```
> restart:
```

faraday.mws (R.M.Kiehn, example created 9/1/97)
Using MAPLE as a symbolic math calculator to compute
Maxell-Faraday formulas from an arbitrary 1-form of Action on 4D space time.
The space-time is not constrained by any metric, group, or connection.
The program allows the computation of the vector part and scalar part (the helicity) of the torsion current,
R. M. Kiehn,et.al.Phys Rev A, 43 , (1991) p. 5665
or
http://www.uh.edu/~rkiehn/pdf/maxwell.pdf
(which, if it exists, is the equivalent to Poincare formulation of the B3 concept
as given by Evans in
"Helicity and the Electromagnetic Field" Eq 55
www.europa.com/~rsc/physics/B3/evans

```
>  with(liesymm):with(linalg):with(plots):
Warning, new definition for close
Warning, new definition for norm
Warning, new definition for trace
```

```
> setup(x,y,z,t);
```

$$[x, y, z, t]$$

The independent variables are assumed as {x,y,z,t}
Forms are intitialized below:

```
> defform(x=0,y=0,z=0,t=0,a=const,b=const,c=const,k=const,mu=const,m=const,alpha=const);
```

$$\text{defform}(x = 0, y = 0, z = 0, t = 0, a = const, b = const, c = const, k = const, \mu = const, m = const, \alpha = const)$$

The differential position vector on the domain:

```
> dR:=[d(x),d(y),d(z),d(t)];
```

$$dR := [\,\text{d}(x), \text{d}(y), \text{d}(z), \text{d}(t)\,]$$

Specify the four functions that are the covariant components of the Action 1-form.

```
> A1:=Ax(x,y,z,t);A2:=Ay(x,y,z,t);A2:=Ay(x,y,z,t);A4:=phi(x,y,z,t);
>
```

$$A1 := \text{Ax}(x, y, z, t)$$
$$A2 := \text{Ay}(x, y, z, t)$$
$$A2 := \text{Ay}(x, y, z, t)$$
$$A4 := \phi(x, y, z, t)$$

Skip the next line for abstract formulas, otherwise insert explicit functional forms

```
> A1:=(B*cos(y)+C*sin(z));A2:=(C*cos(z)+A*sin(x));A3:=(A*cos(x)+B*sin(y));A4:=c;
```

$$A1 := B\cos(y) + C\sin(z)$$
$$A2 := C\cos(z) + A\sin(x)$$
$$A3 := A\cos(x) + B\sin(y)$$
$$A4 := c$$

The above example is a modification of Arnold's famous ABC flow in hydrodynamics, which is known to generate chaotic evolutionary trajectories. You can change the functional forms and let MAPLE due to symbolic math.

```
> Action:=A1*d(x)+A2*d(y)+A3*d(z)-A4*d(t);
```

$$Action := (B\cos(y) + C\sin(z))\,\text{d}(x) + (C\cos(z) + A\sin(x))\,\text{d}(y) + (A\cos(x) + B\sin(y))\,\text{d}(z) - c\,\text{d}(t)$$

```
> F:=wcollect(d(Action));
```

$$F := (-A\sin(x) - C\cos(z))\,((\mathrm{d}(x))\ \&\wedge\ (\mathrm{d}(z))) + (B\cos(y) + C\sin(z))\,((\mathrm{d}(y))\ \&\wedge\ (\mathrm{d}(z))) + (A\cos(x) + B\sin(y))\,((\mathrm{d}(x))\ \&\wedge\ (\mathrm{d}(y)))$$

Vector potential Av

```
> Av:=[A1,A2,A3];
```

$$Av := [B\cos(y) + C\sin(z),\ C\cos(z) + A\sin(x),\ A\cos(x) + B\sin(y)]$$

Magnetic field vector Bv

```
> Bv:=curl(Av,[x,y,z]);
```

$$Bv := [B\cos(y) + C\sin(z),\ C\cos(z) + A\sin(x),\ A\cos(x) + B\sin(y)]$$

Electric Field

```
> E:=[-diff(A4,x)-diff(Av[1],t),-diff(A4,y)-diff(Av[2],t),-diff(A4,z)-diff(Av[3],t)];
```

$$E := [0, 0, 0]$$

The E and B fields satisfy Maxwell's equations, curlE+partialdB/dt = 0 and div B = 0

