```
> restart: with(linalg):with(plots):with(liesymm):with(difforms):
  setup(ct,phi,theta,r):
  defform(ct=0,Z=0,phi=0,theta=0,r=0,p=const,q=const,a=const,Y=0,X=0
  ,E=0);
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

Cartan Connection coefficients from the Jacobian Matrix as the Basis Frame

(C) R.M.Kiehn 1999

a4sphjac.mws

```
> dim:=4;coord := [ct,r, theta,
  phi];LGUN:=array([[-1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]]);
```

$$dim := 4$$

$$coord := [ct, r, \theta, \phi]$$

$$LGUN := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Position vector to a point in terms of spherical coordinates

```
> MR:=[ct,r*sin(theta)*cos(phi),r*sin(theta)*sin(phi),r*cos(theta)];
```

$$MR := [ct, r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)]$$

```
> JAC:=jacobian(MR,coord);DET:=simplify(det(JAC));
```

$$JAC := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ 0 & \sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ 0 & \cos(\theta) & -r\sin(\theta) & 0 \end{bmatrix}$$

$$DET := \sin(\theta)\, r^2$$

Specify the Frame Matrtix

```
> FF:=evalm(JAC);
```

$$FF := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ 0 & \sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ 0 & \cos(\theta) & -r\sin(\theta) & 0 \end{bmatrix}$$

```
> GG:=evalm(simplify(inverse(FF)));
```

$$GG := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ 0 & \frac{\cos(\phi)\cos(\theta)}{r} & \frac{\sin(\phi)\cos(\theta)}{r} & -\frac{\sin(\theta)}{r} \\ 0 & -\frac{\sin(\phi)}{r\sin(\theta)} & \frac{\cos(\phi)}{r\sin(\theta)} & 0 \end{bmatrix}$$

```
> DGG:=evalm(d(GG)):
> CARTANRIGHT:=simplify(innerprod(-DGG,FF));
```

$$CARTANRIGHT := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r\,d(\theta) & -r\,d(\phi) + r\,d(\phi)\cos(\theta)^2 \\ 0 & \frac{d(\theta)}{r} & \frac{d(r)}{r} & -\cos(\theta)\,d(\phi)\sin(\theta) \\ 0 & \frac{d(\phi)}{r} & \frac{\cos(\theta)\,d(\phi)}{\sin(\theta)} & \frac{\sin(\theta)\,d(r) + \cos(\theta)\,d(\theta)\,r}{r\sin(\theta)} \end{bmatrix}$$

```
> CONJ:=simplify(innerprod(FF,CARTANRIGHT,-GG)):
```

The Left Cartan matrix is the Shipov Connection (IMO)

```
> CARTANLEFT:=simplify(innerprod(FF,DGG));
```

$CARTANLEFT :=$

$$[0, 0, 0, 0]$$

$$\Big[0, -(\cos(\phi)^2\cos(\theta)^2\sin(\theta)\,d(r) + \sin(\theta)\,d(r) - \cos(\phi)^2\,d(r)\sin(\theta) + \cos(\theta)\,d(\theta)\,r - \cos(\phi)^2\cos(\theta)\,d(\theta)\,r)/(r\sin(\theta)), (r\,d(\phi)\sin(\theta) - \sin(\phi)\cos(\theta)^2\sin(\theta)\cos(\phi)\,d(r) + \cos(\phi)\sin(\phi)\,d(r)\sin(\theta) + \cos(\phi)\sin(\phi)\cos(\theta)\,d(\theta)\,r)/(r\sin(\theta)), \frac{\cos(\phi)(-r\,d(\theta) + \sin(\theta)\cos(\theta)\,d(r))}{r}\Big]$$

$$\Big[0, -(\sin(\phi)\cos(\theta)^2\sin(\theta)\cos(\phi)\,d(r) + r\,d(\phi)\sin(\theta) - \cos(\phi)\sin(\phi)\,d(r)\sin(\theta) - \cos(\phi)\sin(\phi)\cos(\theta)\,d(\theta)\,r)/(r\sin(\theta)), (-\cos(\theta)^2\sin(\theta)\,d(r) + \cos(\phi)^2\cos(\theta)^2\sin(\theta)\,d(r) - \cos(\phi)^2\,d(r)\sin(\theta) - \cos(\phi)^2\cos(\theta)\,d(\theta)\,r)/(r\sin(\theta)), \frac{\sin(\phi)(-r\,d(\theta) + \sin(\theta)\cos(\theta)\,d(r))}{r}\Big]$$

$$\Big[0, \frac{(\sin(\theta)\cos(\theta)\,d(r) + r\,d(\theta))\cos(\phi)}{r}, \frac{(\sin(\theta)\cos(\theta)\,d(r) + r\,d(\theta))\sin(\phi)}{r}, \frac{d(r)(-1 + \cos(\theta)^2)}{r}\Big]$$

