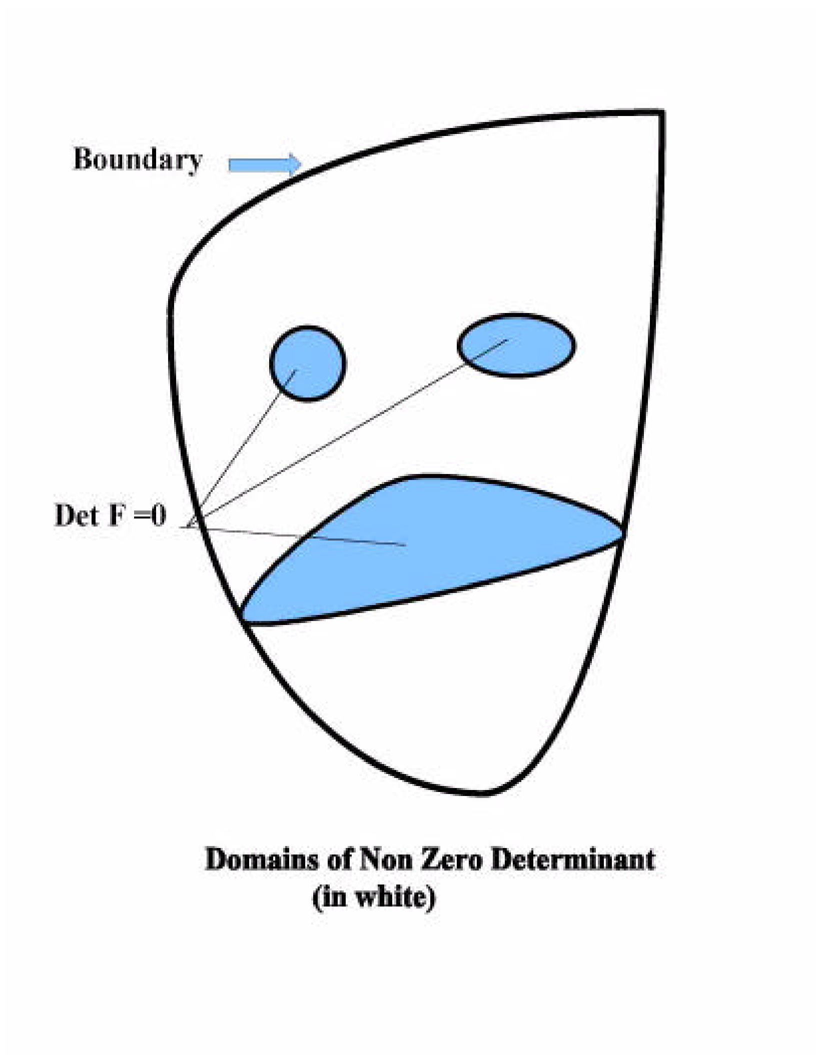

Figure 2:

13 free parameters, subjected in the case at hand to the additional non-zero determinant condition. (It is a special case to consider only those matrices of real functions which are of determinant =1, reducing the number of free parameters to 12) The matrix product of any two elements of the Affine group produce another matrix of the same format. The inverse matrices of the group also have the same format. Affine matrices are said to preserve parallelism. (Problem: Prove pAffine matrices preserve parallelism.)

Note that the matrices of the form

$$[w\mathbf{A}] = \begin{bmatrix} F_1^1 & F_2^1 & F_3^1 & 0 \\ F_1^2 & F_2^2 & F_3^2 & 0 \\ F_1^3 & F_2^3 & F_3^3 & 0 \\ P & Q & R & E \end{bmatrix} \tag{24}$$

form a different sub group (of the general linear group) of matrices with 13 free parameters (or 12 parameters for fixed determinant). These two distinct groups are defined herein as the "particle Affine group" $[p\mathbf{A}]$ and the "wave Affine group" $[w\mathbf{A}]$ for reasons that will become apparent below. In both cases, the determinant = 0 case is excluded. Hence the element E cannot be zero, and the 3x3 subdeterminant ($Vol3$) is not zero. Note that the transpose of $[p\mathbf{A}] \subset [w\mathbf{A}]$ and the transpose of $[w\mathbf{A}] \subset [p\mathbf{A}]$, but the groups are mutually exclusive. The two groups have an intersection for matrices of the form,

$$[e\mathbf{A}] = \begin{bmatrix} F_1^1 & F_2^1 & F_3^1 & 0 \\ F_1^2 & F_2^2 & F_3^2 & 0 \\ F_1^3 & F_2^3 & F_3^3 & 0 \\ 0 & 0 & 0 & E \end{bmatrix}. \tag{25}$$

These matrices form a subgroup of 10 free parameters. With a few more constraints, this group has been used to define the Euclidean group, which preserves both parallelism and orthogonality.