Parity (Second Poncare invariant). For the ABC 4-potential, there is no E field, hence the system would be equivalent to a Lorentz plasma state, in the sense that the magnetic energy is always greater that the electric energy density.

```
> EdotB:=factor(simplify(inner(E,Bv)));
```

$$EdotB := 0$$

The Torsion current.

```
> ExAv:=crossprod(E,Av);Bphi:=[Bv[1]*A4,Bv[2]*A4,Bv[3]*A4];
```

$$ExAv := [0, 0, 0]$$

$$Bphi := [(B\cos(y) + C\sin(z))\,c,\ (C\cos(z) + A\sin(x))\,c,\ (A\cos(x) + B\sin(y))\,c]$$

The vector part of the Torsion current is the Poincare form of B3

```
> TORSB3:=evalm(ExAv+A4*Bv);
```

$$TORSB3 := [(B\cos(y) + C\sin(z))\,c,\ (C\cos(z) + A\sin(x))\,c,\ (A\cos(x) + B\sin(y))\,c]$$

The Helicity density is given by the expression

```
> AdotB:=simplify(inner(Av,Bv));
```

$$AdotB := 2\,B\cos(y)\,C\sin(z) + C^2 + 2\,C\cos(z)\,A\sin(x) + A^2 + 2\,A\cos(x)\,B\sin(y) + B^2$$

The 4 vector of Torsion current becomes:

```
> TORSION:=[TORSB3[1],TORSB3[2],TORSB3[3],AdotB];
```

$$TORSION := [(B\cos(y) + C\sin(z))\,c,\ (C\cos(z) + A\sin(x))\,c,\ (A\cos(x) + B\sin(y))\,c,\ 2\,B\cos(y)\,C\sin(z) + C^2 + 2\,C\cos(z)\,A\sin(x) + A^2 + 2\,A\cos(x)\,B\sin(y) + B^2]$$

Divergence of the Torsion current. If the E.B Poincare invariant is zero then the Torsion current satisfies a conservation law. 4divT = 0.

```
> DIVT:=factor(diverge(TORSION,[x,y,z,t]));
```

$$DIVT := 0$$

An evolutionary dynamical system.

```
> VV:=[U(x,y,z,t),V(x,y,z,t),W(x,y,z,t)];
```

$$VV := [\mathrm{U}(x, y, z, t),\ \mathrm{V}(x, y, z, t),\ \mathrm{W}(x, y, z, t)]$$

The Lorentz force.

```
> FL:=evalm(E+crossprod(VV,Bv));P:=innerprod(VV,E);
```

$$FL := [\mathrm{V}(x, y, z, t)\,(A\cos(x) + B\sin(y)) - \mathrm{W}(x, y, z, t)\,(C\cos(z) + A\sin(x)),$$
$$\mathrm{W}(x, y, z, t)\,(B\cos(y) + C\sin(z)) - \mathrm{U}(x, y, z, t)\,(A\cos(x) + B\sin(y)),$$

$U(x, y, z, t)\,(C\cos(z) + A\sin(x)) - V(x, y, z, t)\,(B\cos(y) + C\sin(z))]$

$$P := 0$$

Check for evolution in the direction of the Torsion field.

```
> FLIRR:=evalm(E*AdotB+crossprod(TORSB3,Bv));PIRR:=innerprod(TORSB3,E);ZZIR:=simplify(TORSB3[1]*A1+TORSB3[2]*A2+TORSB3[3]*A3-AdotB*A4);
```

$$FLIRR := [0, 0, 0]$$

$$PIRR := 0$$

$$ZZIR := 0$$

When all three terms (with suffix IRR) vanish, it means that the torsion vector is a characteristic vector of the 1-form of action. Then the Lie derivative of the Action is zero, and the Lie derivative of the 2-form dA is zero. Such evolution corresponds to an adiabatic process. All components of the Cartan topology are preserved. The Torsion vector field in this case is more than a Hamiltonian field,
it is a characteristic field! (due to the fact that E.B=0). Characteristic fields are typical of classical "wave" solutions that admit discontinuities.

