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HOLDER NORMS AND ADJOINT CURRENTS IN 4D

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Definitions:

On a variety of independent variables (x,y,z,t), consider a 4 component vector VV with arbitrary component functions U,V,W,C.

Define the Holder Norm, of signature (a,b,c,e) degree p and homogeneity index n, as the function:

$$\text{HolderNorm} := (a U(x, y, z, t)^p + b V(x, y, z, t)^p + c W(x, y, z, t)^p + e C(x, y, z, t)^p)^{\left(\frac{n}{p}\right)}$$

Define a vector VVN homogeneous of degree n, by dividing the vector VV by the Holder Norm.

Define the Adjoint matrix as the matrix of cofactors transposed of the Jacobian matrix computed from VVN

Define the Adjoint current as the vector equal to the product of the Adoint matrix times the vector VVN.

THEOREM: For any vector VV made homogeneous of degree zero by dividing each component by a Holder norm of any signature and any exponent p, but with homogeneity index n=1, then (part 1) the associated Jacobian matrix has zero determinant and (part 2) the adjoint current has zero divergence with respect to the 4 independent variables.

PROOF:

[> with(linalg):

[Construct a vector with 4 arbitrary functions on a 4D variety.

[> UU:=U(x,y,z,t);VV:=V(x,y,z,t);WW:=W(x,y,z,t);CC:=C(x,y,z,t);

$$UU := U(x, y, z, t)$$

$$VV := V(x, y, z, t)$$

$$WW := W(x, y, z, t)$$

$$CC := C(x, y, z, t)$$

[Construct the Holder norm with arbitrary signature (a,b,c,e) and with homogeneity index n and power p.

[> HolderNorm:=(a*UU^p+b*VV^p+c*WW^p+e*CC^p)^(n/p);

$$\text{HolderNorm} := (a U(x, y, z, t)^p + b V(x, y, z, t)^p + c W(x, y, z, t)^p + e C(x, y, z, t)^p)^{\left(\frac{n}{p}\right)}$$

[Normalize the original vector by dividing each component by the Holder norm.

[> VVN:=[UU/HolderNorm,VV/HolderNorm,WW/HolderNorm,CC/HolderNorm];

$$\text{VVN} := \left[\begin{array}{l} \frac{U(x, y, z, t)}{(a U(x, y, z, t)^p + b V(x, y, z, t)^p + c W(x, y, z, t)^p + e C(x, y, z, t)^p)^{\left(\frac{n}{p}\right)}, \\ \frac{V(x, y, z, t)}{(a U(x, y, z, t)^p + b V(x, y, z, t)^p + c W(x, y, z, t)^p + e C(x, y, z, t)^p)^{\left(\frac{n}{p}\right)}, \\ \frac{W(x, y, z, t)}{(a U(x, y, z, t)^p + b V(x, y, z, t)^p + c W(x, y, z, t)^p + e C(x, y, z, t)^p)^{\left(\frac{n}{p}\right)}, \\ \frac{C(x, y, z, t)}{(a U(x, y, z, t)^p + b V(x, y, z, t)^p + c W(x, y, z, t)^p + e C(x, y, z, t)^p)^{\left(\frac{n}{p}\right)}} \end{array} \right]$$

[Compute the JAcobian matrix of the homogeneous vector, and the determinant of the JAcobian matrix.

$$-\left(\frac{\partial}{\partial x} W(x, y, z, t)\right)\left(\frac{\partial}{\partial y} C(x, y, z, t)\right)\left(\frac{\partial}{\partial z} U(x, y, z, t)\right)\left(\frac{\partial}{\partial t} V(x, y, z, t)\right)\right) \Bigg/ \left((a U(x, y, z, t)^p + b V(x, y, z, t)^p + c W(x, y, z, t)^p + e C(x, y, z, t)^p)^{\left(\frac{n}{p}\right)^4} \right)$$

[So determinant vanishes for n=1, any signature, and exponent p. QED part 1

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[Compute the Adjoint matrix and the current = [Adjoint matrix of JAcobian] | VVN> any n, any p, any signature.

