

```

> restart;
>
with(liesymm):with(linalg):setup(x,y,z,t);deforms(a=const,b=const,c=const,d=const,p=const,n=const);
Warning, new definition for close
Warning, new definition for norm
Warning, new definition for trace

```

[x, y, z, t]
deforms(a = const, b = const, c = const, d = const, p = const, n = const)

HOLDER NORMS

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Euclidean Three Dimensions

In the theory of implicit surfaces in 3 dimensions, the Jacobian matrix of the Gauss Map can be used to construct the Shape Matrix of the surface.

That is, for an implicit 2 surface in 3 dimensions, defined by the constraint, $\phi(x,y,z) = 0$, construct the 3 component gradient, which is orthogonal to the tangent vectors of the surface.

Divide the gradient field by the euclidean homogeneous degree 1, isotropic, norm.

$\lambda = (\text{square root of the sum of gradient components squared with positive unit coefficients}).$

The renormalized vector is now homogeneous of degree zero.

Next construct the Jacobian matrix [JAC] of this "normalized" homogeneous vector.

The matrix elements consist of the partial derivatives of the normalized components of the vector.

Each matrix element is a function of the N=3 independent variables (x,y,z)

Form the Characteristic Polynomial of this matrix and determine the Similarity Invariants.

The similarity invariants are the sum of the eigen values, the sum of the products of pairs of the eigen values, etc.

The similarity invariants are functions on the domain of independent variables.

They are called similarity invariants, for if the matrix [JAC] is changed into a new matrix by means of an invertible transformation, [T],

such that $[JAC\ new] = [T] [JAC\ old] [T\ inverse]$, then the similarity invariants of the new matrix are the same as the similarity invariants of the old matrix.

The Gauss construction always has one zero eigenvalue. hence the determinant of the Jacobian is zero. The Jacobian matrix, when diagonalized, has 1 zero along its diagonal, and two other, usually non-zero, diagonal entries.

The fact that the determinant of the matrix vanishes is a clue to the existence of a "global" 2 surface, and is due to the fact that the divisor that defines the "unit length" is homogeneous of degree 1.

The trace of the matrix (some of the two non-zero eigenvalues) yields twice the **mean curvature**. (the first order similarity invariant) of the surface.

The trace of the adjoint matrix (or the product of the remaining two eigen values) always yields the **Gauss sectional curvature**. (the second order similarity invariant).

If the mean curvature vanishes, then the surface is called a minimal surface.

Three Dimensions (not necessarily Euclidean)

A more general definition of the normalization denominator is given by the Holder Norm, constructed in terms of the components of the vector by the formula

>
 > **lambda := (a*U^p+b*V^p+c*W^p)^(n/p);**

$$\lambda := (a U^p + b V^p + c W^p)^{\left(\frac{n}{p}\right)}$$

The constant coefficients, a,b,c can have any values (plus or minus).

The determinant of the Jacobian matrix constructed from the renormalized vector **always** vanishes for Holder divisors which are homogeneous of degree 1 (any p, any a,b,c, but n=1)

$$\lambda := (a U^p + b V^p + c W^p)^{\left(\frac{n}{p}\right)}$$

The orthogonal field to a classic implicit surface yields a differential (exact) 1-form constructed from the gradient of the function whose zero set defines the surface.

> **A:=d(Phi(x,y,z));**

$$A := \left(\frac{\partial}{\partial z} \Phi(x, y, z)\right) d(z) + \left(\frac{\partial}{\partial x} \Phi(x, y, z)\right) d(x) + \left(\frac{\partial}{\partial y} \Phi(x, y, z)\right) d(y)$$

This exact 1-form can be made homogeneous of degree 1-n by dividing by the Holder norm .

However, the Jacobian process and its similarity constructions works for any arbitrary 1-form., A, exact or not.

> **A:=U*d(x)+V*d(y)+W*d(z);**

$$A := U d(x) + V d(y) + W d(z)$$

The procedure is to divide the arbitrary 1-form by the Holder norm, then construct the Jacobian matrix of partial derivatives.

The resulting matrix is a covariant tensor of second rank, but it is not necessarily symmetric.

The Jacobian matrix, as before, has a set of similarity invariants.

If the Holder norm is homogeneous of degree 1 (n= 1 any p, a,b,c...) then the determinant of the resultin Jacobian matrix is zero.

The result implies that the inverse matrix for the Jacobian matrix does not exist.

However, there always exists an algebraically well defined "adjoint" matrix (the matrix of cofactors transposed) which is a contravariant tensor of second rank) even though the inverse does not exist. This adjoint matrix is not necessarily symmetric, but may be used as a "raising" tensor, similar to the use of the inverse metric in Riemannian spaces with metrc. However, it is to be emphasized that the spaces being considered need not be Riemannian, and no (symmetric and invertable) metric has been assumed to constrain the space of interest.

The "Dual" Current J starting from a covariant 1-form A

When the covariant components of the 1-form are premultiplied by the adjoint matrix, a contravariant "adjoint dual" field is produced.

