```
> restart;
> 
  with(liesymm):with(linalg):setup(x,y,z,t,s,Ct);deforms(a=const,b=const,c=const,d
  =const,p=const,n=const,k=const,omega=const);
Warning, new definition for close
Warning, new definition for norm
Warning, new definition for trace
```

$$[x, y, z, t, s, Ct]$$

$$\mathrm{deforms}(a = const, b = const, c = const, d = const, p = const, n = const, k = const, \omega = const)$$

HOLDER NORMS

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Euclidean Three Dimensions

In the theory of implicit surfaces in 3 dimensions, the Jacobian matrix of the Gauss Map can be used to construct the Shape Matrix of the surface.
That is, for an implicit 2 surface in 3 dimensions, defined by the constraint, phi(x,y,z) = 0,
construct the 3 component gradient, which is orthogonal to the tangent vectors of the surface.
Divide the gradient field by the euclidean homogeneous degree 1, isotropic, norm. lambda.
lambda = (square root of the sum of gradient components squared with positive unit coefficients).
The renormalized vector is now homogeneous of degree zero.
Next construct the Jacobian matrix [JAC] of this "normalized" homogeneous vector.
The matrix elements consist of the partial derivatives of the normalized components of the gradient co vector.
Each matrix element is a function of the N=3 independent variables (x,y,z)
Form the Characteristic Polynomial of this matrix and determine the Similarity Invariants.
The similarity invariants are the sum of the eigen values, the sum of the products of pairs of the eigen values, etc.
The similarity invariants are functions on the domain of independent variables.
They are called similarity invariants, for if the matrix [JAC] is changed into a new matrix by means of an invertible transformation, [T],
such that [JAC new] = [T] [JAC old] [T inverse], then the similarity invariants of the new matrix are the same as the similarity invariants of the old matrix.
The Gauss construction always has one zero eigenvalue. hence the determinant of the Jacobian is zero.
The Jacobian matrix, when diagonalized, has 1 zero along its diagonal, and two other, ususally non--zero, diagonal entries.
The fact that the determinant of the matrix vanishes is a clue to the existence of a "global" 2 surface,
and is due to the fact that the divisor that defines the "unit length" is homogeneous of degree 1.
The singular 3 x 3 matrix plays the role of the shape matrix.
The trace of the matrix (sum of the two non-zero eigenvalues) yields twice the **mean curvature.** (the first order similarity invariant) of the surface.
The trace of the adjoint matrix (or the product of the remaining two eigen values) always yields the Adjoint curvature which for 2 dimensional surfaces in 3 dimensions is equal to the **Gauss sectional curvature**. (the second order similarity invariant in 3 dimensions).
If the mean curvature vanishes, then the 2-surface is called a **minimal surface**.
***

Three Dimensions (not necessarily Euclidean)

A more general definition of the normalization denominator is given by the Holder Norm, consrtructed in terms of the components of the vector by the formula

```
>
>lambda:=(a*U^p+b*V^p+c*W^p)^(n/p);
```

$$\lambda := (a\, U^p + b\, V^p + c\, W^p)^{\left(\frac{n}{p}\right)}$$

The constant coefficients, a,b,c can have any values (plus or minus).
The determinant of the Jacobian matrix constructed from the renormalized vector **always** vanishes for Holder divisors which are homogeneous of degree 1 (any p, any a,b,c, but n=1)

$$\lambda := (a\, U^p + b\, V^p + c\, W^p)^{\left(\frac{n}{p}\right)}$$

The orthogonal field to a classic implicit surface yields a differential (exact) 1-form constructed from the gradient of the function whose zero set defines the surface.

```
> A:=d(Phi(x,y,z));
```

$$A := \left(\frac{\partial}{\partial x}\Phi(x, y, z)\right)\mathrm{d}(x) + \left(\frac{\partial}{\partial y}\Phi(x, y, z)\right)\mathrm{d}(y) + \left(\frac{\partial}{\partial z}\Phi(x, y, z)\right)\mathrm{d}(z)$$

This exact 1-form can be made homogeneous of degree 1-n by dividing by the Holder norm .
However, the Jacobian process and its similarity constructions works for any arbitrary 1-form., A, be it exact or not.

```
> A:=U*d(x)+V*d(y)+W*d(z);
```

$$A := U\,\mathrm{d}(x) + V\,\mathrm{d}(y) + W\,\mathrm{d}(z)$$

The procedure is to divide the arbitrary 1-form by the Holder norm, then construct the Jacobian matrix of partial derivatives. The resulting matrix is a covaraint tensor of second rank, but it is not necessarily symmetric.
The Jacobian matrix, as before, has a set of similarity invariants.
If the Holder norm is homogeneous of degree 1 (n= 1 any p, a,b,c...) then the determinant of the resulting Jacobian matrix is zero. The Jacobian is a singular matrix that describe a global surface in the three dimensional space.
The result implies that the inverse matrix for the Jacobian matrix does not exist.
The induced symmetric metric on xyz, constructed from the pullback [g]=[Jtranspose][J], is also singular. Hence a global inverse metric does not exist, and a "raising tensor" has to be found by other means.
However, there always exists an algebraically well defined "adjoint" matrix (the matrix of cofactors transposed) which is a contravariant tensor of second rank) even though the inverse does not exist. This adjoint matrix is not necessarily symmetric, but may be used as a "raising" tensor, similar to the use of the inverse metric in Riemannian spaces with metrc. However, it is to be emphasized that the spaces being considered need not be Riemannian, and no (symmetric and invertable) metric has been assumed to constrain the space of interest.

Preferred method: Starting from a covariant 1-form A, find the contravariant J

When the covariant components of the 1-form are premultiplied by the adjoint matrix, a contravariant "adjoint dual" field is produced.
This dual field can be represented by an N-1 form, **J.** When the Holder norm is homogeneous of degree 1, then the exterior derivative of J vanishes. A physicist would say that the "divergence of the current **J**" is zero. Such statements lead to topological conservation laws.
This Jacobian process works in all dimensions, and does not utilize a metric in the usual sense.
Note that for a given 1-form A there are many J with dJ = 0. The Holder norm is like an integrating

factor, but it is not unique.
In physics of electromagnetism, this suggests (the heretical statement) that the 1-form of potentials A is more fundamental than the N-1 form of conserved currents J.
Guided by electromagnetic experience, it is of interest to compute the "interaction" density, which is defined as the exterior product of the 1-form A and its dual (non-unique) N-1 form current, J
The coefficient of the N-form is equal to the "inner product" of the co and contra vectors, A and J.
It is possible to start with a 1-form (homgeneous of degree 1) which is closed. Hence no E or B field equivalents exist. However, this 1-form when made homogeneous of degree zero leads to a closed current N-1 form, J. The ohmic disspiation, JdotE is zero, which indicates that the induced current is a "super" current.
**
It appears remarkable that there can be different A with different Jacobian matrices that can produce the same J and yield different similarity invariants.