```
> simplify(evalm(CARTANLEFT-CONJ));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

CARTANLEFT IS THE negative CONJUGATE OF CARTANRIGHT

NOW use tensor methods

First compute the differentials of the inverse matrix

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do d1GG[i,j,k] := (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]

which is the right Cartan matrix

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim
  do  s:=0;for m from 1 to dim do s := s+(d1GG[a,m,k]*FF[m,b]);
  CC[a,b,k]:=simplify(-s) od od od od ;
>
```

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim
  do if CC[a,b,k]=0 then else print(`Cabk`(a-1,b-1,k-1)=CC[a,b,k])
  fi od od od ;
```

THE non zero CARTAN CONNECTION coefficients. C(abk) index (1,-1,-1)

$$\text{Cabk}(2, 1, 2) = \frac{1}{r}$$

$$\text{Cabk}(3, 1, 3) = \frac{1}{r}$$

$$\text{Cabk}(1, 2, 2) = -r$$

$$\text{Cabk}(2, 2, 1) = \frac{1}{r}$$

$$\text{Cabk}(3, 2, 3) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\text{Cabk}(1, 3, 3) = -r + \cos(\theta)^2\, r$$

$$\text{Cabk}(2, 3, 3) = -\cos(\theta)\sin(\theta)$$

$$\text{Cabk}(3, 3, 1) = \frac{1}{r}$$

$$\text{Cabk}(3, 3, 2) = \frac{\cos(\theta)}{\sin(\theta)}$$

These results agree with matrix method above.

Now compute the Anti symmetric [bk] components of the Cartan connection:

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim
  do  s := (CC[i,j,k]-CC[i,k,j])/2; TTCCS[i,j,k]:=s od od od  ;
```

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do if TTCCS[i,j,k]=0 then else
  print(`CartanaffineTorsion`(i-1,k-1,j-1)=TTCCS[i,k,j]) fi od od od
  ;
```

If no entries appear here, there is no affine torsion

Next construct the induced metric on the initial state (ct,r,theta,phi)

Christoffel Connection coefficients from the metric

```
> metric:=simplify(innerprod(transpose(FF),-LGUN,FF));
```