Other types of constraints can be placed upon the Frame matrices. For example the 4D frame matrix might be constrained to be symmetric. Hence there are only 10 free parameters (that must also satisfy a non-zero determinant condition). If the 3x3 subspace of the (p or w) Affine group is constrained to be symmetric, then the modified Affine Frame matrix also has 10 free parameters (matrix elements). Note that these three 10 parameter groups are not the same.

1.3.1 Index Constraints

Another method of forming equivalence classes of Frame matrices is to assert that they must preserve the index of a quadratic form. That is

$$[\mathbf{F}]^{transpose} \circ [\boldsymbol{\eta}_i] \circ [\mathbf{F}] = [\boldsymbol{\eta}_i] \tag{26}$$

where $[\boldsymbol{\eta}_i]$ is a diagonal matrix with plus or minus unit values. The euclidean index matrix, $[\boldsymbol{\eta}_0]$ is the identity matrix (all matrix elements are +1). The index is defined as the number, i, of negative entries. On a 4D variety, a

Minkowski matrix of index $i = 1$, $[\boldsymbol{\eta}_1]$, has one negative entry. A Minkowski matrix of index $i = 3$, $[\boldsymbol{\eta}_3]$, has one positive entry. Using the index matrix to construct a quadratic form follows the matrix rules

$$\langle \mathbf{X}| \circ [\boldsymbol{\eta}_1] \circ |\mathbf{X}\rangle = +X^2 + Y^2 + Z^2 - (Ct)^2, \tag{27}$$

or

$$\langle \mathbf{X}| \circ [\boldsymbol{\eta}_3] \circ |\mathbf{X}\rangle = -X^2 - Y^2 - Z^2 + (Ct)^2, \tag{28}$$

The two quadratic forms are distinct in a topological sense. On a real space of 4D the non-zero quadratic forms with a index 1 form a connected surface of 1 sheet. For a index matrix of one positive unity, the non-zero quadratic forms imply a disconnected surface of three sheets. The two quadratic forms have an intersection on their zero sets.

Consider a function $\Phi(x, y, z, Ct)$ whose zero set forms a hypersurface upon which the solutions to Maxwell's equations are not unique. The surface represents propagating discontinuities in the electric field amplitudes. Fock and Luneberg exploit this concept as a definition of an electromagnetic signal. A simple format of this contraint is given by the quartic Eikonal expression, with quadratic factors

$$\pm(\partial\Phi/\partial x)^2 \pm (\partial\Phi/\partial y)^2 \pm (\partial\Phi/\partial z)^2 \mp ((\partial\Phi/\partial Ct))^2 = 0. \tag{29}$$

(Problem: Work out the zero determinant condition for an arbitrary constitutive tensor, and relate to the Eikonal as a Kummer surface).

It is of interest to physical systems to study those cases where the Frame matrix is such that the propagating discontinuity is preserved. Fock has demonstrated that the Lorentz transformation is the only linear transformation group that preserves the index i = 1, or i = 3 of a quadratic form.

Suppose that the basis Frame is of the equivalence class such that the quadratic form of index zero is an invariant. Then

$$[\mathbf{F}]^{transpose} \circ [\mathbf{I}] \circ [\mathbf{F}] = [\mathbf{I}] \tag{30}$$

from which it follows that

$$[\mathbf{F}]^{transpose} = [\mathbf{F}]^{inverse}. \tag{31}$$

Such matrices are defined to be elements of the Orthogonal Group.

Matrices such that

$$[\mathbf{F}]^{transpose} \circ [\boldsymbol{\eta}_1] \circ [\mathbf{F}] = [\boldsymbol{\eta}_1] \tag{32}$$

are defined to be elements of the Lorentz group.

1.3.2 Integrable Affine systems

If the groups described above are also integrable, further constraints must be placed upon the functions of the basis frame. For example, the Vierbiens of the format of the particle-Affine group form a set of 1-forms,

$$|\varpi^a\rangle = [p\mathbf{A}(\mathbf{x})] \circ |dx^j\rangle \tag{33}$$

$$= \begin{bmatrix} F_1^1 & F_2^1 & F_3^1 & U \\ F_1^2 & F_2^2 & F_3^2 & V \\ F_1^3 & F_2^3 & F_3^3 & W \\ 0 & 0 & 0 & E \end{bmatrix} |dx^j\rangle \tag{34}$$

$$= \left| \begin{matrix} \{[\text{-}\ ] \bullet |dx^j\rangle + V^j dt\} \\ \{Edt\} \end{matrix} \right\rangle \tag{35}$$

To be integrable to exactness, the exterior derivative of each vierbein must vanish. In particular, the expansion E must be constant or a function of t only. (Uniform expansion). In the more general case, the "velocity" field, $\mathbf{V}$, need not uniform.