Alternate method: Starting from a contravariant J, find the associated A is not as useful.

Note that given a covariant A it is possible to construct many J,
but given a contravariant J the procedure to construct an A is not as well defined.
The procedure analogous to that described above, would start with a contravariant J, renormalizing with the Holder norm,, and then constructing a Jacobian matrix of partial derivatives, [JAC(J)]. This matrix construction yields a **mixed** tensor of rank 2.
Note that [JAC(A)] is not equal to [JAC(J)]. They are both second rank tensors, but the latter is a mixed tensor, while the formed is a covariant tensor of second rank.
The mixed tensor (or its adjoint) **cannot** be used as a "lowering" tensor (like a metric in Riemannian spaces) to construct a covariant A field from the contravariant J field.
A further assumption has to be made to achieve the 1-form A dual to the N-1 form J.
**
A possibility that is often used is to construct the metric field, [g] = [JAC(J) tranpose] [JAC(J)]
The idea follows from the assertion that the map from the variety to the contravariant vector field is a map to another vector space with a euclidean metric. Then the pullback of the metric on the vector space induces a metric on the variety.
Then given J, an associated "dual" A follows from A = [g] J.
Note that this procedure loses the anti-symmetry features of the Jacobian, as g is a symmetric matrix.

IT IS NOT TRUE THAT THE TWO PROCEDURES PRODUCE CLOSURE. THAT IS
[g][JAC[A)] is not necessarily the identity. Also JAC(A) [g] is not necessarily the identity.
Moreover, it is possible that the induced metric is not singular except on the "surface".
For Monge surfaces, where a component of the normal field is constant, the determinant of the induded metric is zero.
For a "non-linear" surface the induced metric is singular on the surface, and not singular otherwise.

>

PROCEDURE 1: Start with a 1 form A

Let us start with an abstract example: Note that for any a,b,c,p, if n = 1 such that the divisor is homogenous of degree 1,
then the determinant of the Jacobian of the rescaled co-vector is zero.

It is also true that if the rescaled co-vector whose components are used to construct the 1-form A , is premultiplied by the adjoint of the jacobian matrix, then a "dual" contra-vector J is constructed which has zero divergence. To prove these statements, constuct the component functions:

```
> U:=u(x,y,z);V:=v(x,y,z);W:=w(x,y,z);
```

$$U := u(x, y, z)$$
$$V := v(x, y, z)$$
$$W := w(x, y, z)$$

and the rescaled components of the 1 form A:

```
> NormedPotentials:=evalm([U,V,W]/(lambda));
```

$$NormedPotentials := \left[\frac{u(x,y,z)}{(a\,u(x,y,z)^p + b\,v(x,y,z)^p + c\,w(x,y,z)^p)^{\left(\frac{n}{p}\right)}}, \frac{v(x,y,z)}{(a\,u(x,y,z)^p + b\,v(x,y,z)^p + c\,w(x,y,z)^p)^{\left(\frac{n}{p}\right)}}, \frac{w(x,y,z)}{(a\,u(x,y,z)^p + b\,v(x,y,z)^p + c\,w(x,y,z)^p)^{\left(\frac{n}{p}\right)}}\right]$$

Construct the Jacobian matrix by constucting the array of partial derivatives, and show that for n=1 , the determinant vanishes.

```
> JACA:=jacobian(NormedPotentials,[x,y,z]):DET:=factor(det(JACA));
```

$$DET := -\left((-1+n)\left(\left(\frac{\partial}{\partial x}u(x,y,z)\right)\left(\frac{\partial}{\partial y}v(x,y,z)\right)\left(\frac{\partial}{\partial z}w(x,y,z)\right) + \left(\frac{\partial}{\partial x}w(x,y,z)\right)\left(\frac{\partial}{\partial y}u(x,y,z)\right)\left(\frac{\partial}{\partial z}v(x,y,z)\right) - \left(\frac{\partial}{\partial x}u(x,y,z)\right)\left(\frac{\partial}{\partial z}v(x,y,z)\right)\left(\frac{\partial}{\partial y}w(x,y,z)\right) - \left(\frac{\partial}{\partial x}v(x,y,z)\right)\left(\frac{\partial}{\partial y}u(x,y,z)\right)\left(\frac{\partial}{\partial z}w(x,y,z)\right) - \left(\frac{\partial}{\partial x}w(x,y,z)\right)\left(\frac{\partial}{\partial z}u(x,y,z)\right)\left(\frac{\partial}{\partial y}v(x,y,z)\right) + \left(\frac{\partial}{\partial x}v(x,y,z)\right)\left(\frac{\partial}{\partial z}u(x,y,z)\right)\left(\frac{\partial}{\partial y}w(x,y,z)\right)\right)\right) \Big/ \left((a\,u(x,y,z)^p + b\,v(x,y,z)^p + c\,w(x,y,z)^p)^{\left(\frac{n}{p}\right)}\right)^3$$

```
> GMN:=innerprod(transpose(JACA),JACA):
>
```

The determinant has a factor (n-1), hence for n=1 the determinant vanishes, for ANY co vector field scaled by a Holder norm with any a,b,c,p

Now construct the adjoint vector and show that this vector current J "dual to" A has a zero divergence (Jacobi's theorem).

The adjoint vector is constructed by multiplying the renormalized co-vector components by the adjoint matrix of the Jacobian matrix [JAC(A)].

The adjoint matrix is the matrix of cofactors transposed, and would be equal to the inverse of the original matrix times the determinant of the original matrix -- if the determinant is non-zero. The adjoint construction exists whether of not the inverse exists.

```
> ADJ:=adjoint(JACA):CurrentJ:=(innerprod(ADJ,NormedPotentials));DIVJ:=factor(dive
  rge(CurrentJ,[x,y,z]));Current:=J[1]*d(y)&^d(z)-J[2]*d(z)&^d(x)+J[3]*d(x)&^d(y):
```

$$CurrentJ := \left[-\left(\left(\left(\frac{\partial}{\partial z}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial y}v(x,y,z)\right) - \left(\frac{\partial}{\partial y}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial z}v(x,y,z)\right)\right.\right.\right.$$

