```
> restart;
> with(liesymm):with(linalg):setup(x,y,z,t,s,Ct);deforms(a=const,b=const,c=const,d=const,p=const,n=const,k=const,omega=const,e=const);
Warning, new definition for close
Warning, new definition for norm
Warning, new definition for trace
```

$$[x, y, z, t, s, Ct]$$

$$\mathrm{deforms}(a = const, b = const, c = const, d = const, p = const, n = const, k = const, \omega = const, e = const)$$

Classic Implicit Surfaces in 3D using HOLDER NORMS

R. M. Kiehn

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The similarity invariants for the Jacobian matrix of the surface normal scaled by the Holder norm with a=b=c=1,n=1,p=2
yields the classic partial differential equations for the Mean and Gauss curvature. The surface is defined by the zero set of some function Phi(x,y,z). That is, the implicit surface is defined by the equation Phi(x,y,z) = 0.

```
> Phi:= f(x,y,z);
```

$$\Phi := \mathrm{f}(x, y, z)$$

```
>
```

Construct the Normal field to the implicit surface as the gradient of Phi(x,y,z) with respect to (x,y,z)

```
> A1:=diff(Phi,x);A2:=diff(Phi,y);A3:=diff(Phi,z);
```

$$A1 := \frac{\partial}{\partial x}\mathrm{f}(x, y, z)$$

$$A2 := \frac{\partial}{\partial y}\mathrm{f}(x, y, z)$$

$$A3 := \frac{\partial}{\partial z}\mathrm{f}(x, y, z)$$

```
> A:=[A1,A2,A3];
```

$$A := \left[\frac{\partial}{\partial x}\mathrm{f}(x, y, z), \frac{\partial}{\partial y}\mathrm{f}(x, y, z), \frac{\partial}{\partial z}\mathrm{f}(x, y, z)\right]$$

Construct the Holder Norm and divide each component of the gradient by the Holder norm.

```
> lambda:=(a*A[1]^p+b*A[2]^p+c*A[3]^p)^(n/p);a:=1;b:=1;c:=1;
```

Force lambda to be homogeneous of degree 1 by choosing n = 1, specialize the degree to p =2, and euclidean signature 1,1,1 to make a correspondence with the Gauss map and classic implicit surface theory.

```
> n:=1;p:=2;
```

$$\lambda := \left(a\left(\frac{\partial}{\partial x}\mathrm{f}(x, y, z)\right)^p + b\left(\frac{\partial}{\partial y}\mathrm{f}(x, y, z)\right)^p + c\left(\frac{\partial}{\partial z}\mathrm{f}(x, y, z)\right)^p\right)^{\left(\frac{n}{p}\right)}$$

$$a := 1$$

$$b := 1$$

$$c := 1$$

$$n := 1$$

$$p := 2$$

```
> NA:=evalm([A1,A2,A3]/lambda);
```

$$NA := \left[\frac{\frac{\partial}{\partial x} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}, \frac{\frac{\partial}{\partial y} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}, \frac{\frac{\partial}{\partial z} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}\right]$$

\>

Compute the Jacobian matrix of the normalized gradient

```
> JAC:=(jacobian(NA,[x,y,z]));DET:=factor(det(JAC));
```

$JAC :=$

$$\left[-\frac{1}{2}\left(\frac{\partial}{\partial x} f(x, y, z)\right) \left(2\left(\frac{\partial}{\partial x} f(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2} f(x, y, z)\right) + 2\left(\frac{\partial}{\partial y} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x, y, z)\right) + 2\left(\frac{\partial}{\partial z} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x} f(x, y, z)\right)\right) \Big/ \left(\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2\right)^{(3/2)} + \frac{\frac{\partial^2}{\partial x^2} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}, -\frac{1}{2}\left(\frac{\partial}{\partial x} f(x, y, z)\right) \left(2\left(\frac{\partial}{\partial x} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x, y, z)\right) + 2\left(\frac{\partial}{\partial y} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y^2} f(x, y, z)\right) + 2\left(\frac{\partial}{\partial z} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y} f(x, y, z)\right)\right) \Big/ \left(\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2\right)^{(3/2)} + \frac{\frac{\partial^2}{\partial y\,\partial x} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}, -\frac{1}{2}\left(\frac{\partial}{\partial x} f(x, y, z)\right) \left(2\left(\frac{\partial}{\partial x} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x} f(x, y, z)\right) + 2\left(\frac{\partial}{\partial y} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y} f(x, y, z)\right) + 2\left(\frac{\partial}{\partial z} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2} f(x, y, z)\right)\right) \Big/ \left(\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2\right)^{(3/2)} + \frac{\frac{\partial^2}{\partial z\,\partial x} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} f(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}\right]$$