In the more general (integrable) Affine case, det[-] $\neq 0$, and the three 1-forms with components determined by the elements of [-] must be closed (to within a factor):

$$\sigma^k = \{A_j^{\square} dx^j + V^k dt\} \; such \, that \; d\sigma^k \Rightarrow 0. \tag{36}$$

There are many affine basis frames that are not uniquely integrable.

1.3.3 Semi groups of integrable Frames without inverse

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1.3.4 Groups of invertible frames which are not integrable

Under construction.

. The non-zero determinant constraint permits the construction of the inverse matrix of the Basis Frame. The inverse of $[\mathbf{F}(\mathbf{x})]$ permits the definition of one a set of Vierbeins, or differential 1-forms by means, of the equation:

$$|\varpi^a\rangle = [\mathbf{G}] \circ |d\mathbf{R}\rangle = \left| \begin{matrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \omega \end{matrix} \right\rangle \tag{37}$$

There exists another (usually different) set of Vierbeins defined by the transpose (or pullback) of the Frame Matrix.

$$|p^a\rangle = [\mathbf{F}]^{transpose} \circ |d\mathbf{R}\rangle = \left| \begin{array}{c} p^1 \\ p^2 \\ p^3 \\ s \end{array} \right\rangle \tag{38}$$

This second set of Vierbeins exists even if the determinant of the Frame matrix is zero. A necessary condition for the integrability of the Vierbiens is that $d\,|p^a\rangle = 0$. This result is valid whether or not an inverse Jacobian exits. It is to be associated with the existence of a semi group.

The two sets of vierbeins are the same to within scaling factors when the Frame matrix is an element of the orthogonal group. They are identical when the Frame matrix is an element of the orthonormal group, such that $[\mathbf{F}]^{transpose} = [\mathbf{F}]^{-1} = [\mathbf{G}(\mathbf{x})]$

The two sets of vierbeins may exhibit dynamical behavior that are different from one another. That is the Vierbiens defined by the transpose may precess about the Vierbeins defined by the inverse.

Either set of Vierbeins may or may not be integrable in the sense of Frobenius. That is, each 1-form that makes up the Vierbein set may not be an exact perfect differential, or even proportional to an exact perfect differential of a single function. In general

$$\varpi^a \hat{}\, d(\varpi^a) \neq 0 \qquad p^a \hat{}\, d(p^a) \neq 0.$$

However, the criteria for the integrability of the transpose Vierbeins is more general than the integrability for the inverse Vierbeins, for the transpose condition is valid for semi groups.

1.3.5 The $[\mathbf{F}]^{transpose}$ Vierbein set

The transpose Vierbein set has the advantages that displacements in domains where the determinant of the Frame matrix is zero are *not* excluded. As

$$d\,|p^a\rangle = \{d[\mathbf{F}]^{transpose}\}\hat{}\ |d\mathbf{R}\rangle = \{d\left[\widetilde{\mathbf{F}}\right]\}\hat{}\ |d\mathbf{R}\rangle\,, \tag{39}$$

the Frobenius condition of integrability amounts to

$$p^a \hat{}\, d(p^a) = \langle d\mathbf{R}|\,\hat{}\,[\mathbf{F}]\hat{}\,d\,|p^a\rangle = \langle d\mathbf{R}|\,\hat{}\,[\mathbf{F}]\hat{}\,\{d\left[\widetilde{\mathbf{F}}\right]\}\hat{}\ |d\mathbf{R}\rangle \Rightarrow 0. \tag{40}$$

The global integrability condition requires that matrix $[\mathbf{F}]\hat{}\{d\left[\widetilde{\mathbf{F}}\right]\}$ be anti-symmetric (or zero). Hence, for integrability of the tranpose (pullback) vierbeins, it is necessary that

$$Frobenius\ Integrability\ condition:\ [\mathbf{Fr}] = [\mathbf{F}]\hat{}\{d\left[\widetilde{\mathbf{F}}\right]\} = -\{d[\mathbf{F}]\}\hat{}\ \left[\widetilde{\mathbf{F}}\right]. \tag{41}$$

All of these functions can be computed explicity for C1 systems, given a representative of the Frame Field. The matrix $[\mathbf{Fr}(x)]$ defined over the whole domain is called herein the the Frobenius matrix of 1-forms.