$$-\left(\frac{\partial}{\partial z}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial y}w(x,y,z)\right)+\left(\frac{\partial}{\partial y}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial z}w(x,y,z)\right)$$
$$-\left(\frac{\partial}{\partial y}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial z}w(x,y,z)\right)+\left(\frac{\partial}{\partial z}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial y}w(x,y,z)\right)\Bigg)\Bigg/$$
$$\left(\left(a\,u(x,y,z)^p+b\,v(x,y,z)^p+c\,w(x,y,z)^p\right)^{\left(\frac{n}{p}\right)}\right)^3, \Bigg(\left(\frac{\partial}{\partial z}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial x}v(x,y,z)\right)$$
$$-\left(\frac{\partial}{\partial x}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial z}v(x,y,z)\right)+\left(\frac{\partial}{\partial z}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial x}w(x,y,z)\right)$$
$$-\left(\frac{\partial}{\partial x}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial z}w(x,y,z)\right)-\left(\frac{\partial}{\partial z}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial x}w(x,y,z)\right)$$
$$+\left(\frac{\partial}{\partial x}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial z}w(x,y,z)\right)\Bigg)\Bigg/\left(\left(a\,u(x,y,z)^p+b\,v(x,y,z)^p+c\,w(x,y,z)^p\right)^{\left(\frac{n}{p}\right)}\right)^3, -\Bigg($$
$$-\left(\frac{\partial}{\partial x}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial y}w(x,y,z)\right)+\left(\frac{\partial}{\partial y}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial x}w(x,y,z)\right)$$
$$+\left(\frac{\partial}{\partial y}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial x}v(x,y,z)\right)-\left(\frac{\partial}{\partial x}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial y}v(x,y,z)\right)$$
$$-\left(\frac{\partial}{\partial y}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial x}w(x,y,z)\right)+\left(\frac{\partial}{\partial x}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial y}w(x,y,z)\right)\Bigg)\Bigg/$$
$$\left(\left(a\,u(x,y,z)^p+b\,v(x,y,z)^p+c\,w(x,y,z)^p\right)^{\left(\frac{n}{p}\right)}\right)^3\Bigg]$$

$$DIVJ := -3\Bigg((-1+n)\Bigg(\left(\frac{\partial}{\partial x}u(x,y,z)\right)\left(\frac{\partial}{\partial y}v(x,y,z)\right)\left(\frac{\partial}{\partial z}w(x,y,z)\right)$$
$$+\left(\frac{\partial}{\partial x}w(x,y,z)\right)\left(\frac{\partial}{\partial y}u(x,y,z)\right)\left(\frac{\partial}{\partial z}v(x,y,z)\right)-\left(\frac{\partial}{\partial x}u(x,y,z)\right)\left(\frac{\partial}{\partial z}v(x,y,z)\right)\left(\frac{\partial}{\partial y}w(x,y,z)\right)$$
$$-\left(\frac{\partial}{\partial x}v(x,y,z)\right)\left(\frac{\partial}{\partial y}u(x,y,z)\right)\left(\frac{\partial}{\partial z}w(x,y,z)\right)-\left(\frac{\partial}{\partial x}w(x,y,z)\right)\left(\frac{\partial}{\partial z}u(x,y,z)\right)\left(\frac{\partial}{\partial y}v(x,y,z)\right)$$
$$+\left(\frac{\partial}{\partial x}v(x,y,z)\right)\left(\frac{\partial}{\partial z}u(x,y,z)\right)\left(\frac{\partial}{\partial y}w(x,y,z)\right)\Bigg)\Bigg)\Bigg/\left(\left(a\,u(x,y,z)^p+b\,v(x,y,z)^p+c\,w(x,y,z)^p\right)^{\left(\frac{n}{p}\right)}\right)^3$$

The vector J is not zero, but its divergence has a factor of (n-1) , hence the divergence of J is always zero for holder norms which are homogeneous of degree 1. This result is global, hence it would appear that the construction implies that J (for n=1) is exact.: J = dG. Note that the zero divergence result is independent from a,b,c,p for n=1

\>

The "Interaction" coefficient can be computed as

```
> Interaction:=factor(innerprod(CurrentJ,NormedPotentials));
```

$$Interaction := -\Bigg(u(x,y,z)\left(\frac{\partial}{\partial z}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial y}v(x,y,z)\right)$$
$$-u(x,y,z)\left(\frac{\partial}{\partial y}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial z}v(x,y,z)\right)-u(x,y,z)\left(\frac{\partial}{\partial z}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial y}w(x,y,z)\right)$$
$$+u(x,y,z)\left(\frac{\partial}{\partial y}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial z}w(x,y,z)\right)-\left(\frac{\partial}{\partial y}v(x,y,z)\right)u(x,y,z)^2\left(\frac{\partial}{\partial z}w(x,y,z)\right)$$

$$+\left(\frac{\partial}{\partial z}v(x,y,z)\right)u(x,y,z)^2\left(\frac{\partial}{\partial y}w(x,y,z)\right)-v(x,y,z)\left(\frac{\partial}{\partial z}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial x}v(x,y,z)\right)$$
$$+v(x,y,z)\left(\frac{\partial}{\partial x}u(x,y,z)\right)w(x,y,z)\left(\frac{\partial}{\partial z}v(x,y,z)\right)-v(x,y,z)\left(\frac{\partial}{\partial z}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial x}w(x,y,z)\right)$$
$$+v(x,y,z)\left(\frac{\partial}{\partial x}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial z}w(x,y,z)\right)+\left(\frac{\partial}{\partial z}u(x,y,z)\right)v(x,y,z)^2\left(\frac{\partial}{\partial x}w(x,y,z)\right)$$
$$-\left(\frac{\partial}{\partial x}u(x,y,z)\right)v(x,y,z)^2\left(\frac{\partial}{\partial z}w(x,y,z)\right)-w(x,y,z)\left(\frac{\partial}{\partial x}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial y}w(x,y,z)\right)$$
$$+w(x,y,z)\left(\frac{\partial}{\partial y}v(x,y,z)\right)u(x,y,z)\left(\frac{\partial}{\partial x}w(x,y,z)\right)+\left(\frac{\partial}{\partial y}u(x,y,z)\right)w(x,y,z)^2\left(\frac{\partial}{\partial x}v(x,y,z)\right)$$
$$-\left(\frac{\partial}{\partial x}u(x,y,z)\right)w(x,y,z)^2\left(\frac{\partial}{\partial y}v(x,y,z)\right)-w(x,y,z)\left(\frac{\partial}{\partial y}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial x}w(x,y,z)\right)$$
$$+w(x,y,z)\left(\frac{\partial}{\partial x}u(x,y,z)\right)v(x,y,z)\left(\frac{\partial}{\partial y}w(x,y,z)\right)\Bigg)\Bigg/\left(\left(a\,u(x,y,z)^p+b\,v(x,y,z)^p+c\,w(x,y,z)^p\right)^{\left(\frac{n}{p}\right)}\right)^4$$

```
>
```

The result is of interest for the classic specialization of the Holder norm to a=1,b=1,c=1,p=2, n=1 (which is the Gauss Map.

For then the interaction A dot J is equal to the Adjoint curvature (trace of the Adjoint matrix to the JAcobian) which in 3 dimensions is exactly equal to the Gauss curvature !!!!