$$\left[-\frac{1}{2}\left(\frac{\partial}{\partial y} f(x, y, z)\right)\right.$$

$$\left(2\left(\frac{\partial}{\partial x} f(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2} f(x, y, z)\right)+2\left(\frac{\partial}{\partial y} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x, y, z)\right)+2\left(\frac{\partial}{\partial z} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x} f(x, y, z)\right)\right)$$

$$\Big/\left(\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2\right)^{(3/2)}$$

$$+\frac{\frac{\partial^2}{\partial y\,\partial x} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}, -\frac{1}{2}\left(\frac{\partial}{\partial y} f(x, y, z)\right)$$

$$\left(2\left(\frac{\partial}{\partial x} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x, y, z)\right)+2\left(\frac{\partial}{\partial y} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y^2} f(x, y, z)\right)+2\left(\frac{\partial}{\partial z} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y} f(x, y, z)\right)\right)$$

$$\Big/\left(\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2\right)^{(3/2)}$$

$$+\frac{\frac{\partial^2}{\partial y^2} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}, -\frac{1}{2}\left(\frac{\partial}{\partial y} f(x, y, z)\right)$$

$$\left(2\left(\frac{\partial}{\partial x} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x} f(x, y, z)\right)+2\left(\frac{\partial}{\partial y} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y} f(x, y, z)\right)+2\left(\frac{\partial}{\partial z} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2} f(x, y, z)\right)\right)\Big/$$

$$\left(\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2\right)^{(3/2)}$$

$$\left.+\frac{\frac{\partial^2}{\partial z\,\partial y} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}\right]$$

$$\left[-\frac{1}{2}\left(\frac{\partial}{\partial z} f(x, y, z)\right)\right.$$

$$\left(2\left(\frac{\partial}{\partial x} f(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2} f(x, y, z)\right)+2\left(\frac{\partial}{\partial y} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x, y, z)\right)+2\left(\frac{\partial}{\partial z} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x} f(x, y, z)\right)\right)$$

$$\Big/\left(\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2\right)^{(3/2)}$$

$$+\frac{\frac{\partial^2}{\partial z\,\partial x} f(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2}}, -\frac{1}{2}\left(\frac{\partial}{\partial z} f(x, y, z)\right)$$

$$\left(2\left(\frac{\partial}{\partial x} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x, y, z)\right)+2\left(\frac{\partial}{\partial y} f(x, y, z)\right)\left(\frac{\partial^2}{\partial y^2} f(x, y, z)\right)+2\left(\frac{\partial}{\partial z} f(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y} f(x, y, z)\right)\right)$$

$$\Big/\left(\left(\frac{\partial}{\partial x} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial y} f(x, y, z)\right)^2+\left(\frac{\partial}{\partial z} f(x, y, z)\right)^2\right)^{(3/2)}$$

$$+\frac{\frac{\partial^2}{\partial z\,\partial y}\mathrm{f}(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2}},\ -\frac{1}{2}\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)\left(2\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\mathrm{f}(x,y,z)\right)+2\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\mathrm{f}(x,y,z)\right)+2\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\mathrm{f}(x,y,z)\right)\right)\Big/\left(\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2\right)^{(3/2)}$$

$$+\frac{\frac{\partial^2}{\partial z^2}\mathrm{f}(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2}}\Bigg]$$

$$DET := 0$$

```
> MEAN_CURVATURE:=factor(trace(JAC)/2);
```

$$MEAN\_CURVATURE := \frac{1}{2}\left(-2\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\mathrm{f}(x,y,z)\right)-2\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\mathrm{f}(x,y,z)\right)+\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2\left(\frac{\partial^2}{\partial x^2}\mathrm{f}(x,y,z)\right)+\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2\left(\frac{\partial^2}{\partial x^2}\mathrm{f}(x,y,z)\right)-2\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\mathrm{f}(x,y,z)\right)+\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2\left(\frac{\partial^2}{\partial y^2}\mathrm{f}(x,y,z)\right)+\left(\frac{\partial^2}{\partial y^2}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial^2}{\partial z^2}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial^2}{\partial z^2}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2\right)\Big/\left(\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2\right)^{(3/2)}$$