This dual field can be represented by an N-1 form, J. When the Holder norm is homogeneous of degree 1, then the exterior derivative of J vanishes. A physicist would say that the "divergence of the current J" is zero. Such statements lead to topological conservation laws.

This Jacobian process works in all dimensions, and does not utilize a metric in the usual sense.

Note that for a given 1-form A there are many J with dJ = 0. The Holder norm is like an integrating factor, but it is not unique.

In physics of electromagnetism, this suggests (the heretical statement) that the 1-form of potentials A is more fundamental than the N-1 form of conserved currents J.

Guided by electromagnetic experience, it is of interest to compute the "interaction" density, which is defined as the exterior product of the 1-form A and its dual (non-unique) N-1 form current, J

The coefficient of the N-form is equal to the "inner product" of the co and contra vectors, A and J.

Starting from a contravariant J, find the associated A

Note that given a covariant A it is possible to construct many J,

but given a contravariant J the procedure to construct an A is not as well defined.

The procedure analogous to that described above, would start with a contravariant J, renormalizing with the Holder norm,, and then constructing a Jacobian matrix of mixed partial derivatives, [JAC(J)]. This matrix construction yields a **mixed** tensor of rank 2.

Note that [JAC(A)] is not equal to [JAC(J)]. They are both second rank tensors, but the latter is a mixed tensor, while the former is a covariant tensor of second rank.

The mixed tensor (or its adjoint) **cannot** be used as a "lowering" tensor (like a metric in Riemannian spaces) to construct a covariant A field from the contravariant J field.

A further assumption has to be made to achieve the A dual to the J.

A possibility that is often used is to construct the metric field, [g] = [JAC(J) transpose] [JAC(J)]

Then given J, an associated "dual" A follows from A = [g] J.

Note that this procedure loses the anti-symmetry features of the Jacobian, as g is a symmetric matrix.

>

Example 1: Start with a 1 form A

Let us start with an abstract example: Note that for any a,b,c,p, if n = 1 such that the divisor is homogenous of degree 1, then the determinant of the Jacobian of the rescaled co-vector is zero.

It is also true that if the rescaled co-vector whose components are used to construct the 1-form A, is premultiplied by the adjoint of the jacobian matrix, then a "dual" contra-vector J is constructed which has zero divergence. To prove these statements, construct the component functions:

> **U:=u(x,y,z);V:=v(x,y,z);W:=w(x,y,z);**

$$U := u(x, y, z)$$

$$V := v(x, y, z)$$

$$W := w(x, y, z)$$

and the rescaled components of the 1 form A:

> **Potentials:=evalm([U,V,W]/(lambda));**

$$\text{Potentials} := \left[\begin{array}{c} \frac{u(x, y, z)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)}, \quad \frac{v(x, y, z)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)}, \\ \frac{w(x, y, z)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)}} \end{array} \right]$$

Construct the Jacobian matrix by constructing the array of partial derivatives, and show that for n=1, the determinant vanishes.

> **JAC(A):=jacobian(Potentials,[x,y,z]);DET:=factor(det(JAC(A)));**

JAC(u(x, y, z) d(x) + v(x, y, z) d(y) + w(x, y, z) d(z)) :=

$$\left[\begin{array}{c} u(x, y, z) n \left(\frac{a u(x, y, z)^p p \left(\frac{\partial}{\partial x} u(x, y, z) \right)}{u(x, y, z)} + \frac{b v(x, y, z)^p p \left(\frac{\partial}{\partial x} v(x, y, z) \right)}{v(x, y, z)} + \frac{c w(x, y, z)^p p \left(\frac{\partial}{\partial x} w(x, y, z) \right)}{w(x, y, z)} \right) \\ - \frac{\left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p}\right)} p \left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)}{\frac{\partial}{\partial x} u(x, y, z)} \\ + \frac{\left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p}\right)}}{\left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)} \end{array} \right]$$