This result is valid in any dimension. The trace of the Adjoint matrix is equal to the interaction for the Gauss map;

The special cases for a=b=c=n=1, p=2 are given explicity below: The fact that Gnet is zero proves the algebraic equivalence of the interaction and the similarity invaraint of degree N-1.

```
> Adjointcurv:=simplify(factor(trace(ADJ))):Mean:=factor(trace(JACA)):Net:=factor(
  Adjointcurv-Interaction):Gnet:=factor(subs(a=1,b=1,c=1,p=2,n=1,Net));
```

$$Gnet := 0$$

The usual differential form processes can be applied to the 1-form A to construct the fields (Vorticity = dA) and the Helicity (A^dA)

```
> A1:=U;A2:=V;A3:=W;
>
```

$$A1 := u(x,y,z)$$
$$A2 := v(x,y,z)$$
$$A3 := w(x,y,z)$$

```
> Action:=subs(a=1,b=1,c=1,n=1,p=2,A1*d(x)+A2*d(y)+A3*d(z));
```

$$Action := u(x,y,z)\,d(x)+v(x,y,z)\,d(y)+w(x,y,z)\,d(z)$$

```
> Vorticity:=wcollect(d(Action));Helicity:=simplify(wcollect(Action&^Vorticity));
```

$$Vorticity := \left(-\left(\frac{\partial}{\partial x}w(x,y,z)\right)+\left(\frac{\partial}{\partial z}u(x,y,z)\right)\right)(d(z)\ \&\wedge\ d(x))$$
$$+\left(\left(\frac{\partial}{\partial y}u(x,y,z)\right)-\left(\frac{\partial}{\partial x}v(x,y,z)\right)\right)(d(y)\ \&\wedge\ d(x))+\left(-\left(\frac{\partial}{\partial y}w(x,y,z)\right)+\left(\frac{\partial}{\partial z}v(x,y,z)\right)\right)(d(z)\ \&\wedge\ d(y))$$

$$Helicity := -\Bigg(-v(x,y,z)\left(\frac{\partial}{\partial x}w(x,y,z)\right)+v(x,y,z)\left(\frac{\partial}{\partial z}u(x,y,z)\right)-w(x,y,z)\left(\frac{\partial}{\partial y}u(x,y,z)\right)$$

$$+ w(x,y,z)\left(\frac{\partial}{\partial x} v(x,y,z)\right) + u(x,y,z)\left(\frac{\partial}{\partial y} w(x,y,z)\right) - u(x,y,z)\left(\frac{\partial}{\partial z} v(x,y,z)\right)\Bigg) \&\text{^}(d(z), d(y), d(x))$$

It is important to note that the renormalized 1-form of potentials A can be of Pfaff dimension greater than 2, even though the renormalization is homogeneous of degree 1. That is, the topological torsion , A^dA is not necessarily zero. In dimension 3, the set of points that yields a surface of zero Helicity si a surface where the curl of the potentials is orthogonal to the vector of potentials. Such surfaces are called Lamb surfaces in the theory of streamline hydrodynamics. This surface of zero helicity separates domains of positive from negative helicity.
Note that the equivalence between the interaction N form A^J and the Adjoint curvature exists whether or not the 1-form A satisfies the Frobenius integrability conditions,

\>

A simple 3D surface Example:

The similarity invariants for the Jacobian matrix of the surface normal scaled by the Holder norm with a=b=c=1,n=1,p=2
fields the classic partial differential equations for the mean and Gauss curvature.

```
> Phi:=z-phi(x,y);
```

$$\Phi := z - \phi(x,y)$$

```
> A:=grad(Phi,[x,y,z]);
```

$$A := \left[-\left(\frac{\partial}{\partial x}\phi(x,y)\right), -\left(\frac{\partial}{\partial y}\phi(x,y)\right), 1\right]$$

```
> magn:=innerprod(A,A);
```

$$magn := \left(\frac{\partial}{\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial}{\partial y}\phi(x,y)\right)^2 + 1$$

```
> NA:=evalm(A/magn^(1/2));
```

$$NA := \left[-\frac{\frac{\partial}{\partial x}\phi(x,y)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial}{\partial y}\phi(x,y)\right)^2 + 1}}, -\frac{\frac{\partial}{\partial y}\phi(x,y)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial}{\partial y}\phi(x,y)\right)^2 + 1}}, \frac{1}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial}{\partial y}\phi(x,y)\right)^2 + 1}}\right]$$

```
> JAC:=jacobian(NA,[x,y,z]):JACAA:=jacobian(A,[x,y,z]):GMN:=innerprod(transpose(JACAA),JACAA)/magn;
```

$$GMN := \frac{\begin{bmatrix} \left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)^2 + \left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2, & \left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right) + \left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right), & 0 \\ \left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right) + \left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right), & \left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)^2, & 0 \\ 0, & 0, & 0 \end{bmatrix}}{\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial}{\partial y}\phi(x,y)\right)^2 + 1}$$

The induced metric for the simple surface forms a 2x2 partition, with all other values zero. The two by two space is not flat.

```
> E:=-evalm(([diff(A[1],t),diff(A[2],t),diff(A[3],t)]));
```

$$E := -[0, 0, 0]$$

```
> B:=curl(A,[x,y,z]);
```

$$B := [0, 0, 0]$$

```
>
>
> MEAN_CURVATURE:=factor(trace(JAC)/2);
```

$$MEAN\_CURVATURE := -\frac{1}{2}\left(-2\left(\frac{\partial}{\partial x}\phi(x,y)\right)\left(\frac{\partial}{\partial y}\phi(x,y)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)+\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2 + \left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)+\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\right) \Bigg/ \left(\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+1\right)^{(3/2)}$$

Note the classic formula for the mean curvature of a Monge surface is obtained.

```
> ADJAC:=adjoint(JAC);
```

$$ADJAC := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\left(\frac{\partial}{\partial x}\phi(x,y)\right)\left(-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\right)}{\left(\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+1\right)^2} & -\frac{\left(\frac{\partial}{\partial y}\phi(x,y)\right)\left(-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\right)}{\left(\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+1\right)^2} & \frac{-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)}{\left(\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+1\right)^2} \end{bmatrix}$$

```
> GAUSS_ADJOINT_CURVATURE:=factor(trace(ADJAC));
```

$$GAUSS\_ADJOINT\_CURVATURE := \frac{-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)}{\left(\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+1\right)^2}$$

Note that the classic formula for the Gauss curvature of a Monge surface is obtained.

```
> CurrentJ:=innerprod(ADJAC,NA);Interaction:=factor(innerprod(CurrentJ,NA));DivJ:=
  diverge(CurrentJ,[x,y,z]);JZ:=factor(CurrentJ[3]):
```

$$CurrentJ := \left[0, 0, \left(-\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right) - \left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\right) \Bigg/ \left(\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+1\right)^{(5/2)}\right]$$

$$Interaction := \frac{-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)}{\left(\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+1\right)^2}$$

$$DivJ := 0$$

```
> DISS:=factor(innerprod(E,[CurrentJ[1],CurrentJ[2],CurrentJ[3]]));
```

$$DISS := 0$$

IT seems that the current so constructed is a Superconducting current without dissipation, J is perpendicular to E