Note the classic formula for the mean curvature of a 3D implicit surface in xyz is obtained.

```
> S2:=factor(trace(innerprod(JAC,JAC))):
  Gauss:=factor(-(1/2)*(-trace(JAC)*trace(JAC)+S2));
```

$$Gauss := \left(-\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2\left(\frac{\partial^2}{\partial z\,\partial y}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2\left(\frac{\partial^2}{\partial y^2}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\mathrm{f}(x,y,z)\right)+2\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\mathrm{f}(x,y,z)\right)+2\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\mathrm{f}(x,y,z)\right)-2\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\mathrm{f}(x,y,z)\right)-2\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial y^2}\mathrm{f}(x,y,z)\right)-\left(\frac{\partial^2}{\partial z\,\partial x}\mathrm{f}(x,y,z)\right)^2\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2-\left(\frac{\partial^2}{\partial y\,\partial x}\mathrm{f}(x,y,z)\right)^2\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2+2\left(\frac{\partial^2}{\partial y\,\partial x}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\mathrm{f}(x,y,z)\right)+\left(\frac{\partial^2}{\partial x^2}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial y^2}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial^2}{\partial x^2}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2-2\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\mathrm{f}(x,y,z)\right)\right)\Big/\left(\left(\frac{\partial}{\partial x}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\mathrm{f}(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\mathrm{f}(x,y,z)\right)^2\right)^2$$

```
> ADJAC:=adjoint(JAC):
```

```
> ADJOINT_CURVATURE:=factor(trace(ADJAC));
```

$$ADJOINT\_CURVATURE := \Big(-\left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right)^2 \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right)^2 + \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right)^2 \left(\frac{\partial^2}{\partial y^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z^2} \mathrm{f}(x, y, z)\right)$$

$$+ 2 \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right)$$

$$+ 2 \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right)$$

$$- 2 \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z^2} \mathrm{f}(x, y, z)\right)$$

$$- 2 \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial y^2} \mathrm{f}(x, y, z)\right) - \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right)^2 \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right)^2$$

$$- \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right)^2 \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right)^2 + 2 \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right)$$

$$+ \left(\frac{\partial^2}{\partial x^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial y^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right)^2 + \left(\frac{\partial^2}{\partial x^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right)^2$$

$$- 2 \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial x^2} \mathrm{f}(x, y, z)\right) \Big) \Big/$$

$$\left(\left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right)^2 \right)^2$$

```
>
```

Note that the classic formula for the ADJOINT Gauss curvature of a 3D implicit surface in xyz is obtained.

Now define the adjoint current, and show that in 3D the adjoint current times the homogeneous vector is equal to the trace of the adjoint matrix. Moreover show that the 3D divergence of the adjoint current so defined is globally zero.

```
> CurrentJ:=innerprod(ADJAC,NA);Interaction:=innerprod(CurrentJ,NA);DivJ:=facto
  r(diverge(CurrentJ,[x,y,z]));
```

$$CurrentJ := \Big[\Big(\left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right)$$

$$+ \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right) - \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z^2} \mathrm{f}(x, y, z)\right)$$

$$- \left(\frac{\partial^2}{\partial y^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right) + \left(\frac{\partial^2}{\partial y^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right)$$

$$- \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right)^2 \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right) \Big) \Big/ \left(\left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right)^2 \right)^{(3/2)}, \Big($$

$$\left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right) + \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right)$$

$$+ \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial x^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z^2} \mathrm{f}(x, y, z)\right) - \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right)^2$$

$$- \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right) - \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial x^2} \mathrm{f}(x, y, z)\right) \Big) \Big/$$

$$\left(\left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right)^2 + \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right)^2 + \left(\frac{\partial}{\partial z} \mathrm{f}(x, y, z)\right)^2 \right)^{(3/2)}, -\Big(-\left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right)$$

$$- \left(\frac{\partial^2}{\partial z \partial y} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial y \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right) + \left(\frac{\partial^2}{\partial y^2} \mathrm{f}(x, y, z)\right) \left(\frac{\partial^2}{\partial z \partial x} \mathrm{f}(x, y, z)\right) \left(\frac{\partial}{\partial x} \mathrm{f}(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial z\,\partial y} f(x,y,z)\right)\left(\frac{\partial}{\partial y} f(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2} f(x,y,z)\right)-\left(\frac{\partial^2}{\partial y^2} f(x,y,z)\right)\left(\frac{\partial}{\partial z} f(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2} f(x,y,z)\right)$$