[> **ADJ:=adj(JJ) :**

[> **Current:=subs(innerprod(ADJ,VVN)) ;**

$$\begin{aligned} \text{Current} := & \left[-\left(-\left(\frac{\partial}{\partial y} U(x, y, z, t) \right) C(x, y, z, t) \left(\frac{\partial}{\partial t} V(x, y, z, t) \right) \left(\frac{\partial}{\partial z} W(x, y, z, t) \right) \right. \right. \\ & + \left(\frac{\partial}{\partial y} W(x, y, z, t) \right) V(x, y, z, t) \left(\frac{\partial}{\partial t} U(x, y, z, t) \right) \left(\frac{\partial}{\partial z} C(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} C(x, y, z, t) \right) V(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t) \right) \left(\frac{\partial}{\partial t} W(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} U(x, y, z, t) \right) V(x, y, z, t) \left(\frac{\partial}{\partial z} W(x, y, z, t) \right) \left(\frac{\partial}{\partial t} C(x, y, z, t) \right) \\ & - \left(\frac{\partial}{\partial y} V(x, y, z, t) \right) U(x, y, z, t) \left(\frac{\partial}{\partial z} W(x, y, z, t) \right) \left(\frac{\partial}{\partial t} C(x, y, z, t) \right) \\ & - \left(\frac{\partial}{\partial y} C(x, y, z, t) \right) U(x, y, z, t) \left(\frac{\partial}{\partial z} V(x, y, z, t) \right) \left(\frac{\partial}{\partial t} W(x, y, z, t) \right) \\ & - \left(\frac{\partial}{\partial y} W(x, y, z, t) \right) U(x, y, z, t) \left(\frac{\partial}{\partial z} V(x, y, z, t) \right) \left(\frac{\partial}{\partial t} C(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} W(x, y, z, t) \right) C(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t) \right) \left(\frac{\partial}{\partial t} V(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} V(x, y, z, t) \right) C(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t) \right) \left(\frac{\partial}{\partial t} W(x, y, z, t) \right) \\ & - \left(\frac{\partial}{\partial y} W(x, y, z, t) \right) C(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t) \right) \left(\frac{\partial}{\partial t} V(x, y, z, t) \right) \\ & - \left(\frac{\partial}{\partial y} V(x, y, z, t) \right) C(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t) \right) \left(\frac{\partial}{\partial t} W(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} U(x, y, z, t) \right) C(x, y, z, t) \left(\frac{\partial}{\partial z} V(x, y, z, t) \right) \left(\frac{\partial}{\partial t} W(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} V(x, y, z, t) \right) W(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t) \right) \left(\frac{\partial}{\partial t} C(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} C(x, y, z, t) \right) W(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t) \right) \left(\frac{\partial}{\partial t} V(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} U(x, y, z, t) \right) W(x, y, z, t) \left(\frac{\partial}{\partial z} V(x, y, z, t) \right) \left(\frac{\partial}{\partial t} C(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} C(x, y, z, t) \right) U(x, y, z, t) \left(\frac{\partial}{\partial z} V(x, y, z, t) \right) \left(\frac{\partial}{\partial t} W(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} V(x, y, z, t) \right) U(x, y, z, t) \left(\frac{\partial}{\partial z} W(x, y, z, t) \right) \left(\frac{\partial}{\partial t} C(x, y, z, t) \right) \\ & + \left(\frac{\partial}{\partial y} W(x, y, z, t) \right) U(x, y, z, t) \left(\frac{\partial}{\partial z} V(x, y, z, t) \right) \left(\frac{\partial}{\partial t} C(x, y, z, t) \right) \end{aligned}$$