1.3.6 The $[\mathbf{F}]^{-1}$ Vierbein set

The inverse vierbein set is integrable when

$$\varpi^a\hat{}d(\varpi^a) = \langle d\mathbf{R}|\hat{}\left[\widetilde{\mathbf{G}}\right]\hat{}d\,|p^a\rangle = \langle d\mathbf{R}|\hat{}\left[\widetilde{\mathbf{G}}\right]\hat{}\{d[\mathbf{G}]\hat{}\,|d\mathbf{R}\rangle \Rightarrow 0 \tag{42}$$

Hence, the matrix $\left[\widetilde{\mathbf{G}}\right]\{d[\mathbf{G}]\}$ must be antisymmetric or zero. This condition requires that the inverse Frame Field exist, and therefor is limited to displacements in those regions where the determinant of $[\mathbf{F}(\mathbf{x})]$ is not zero. In that which follows in this section, the domain is assumed to limited to the non-zero determinant subspace. Note that for orthonormal frames, the two types of Vierbeins are the same. For conformal Frame fields, the two sets of vierbeins are proportional to one another; that is $\varpi^a = \lambda^a(x)\,p^a$. and $\varpi^a\hat{}p^a = 0$.

However, if the Frame field is not orthogonal then the two sets of vierbeins have an intersection, $\varpi^a\hat{}p^a \neq 0$. It is at this point that the concept of a twisted (gyroscopic-like) space arises. Evolution of one set of vierbeins can gyrate about the other set of vierbeins.

1.3.7 The Cartan Connection

Over the global domain of $\det[\mathbf{F}(\mathbf{x})] \neq 0$, it is possible to compute the exterior differentials of the functions that make up the basis frame, and find that (using $[\mathbf{F}(\mathbf{x})]\circ[\mathbf{F}(\mathbf{x})]^{-1} = [\mathbf{F}(\mathbf{x})]\circ[\mathbf{G}(\mathbf{x})] = [\mathbf{I}]$)

$$d[\mathbf{F}(\mathbf{x})] = [\mathbf{F}(\mathbf{x})]\circ[\mathbf{C}(\mathbf{x})]\ \textit{where} \tag{43}$$

$$[\mathbf{C}(\mathbf{x})] = -d[\mathbf{G}(\mathbf{x})]\circ[\mathbf{F}(\mathbf{x})]\,. \tag{44}$$

The matrix $[\mathbf{C}(\mathbf{x})]$ is the Cartan connection matrix of 1-forms. The Cartan matrix is a matrix of 1-forms defined over the restricted space $\{\mathbf{x}\}$ where the Frobenius matrix $[\mathbf{Fr}(x)]$ is defined over the whole domain. The fundamental idea is that the process of exterior differentiation of the basis vectors of the Frame is closed, on the restricted domain. The differentials of any basis vector, on the restricted domain, is a linear combination of all of the elements of the basis Frame, whatever it may be.

Application of the exterior derivative to the equation for $|d\mathbf{R}\rangle$ and $d[\mathbf{F}]$, and assuming that all functions are twice differentiable, yields the Cartan structural equations:

$$[\mathbf{F}]\circ\{\,|d\varpi^a\rangle + [\mathbf{C}]\circ|\varpi^a\rangle\} = 0, \tag{45}$$

and

$$[\mathbf{F}] \circ \{d[\mathbf{C}] + [\mathbf{C}] \,\hat{}\, [\mathbf{C}]\} = 0. \tag{46}$$

As $[\mathbf{F}]$ is never zero, then the factors must be zero, and the final result is that

$$\{\, |d\varpi^a\rangle + [\mathbf{C}] \circ |\varpi^a\rangle\} = 0 \qquad \textit{and} \qquad \{d[\mathbf{C}] + [\mathbf{C}] \,\hat{}\, [\mathbf{C}]\} = 0. \tag{47}$$

These equations define an An, and in this case a A4 space.

Note that by partitioning $[\mathbf{F}]$into "interior" basis vectors (columns) and "exterior" basis vectors (columns), induces partitions of $[\mathbf{C}]$and $|\varpi^a\rangle$ such that

$$[\mathbf{C}] = \begin{bmatrix} \mathbf{\Gamma} & \gamma \\ h & \text{-} \end{bmatrix} \qquad \textit{and} \qquad |\varpi^a\rangle = \left| \begin{matrix} \varpi \\ \omega \end{matrix} \right\rangle \tag{48}$$