$$\begin{aligned}
& \left. \begin{aligned}
& + \frac{\frac{\partial}{\partial z} v(x, y, z)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)}} \right] \\
& \left[- \frac{w(x, y, z) n \left(\frac{a u(x, y, z)^p p \left(\frac{\partial}{\partial x} u(x, y, z) \right)}{u(x, y, z)} + \frac{b v(x, y, z)^p p \left(\frac{\partial}{\partial x} v(x, y, z) \right)}{v(x, y, z)} + \frac{c w(x, y, z)^p p \left(\frac{\partial}{\partial x} w(x, y, z) \right)}{w(x, y, z)} \right)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)} p (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)} \right. \\
& \left. + \frac{\frac{\partial}{\partial x} w(x, y, z)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)}} \right], \\
& \left[- \frac{w(x, y, z) n \left(\frac{a u(x, y, z)^p p \left(\frac{\partial}{\partial y} u(x, y, z) \right)}{u(x, y, z)} + \frac{b v(x, y, z)^p p \left(\frac{\partial}{\partial y} v(x, y, z) \right)}{v(x, y, z)} + \frac{c w(x, y, z)^p p \left(\frac{\partial}{\partial y} w(x, y, z) \right)}{w(x, y, z)} \right)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)} p (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)} \right. \\
& \left. + \frac{\frac{\partial}{\partial y} w(x, y, z)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)}} \right], \\
& \left[- \frac{w(x, y, z) n \left(\frac{a u(x, y, z)^p p \left(\frac{\partial}{\partial z} u(x, y, z) \right)}{u(x, y, z)} + \frac{b v(x, y, z)^p p \left(\frac{\partial}{\partial z} v(x, y, z) \right)}{v(x, y, z)} + \frac{c w(x, y, z)^p p \left(\frac{\partial}{\partial z} w(x, y, z) \right)}{w(x, y, z)} \right)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)} p (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)} \right. \\
& \left. + \frac{\frac{\partial}{\partial z} w(x, y, z)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)}} \right] \\
& DET := \left((n-1) \left(\left(\frac{\partial}{\partial x} v(x, y, z) \right) \left(\frac{\partial}{\partial y} u(x, y, z) \right) \left(\frac{\partial}{\partial z} w(x, y, z) \right) - \left(\frac{\partial}{\partial x} v(x, y, z) \right) \left(\frac{\partial}{\partial z} u(x, y, z) \right) \left(\frac{\partial}{\partial y} w(x, y, z) \right) \right. \right. \\
& \left. - \left(\frac{\partial}{\partial x} w(x, y, z) \right) \left(\frac{\partial}{\partial z} v(x, y, z) \right) \left(\frac{\partial}{\partial y} u(x, y, z) \right) + \left(\frac{\partial}{\partial x} w(x, y, z) \right) \left(\frac{\partial}{\partial y} v(x, y, z) \right) \left(\frac{\partial}{\partial z} u(x, y, z) \right) \right. \\
& \left. + \left(\frac{\partial}{\partial x} u(x, y, z) \right) \left(\frac{\partial}{\partial z} v(x, y, z) \right) \left(\frac{\partial}{\partial y} w(x, y, z) \right) - \left(\frac{\partial}{\partial x} u(x, y, z) \right) \left(\frac{\partial}{\partial y} v(x, y, z) \right) \left(\frac{\partial}{\partial z} w(x, y, z) \right) \right) \Big/ \\
& \left((a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)^3} \right)
\end{aligned}$$

The determinant has a factor (n-1), hence for n=1 the determinant vanishes, for ANY co vector field.

Now construct the adjoint vector and show that this vector current J "dual to" A has a zero divergence (Jacobi's theorem).

The adjoint vector is constructed by multiplying the renormalized co-vector components by the adjoint matrix of the Jacobian

matrix [JAC(A)].

The adjoint matrix is the matrix of cofactors transposed, and would be equal to the inverse of the original matrix times the determinant of the original matrix -- if the determinant is non-zero. The adjoint construction exists whether or not the inverse exists.

> **ADJ:=adjoint(JAC(A));J:=(innerprod(ADJ,Potentials));DIVJ:=factor(diverge(J,[x,y,z]));**

$$\begin{aligned}
 J := & \left[\left(\left(\frac{\partial}{\partial y} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) + \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) \right. \right. \\
 & - \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial z} w(x, y, z) \right) - \left(\frac{\partial}{\partial z} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial y} v(x, y, z) \right) \\
 & \left. + \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial z} w(x, y, z) \right) - \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) \right] / \\
 & \left((a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)^3}, - \left(\left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \right. \right. \\
 & - \left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial z} w(x, y, z) \right) - \left(\frac{\partial}{\partial z} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) \\
 & \left. + \left(\frac{\partial}{\partial x} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) - \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \right. \\
 & \left. + \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial z} w(x, y, z) \right) \right) / \left((a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)^3}, \left(\right. \right. \\
 & - \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) + \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) \\
 & \left. + \left(\frac{\partial}{\partial x} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial y} v(x, y, z) \right) - \left(\frac{\partial}{\partial y} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) \right. \\
 & \left. - \left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) + \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \right) / \\
 & \left((a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)^3} \right) \\
 DIVJ := & 3 \left((n-1) \left(\left(\frac{\partial}{\partial x} v(x, y, z) \right) \left(\frac{\partial}{\partial y} u(x, y, z) \right) \left(\frac{\partial}{\partial z} w(x, y, z) \right) - \left(\frac{\partial}{\partial x} v(x, y, z) \right) \left(\frac{\partial}{\partial z} u(x, y, z) \right) \left(\frac{\partial}{\partial y} w(x, y, z) \right) \right. \right. \\
 & - \left(\frac{\partial}{\partial x} w(x, y, z) \right) \left(\frac{\partial}{\partial z} v(x, y, z) \right) \left(\frac{\partial}{\partial y} u(x, y, z) \right) + \left(\frac{\partial}{\partial x} w(x, y, z) \right) \left(\frac{\partial}{\partial y} v(x, y, z) \right) \left(\frac{\partial}{\partial z} u(x, y, z) \right) \\
 & \left. + \left(\frac{\partial}{\partial x} u(x, y, z) \right) \left(\frac{\partial}{\partial z} v(x, y, z) \right) \left(\frac{\partial}{\partial y} w(x, y, z) \right) - \left(\frac{\partial}{\partial x} u(x, y, z) \right) \left(\frac{\partial}{\partial y} v(x, y, z) \right) \left(\frac{\partial}{\partial z} w(x, y, z) \right) \right) / \\
 & \left((a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)^3} \right)
 \end{aligned}$$