$$+\left(\frac{\partial^2}{\partial y\,\partial x} f(x,y,z)\right)^2\left(\frac{\partial}{\partial z} f(x,y,z)\right)\Bigg) \Bigg/ \left(\left(\left(\frac{\partial}{\partial x} f(x,y,z)\right)^2+\left(\frac{\partial}{\partial y} f(x,y,z)\right)^2+\left(\frac{\partial}{\partial z} f(x,y,z)\right)^2\right)^{(3/2)}\right]$$

$$Interaction := \Bigg(-\left(\frac{\partial}{\partial x} f(x,y,z)\right)^2\left(\frac{\partial^2}{\partial z\,\partial y} f(x,y,z)\right)^2+\left(\frac{\partial}{\partial x} f(x,y,z)\right)^2\left(\frac{\partial^2}{\partial y^2} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2} f(x,y,z)\right)$$

$$+2\left(\frac{\partial}{\partial x} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y} f(x,y,z)\right)\left(\frac{\partial}{\partial z} f(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x,y,z)\right)$$

$$+2\left(\frac{\partial}{\partial x} f(x,y,z)\right)\left(\frac{\partial}{\partial y} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y} f(x,y,z)\right)$$

$$-2\left(\frac{\partial}{\partial x} f(x,y,z)\right)\left(\frac{\partial}{\partial y} f(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2} f(x,y,z)\right)$$

$$-2\left(\frac{\partial}{\partial x} f(x,y,z)\right)\left(\frac{\partial}{\partial z} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x} f(x,y,z)\right)\left(\frac{\partial^2}{\partial y^2} f(x,y,z)\right)-\left(\frac{\partial^2}{\partial z\,\partial x} f(x,y,z)\right)^2\left(\frac{\partial}{\partial y} f(x,y,z)\right)^2$$

$$-\left(\frac{\partial^2}{\partial y\,\partial x} f(x,y,z)\right)^2\left(\frac{\partial}{\partial z} f(x,y,z)\right)^2+2\left(\frac{\partial^2}{\partial y\,\partial x} f(x,y,z)\right)\left(\frac{\partial}{\partial y} f(x,y,z)\right)\left(\frac{\partial}{\partial z} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x} f(x,y,z)\right)$$

$$+\left(\frac{\partial^2}{\partial x^2} f(x,y,z)\right)\left(\frac{\partial^2}{\partial y^2} f(x,y,z)\right)\left(\frac{\partial}{\partial z} f(x,y,z)\right)^2+\left(\frac{\partial^2}{\partial x^2} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2} f(x,y,z)\right)\left(\frac{\partial}{\partial y} f(x,y,z)\right)^2$$

$$-2\left(\frac{\partial}{\partial z} f(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y} f(x,y,z)\right)\left(\frac{\partial}{\partial y} f(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2} f(x,y,z)\right)\Bigg) \Bigg/$$

$$\left(\left(\frac{\partial}{\partial x} f(x,y,z)\right)^2+\left(\frac{\partial}{\partial y} f(x,y,z)\right)^2+\left(\frac{\partial}{\partial z} f(x,y,z)\right)^2\right)^2$$

$$DivJ := 0$$

It is now apparent that the interaction between the potentials and the conserved current is equal to the Adjoint curvature of the simple surface.

EXAMPLE

\>

The Monkey saddle as an implicit surface phi(x,y,z) mapped to zero.

```
>
> Phi:=z-x^3+3*x*y^2;
```

$$\Phi := z - x^3 + 3\,x\,y^2$$

\>

Construct the Normal field to the implicit surface as the gradient of Phi(x,y,z) with respect to (x,y,z)

```
> A1:=diff(Phi,x);A2:=diff(Phi,y);A3:=diff(Phi,z);
```

$$A1 := -3\,x^2 + 3\,y^2$$

$$A2 := 6\,x\,y$$

$$A3 := 1$$

```
> A:=[A1,A2,A3];
```

$$A := [-3\,x^2 + 3\,y^2,\ 6\,x\,y,\ 1]$$

Construct the Holder Norm and divide each component of the gradient by the Holder norm.

```
> lambda:=(a*A[1]^p+b*A[2]^p+c*A[3]^p)^(n/p);a:=1;b:=1;c:=1;
```