The structural equations on the partitions lead to Cartan structural equations for the partitions of the form

$$|d\varpi\rangle + [\mathbf{\Gamma}] \circ |\varpi\rangle = |\mathbf{\Sigma}\rangle \equiv -\omega\,\hat{}\, |\boldsymbol{\gamma}\rangle \tag{49}$$

and

$$d[\mathbf{\Gamma}] + [\mathbf{\Gamma}] \,\hat{}\, [\mathbf{\Gamma}] = [\mathbf{\Theta}] \equiv -\, |\boldsymbol{\gamma}\rangle \,\hat{}\, \langle \mathbf{h}| \tag{50}$$

as well as a third structural equation,

$$|d\gamma\rangle + [\mathbf{\Gamma}] \circ |\gamma\rangle = |\mathbf{\Phi}\rangle \equiv -\text{-} \,\hat{}\, |\gamma\rangle \tag{51}$$

The non-zero right hand sides of these partitioned equations of structure are known as

1. The Cartan Torsion 2-forms, a vector of 2-forms $|\mathbf{\Sigma}\rangle$, often related to affine motions of particles.

2. The Cartan Curvature 2-forms, a matrix of 2-forms $[\mathbf{\Theta}]$.

Apparently Cartan was led to the curvature 2-forms from differential geometry. However, it would appear that the torsion 2-forms came from "out of the blue". (Eisenhart claims that it was after his work on paths generated by continuous transitive groups that Cartan and Schouten developed the "torsion 2-form" ideas. I do not have a reference as to when the concept of Torsion 2-forms appeared.)

However, there is another equation of structure that becomes evident from the partition method:

3. The Expansion Torsion 2-forms, a vector of 2-forms, $|\mathbf{\Phi}\rangle$, often related to expanding twisted wave motions. (If the exterior basis vectors that make up $[\mathbf{F}]$ are normalized, then the Expansion (or Wave Affine) torsion 2-forms vanish. The Gauss map, for example, assumes that the vector to a surface is a unit vector. This assumption will make the Expansion two forms vanish, as the assumption implies that in the partition, - $= 0$.

For an A4 space all of the RHS 2-forms vanish. However, the subspaces of A4 do not have to be A3 or A2 subspaces, and these facts allow the basis frames to be put into classes. The classes can be described in terms of the Pfaff dimension, or class, of each of the Vierbeins.

2 Choosing a Basis Frame

If the basis matrices of functions are further restricted to those which have positive determinant, then the basis matrices have an identity element, and an inverse, and therefore the choice of functions leads to an element of the general linear group $GL4^+$. A particular choice of functions may form a matrix subgroup of $GL4^+$. This procedure has become quite popular in what is called Gauge Theory.

2.0.8 Three special Basis Frames

Of particular interest to certain physical situations are those subgroups that preserve the Minkowski metrics, and the Euclidean metric, at least to within a conformal factor. The projective orthogonal subgroup [**O**] preserves the Euclidean metric to within a factor, and the projective Lorentz group [**L**] preserves the Minkowski metric to within a factor, λ. That is for

$$[\mathbf{E}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad and \quad [\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{52}$$

the following equations are true:

$$\lambda\,[\mathbf{E}] = [\mathbf{O}]^{transpose} \circ [\mathbf{E}] \circ [\mathbf{O}] \qquad and \qquad \lambda\,[\mathbf{M}] = [\mathbf{L}]^{transpose} \circ [\mathbf{M}] \circ [\mathbf{L}]\,. \tag{53}$$

When $\lambda = 1$, the Expansion Torsion 2-forms, $|\mathbf{\Phi}\rangle$ vanish. Hence a fixation on the Special subgroups ($det = 1$) such as SO3, U, etc. eliminate some possible torsion effects.

Another group of interest on the 4D variety is the Symplectic group which statisfies the structural equations

$$[\mathbf{K}] = [\mathbf{S}]^{transpose} \circ [\mathbf{K}] \circ [\mathbf{S}] \tag{54}$$

where

$$[\mathbf{K}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{55}$$

2.0.9 Lorentz group preserves Singular Solutions to Electrodynamics

The point of the projective Lorentz group is that if the Eikonal PDE

$$(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2 - (\partial\phi/\partial ct)^2 = 0 \tag{56}$$

is zero in one frame, it is zero in all frames that can be mapped to that frame by a projective Lorentz transformation. V. Fock has shown that the Eikonal equation is satisfied by those points of the variety upon which the amplitudes of the solutions to Maxwell's equations are not uniquely defined. In otherwords, the Eikonal equation determines the point set of (propagating) discontinuities of the PDE's that represent the electromagnetic field.

If the Eikonal PDE is preserved, then a propagating singularity in one frame is viewed as a propagating singularity in another frame. That is what V. Fock and Luneberg mean by an "electromagnetic signal". A signal (discontinuity) to one Lorentz observer is a signal to a second Lorentz observer. The electric field in front of the propagating discontinuity is zero, but in back of the discontinuity it is finite – dark in front, bright in back. Note that the projective Lorentz transformation is not necessarily linear.