The vector J is not zero, but its divergence has a factor of (n-1), hence the divergence of J is always zero for holder norms which are homogeneous of degree 1. This result is global, hence it would appear that the construction implies that J (for n=1) is exact.: J = dG. Note that the zero divergence result is independent from a,b,c,p for n=1

>

The "Interaction" coefficient can be computed as

> **Interaction:=factor(innerprod(J,Potentials));**

$$\begin{aligned}
\text{Interaction} := & - \left(-u(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) \right. \\
& - u(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) + u(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial z} w(x, y, z) \right) \\
& + u(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial y} v(x, y, z) \right) - \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z)^2 \left(\frac{\partial}{\partial z} w(x, y, z) \right) \\
& + \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z)^2 \left(\frac{\partial}{\partial y} w(x, y, z) \right) + \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z)^2 \left(\frac{\partial}{\partial x} w(x, y, z) \right) \\
& - \left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z)^2 \left(\frac{\partial}{\partial z} w(x, y, z) \right) - v(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) \\
& + v(x, y, z) \left(\frac{\partial}{\partial x} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) - v(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \\
& + v(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial z} w(x, y, z) \right) + w(x, y, z) \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \\
& - w(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) - \left(\frac{\partial}{\partial x} u(x, y, z) \right) w(x, y, z)^2 \left(\frac{\partial}{\partial y} v(x, y, z) \right) \\
& + \left(\frac{\partial}{\partial y} u(x, y, z) \right) w(x, y, z)^2 \left(\frac{\partial}{\partial x} v(x, y, z) \right) + w(x, y, z) \left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) \\
& \left. - w(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \right) / \left((a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)} \right)^4
\end{aligned}$$

The result is of interest for the classic specialization of the Holder norm to a=1,b=1,c=1,p=2, n=1.

For then the interaction A dot J is equal to the Gauss curvature !!!!

The formulas for the similarity invariants lead to complex partial differential equation. The special cases for a=b=c=n=1, p=2 are given explicitly below:

> Gauss:=simplify(factor(trace(ADJ))):Mean:=factor(trace(JAC(A))):MeanEuclidean:=simplify(subs(a=1,b=1,c=1,p=2,n=1,factor(trace(JAC(A)))));GaussEuclidean:=factor(subs(a=1,b=1,c=1,p=2,n=1,Gauss));

$$\begin{aligned}
\text{MeanEuclidean} := & - \left(u(x, y, z) v(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) + u(x, y, z) w(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \right. \\
& - \left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z)^2 - \left(\frac{\partial}{\partial x} u(x, y, z) \right) w(x, y, z)^2 + v(x, y, z) u(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) \\
& + v(x, y, z) w(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) - \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z)^2 - \left(\frac{\partial}{\partial y} v(x, y, z) \right) w(x, y, z)^2 \\
& + w(x, y, z) u(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) + w(x, y, z) v(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) - \left(\frac{\partial}{\partial z} w(x, y, z) \right) u(x, y, z)^2 \\
& \left. - \left(\frac{\partial}{\partial z} w(x, y, z) \right) v(x, y, z)^2 \right) / (u(x, y, z)^2 + v(x, y, z)^2 + w(x, y, z)^2)^{(3/2)}
\end{aligned}$$

$$\begin{aligned}
\text{GaussEuclidean} := & \left(\left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z)^2 \left(\frac{\partial}{\partial z} w(x, y, z) \right) + \left(\frac{\partial}{\partial x} u(x, y, z) \right) w(x, y, z)^2 \left(\frac{\partial}{\partial y} v(x, y, z) \right) \right. \\
& - u(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial y} v(x, y, z) \right) + u(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) \\
& \left. + u(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) + w(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) \right)
\end{aligned}$$

$$\begin{aligned}
& -w(x, y, z) \left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) + w(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \\
& -w(x, y, z) \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) + v(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) \\
& -u(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) \left(\frac{\partial}{\partial z} w(x, y, z) \right) - v(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) \left(\frac{\partial}{\partial z} w(x, y, z) \right) \\
& -v(x, y, z) \left(\frac{\partial}{\partial x} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) + v(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) w(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) \\
& - \left(\frac{\partial}{\partial y} u(x, y, z) \right) w(x, y, z)^2 \left(\frac{\partial}{\partial x} v(x, y, z) \right) + \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z)^2 \left(\frac{\partial}{\partial z} w(x, y, z) \right) \\
& - \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z)^2 \left(\frac{\partial}{\partial y} w(x, y, z) \right) - \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z)^2 \left(\frac{\partial}{\partial x} w(x, y, z) \right) \Big/ \\
& (u(x, y, z)^2 + v(x, y, z)^2 + w(x, y, z)^2)^2
\end{aligned}$$