$$metric := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 + r^2 \cos(\theta)^2 \end{bmatrix}$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do d1gun[i,j,k] := (diff(metric[i,j],coord[k])) od od od:
> for i from 1 to dim do for j from i to dim do for k from 1 to dim
  do C1S[i,j,k] := 0  od od od; for i from 1 to dim do for j from 1
  to dim do for k from 1 to dim do C1S[i,j,k] :=
  1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim
  do s := 0; for m to dim do s := s+metricinverse[k,m]*C1S[i,j,m]
  od; C2S[k,i,j] := simplify(factor(s),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do if C2S[i,j,k]=0 then  else
  print(`Gamma2`(i-1,j-1,k-1)=C2S[i,j,k]) fi od od od;
```

$$\Gamma2(1, 2, 2) = -r$$

$$\Gamma2(1, 3, 3) = -r + \cos(\theta)^2 r$$

$$\Gamma2(2, 1, 2) = \frac{1}{r}$$

$$\Gamma2(2, 2, 1) = \frac{1}{r}$$

$$\Gamma2(2, 3, 3) = -\cos(\theta)\sin(\theta)$$

$$\Gamma2(3, 1, 3) = \frac{1}{r}$$

$$\Gamma2(3, 2, 3) = -\frac{\cos(\theta)\sin(\theta)}{-1 + \cos(\theta)^2}$$

$$\Gamma2(3, 3, 1) = \frac{1}{r}$$

$$\Gamma 2(3,3,2) = -\frac{\cos(\theta)\sin(\theta)}{-1+\cos(\theta)^2}$$

The non zero Christoffel Connection coefficients 2nd kind Gamma2(a,b,k) index (1,-1,-1)

>

NOTE THAT for the Jacobian basis frame, the Cartan Right matrix is EQUAL to the Christoffel CONNECTION matrix. This conjecture works, but I have not proved it abstractly.

NOw compute the Tabk as

Tright(abk) = Cartanright(abk) - Gamma(abk)

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do s:=0; s := (CC[i,j,k]-C2S[i,j,k]); CCTR[i,j,k]:=simplify(s)  od
  od od   ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do if (C2S[i,j,k]=0 and CC[i,j,k]=0) then else
  print(`T`(i-1,j-1,k-1)=simplify(CCTR[i,j,k])) fi od od od ;
```

$$T(1,2,2) = 0$$
$$T(1,3,3) = 0$$
$$T(2,1,2) = 0$$
$$T(2,2,1) = 0$$
$$T(2,3,3) = 0$$
$$T(3,1,3) = 0$$
$$T(3,2,3) = 0$$
$$T(3,3,1) = 0$$
$$T(3,3,2) = 0$$

The "Right Rotation Coefficients" vanish if the frame is an integrable Jacobian matrix. There is no AFFINE torsion in the sense of an anti-symmetric component of the connection.

The Anti-symmetric part of the RIGHT CARTAN MATRIX vanishes for a frame constructed from the Jacobian matrix of a map.

Now compute the Shipov Delta connection, which is the Left Cartan matrix

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do d1GG[a,j,k] := simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrixx product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do  s:=0;for m to dim do s := s+FF[i,m]*(d1GG[m,j,k]);
  DD[i,j,k]:=simplify(s) od od od od ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do if DD[i,j,k]=0 then else print(`Delta`(i-1,j-1,k-1)=DD[i,j,k])
  fi od od od ;
```

NON-ZERO SHIPOV CONNECTION coefficients

Cartan left matrix =Delta(ijk) index (1,-1,-1)

$$\Delta(1,1,1) = -\frac{\cos(\phi)^2\cos(\theta)^2 + 1 - \cos(\phi)^2}{r}$$

$$\Delta(1,1,2) = -\frac{(-1+\cos(\phi)^2)\cos(\theta)\sin(\theta)}{-1+\cos(\theta)^2}$$

$$\Delta(1,2,1) = -\frac{\cos(\phi)\sin(\phi)(-1+\cos(\theta)^2)}{r}$$

$$\Delta(1,2,2) = -\frac{\sin(\theta)\sin(\phi)\cos(\phi)\cos(\theta)}{-1+\cos(\theta)^2}$$

$$\Delta(1,2,3) = 1$$

$$\Delta(1,3,1) = \frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$\Delta(1,3,2) = -\cos(\phi)$$

$$\Delta(2,1,1) = -\frac{\cos(\phi)\sin(\phi)(-1+\cos(\theta)^2)}{r}$$

$$\Delta(2,1,2) = -\frac{\sin(\theta)\sin(\phi)\cos(\phi)\cos(\theta)}{-1+\cos(\theta)^2}$$

$$\Delta(2,1,3) = -1$$

$$\Delta(2,2,1) = \frac{-\cos(\theta)^2 + \cos(\phi)^2\cos(\theta)^2 - \cos(\phi)^2}{r}$$

$$\Delta(2,2,2) = \frac{\sin(\theta)\cos(\phi)^2\cos(\theta)}{-1+\cos(\theta)^2}$$

$$\Delta(2,3,1) = \frac{\cos(\theta)\sin(\phi)\sin(\theta)}{r}$$

$$\Delta(2,3,2) = -\sin(\phi)$$

$$\Delta(3, 1, 1) = \frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$\Delta(3, 1, 2) = \cos(\phi)$$

$$\Delta(3, 2, 1) = \frac{\cos(\theta)\sin(\phi)\sin(\theta)}{r}$$

$$\Delta(3, 2, 2) = \sin(\phi)$$

$$\Delta(3, 3, 1) = \frac{-1 + \cos(\theta)^2}{r}$$

These values agree with the matrix methods.

The anti-symmetric part of the Shipov (Left) CARTAN Connection

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim
  do  s := (DD[i,j,k]-DD[i,k,j])/2; TTS[i,j,k]:=simplify(s) od od od
  ;
```