2.1 A Basis Frame from an Action Principle

Suppose that the existence of a physical system implies that the variety supports a 1-form of Action, A. Then it is possible to construct algebraically (in many ways) a set of N-1 (equal to 3 in the examples) linearly independent associated vectors $\mathbf{e}^i$ such that

$$i(\mathbf{e}^i)A = 0. \tag{57}$$

Note that these associated vectors $\mathbf{e}^i$ can be scaled by arbitrary functions. These three 4 component vectors can make up the interior partition of a Basis Frame matrix

$$[\mathbf{F}] = \begin{bmatrix} \mathbf{e} & \mathbf{n} \end{bmatrix} \tag{58}$$

The exterior vector $\mathbf{n}$ can be computed by the Grassman adjoint method, and will be proportional to the components that make up the 1-form, A. The construction is such that $\det[\mathbf{F}] \neq 0$.

Hence the existence of a 1-form on a variety will induce a basis frame, and therefor an A4 space. It follows that the Action 1-form can also serve as the integrand in the calculus of variations. Therefore, a Lagrange least action principle can induce an A4 space.

2.1.1 An Example - A Monge surface in A4

Consider the 1-form of Action defined as the exterior differential of "potential" functions, $\phi(x, y, z, s)$ such that

$$A = (\partial\phi/\partial x)dx + (\partial\phi/\partial y)dy + (\partial\phi/\partial z)dz + (\partial\phi/\partial s)ds \tag{59}$$

The Pfaff dimension of this 1-form is unity. Find the associated vectors relative to the 1-form as

$$[\mathbf{F}] = \begin{bmatrix} (\partial\phi/\partial s) & 0 & 0 & -(\partial\phi/\partial x) \\ 0 & (\partial\phi/\partial s) & 0 & -(\partial\phi/\partial y) \\ 0 & 0 & (\partial\phi/\partial s) & -(\partial\phi/\partial z) \\ (\partial\phi/\partial x) & (\partial\phi/\partial y) & (\partial\phi/\partial z) & (\partial\phi/\partial s) \end{bmatrix}$$

More examples to follow (08/23/99)

3 APPENDIX A

3.1 Examples in Two dimensions

Consider a two dimensional matrix $[G]$ of 4 elements over a base space of two dimensions, $\{u, v\}$. Assume that an inverse $[F]$ exists such that

$$[F] \circ [G] = [I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{60}$$

For

$$[F] = \begin{bmatrix} a(u,v) & b(u,v) \\ c(u,v) & e(u,v) \end{bmatrix} \tag{61}$$

the inverse exists when $\det[F] = 1/\rho(u,v) = (ae - bc) \neq 0$. The collection of such matrices of functions forms the General Linear group, with two disconnected components, $\det[F] > 0$ and $\det[F] < 0$. The component with $\det[F] > 0$ contains the identity, and therefor forms a subgroup. The collection of 1-forms,

$$\left| \begin{matrix} \sigma \\ \omega \end{matrix} \right\rangle = [G] \circ \left| \begin{matrix} du \\ dv \end{matrix} \right\rangle \tag{62}$$

$$= \rho(u,v) \begin{bmatrix} e(u,v) & -b(u,v) \\ -c(u,v) & a(u,v) \end{bmatrix} \circ \left| \begin{matrix} du \\ dv \end{matrix} \right\rangle \tag{63}$$

is always integrable in the sense of Frobenius, for the forms depend only on two variables, $\{u, v\}$. (This is usually not true in higher dimensions).

For an arbitrary repere mobile (Basis Frame) in two dimensions, the Cartan Connection matrix is defined as

$$d\,[F] = [F] \circ [C] \tag{64}$$

where

$$[C] = \{-d\,[G] \circ [F]\} \tag{65}$$

$$= \begin{bmatrix} d(\rho e(u,v)) & -d(\rho b(u,v)) \\ -d(\rho c(u,v)) & d(\rho a(u,v)) \end{bmatrix} \circ \begin{bmatrix} a(u,v) & b(u,v) \\ c(u,v) & e(u,v) \end{bmatrix} \tag{66}$$

Consider the affine subgroup of the GL group, a group which is represented by matrices of the type

$$[pAffine] = \begin{bmatrix} a(u,v) & b(u,v) \\ 0 & e(u,v) \end{bmatrix}. \quad (67)$$

The fixed point of the pAffine map is $\left| \begin{array}{c} 1/a \\ 0 \end{array} \right\rangle$.