The usual differential form processes can be applied to the 1-form A to construct the fields (Vorticity = dA) and the Helicity (A∧dA)

```

> A1:=factor(Potentials[1]):A2:=factor(Potentials[2]):A3:=factor(Potentials[3]):
>
> Action:=innerprod([A1,A2,A3],[d(x),d(y),d(z)]);

```

$$Action := \frac{u(x, y, z) d(x) + v(x, y, z) d(y) + w(x, y, z) d(z)}{(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)}}$$

```

> Vorticity:=wcollect(d(Action));Helicity:=simplify(wcollect(Action&^Vorticity));

```

$$\begin{aligned}
Vorticity := & \left(\left(n a u(x, y, z)^p \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) w(x, y, z) + n b v(x, y, z)^p \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) w(x, y, z) \right. \right. \\
& + n c w(x, y, z)^p \left(\frac{\partial}{\partial z} w(x, y, z) \right) u(x, y, z) v(x, y, z) - \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) w(x, y, z) a u(x, y, z)^p \\
& - \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) w(x, y, z) b v(x, y, z)^p - \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) w(x, y, z) c w(x, y, z)^p \Big/ \left(\right. \\
& (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)} u(x, y, z) w(x, y, z) (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p) \Big) + \left(\right. \\
& -n a u(x, y, z)^p \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) w(x, y, z) - n b v(x, y, z)^p \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z) w(x, y, z) \\
& - n c w(x, y, z)^p \left(\frac{\partial}{\partial y} w(x, y, z) \right) u(x, y, z) v(x, y, z) + \left(\frac{\partial}{\partial y} w(x, y, z) \right) u(x, y, z) v(x, y, z) a u(x, y, z)^p \\
& + \left(\frac{\partial}{\partial y} w(x, y, z) \right) u(x, y, z) v(x, y, z) b v(x, y, z)^p + \left(\frac{\partial}{\partial y} w(x, y, z) \right) u(x, y, z) v(x, y, z) c w(x, y, z)^p \Big/ \left(\right. \\
& (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{n}{p}\right)} u(x, y, z) v(x, y, z) (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p) \Big) \Big) \\
& (d(y) \wedge d(z)) + \left(\left(n a u(x, y, z)^p \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) w(x, y, z) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + n b v(x, y, z)^p \left(\frac{\partial}{\partial y} v(x, y, z) \right) u(x, y, z) w(x, y, z) + n c w(x, y, z)^p \left(\frac{\partial}{\partial y} w(x, y, z) \right) u(x, y, z) v(x, y, z) \\
& - \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) w(x, y, z) a u(x, y, z)^p - \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) w(x, y, z) b v(x, y, z)^p \\
& - \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) w(x, y, z) c w(x, y, z)^p \Big/ \left(\left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p} \right)} v(x, y, z) w(x, y, z) \left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right) \right) + \left(\right. \\
& - n a u(x, y, z)^p \left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z) w(x, y, z) - n b v(x, y, z)^p \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) w(x, y, z) \\
& - n c w(x, y, z)^p \left(\frac{\partial}{\partial x} w(x, y, z) \right) u(x, y, z) v(x, y, z) + \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) w(x, y, z) a u(x, y, z)^p \\
& + \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) w(x, y, z) b v(x, y, z)^p + \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) w(x, y, z) c w(x, y, z)^p \Big/ \left(\left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p} \right)} \right. \\
& \left. \left. \left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p} \right)} u(x, y, z) w(x, y, z) \left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right) \right) \right) \\
& (d(x) \ \&^{\wedge} \ d(y)) + \left(\left(n a u(x, y, z)^p \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) w(x, y, z) \right. \right. \\
& + n b v(x, y, z)^p \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z) w(x, y, z) + n c w(x, y, z)^p \left(\frac{\partial}{\partial z} w(x, y, z) \right) u(x, y, z) v(x, y, z) \\
& - \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) w(x, y, z) a u(x, y, z)^p - \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) w(x, y, z) b v(x, y, z)^p \\
& - \left. \left. \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z) w(x, y, z) c w(x, y, z)^p \right) \Big/ \left(\left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p} \right)} \right. \right. \\
& \left. \left. \left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p} \right)} v(x, y, z) w(x, y, z) \left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right) \right) \right) + \left(\right. \\
& - n a u(x, y, z)^p \left(\frac{\partial}{\partial x} u(x, y, z) \right) v(x, y, z) w(x, y, z) - n b v(x, y, z)^p \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) w(x, y, z) \\
& - n c w(x, y, z)^p \left(\frac{\partial}{\partial x} w(x, y, z) \right) u(x, y, z) v(x, y, z) + \left(\frac{\partial}{\partial x} w(x, y, z) \right) u(x, y, z) v(x, y, z) a u(x, y, z)^p \\
& + \left(\frac{\partial}{\partial x} w(x, y, z) \right) u(x, y, z) v(x, y, z) b v(x, y, z)^p + \left(\frac{\partial}{\partial x} w(x, y, z) \right) u(x, y, z) v(x, y, z) c w(x, y, z)^p \Big/ \left(\left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p} \right)} \right. \\
& \left. \left. \left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right)^{\left(\frac{n}{p} \right)} u(x, y, z) v(x, y, z) \left(a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p \right) \right) \right) \\
& (d(x) \ \&^{\wedge} \ d(z))
\end{aligned}$$