It is now apparent that the interaction between the potentials and the conserved current is equal to the (Gauss) ADJOINT curvature of the Monge surface in Euclidean space.

```
> Adjointcurv:=simplify(factor(trace(ADJAC))):Mean:=factor(trace(JAC));Net:=factor
  (Adjointcurv-Interaction):Gnet:=factor(subs(a=1,b=1,c=1,p=2,n=1,Net));
```

$$Mean := -\left(-2\left(\frac{\partial}{\partial x}\phi(x,y)\right)\left(\frac{\partial}{\partial y}\phi(x,y)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right)+\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right)\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial x^2}\phi(x,y)\right) + \left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial^2}{\partial y^2}\phi(x,y)\right)\right) \Big/ \left(\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2+1\right)^{(3/2)}$$

$$Gnet := 0$$

```
> Vorticity:=curl([A[1],A[2],A[3]],[x,y,z]);Helicity:=innerprod([A[1],A[2],A[3]],V
  orticity);
```

$$Vorticity := [0, 0, 0]$$

$$Helicity := 0$$

NOTE that the Helicity of a simple surface is zero, and the conserved current is NOT orthogonal to the surface,

```
>
>
>
```

A non-linear Implicit surface:

The similarity invariants for the Jacobian matrix of the surface normal scaled by the Holder norm with a=b=c=1,n=1,p=2
fields the classic partial differential equations for the mean and Gauss curvature.

```
> Phi:=phi(x,y,z);
```

$$\Phi := \phi(x,y,z)$$

```
> A:=grad(Phi,[x,y,z]);
```

$$A := \left[\frac{\partial}{\partial x}\phi(x,y,z), \frac{\partial}{\partial y}\phi(x,y,z), \frac{\partial}{\partial z}\phi(x,y,z)\right]$$

```
> magn:=innerprod(A,A);
```

$$magn := \left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2$$

```
> NA:=evalm(A/magn^(1/2));
```

$$NA := \left[\frac{\frac{\partial}{\partial x}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}}, \frac{\frac{\partial}{\partial y}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}}, \frac{\frac{\partial}{\partial z}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}} \right]$$

```
> JAC:=jacobian(NA,[x,y,z]);
```

$$JAC :=$$

$$\left[-\frac{1}{2}\left(\frac{\partial}{\partial x}\phi(x,y,z)\right) \left(2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\right) \Big/ \left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)} + \frac{\frac{\partial^2}{\partial x^2}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}}, -\frac{1}{2}\left(\frac{\partial}{\partial x}\phi(x,y,z)\right) \left(2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)\right) \Big/ \left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)} + \frac{\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}}, -\frac{1}{2}\left(\frac{\partial}{\partial x}\phi(x,y,z)\right) \left(2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right)\right) \Big/ \left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)} + \frac{\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}} \right]$$

$$\left[-\frac{1}{2}\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\right)\Big/\left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)}+\frac{\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}},\right.$$

$$-\frac{1}{2}\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)\right)\Big/\left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)}+\frac{\frac{\partial^2}{\partial y^2}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}},$$

$$\left.-\frac{1}{2}\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right)\right)\Big/\left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)}+\frac{\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}}\right]$$

$$\left[-\frac{1}{2}\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)+2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\right)\Big/\left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)}+\frac{\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2}},-\frac{1}{2}\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\right.$$

$$\left(2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)+2\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)+2\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\right)$$

$$\Big/\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^{(3/2)}$$

$$+\frac{\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2}},\ -\frac{1}{2}\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)$$

$$\left(2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)+2\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)+2\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\right)$$

$$\Big/\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^{(3/2)}$$

$$+\frac{\frac{\partial^2}{\partial z^2}\phi(x, y, z)}{\sqrt{\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2}}\Bigg]$$

```
> MEAN_CURVATURE:=factor(trace(JAC)/2);
```

$$MEAN\_CURVATURE := -\frac{1}{2}\left(2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\right.$$

$$+2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)-\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2$$

$$-\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2+2\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2-\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2$$

$$\left.-\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2\right)\Big/\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^{(3/2)}$$

Note the classic formula for the mean curvature of a Monge surface is obtained.

```
> ADJAC:=adjoint(JAC);
```

$$ADJAC :=$$

$$\left[-\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(-\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\right.\right.$$

$$-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)$$

$$\left.+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\right)\Big/\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2,\ -$$

$$\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(-\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\right.$$

$$-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\Bigg) \Bigg/ \left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2, -$$

$$\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\Bigg(-\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\Bigg) \Bigg/ \left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2\Bigg]$$

$$\Bigg[\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\Bigg(-\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)^2\Bigg) \Bigg/ \left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2, \left(\frac{\partial}{\partial y}\phi(x, y, z)\right)$$

$$\Bigg(-\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)^2\Bigg) \Bigg/$$

$$\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2, \left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\Bigg($$

$$-\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)^2\Bigg) \Bigg/$$

$$\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2\Bigg]$$

$$\Bigg[\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\Bigg(-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\Bigg) \Bigg/$$

$$\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2, \left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\Bigg(-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\Bigg) \Bigg/$$

$$\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2, \left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\Bigg(-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)$$

$$+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\Bigg) \Bigg/$$

$$\left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2\Bigg]$$

```
> ADJOINT_GAUSS_CURVATURE:=factor(trace(ADJAC));
```

$$ADJOINT\_GAUSS\_CURVATURE := -\Bigg(-2\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)$$

$$-2\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)$$

$$-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2$$

$$+2\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)$$

$$+2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2$$

$$+2\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x, y, z)\right)$$

$$-2\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)+\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2$$

$$-\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2-\left(\frac{\partial^2}{\partial y^2}\phi(x, y, z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x, y, z)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2$$

$$+\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x, y, z)\right)^2\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\Bigg) \Bigg/ \left(\left(\frac{\partial}{\partial x}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x, y, z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x, y, z)\right)^2\right)^2$$

Note that the classic formula for the ADJOINT Gauss curvature of a Monge surface is obtained.

```
> CurrentJ:=innerprod(ADJAC,NA);Interaction:=innerprod(CurrentJ,NA);DivJ:=factor(d
  iverge(CurrentJ,[x,y,z]));
```

$$
\begin{aligned}
\mathit{CurrentJ} := \Bigg[& -\Bigg(-\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right) \\
& -\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)-\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right) \\
& +\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right) \\
& +\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)^2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\Bigg) \Bigg/ \left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)}, \Bigg(\\
& -\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)+\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right) \\
& +\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right)-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right) \\
& +\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)-\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)^2\Bigg) \Bigg/ \\
& \left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)}, \Bigg(-\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)^2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right) \\
& -\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right) \\
& -\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)+\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right) \\
& +\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\Bigg) \Bigg/ \\
& \left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^{(3/2)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Interaction} := -\Bigg(& -2\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right) \\
& -2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right) \\
& -\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2 \\
& +2\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right) \\
& +2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right)+\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right)^2\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2 \\
& +2\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z\,\partial y}\phi(x,y,z)\right) \\
& -2\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)+\left(\frac{\partial^2}{\partial z\,\partial x}\phi(x,y,z)\right)^2\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2 \\
& -\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2-\left(\frac{\partial^2}{\partial y^2}\phi(x,y,z)\right)\left(\frac{\partial^2}{\partial x^2}\phi(x,y,z)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2
\end{aligned}
$$