Now consider the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (68)$$

and the groups that preserve these structures:

Orthogonal Frames: $[O]$ $[E] = [O]^{transpose} \circ [E] \circ [O]$ $AC + BD = Inv$

"Lorentz" Frames: $[L]$ $[M] = [L]^{transpose} \circ [M] \circ [L]$ $AC - BD = Inv$

Symplectic Frames: $[S]$ $[K] = [S]^{transpose} \circ [K] \circ [S]$ $AD - BC = Inv$

Expansion Frames: $[Z]$ $[J] = [Z]^{transpose} \circ [J] \circ [Z]$ $AD + BC = Inv$

where [A,B] and [C,D] are two arbitrary vectors

These are subgroups of the General Linear group, and have specific functional forms such as to satisfy the structural constraints. The subgroups have two components, the connected component with determinant > 0 that contains the identity, and the second component with determinant < 0

The orthogonal matrices of determinant plus and minus 1 have the form

$$[O]^{+} = \begin{bmatrix} \cos(\theta(u,v)) & \sin(\theta(u,v)) \\ -\sin(\theta(u,v)) & \cos(\theta(u,v)) \end{bmatrix}, \quad (69)$$

$$[O]^{-} = \begin{bmatrix} \cos(\theta(u,v)) & \sin(\theta(u,v)) \\ \sin(\theta(u,v)) & -\cos(\theta(u,v)) \end{bmatrix} \quad (70)$$

where the matrices of determinant minus one are reflexive (involutary)

The Minkowski matrices of determinant plus and minus 1 have two formats

$$[L]^{+} = \begin{bmatrix} 1/\cos(\theta(u,v)) & \tan(\theta(u,v)) \\ \tan(\theta(u,v)) & 1/\cos(\theta(u,v)) \end{bmatrix}, \quad (71)$$

$$[L]^{-} = \begin{bmatrix} 1/\cos(\theta(u,v)) & \tan(\theta(u,v)) \\ -\tan(\theta(u,v)) & -1/\cos(\theta(u,v)) \end{bmatrix} \quad (72)$$

$$[L]^{+} = \begin{bmatrix} \cosh(\theta(u,v)) & \sinh(\theta(u,v)) \\ \sinh(\theta(u,v)) & \cosh(\theta(u,v)) \end{bmatrix}, \quad (73)$$

$$[L]^{-} = \begin{bmatrix} \cosh(\theta(u,v)) & \sinh(\theta(u,v)) \\ -\sinh(\theta(u,v)) & -\cosh(\theta(u,v)) \end{bmatrix} \quad (74)$$

and the matrices of determinant minus one are reflexive (involutary).

The only constraint on the Symplectic matrices in two variables is that they be of determinant +1

$$[S]^{+1} = \begin{bmatrix} a & b \\ c & (1+bc)/a \end{bmatrix} \tag{75}$$

Hence the $[O]^{+1}$, and $[L]^{+1}$ are subgroups of the Symplectic group in two dimensions. The Symplectic group on two dimensions has three parameters, while the Orthogonal group and the Lorentz group have one free parameter, $\theta(u,v)$. (As a note, the three groups $[O(2n,\mathrm{R})]$, $[S(2n,\mathrm{R})]$, $[GL(n,\mathrm{C})]$ have a common intersection which is the Unitary group $[U(n,\mathrm{C})]$. This works in even dimensional spaces)

3.1.1 Orthonormal Subgroup in 2D

For $[O]^{+1}$, the inverse matrix with determinant $= +1$ is

$$[G] = \begin{bmatrix} \cos(\theta(u,v)) & -\sin(\theta(u,v)) \\ \sin(\theta(u,v)) & \cos(\theta(u,v)) \end{bmatrix}, \tag{76}$$

such that the Cartan matrix becomes:

$$[C] = \{- \begin{bmatrix} -\sin(\theta(u,v))d\theta & -\cos(\theta(u,v))d\theta \\ \cos(\theta(u,v))d\theta & \sin(\theta(u,v))d\theta \end{bmatrix} \circ \begin{bmatrix} \cos(\theta(u,v)) & \sin(\theta(u,v)) \\ -\sin(\theta(u,v)) & \cos(\theta(u,v)) \end{bmatrix} \} \tag{77}$$

$$= \begin{bmatrix} 0 & d\theta \\ -d\theta & 0 \end{bmatrix} \tag{78}$$

Hence $d\,[C] = 0$, and $[C]\,\hat{}\,[C] = 0$, proving that the Cartan curvature 2-forms vanish.