$$\begin{aligned}
\text{Helicity} := & - \ \&^{\wedge} (d(x), d(y), d(z)) \left(-v(x, y, z)^{(2+p)} u(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) w(x, y, z) b \right. \\
& + u(x, y, z)^{(2+p)} v(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) w(x, y, z) a - w(x, y, z)^{(2+p)} v(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) c \\
& + v(x, y, z)^{(2+p)} w(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) u(x, y, z) b - u(x, y, z)^{(2+p)} w(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) v(x, y, z) a
\end{aligned}$$

$$\begin{aligned}
& + v(x, y, z)^{(1+p)} \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z)^2 w(x, y, z) b + u(x, y, z)^{(1+p)} \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) w(x, y, z)^2 a \\
& + u(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z)^{(1+p)} w(x, y, z)^2 b + v(x, y, z) \left(\frac{\partial}{\partial z} v(x, y, z) \right) u(x, y, z)^2 w(x, y, z)^{(1+p)} c \\
& - w(x, y, z) \left(\frac{\partial}{\partial y} w(x, y, z) \right) u(x, y, z)^2 v(x, y, z)^{(1+p)} b - w(x, y, z)^{(1+p)} \left(\frac{\partial}{\partial y} w(x, y, z) \right) u(x, y, z)^2 v(x, y, z) c \\
& + w(x, y, z) \left(\frac{\partial}{\partial x} w(x, y, z) \right) u(x, y, z)^{(1+p)} v(x, y, z)^2 a + w(x, y, z)^{(1+p)} \left(\frac{\partial}{\partial x} w(x, y, z) \right) u(x, y, z) v(x, y, z)^2 c \\
& - v(x, y, z) \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z)^{(1+p)} w(x, y, z)^2 a - v(x, y, z)^{(1+p)} \left(\frac{\partial}{\partial x} v(x, y, z) \right) u(x, y, z) w(x, y, z)^2 b \\
& - u(x, y, z)^{(1+p)} \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z)^2 w(x, y, z) a - u(x, y, z) \left(\frac{\partial}{\partial z} u(x, y, z) \right) v(x, y, z)^2 w(x, y, z)^{(1+p)} c \\
& + w(x, y, z)^{(2+p)} u(x, y, z) \left(\frac{\partial}{\partial y} u(x, y, z) \right) v(x, y, z) c \Big/ \left(w(x, y, z) v(x, y, z) u(x, y, z) \right. \\
& \left. (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p)^{\left(\frac{2n}{p}\right)} (a u(x, y, z)^p + b v(x, y, z)^p + c w(x, y, z)^p) \right)
\end{aligned}$$

It is important to note that the renormalized 1-form of potentials A can be of Pfaff dimension greater than 2, even though the renormalization is homogeneous of degree 1. That is, the topological torsion, $A \wedge dA$ is not necessarily zero. In dimension 3, the set of points that yields a surface of zero Helicity is a surface where the curl of the potentials is orthogonal to the vector of potentials. Such surfaces are called Lamb surfaces in the theory of streamline hydrodynamics. This surface of zero helicity separates domains of positive from negative helicity.

A Monge surface Example:

The similarity invariants for the Jacobian matrix of the surface normal scaled by the Holder norm with $a=b=c=1, n=1, p=2$ fields the classic partial differential equations for the mean and Gauss curvature.

> **Phi := z - phi(x, y);**

$$\Phi := z - \phi(x, y)$$

> **A := grad(Phi, [x, y, z]);**

$$A := \left[-\left(\frac{\partial}{\partial x} \phi(x, y)\right), -\left(\frac{\partial}{\partial y} \phi(x, y)\right), 1 \right]$$

> **magn := innerprod(A, A);**

$$magn := \left(\frac{\partial}{\partial x} \phi(x, y)\right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y)\right)^2 + 1$$

> **NA := evalm(A/magn^(1/2));**

$$NA := \left[\begin{array}{c} -\frac{\frac{\partial}{\partial x} \phi(x, y)}{\sqrt{\left(\frac{\partial}{\partial x} \phi(x, y)\right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y)\right)^2 + 1}}, -\frac{\frac{\partial}{\partial y} \phi(x, y)}{\sqrt{\left(\frac{\partial}{\partial x} \phi(x, y)\right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y)\right)^2 + 1}}, \\ \frac{1}{\sqrt{\left(\frac{\partial}{\partial x} \phi(x, y)\right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y)\right)^2 + 1}} \end{array} \right]$$

> **JAC := jacobian(NA, [x, y, z]);**

JAC :=

$$MEAN_CURVATURE := -\frac{1}{2} \left(-2 \left(\frac{\partial}{\partial x} \phi(x, y) \right) \left(\frac{\partial}{\partial y} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right) + \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \right) / \left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^{3/2}$$

Note the classic formula for the mean curvature of a Monge surface is obtained.