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do if TTS[i,j,k]=0 then else
  print(`ShipovTorsion`(i-1,k-1,j-1)=TTS[i,k,j]) fi od od od ;
```

$$\mathrm{ShipovTorsion}(1, 2, 1) = \frac{1}{2}(-\cos(\theta)\sin(\theta)\,r + \cos(\theta)\sin(\theta)\,r\cos(\phi)^2 - \cos(\phi)\sin(\phi)$$
$$+ 2\cos(\phi)\sin(\phi)\cos(\theta)^2 - \cos(\phi)\sin(\phi)\cos(\theta)^4) \Big/ ((-1 + \cos(\theta)^2)\,r)$$

$$\mathrm{ShipovTorsion}(1, 3, 1) = \frac{1}{2}\frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$\mathrm{ShipovTorsion}(1, 1, 2) = -\frac{1}{2}(-\cos(\theta)\sin(\theta)\,r + \cos(\theta)\sin(\theta)\,r\cos(\phi)^2 - \cos(\phi)\sin(\phi)$$
$$+ 2\cos(\phi)\sin(\phi)\cos(\theta)^2 - \cos(\phi)\sin(\phi)\cos(\theta)^4) \Big/ ((-1 + \cos(\theta)^2)\,r)$$

$$\mathrm{ShipovTorsion}(1, 3, 2) = -\frac{1}{2}\cos(\phi) - \frac{1}{2}$$

$$\mathrm{ShipovTorsion}(1, 1, 3) = -\frac{1}{2}\frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$\mathrm{ShipovTorsion}(1, 2, 3) = \frac{1}{2} + \frac{1}{2}\cos(\phi)$$

$$\mathrm{ShipovTorsion}(2, 2, 1) = \frac{1}{2}(\sin(\theta)\sin(\phi)\cos(\phi)\cos(\theta)\,r + \cos(\theta)^2 - \cos(\theta)^4$$
$$- 2\cos(\phi)^2\cos(\theta)^2 + \cos(\phi)^2\cos(\theta)^4 + \cos(\phi)^2) \Big/ ((-1 + \cos(\theta)^2)\,r)$$

$$\mathrm{ShipovTorsion}(2, 3, 1) = \frac{1}{2}\frac{r + \sin(\phi)\cos(\theta)\sin(\theta)}{r}$$

$$\mathrm{ShipovTorsion}(2, 1, 2) = -\frac{1}{2}(\sin(\theta)\sin(\phi)\cos(\phi)\cos(\theta)\,r + \cos(\theta)^2 - \cos(\theta)^4$$

$$-2\cos(\phi)^2\cos(\theta)^2+\cos(\phi)^2\cos(\theta)^4+\cos(\phi)^2)\Big/((-1+\cos(\theta)^2)\,r)$$

$$\text{ShipovTorsion}(2,3,2)=-\frac{1}{2}\sin(\phi)$$

$$\text{ShipovTorsion}(2,1,3)=-\frac{1}{2}\frac{r+\sin(\phi)\cos(\theta)\sin(\theta)}{r}$$

$$\text{ShipovTorsion}(2,2,3)=\frac{1}{2}\sin(\phi)$$

$$\text{ShipovTorsion}(3,2,1)=\frac{1}{2}\frac{-\cos(\phi)\,r+\sin(\phi)\cos(\theta)\sin(\theta)}{r}$$

$$\text{ShipovTorsion}(3,3,1)=\frac{1}{2}\frac{-1+\cos(\theta)^2}{r}$$

$$\text{ShipovTorsion}(3,1,2)=-\frac{1}{2}\frac{-\cos(\phi)\,r+\sin(\phi)\cos(\theta)\sin(\theta)}{r}$$

$$\text{ShipovTorsion}(3,1,3)=-\frac{1}{2}\frac{-1+\cos(\theta)^2}{r}$$

Now compute the Tijk assuming the formula

T(ijk) = Cartanleft(ijk) - Gamma(ijk)

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do s:=0; s := (DD[i,j,k]-C2S[i,j,k]); SHIPTR[i,j,k]:=simplify(s)
  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim
  do if (C2S[i,j,k]=0 and DD[i,j,k]=0) then else
  print(`T`(i-1,j-1,k-1)=simplify(SHIPTR[i,j,k])) fi od od od ;
```