$$\begin{aligned}
& -\left(\frac{\partial}{\partial x} W(x, y, z, t)\right) V(x, y, z, t) \left(\frac{\partial}{\partial y} U(x, y, z, t)\right) \left(\frac{\partial}{\partial z} C(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} C(x, y, z, t)\right) V(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t)\right) \left(\frac{\partial}{\partial y} W(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} U(x, y, z, t)\right) V(x, y, z, t) \left(\frac{\partial}{\partial z} W(x, y, z, t)\right) \left(\frac{\partial}{\partial y} C(x, y, z, t)\right) \\
& +\left(\frac{\partial}{\partial x} W(x, y, z, t)\right) U(x, y, z, t) \left(\frac{\partial}{\partial y} V(x, y, z, t)\right) \left(\frac{\partial}{\partial z} C(x, y, z, t)\right) \\
& +\left(\frac{\partial}{\partial x} V(x, y, z, t)\right) U(x, y, z, t) \left(\frac{\partial}{\partial z} W(x, y, z, t)\right) \left(\frac{\partial}{\partial y} C(x, y, z, t)\right) \\
& +\left(\frac{\partial}{\partial x} C(x, y, z, t)\right) U(x, y, z, t) \left(\frac{\partial}{\partial z} V(x, y, z, t)\right) \left(\frac{\partial}{\partial y} W(x, y, z, t)\right) \\
& +\left(\frac{\partial}{\partial x} V(x, y, z, t)\right) C(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t)\right) \left(\frac{\partial}{\partial y} W(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} V(x, y, z, t)\right) C(x, y, z, t) \left(\frac{\partial}{\partial y} U(x, y, z, t)\right) \left(\frac{\partial}{\partial z} W(x, y, z, t)\right) \\
& +\left(\frac{\partial}{\partial x} U(x, y, z, t)\right) C(x, y, z, t) \left(\frac{\partial}{\partial y} V(x, y, z, t)\right) \left(\frac{\partial}{\partial z} W(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} U(x, y, z, t)\right) C(x, y, z, t) \left(\frac{\partial}{\partial z} V(x, y, z, t)\right) \left(\frac{\partial}{\partial y} W(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} W(x, y, z, t)\right) C(x, y, z, t) \left(\frac{\partial}{\partial z} U(x, y, z, t)\right) \left(\frac{\partial}{\partial y} V(x, y, z, t)\right) \\
& +\left(\frac{\partial}{\partial x} W(x, y, z, t)\right) C(x, y, z, t) \left(\frac{\partial}{\partial y} U(x, y, z, t)\right) \left(\frac{\partial}{\partial z} V(x, y, z, t)\right) \Big/ \\
& \left[\left(a U(x, y, z, t)^p + b V(x, y, z, t)^p + c W(x, y, z, t)^p + e C(x, y, z, t)^p \right)^{\left(\frac{n}{p}\right)^4} \right]
\end{aligned}$$

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Compute the divergence of this Current

> **DIVCurrent:=factor(diverge(Current,[x,y,z,t]));**

$$\begin{aligned}
DIVCurrent := & -4 \left((-1+n) \left(-\left(\frac{\partial}{\partial x} C(x, y, z, t)\right) \left(\frac{\partial}{\partial y} U(x, y, z, t)\right) \left(\frac{\partial}{\partial z} V(x, y, z, t)\right) \left(\frac{\partial}{\partial t} W(x, y, z, t)\right) \right. \right. \\
& -\left(\frac{\partial}{\partial x} C(x, y, z, t)\right) \left(\frac{\partial}{\partial y} W(x, y, z, t)\right) \left(\frac{\partial}{\partial z} U(x, y, z, t)\right) \left(\frac{\partial}{\partial t} V(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} V(x, y, z, t)\right) \left(\frac{\partial}{\partial y} C(x, y, z, t)\right) \left(\frac{\partial}{\partial t} W(x, y, z, t)\right) \left(\frac{\partial}{\partial z} U(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} C(x, y, z, t)\right) \left(\frac{\partial}{\partial y} V(x, y, z, t)\right) \left(\frac{\partial}{\partial t} U(x, y, z, t)\right) \left(\frac{\partial}{\partial z} W(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} U(x, y, z, t)\right) \left(\frac{\partial}{\partial y} C(x, y, z, t)\right) \left(\frac{\partial}{\partial t} V(x, y, z, t)\right) \left(\frac{\partial}{\partial z} W(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} V(x, y, z, t)\right) \left(\frac{\partial}{\partial y} U(x, y, z, t)\right) \left(\frac{\partial}{\partial z} W(x, y, z, t)\right) \left(\frac{\partial}{\partial t} C(x, y, z, t)\right) \\
& -\left(\frac{\partial}{\partial x} U(x, y, z, t)\right) \left(\frac{\partial}{\partial y} V(x, y, z, t)\right) \left(\frac{\partial}{\partial t} W(x, y, z, t)\right) \left(\frac{\partial}{\partial z} C(x, y, z, t)\right) \\
& +\left(\frac{\partial}{\partial x} U(x, y, z, t)\right) \left(\frac{\partial}{\partial y} V(x, y, z, t)\right) \left(\frac{\partial}{\partial z} W(x, y, z, t)\right) \left(\frac{\partial}{\partial t} C(x, y, z, t)\right) \\
& \left. -\left(\frac{\partial}{\partial x} U(x, y, z, t)\right) \left(\frac{\partial}{\partial y} W(x, y, z, t)\right) \left(\frac{\partial}{\partial z} V(x, y, z, t)\right) \left(\frac{\partial}{\partial t} C(x, y, z, t)\right) \right)
\end{aligned}$$