> **ADJAC:=adjoint(JAC);**

ADJAC :=

[0, 0, 0]

[0, 0, 0]

$$\left[\begin{array}{l} \left(\frac{\partial}{\partial x} \phi(x, y) \right) \left(-\left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \right) \\ - \frac{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^2}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^2}, \\ \left(\frac{\partial}{\partial y} \phi(x, y) \right) \left(-\left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \right) \\ - \frac{\left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right)}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^2}, \frac{\left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right)}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^2} \end{array} \right]$$

> **GAUSS_CURVATURE:=factor(trace(ADJAC));**

$$GAUSS_CURVATURE := \frac{-\left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right)}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^2}$$

Note that the classic formula for the Gauss curvature of a Monge surface is obtained.

> **Current:=innerprod(ADJAC,NA);Interaction:=innerprod(Current,NA);DivJ:=diverge(Current,[x,y,z]);**

$$Current := \left[0, 0, \left(-\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) - \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) - \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \right) / \left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^{5/2} \right]$$

$$Interaction := \frac{-\left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right)}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^2}$$

DivJ := 0

It is now apparent that the interaction between the potentials and the conserved current is equal to the Gauss curvature of the Monge surface in

Euclidean space.

> **Vorticity:=curl(NA,[x,y,z]);Helicity:=innerprod(NA,Vorticity);**

$$\begin{aligned}
 \text{Vorticity} := & \left[\begin{aligned} & -\frac{1}{2} \frac{2 \left(\frac{\partial}{\partial x} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right) + 2 \left(\frac{\partial}{\partial y} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right)}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^{3/2}}, \\ & \frac{1}{2} \frac{2 \left(\frac{\partial}{\partial x} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) + 2 \left(\frac{\partial}{\partial y} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^{3/2}}, \\ & \frac{1}{2} \frac{\left(\frac{\partial}{\partial y} \phi(x, y) \right) \left(2 \left(\frac{\partial}{\partial x} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) + 2 \left(\frac{\partial}{\partial y} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right) \right)}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^{3/2}}, \\ & -\frac{1}{2} \frac{\left(\frac{\partial}{\partial x} \phi(x, y) \right) \left(2 \left(\frac{\partial}{\partial x} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right) + 2 \left(\frac{\partial}{\partial y} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \right)}{\left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^{3/2}} \end{aligned} \right] \\
 & \text{Helicity} := 0
 \end{aligned}$$

[NOTE that the Helicity of a Monge surface is zero, and the conserved current is NOT orthogonal to the surface,

> **crossprod(Current,NA);**

$$\begin{aligned}
 & \left[\left(\left(-\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) - \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 \right. \right. \\
 & \quad \left. \left. + \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) - \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \right) \left(\frac{\partial}{\partial y} \phi(x, y) \right) \right) / \\
 & \quad \left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^3, - \left(\left(-\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 \right. \right. \\
 & \quad \left. \left. + \left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) - \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 \right. \right. \\
 & \quad \left. \left. + \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) - \left(\frac{\partial^2}{\partial y \partial x} \phi(x, y) \right)^2 + \left(\frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left(\frac{\partial^2}{\partial x^2} \phi(x, y) \right) \right) \left(\frac{\partial}{\partial x} \phi(x, y) \right) \right) / \\
 & \quad \left(\left(\frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left(\frac{\partial}{\partial y} \phi(x, y) \right)^2 + 1 \right)^3, 0 \right]
 \end{aligned}$$

Explicit EXAMPLE for a nonlinear surface function. The current J is evaluated for n=1, and has zero divergence. Note that in this example The interaction N form, A~J, is a non-zero volume element almost everywhere. However there exists a surface for which the interaction vanishes.

> **U:=x*y;V:=y*z;W:=z*x;**

$$U := x y$$

$$V := y z$$

$$W := z x$$

>

```
>
> AA:=evalm([U,V,W]/(subs(a=1,b=1,c=1,p=2,n=1,lambda)));
```

$$AA := \left[\frac{xy}{\sqrt{x^2y^2 + y^2z^2 + z^2x^2}}, \frac{yz}{\sqrt{x^2y^2 + y^2z^2 + z^2x^2}}, \frac{zx}{\sqrt{x^2y^2 + y^2z^2 + z^2x^2}} \right]$$

Construct the Jacobin and show that for n=1, the determinant vanishes.

```
> JACWW:=jacobian(AA,[x,y,z]);DETJACAA:=factor(det(JACWW));Mean:=factor(trace(JACW
W)/2);ADJQQ:=adjoint(JACWW);GaussCurvature:=factor(trace(ADJQQ));
```

```
>
>
```

$$DETJACAA := 0$$

$$Mean := \frac{1}{2} \frac{y^3z^2 + z^3x^2 + y^2x^3}{(x^2y^2 + y^2z^2 + z^2x^2)^{(3/2)}}$$

$$GaussCurvature := \frac{zxy(x^2y + y^2z + z^2x)}{(x^2y^2 + y^2z^2 + z^2x^2)^2}$$