The virtual work 1-form

```
> Work:=FL[1]*d(x)+FL[2]*d(y)+FL[3]*d(z)-P*d(t);
```

$$Work := (V(x, y, z, t)\,(A\cos(x) + B\sin(y)) - W(x, y, z, t)\,(C\cos(z) + A\sin(x)))\,d(x) + (W(x, y, z, t)\,(B\cos(y) + C\sin(z)) - U(x, y, z, t)\,(A\cos(x) + B\sin(y)))\,d(y) + (U(x, y, z, t)\,(C\cos(z) + A\sin(x)) - V(x, y, z, t)\,(B\cos(y) + C\sin(z)))\,d(z)$$

If the exterior differential of the Work 1-form vanishes, then the motion induced by the dynamical system satisfies the Helmholtz condition (and vorticity is conserved.) In space time this is implies that the evolution is a symplectomorphism. A subset of symplecto-morphisms are conservative Hamiltonian systems.

```
> Helmholtz:=wcollect(d(Work));
```

$$Helmholtz := \Big(U(x, y, z, t)\,A\cos(x) + \left(\frac{\partial}{\partial x}U(x, y, z, t)\right)C\cos(z) + \left(\frac{\partial}{\partial x}U(x, y, z, t)\right)A\sin(x) - \left(\frac{\partial}{\partial x}V(x, y, z, t)\right)B\cos(y) - \left(\frac{\partial}{\partial x}V(x, y, z, t)\right)C\sin(z) - \left(\frac{\partial}{\partial z}V(x, y, z, t)\right)A\cos(x) - \left(\frac{\partial}{\partial z}V(x, y, z, t)\right)B\sin(y) - W(x, y, z, t)\,C\sin(z) + \left(\frac{\partial}{\partial z}W(x, y, z, t)\right)C\cos(z) + \left(\frac{\partial}{\partial z}W(x, y, z, t)\right)A\sin(x)\Big)((d(x))\,\&\wedge\,(d(z))) + \Big(-\left(\frac{\partial}{\partial z}W(x, y, z, t)\right)B\cos(y) - \left(\frac{\partial}{\partial z}W(x, y, z, t)\right)C\sin(z) - W(x, y, z, t)\,C\cos(z) + \left(\frac{\partial}{\partial z}U(x, y, z, t)\right)A\cos(x) + \left(\frac{\partial}{\partial z}U(x, y, z, t)\right)B\sin(y) + \left(\frac{\partial}{\partial y}U(x, y, z, t)\right)C\cos(z) + \left(\frac{\partial}{\partial y}U(x, y, z, t)\right)A\sin(x) - \left(\frac{\partial}{\partial y}V(x, y, z, t)\right)B\cos(y) - \left(\frac{\partial}{\partial y}V(x, y, z, t)\right)C\sin(z) + V(x, y, z, t)\,B\sin(y)\Big)((d(y))\,\&\wedge\,(d(z))) + \Big(-\left(\frac{\partial}{\partial t}U(x, y, z, t)\right)A\cos(x) - \left(\frac{\partial}{\partial t}U(x, y, z, t)\right)B\sin(y) + \left(\frac{\partial}{\partial t}W(x, y, z, t)\right)B\cos(y) + \left(\frac{\partial}{\partial t}W(x, y, z, t)\right)C\sin(z)\Big)((d(t))\,\&\wedge\,(d(y))) + \Big(-\left(\frac{\partial}{\partial t}V(x, y, z, t)\right)B\cos(y) - \left(\frac{\partial}{\partial t}V(x, y, z, t)\right)C\sin(z) + \left(\frac{\partial}{\partial t}U(x, y, z, t)\right)C\cos(z) + \left(\frac{\partial}{\partial t}U(x, y, z, t)\right)A\sin(x)\Big)((d(t))\,\&\wedge\,(d(z))) + \Big(-\left(\frac{\partial}{\partial y}V(x, y, z, t)\right)A\cos(x) - \left(\frac{\partial}{\partial y}V(x, y, z, t)\right)B\sin(y) - V(x, y, z, t)\,B\cos(y)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{W}(x,y,z,t)\right)C\cos(z)+\left(\frac{\partial}{\partial y}\mathrm{W}(x,y,z,t)\right)A\sin(x)+\mathrm{U}(x,y,z,t)\,A\sin(x)+\left(\frac{\partial}{\partial x}\mathrm{W}(x,y,z,t)\right)B\cos(y)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{W}(x,y,z,t)\right)C\sin(z)-\left(\frac{\partial}{\partial x}\mathrm{U}(x,y,z,t)\right)A\cos(x)-\left(\frac{\partial}{\partial x}\mathrm{U}(x,y,z,t)\right)B\sin(y)\Bigg)((\mathrm{d}(x))\ \&\wedge\ (\mathrm{d}(y)))+$$
$$\left(-\left(\frac{\partial}{\partial t}\mathrm{W}(x,y,z,t)\right)C\cos(z)-\left(\frac{\partial}{\partial t}\mathrm{W}(x,y,z,t)\right)A\sin(x)+\left(\frac{\partial}{\partial t}\mathrm{V}(x,y,z,t)\right)A\cos(x)+\left(\frac{\partial}{\partial t}\mathrm{V}(x,y,z,t)\right)B\sin(y)\right)$$
$$((\mathrm{d}(t))\ \&\wedge\ (\mathrm{d}(x)))$$