$$[\Theta] = d\,[C] + [C]\,\hat{}\,[C] = 0 \tag{79}$$

The Vierbiens have the format

$$\left| \begin{matrix} \sigma \\ \omega \end{matrix} \right\rangle = \begin{bmatrix} \cos(\theta(u,v)) & -\sin(\theta(u,v)) \\ \sin(\theta(u,v)) & \cos(\theta(u,v)) \end{bmatrix} \circ \left| \begin{matrix} du \\ dv \end{matrix} \right\rangle \tag{80}$$

with

$$\left| \begin{matrix} d\sigma \\ d\omega \end{matrix} \right\rangle = \begin{bmatrix} -\sin\theta(u,v))d\theta & -\cos\theta(u,v))d\theta \\ \cos(\theta(u,v))d\theta & -\sin(\theta(u,v))d\theta \end{bmatrix} \hat{} \left| \begin{matrix} du \\ dv \end{matrix} \right\rangle \tag{81}$$

and

$$[C]\,\hat{} \left| \begin{matrix} \sigma \\ \omega \end{matrix} \right\rangle = \begin{bmatrix} 0 & d\theta \\ -d\theta & 0 \end{bmatrix} \hat{} \begin{bmatrix} \cos(\theta(u,v)) & -\sin(\theta(u,v)) \\ \sin(\theta(u,v)) & \cos(\theta(u,v)) \end{bmatrix} \circ \left| \begin{matrix} du \\ dv \end{matrix} \right\rangle \tag{82}$$

such that the first structural equation is satisfied:

$$\left| \begin{array}{c} d\sigma \\ d\omega \end{array} \right\rangle + [C]\, \hat{} \left| \begin{array}{c} \sigma \\ \omega \end{array} \right\rangle = 0 \tag{83}$$

3.1.2 Lorentz Subgroup in 2D

For $[L]^{+1}$, the inverse matrix with determinant $= +1$ is

$$[G] = \begin{bmatrix} \cosh(\theta(u,v)) & -\sinh(\theta(u,v)) \\ -\sinh(\theta(u,v)) & \cosh(\theta(u,v)) \end{bmatrix}, \tag{84}$$

such that the Cartan matrix becomes:

$$[C] = \{- \begin{bmatrix} \sinh(\theta(u,v))d\theta & -\cosh(\theta(u,v))d\theta \\ -\cosh(\theta(u,v))d\theta & \sinh(\theta(u,v))d\theta \end{bmatrix} \circ \begin{bmatrix} \cosh(\theta(u,v)) & \sinh(\theta(u,v)) \\ \sinh(\theta(u,v)) & \cosh(\theta(u,v)) \end{bmatrix}\} \tag{85}$$

$$= \begin{bmatrix} 0 & d\theta \\ -d\theta & 0 \end{bmatrix} \tag{86}$$

Hence $d\,[C] = 0$, and $[C]\, \hat{}\, [C] = 0$, proving that the Cartan curvature 2-forms vanish.

$$[\Theta] = d\,[C] + [C]\, \hat{}\, [C] = 0 \tag{87}$$

The Vierbiens have the format

$$\left| \begin{array}{c} \sigma \\ \omega \end{array} \right\rangle = \begin{bmatrix} \cosh(\theta(u,v)) & -\sinh(\theta(u,v)) \\ -\sinh(\theta(u,v)) & \cosh(\theta(u,v)) \end{bmatrix} \circ \left| \begin{array}{c} du \\ dv \end{array} \right\rangle \tag{88}$$

with

$$\left| \begin{array}{c} d\sigma \\ d\omega \end{array} \right\rangle = \begin{bmatrix} \sinh\theta(u,v))d\theta & -\cosh\theta(u,v))d\theta \\ -\cosh(\theta(u,v))d\theta & \sinh(\theta(u,v))d\theta \end{bmatrix} \hat{} \left| \begin{array}{c} du \\ dv \end{array} \right\rangle \tag{89}$$

and

$$[C]\, \hat{} \left| \begin{array}{c} \sigma \\ \omega \end{array} \right\rangle = \begin{bmatrix} 0 & d\theta \\ -d\theta & 0 \end{bmatrix} \hat{} \begin{bmatrix} \cosh(\theta(u,v)) & -\sinh(\theta(u,v)) \\ -\sinh(\theta(u,v)) & \cosh(\theta(u,v)) \end{bmatrix} \circ \left| \begin{array}{c} du \\ dv \end{array} \right\rangle \tag{90}$$

such that the first structural equation is satisfied:

$$\left| \begin{array}{c} d\sigma \\ d\omega \end{array} \right\rangle + [C]\, \hat{} \left| \begin{array}{c} \sigma \\ \omega \end{array} \right\rangle = 0 \tag{91}$$

3.1.3 The Symplectic Subgroup in 2D

to Follow.

3.2 Examples in Three Dimensions

To follow

3.3 Examples in Four Dimensions

Top follow