Now construct the adjoint vector and show that this vector current J, "dual to" A, has a zero divergence (Jacobi's theorem). The divergence of the adjoint dual is equal to the determinant of the Jacobian matrix generated by the components of the 1-form A

```
> ADJAA:=adjoint(JACWW);JA:=(innerprod(ADJAA,AA));DIVJA:=factor(diverge(JA,[x,y,z]
));InterA:=innerprod(AA,JA);
```

$$JA := \left[\frac{yzx^2}{(x^2y^2 + y^2z^2 + z^2x^2)^{(3/2)}, \frac{y^2zx}{(x^2y^2 + y^2z^2 + z^2x^2)^{(3/2)}, \frac{yz^2x}{(x^2y^2 + y^2z^2 + z^2x^2)^{(3/2)}} \right]$$

$$DIVJA := 0$$

$$InterA := \frac{zxy(x^2y + y^2z + z^2x)}{(x^2y^2 + y^2z^2 + z^2x^2)^2}$$

```
>
```

Four Dimensions

The remarkable feature is that simply by fixing the exponent such that n=1 making the divisor homogeneous of degree 1, but anisotropic with any signature, produces a dual CURRENT defined as the Adjoint pullback of the scaled normal field.

This dual current J has a divergence which is ZERO.!

THIS MEANS THERE IS A FAMILY of CONSERVATION LAWS!!! (a N+1 parameter family for every p,a,b,c,)

```
>
```

```
> ACTION:=wcollect((U*d(x)+V*d(y)+W*d(z)+0*T*d(t))/lambda);F:=d(ACTION);TORS:=fact
or(subs(n=1,p=2,wcollect(ACTION&^F)));
```

```
> INTERACTION:=factor(innerprod(WW,CURRENT));
```

```
> FF:=wcollect(factor(d(ACTION)));
```

```
> HELICITY:=wcollect(factor(wcollect(ACTION&^FF)));POINCARE:=wcollect(FF&^FF);HELN
1:=subs(n=1,HELICITY);POINCn1:=factor(subs(n=4,p=2,POINCARE));
```

```
> AA:=evalm([U/lambda,V/lambda,W/lambda,T/lambda]);
```

```
>
```

```
> ACTION:=factor(AA[1]*d(x)+AA[2]*d(y)+AA[3]*d(z)+AA[4]*d(t));
```

```
> FF:=wcollect(factor(d(ACTION)));
```

```
> ACTION&^FF;
```

```
>
```

```

[ >
[ > restart;
[ > with(liesymm):with(linalg):
[ Homogeneous vector fields: map (x,y,z,t) to four functions, plus fifth function which is a homog
[ polynomial in the other functions. 4d to 5d map, then project to 4d homogeneous vector. Construct Jacobian of the map
[ and Cayley Hamilton features and invariants.
[ > lambda:=(a*U^p+b*V^p+c*W^p+d*T^p)^(n/p);
[ > VECT:=evalm([U,V,W,T]/lambda);
[ >
[ >
[ > JAC:=jacobian(VECT,[U,V,W,T]);GUN:=innerprod(transpose(JAC),JAC);
[ > ADJ:=adjoint(JAC);
[ > Z:=innerprod(ADJ,VECT);
[ > DIVZ:=factor(diverge(Z,[U,V,W,T]));
[
[ >
[ > DIV:=factor(diverge(VECT,[U,V,W,T]));
[ >
[ > DET:=det(JAC);
[ > FFSQ:=innerprod(JAC,JAC);FF4th:=innerprod(FFSQ,FFSQ);FF3rd:=innerprod(JAC,FFSQ);
[ > factor(charpoly(JAC,alpha));factor(charpoly(FFSQ,alpha));FFSQ[1,1];
[ characteric polynomial has n-1 degenerate roots and 1 zero root if n=1,
[ it has 4 degenerate roots otherwise determined by (1/lambda)!!!!
[ this result is valid for any a,b,c,d, n, p
[ When n=2 any a,b,c,d,p the jacobian is involutive in the sense that its square is proportional to the identity times
[ (1/lambda)^2
[ >
[ J is involutive for n=2, has zero div for n = 4, and is of rank 3 (zero det for n= 1)
[ the curl vanishes if isotropic and p=1 or p=2, any n.
[ IS THIS WHY THE QUADRATIC and LINEAR FORM IS SO IMPORTANT?
[ > solve(n^2-2*n+2,n);solve(n^2-3*n+3);
[ >
[ >
[ > CURL:=evalm(transpose(JAC)-evalm(JAC));
[ > FORCE:=innerprod(CURL,VECT);DETCURL:=det(CURL);
[ > zz:=evalm(innerprod(CURL,VECT));
[
[ > TRADJ:=factor(trace(ADJ));
[ >
[ >
[ >

```