Force lambda to be homogeneous of degree 1 by choosing n = 1, specialize the degree to p =2, and euclidean signature 1,1,1 to make a correspondence with the Gauss map and classic implicit surface theory.

```
> n:=1;p:=2;
```

$$\lambda := \sqrt{(-3\,x^2 + 3\,y^2)^2 + 36\,x^2\,y^2 + 1}$$

$$a := 1$$

$$b := 1$$
$$c := 1$$
$$n := 1$$
$$p := 2$$

```
> NA:=evalm([A1,A2,A3]/lambda);
```

$$NA := \left[\frac{-3x^2+3y^2}{\sqrt{(-3x^2+3y^2)^2+36x^2y^2+1}}, 6\frac{xy}{\sqrt{(-3x^2+3y^2)^2+36x^2y^2+1}}, \frac{1}{\sqrt{(-3x^2+3y^2)^2+36x^2y^2+1}}\right]$$

```
>
```

Compute the Jacobian matrix of the normalized gradient

```
> JAC:=(jacobian(NA,[x,y,z]));DET:=factor(det(JAC));
```

$JAC :=$

$$\left[-\frac{1}{2}\frac{(-3x^2+3y^2)(-12(-3x^2+3y^2)x+72xy^2)}{((-3x^2+3y^2)^2+36x^2y^2+1)^{(3/2)}} - 6\frac{x}{\sqrt{(-3x^2+3y^2)^2+36x^2y^2+1}},\right.$$
$$\left.-\frac{1}{2}\frac{(-3x^2+3y^2)(12(-3x^2+3y^2)y+72x^2y)}{((-3x^2+3y^2)^2+36x^2y^2+1)^{(3/2)}} + 6\frac{y}{\sqrt{(-3x^2+3y^2)^2+36x^2y^2+1}}, 0\right]$$
$$\left[-3\frac{xy(-12(-3x^2+3y^2)x+72xy^2)}{((-3x^2+3y^2)^2+36x^2y^2+1)^{(3/2)}} + 6\frac{y}{\sqrt{(-3x^2+3y^2)^2+36x^2y^2+1}},\right.$$
$$\left.-3\frac{xy(12(-3x^2+3y^2)y+72x^2y)}{((-3x^2+3y^2)^2+36x^2y^2+1)^{(3/2)}} + 6\frac{x}{\sqrt{(-3x^2+3y^2)^2+36x^2y^2+1}}, 0\right]$$
$$\left[-\frac{1}{2}\frac{-12(-3x^2+3y^2)x+72xy^2}{((-3x^2+3y^2)^2+36x^2y^2+1)^{(3/2)}}, -\frac{1}{2}\frac{12(-3x^2+3y^2)y+72x^2y}{((-3x^2+3y^2)^2+36x^2y^2+1)^{(3/2)}}, 0\right]$$

$$DET := 0$$

```
> MEAN_CURVATURE:=factor(trace(JAC)/2);
```

$$MEAN\_CURVATURE := -27\frac{x(x^2+y^2)(-x^2+3y^2)}{(9x^4+18x^2y^2+9y^4+1)^{(3/2)}}$$

Note the classic formula for the mean curvature of a 3D implicit surface in xyz is obtained.

```
> S2:=factor(trace(innerprod(JAC,JAC))):
  Gauss:=factor(-(1/2)*(-trace(JAC)*trace(JAC)+S2));
```

$$Gauss := -36\frac{x^2+y^2}{(9x^4+18x^2y^2+9y^4+1)^2}$$

```
> ADJAC:=adjoint(JAC):
> ADJOINT_CURVATURE:=factor(trace(ADJAC));
```

$$ADJOINT\_CURVATURE := -36\frac{x^2+y^2}{(9x^4+18x^2y^2+9y^4+1)^2}$$

```
>
```

Note that the classic formula for the ADJOINT Gauss curvature of a 3D implicit surface in xyz is obtained.

Now define the adjoint current, and show that in 3D the adjoint current times the homogeneous vector is equal to the trace of the adjoint matrix. Moreover show that the 3D divergence of the adjoint current so defined is globally zero.

```
> CurrentJ:=innerprod(ADJAC,NA);Interaction:=innerprod(CurrentJ,NA);DivJ:=facto
  r(diverge(CurrentJ,[x,y,z]));
```

$$CurrentJ := \left[0,\ 0,\ -36\,\frac{9\,x^6 + 27\,y^2\,x^4 + 27\,x^2\,y^4 + 9\,y^6 + x^2 + y^2}{\left(9\,x^4 + 18\,x^2\,y^2 + 9\,y^4 + 1\right)^{(5/2)}}\right]$$

$$Interaction := -36\,\frac{x^2 + y^2}{\left(9\,x^4 + 18\,x^2\,y^2 + 9\,y^4 + 1\right)^2}$$

$$DivJ := 0$$

>