$$+\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y,z)\right)^2\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\Bigg)\Bigg/\left(\left(\frac{\partial}{\partial x}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial y}\phi(x,y,z)\right)^2+\left(\frac{\partial}{\partial z}\phi(x,y,z)\right)^2\right)^2$$

$$DivJ := 0$$

It is now apparent that the interaction between the potentials and the conserved current is equal to the Gauss curvature of the Monge surface in Euclidean space.

```
>
> Adjointcurv:=simplify(factor(trace(ADJAC))):Mean:=factor(trace(JAC)):Net:=factor
  (Adjointcurv-Interaction):Gnet:=factor(subs(a=1,b=1,c=1,p=2,n=1,Net));
```

$$Gnet := 0$$

```
> Vorticity:=curl(A,[x,y,z]);Helicity:=innerprod(A,Vorticity);
```

$$Vorticity := [0, 0, 0]$$

$$Helicity := 0$$

NOTE that the Helicity of a Monge surface is zero, and the conserved current is NOT orthogonal to the surface,

Explicit EXAMPLE: The spherical surface in 3D

The current J is evaluated for n=1, and has zero divergence. Note that in this example The interaction N form, A^J, is a non-zero volume element almost everywhere.

```
> Phi:=x^2+y^2+z^2-1;NN:=grad(Phi,[x,y,z]);U:=NN[1];V:=NN[2];W:=NN[3];CurrentJ:=0:
```

$$\Phi := x^2+y^2+z^2-1$$

$$NN := [2x, 2y, 2z]$$

$$U := 2x$$

$$V := 2y$$

$$W := 2z$$

```
>
> A:=evalm([U,V,W]);
> AA:=evalm([U,V,W]/(subs(a=1,b=1,c=1,p=2,n=1,lambda)));
```

$$A := [2x, 2y, 2z]$$

$$AA := \left[2\frac{x}{\sqrt{4x^2+4y^2+4z^2}},\ 2\frac{y}{\sqrt{4x^2+4y^2+4z^2}},\ 2\frac{z}{\sqrt{4x^2+4y^2+4z^2}}\right]$$

Construct the Jacobin and show that or n=1 , the determinant vanishes.

```
> JACWW:=jacobian(AA,[x,y,z]);DETJACAA:=factor(det(JACWW));Mean:=factor(trace(JACW
  W)/2);ADJQQ:=adjoint(JACWW):AdjointGaussCurvature:=factor(trace(ADJQQ));GMN:=inn
  erprod(transpose(JACWW),JACWW);eigenvalues(GMN);
>
>
```

$JACWW :=$

$$\begin{bmatrix} -8\frac{x^2}{(4x^2+4y^2+4z^2)^{(3/2)}}+2\frac{1}{\sqrt{4x^2+4y^2+4z^2}}, & -8\frac{xy}{(4x^2+4y^2+4z^2)^{(3/2)}}, & -8\frac{xz}{(4x^2+4y^2+4z^2)^{(3/2)}} \\ -8\frac{xy}{(4x^2+4y^2+4z^2)^{(3/2)}}, & -8\frac{y^2}{(4x^2+4y^2+4z^2)^{(3/2)}}+2\frac{1}{\sqrt{4x^2+4y^2+4z^2}}, & -8\frac{yz}{(4x^2+4y^2+4z^2)^{(3/2)}} \\ -8\frac{xz}{(4x^2+4y^2+4z^2)^{(3/2)}}, & -8\frac{yz}{(4x^2+4y^2+4z^2)^{(3/2)}}, & -8\frac{z^2}{(4x^2+4y^2+4z^2)^{(3/2)}}+2\frac{1}{\sqrt{4x^2+4y^2+4z^2}} \end{bmatrix}$$

$$DETJACAA := 0$$

$$Mean := \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$AdjointGaussCurvature := \frac{1}{x^2 + y^2 + z^2}$$

$$GMN := \begin{bmatrix} \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^2} & -\frac{x\,y}{(x^2 + y^2 + z^2)^2} & -\frac{x\,z}{(x^2 + y^2 + z^2)^2} \\ -\frac{x\,y}{(x^2 + y^2 + z^2)^2} & \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^2} & -\frac{y\,z}{(x^2 + y^2 + z^2)^2} \\ -\frac{x\,z}{(x^2 + y^2 + z^2)^2} & -\frac{y\,z}{(x^2 + y^2 + z^2)^2} & \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^2} \end{bmatrix}$$

$$0, \frac{1}{x^2 + y^2 + z^2}, \frac{1}{x^2 + y^2 + z^2}$$

Now construct the adjoint vector and show that this vector current J, "dual to" A, has a zero divergence (Jacobi's theorem). The divergence of the adjoint dual is equal to the determinant of the Jacobian matrix generated by the components of the 1-form A

```
> ADJAA:=adjoint(JACWW):CurrentJ:=(innerprod(ADJAA,AA));DIVJ:=factor(diverge(CurrentJ,[x,y,z]));InterAJ:=innerprod(AA,CurrentJ);INT:=innerprod(A,CurrentJ);
```

$$CurrentJ := \left[\frac{x}{(x^2 + y^2 + z^2)^{(3/2)}}, \frac{y}{(x^2 + y^2 + z^2)^{(3/2)}}, \frac{z}{(x^2 + y^2 + z^2)^{(3/2)}}\right]$$

$$DIVJ := 0$$

$$InterAJ := \frac{1}{x^2 + y^2 + z^2}$$

$$INT := 2\frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

```
>
> #JACJ:=evalm(jacobian(CurrentJ,[x,y,z])):TJACJ:=evalm(transpose(JACJ)):
> #GMN:=evalm(innerprod((transpose(JACJ)),JACJ));factor(GMN[1,1]):DETGMN:=factor(det(GMN));

> Helicity:=innerprod(curl(A,[x,y,z]),A);
```

$$Helicity := 0$$

Explicit 3D EXAMPLE for a nonlinear surface function. The current J is evaluated for n=1, and has zero divergence. Note that in this example The interaction N form, A^J, is a non-zero volume element almost everywhere. However there esists a surface for which the interaction vanishes.

```
> U:=y*z;V:=z*x;W:=x*y;CurrentJ:=0:
```

$$U := y\,z$$

$$V := z\,x$$

$$W := x\,y$$

This field is a gradient field with Phi = xyz.

```
>
```

```
> A:=evalm([U,V,W]);
> AA:=evalm([U,V,W]/(subs(a=1,b=1,c=1,p=2,n=1,lambda)));
```

$$A := [y\,z, z\,x, x\,y]$$

$$AA := \left[\frac{y\,z}{\sqrt{y^2 z^2 + z^2 x^2 + x^2 y^2}}, \frac{z\,x}{\sqrt{y^2 z^2 + z^2 x^2 + x^2 y^2}}, \frac{x\,y}{\sqrt{y^2 z^2 + z^2 x^2 + x^2 y^2}}\right]$$