$$\text{T}(1,1,1)=-\frac{\cos(\phi)^2\cos(\theta)^2+1-\cos(\phi)^2}{r}$$

$$\text{T}(1,1,2)=-\frac{(-1+\cos(\phi)^2)\cos(\theta)\sin(\theta)}{-1+\cos(\theta)^2}$$

$$\text{T}(1,2,1)=-\frac{\cos(\phi)\sin(\phi)\,(-1+\cos(\theta)^2)}{r}$$

$$\text{T}(1,2,2)=-\frac{\sin(\theta)\sin(\phi)\cos(\phi)\cos(\theta)+r-\cos(\theta)^2\,r}{-1+\cos(\theta)^2}$$

$$\text{T}(1,2,3)=1$$

$$\text{T}(1,3,1)=\frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$T(1,3,2) = -\cos(\phi)$$

$$T(1,3,3) = r - \cos(\theta)^2\, r$$

$$T(2,1,1) = -\frac{\cos(\phi)\sin(\phi)\,(-1+\cos(\theta)^2)}{r}$$

$$T(2,1,2) = -\frac{\sin(\theta)\sin(\phi)\cos(\phi)\cos(\theta)\,r - 1 + \cos(\theta)^2}{r\,(-1+\cos(\theta)^2)}$$

$$T(2,1,3) = -1$$

$$T(2,2,1) = \frac{-\cos(\theta)^2 + \cos(\phi)^2\cos(\theta)^2 - \cos(\phi)^2 - 1}{r}$$

$$T(2,2,2) = \frac{\sin(\theta)\cos(\phi)^2\cos(\theta)}{-1+\cos(\theta)^2}$$

$$T(2,3,1) = \frac{\cos(\theta)\sin(\phi)\sin(\theta)}{r}$$

$$T(2,3,2) = -\sin(\phi)$$

$$T(2,3,3) = \cos(\theta)\sin(\theta)$$

$$T(3,1,1) = \frac{\cos(\theta)\cos(\phi)\sin(\theta)}{r}$$

$$T(3,1,2) = \cos(\phi)$$

$$T(3,1,3) = -\frac{1}{r}$$

$$T(3,2,1) = \frac{\cos(\theta)\sin(\phi)\sin(\theta)}{r}$$

$$T(3,2,2) = \sin(\phi)$$

$$T(3,2,3) = \frac{\cos(\theta)\sin(\theta)}{-1+\cos(\theta)^2}$$

$$T(3,3,1) = \frac{-2+\cos(\theta)^2}{r}$$

$$T(3,3,2) = \frac{\cos(\theta)\sin(\theta)}{-1+\cos(\theta)^2}$$

>

The "LEFT Rotation Coefficients"

But it does not seem to make sense (to me) to subtract the Christoffel part from Delta.

>