>

>

>

Each algebraic factor must vanish it the Helmholtz conservation law is to be true. The implication is that there are six partial differential equations that must be satisfied for the evolution to be thermodynamically reversible. Such a constraint is always true if EdotB vanishes, for then there exists a UNIQUE hamiltonian vector field of conservative evolution on the resultant contact manifold of topological dimension 3.

Note that the Poincare formulation of B3 is as a covariant tensor field of rank 3, which is covariant with respect to ALL diffeomorphisms, Lorentz, Galilean, or any map with an inverse and a differentiable Jacobian inverse.

. If E.B=0, the conservation law is valid in all frames of reference that can be mapped c1 onto the original formulation. In this case, the conservation law is retrodictive even though the topology of the range space is Not the same as the topology of the target space.

1. NOTE THAT the Poincare completely covariant formulation of the Topological Torsion tensor is NOT Gauge invariant. Any closed 1-form can be added to the Action and the MAXWELL FIELD EQUATIONS are always GAUGE INVARIANT, but the Topological Torsion Tensor depends upon the Closed, but not Exact contributions (Harmonic, not gradient contributions) to the vector potentials. It depends on the induced Cartan Topology!

However, if the Torsion Current is closed for a given choice of 4 potentials, then it is closed for any closed gauge addition to the 1-form of Action. E.B =0 is not influenced by any closed gauge addition to the original 1-form of Action. The details of the adiabatic Torsion path do depend upon the gauge addition. Hence the path that leaves the action adiabatically invariant before the gauge addition will not necessarily be the same after the gauge addition. The flux quantum number and the helicity can change, but their limit sets do NOT. It is this idea which makes up the concept of gauge invariance..

2. NOTE that the Poincare completely covariant formulation of the Topological Torsion tensor is not dependent upon the existence of complex or a non-abelian set of vector potentials.

As the B3 formalism of Evans seems to require a Non-Abelian vector valued set of potentials, then such a set of constraints implies that additional structure has been imposed upon the electromagnetic domain, other that that given by the topological considerations above.

There is no doubt that longitudinal B fields can be constructed from set of vector potentials

The fundamental question is that of their evolution. Are they truely waves in the simple sense?

I think not. The solutions that lead to waves (propagating discontinuities) with longitudinal components must be of a quaternion nature, and here I agree with Evans that such fields must belong to a non-abelian set.

see Kiehn. et. al. Phys Rev A, 43 , (1991) p. 5665

These waves may be projectively equivalent to the vacuum state, but it is hard for me to believe that they are Lorentz equivalent to the vacuum state. The Lorentz transformation preserves isotropy and homogeneity, which need not be part of the projective group.

See V. Fock "Space Time and Gravitation" 1932 appendix

The upshot of all of this is that I see three dimensional magnetic fields as being a normal property of electromagnetism.
I see the Poincare formuation of B3 as a consequence of non-unique integrability of the 1-form of Action used to construct the electromagnetic field.   I have no problem with longitudinal B.
I see that the Poincare formulation is naturally covariant with respect to ALL diffeomorphisms, so Lorentz covariance is trivial.
The concept of gauge invariance, meaning invariance of the 2-forms (E,B) and 4-forms (E.B) with respect to Additions of Closed 1-forms to the original 1-form describing a physical system is also trivial. (THIS DOES NOT MEAN that the integral of the closed three form of Topological Torsion over a closed domain is zero, or insensitive to gauge additions. Such an integral is always a rational multiple of some scaling parameter, when the 3-form is closed)

What I have yet to derive is the idea that this implies O3 symmetry.
O3 symmetry is compatable with the notions, but so is U1, or any other!

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