Construct the Jacobin and show that or n=1 , the determinant vanishes.

```
> JACWW:=jacobian(AA,[x,y,z]):DETJACAA:=factor(det(JACWW));Mean:=factor(trace(JACW
  W)/2);ADJQQ:=adjoint(JACWW):AdjointGaussCurvature:=factor(trace(ADJQQ));GMN:=inn
  erprod(transpose(JACWW),JACWW);eigenvalues(GMN):
>
>
```

$$DETJACAA := 0$$

$$Mean := -\frac{y\,z\,x\,(x^2 + y^2 + z^2)}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^{(3/2)}}$$

$$AdjointGaussCurvature := 3\,\frac{x^2 y^2 z^2}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2}$$

$$GMN := \begin{bmatrix} \frac{y^2 z^2 (y^2 + z^2)}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} & -\frac{x\,y\,z^4}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} & -\frac{z\,x\,y^4}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} \\ -\frac{x\,y\,z^4}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} & \frac{z^2 x^2 (x^2 + z^2)}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} & -\frac{y\,z\,x^4}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} \\ -\frac{z\,x\,y^4}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} & -\frac{y\,z\,x^4}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} & \frac{x^2 y^2 (x^2 + y^2)}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2} \end{bmatrix}$$

Now construct the adjoint vector and show that this vector current J, "dual to" A, has a zero divergence (Jacobi's theorem). The divergence of the adjoint dual is equal to the determinant of the Jacobian matrix generated by the components of the 1-form A

```
>
> ADJAA:=adjoint(JACWW):CurrentJ:=(innerprod(ADJAA,AA));DIVJ:=factor(diverge(Curre
  ntJ,[x,y,z]));InterA:=innerprod(AA,CurrentJ);INT:=innerprod(A,CurrentJ);
```

$$CurrentJ := \left[\frac{z\,y\,x^2}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^{(3/2)}}, \frac{z\,y^2\,x}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^{(3/2)}}, \frac{z^2\,y\,x}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^{(3/2)}}\right]$$

$$DIVJ := 0$$

$$InterA := 3\,\frac{x^2 y^2 z^2}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^2}$$

$$INT := 3\,\frac{y^2 z^2 x^2}{(y^2 z^2 + z^2 x^2 + x^2 y^2)^{(3/2)}}$$

```
>
```

Explicit EXAMPLE for a non-integrable Heiswnberg surface function. The current J is evaluated for n=1, and has zero divergence. Note that in this example The interaction N form, A^J, is a non-zero volume element almost everywhere. However there esists a surface for which the interaction vanishes.

```
> U:=y;V:=-x;W:=z;CurrentJ:=0:
```

$$U := y$$

$$V := -x$$
$$W := z$$

```
> 
> A:=evalm([U,V,W]);
> AA:=evalm([U,V,W]/(subs(a=1,b=1,c=1,p=2,n=1,lambda)));
```

$$A := [y, -x, z]$$
$$AA := \left[\frac{y}{\sqrt{x^2+y^2+z^2}}, -\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right]$$

Construct the Jacobin and show that or n=1 , the determinant vanishes.

```
> JACWW:=jacobian(AA,[x,y,z]):DETJACAA:=factor(det(JACWW));Mean:=factor(trace(JACWW)/2);ADJQQ:=adjoint(JACWW):AdjointGaussCurvature:=factor(trace(ADJQQ));
> 
> 
```

$$DETJACAA := 0$$
$$Mean := \frac{1}{2}\frac{x^2+y^2}{(x^2+y^2+z^2)^{(3/2)}}$$
$$AdjointGaussCurvature := \frac{z^2}{(x^2+y^2+z^2)^2}$$

Now construct the adjoint vector and show that this vector current J, "dual to" A, has a zero divergence (Jacobi's theorem). The divergence of the adjoint dual is equal to the determinant of the Jacobian matrix generated by the components of the 1-form A

```
> 
> ADJAA:=adjoint(JACWW):CurrentJ:=(innerprod(ADJAA,AA));DIVJ:=factor(diverge(CurrentJ,[x,y,z]));InterA:=innerprod(AA,CurrentJ);INT:=innerprod(A,CurrentJ);
> 
```

$$CurrentJ := \left[\frac{x}{(x^2+y^2+z^2)^{(3/2)}}, \frac{y}{(x^2+y^2+z^2)^{(3/2)}}, \frac{z}{(x^2+y^2+z^2)^{(3/2)}}\right]$$
$$DIVJ := 0$$
$$InterA := \frac{z^2}{(x^2+y^2+z^2)^2}$$
$$INT := \frac{z^2}{(x^2+y^2+z^2)^{(3/2)}}$$

```
> 
> Helicity:=innerprod(curl(A,[x,y,z]),A);
```

$$Helicity := -2\,z$$

Four Dimensions

The same sort of ideas follow in 4 D. However now one must distinguish between the third order similarity invariant (ADJOINT curvature)
and the second order similarity invariant (THE Gauss sectional curvature)

An implicit surface Example:

The Holder norm with a=b=c=1,e=1,n=1,p=2 will be used from the outset. It is equivalent to the 4D Gauss map.

The induced adjoint matrix is used to compute the similarity invariants for the Mean, Gauss and Adjoint curvature.

The Adjoint Curvature is cubic in eigenvalues, and is distinct from the Gaussian curvature which is quadratic in eigenvalues.

```
> J:=0;c:=c;U:=u(x,y,z,t);V:=v(x,y,z,t);W:=w(x,y,z,t);Phi:=phi(x,y,z,t);
>
```

$$J := 0$$

$$c := c$$

$$U := \mathrm{u}(x, y, z, t)$$

$$V := \mathrm{v}(x, y, z, t)$$

$$\mathrm{W} := \mathrm{w}(x, y, z, t)$$

$$\Phi := \phi(x, y, z, t)$$

```
>
> A:=[U,V,W,-Phi];
>
```

$$A := [\mathrm{u}(x, y, z, t), \mathrm{v}(x, y, z, t), \mathrm{w}(x, y, z, t), -\phi(x, y, z, t)]$$

```
> magn:=subs(p=2,(A[1]^p+A[2]^p+A[3]^p+A[4]^p)^(1/p));
```

$$magn := \sqrt{\mathrm{u}(x, y, z, t)^2 + \mathrm{v}(x, y, z, t)^2 + \mathrm{w}(x, y, z, t)^2 + \phi(x, y, z, t)^2}$$

```
> NA4D:=evalm(A/magn);AA:=innerprod(A,[d(x),d(y),d(z),d(t)]);
```

$$NA4D := \left[\frac{\mathrm{u}(x, y, z, t)}{\sqrt{\mathrm{u}(x, y, z, t)^2 + \mathrm{v}(x, y, z, t)^2 + \mathrm{w}(x, y, z, t)^2 + \phi(x, y, z, t)^2}}, \frac{\mathrm{v}(x, y, z, t)}{\sqrt{\mathrm{u}(x, y, z, t)^2 + \mathrm{v}(x, y, z, t)^2 + \mathrm{w}(x, y, z, t)^2 + \phi(x, y, z, t)^2}}, \frac{\mathrm{w}(x, y, z, t)}{\sqrt{\mathrm{u}(x, y, z, t)^2 + \mathrm{v}(x, y, z, t)^2 + \mathrm{w}(x, y, z, t)^2 + \phi(x, y, z, t)^2}}, -\frac{\phi(x, y, z, t)}{\sqrt{\mathrm{u}(x, y, z, t)^2 + \mathrm{v}(x, y, z, t)^2 + \mathrm{w}(x, y, z, t)^2 + \phi(x, y, z, t)^2}}\right]$$

$$AA := \mathrm{u}(x, y, z, t)\,\mathrm{d}(x) + \mathrm{v}(x, y, z, t)\,\mathrm{d}(y) + \mathrm{w}(x, y, z, t)\,\mathrm{d}(z) - \phi(x, y, z, t)\,\mathrm{d}(t)$$

```
> JAC:=jacobian(NA4D,[x,y,z,t]):DET:=factor(det(JAC));
```

$$DET := 0$$

```
> MEAN_CURVATURE:=factor(trace(JAC)/2);
```

$$MEAN\_CURVATURE := \frac{1}{2}\Bigg(-\mathrm{v}(x, y, z, t)\,\mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right) - \mathrm{v}(x, y, z, t)\,\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{w}(x, y, z, t)\right) - \mathrm{v}(x, y, z, t)\,\phi(x, y, z, t)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right) - \left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)\mathrm{u}(x, y, z, t)^2 - \left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)\mathrm{v}(x, y, z, t)^2 - \left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)\mathrm{w}(x, y, z, t)^2 + \left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\mathrm{v}(x, y, z, t)^2 + \left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)^2 + \left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\phi(x, y, z, t)^2 + \left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\mathrm{u}(x, y, z, t)^2 + \left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)^2$$

$$+\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)^2+\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2+\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\mathrm{v}(x,y,z,t)^2$$
$$+\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)^2+\phi(x,y,z,t)\,\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)$$
$$+\phi(x,y,z,t)\,\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)+\phi(x,y,z,t)\,\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)$$
$$-\mathrm{w}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)-\mathrm{w}(x,y,z,t)\,\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$
$$-\mathrm{w}(x,y,z,t)\,\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)-\mathrm{u}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)$$
$$-\mathrm{u}(x,y,z,t)\,\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)-\mathrm{u}(x,y,z,t)\,\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\Bigg)\Bigg/$$
$$(\mathrm{u}(x,y,z,t)^2+\mathrm{v}(x,y,z,t)^2+\mathrm{w}(x,y,z,t)^2+\phi(x,y,z,t)^2)^{(3/2)}$$

Note the classic formula for the mean curvature of a Monge surface is obtained (4th degree in derivatives and linear in eigenvalues) This is the first similarity invariant.

The second similarity invariant can be found from the formula Gauss= -(1/2)(Trace(J) -trace(J*J))

```
> S2:=factor(trace(innerprod(JAC,JAC))):
  Gauss:=factor(-(1/2)*(-trace(JAC)*trace(JAC)+S2));
```

$$Gauss := -\Bigg(-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2-\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$
$$-\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)^2$$
$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)^2+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)$$
$$+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)$$
$$+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)$$
$$+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)$$
$$-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)+\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2$$
$$-\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2+\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2$$
$$+\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)^2+\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)$$
$$-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)+\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$
$$+\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)^2+\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)^2$$

$$
\begin{aligned}
&-\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\\
&+\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\\
&+\mathrm{u}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\\
&-\mathrm{u}(x,y,z,t)\,\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)-\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\\
&-\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2-\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)^2\\
&+\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2-\mathrm{u}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\\
&-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\phi(x,y,z,t)\,\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\\
&+\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)+\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\\
&-\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)+\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\\
&-\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)+\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\phi(x,y,z,t)\,\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\\
&-\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\\
&+\mathrm{u}(x,y,z,t)\,\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)-\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{v}(x,y,z,t)^2\\
&+\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2+\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{v}(x,y,z,t)^2\\
&-\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\\
&+\mathrm{v}(x,y,z,t)\,\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\\
&+\mathrm{v}(x,y,z,t)\,\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\\
&-\mathrm{v}(x,y,z,t)\,\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\\
&-\mathrm{v}(x,y,z,t)\,\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\\
&-\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\\
&-\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\\
&-\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\,\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)
\end{aligned}
$$

$$+ w(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$- w(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)$$
$$+ w(x,y,z,t)\,v(x,y,z,t)\left(\frac{\partial}{\partial t}v(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$+ \left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\phi(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)$$
$$+ \left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\phi(x,y,z,t)\,v(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)$$
$$- w(x,y,z,t)\,v(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)$$
$$- \left(\frac{\partial}{\partial z}w(x,y,z,t)\right)\phi(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial t}v(x,y,z,t)\right)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$- w(x,y,z,t)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial y}v(x,y,z,t)\right) + \left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)w(x,y,z,t)^2\Bigg) \Big/$$

$$(u(x,y,z,t)^2+v(x,y,z,t)^2+w(x,y,z,t)^2+\phi(x,y,z,t)^2)^2$$

NOte that the second similarity invariant is of sixth degree in derivatives, and second degree in eigen values

```
> ADJAC:=adjoint(JAC):
> ADJOINT_CURVATURE:=factor(trace(ADJAC));
```

$$
\begin{aligned}
\mathit{ADJOINT\_CURVATURE} := -\Bigg(&-\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)\phi(x,y,z,t)^2\\
&+\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)\phi(x,y,z,t)^2\\
&-\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)v(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)\\
&-\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)\\
&+v(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)u(x,y,z,t)\\
&+v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\\
&+v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)\\
&+v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\\
&-v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\\
&-v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)\\
&-v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\\
&-v(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)\\
&+v(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)\\
&+v(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial t}v(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\\
&+v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)\\
&+v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\\
&-v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\\
&+v(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\\
&-v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)
\end{aligned}
$$

$$- v(x, y, z, t)\, u(x, y, z, t) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) u(x, y, z, t)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) u(x, y, z, t)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) w(x, y, z, t)$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) u(x, y, z, t)$$

$$+ \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) v(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right)$$

$$- \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) u(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) u(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) v(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) u(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) u(x, y, z, t)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) u(x, y, z, t)$$

$$+ \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \phi(x, y, z, t)^2$$

$$- \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \phi(x, y, z, t)^2$$

$$+ u(x, y, z, t) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$+ u(x, y, z, t) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right)$$

$$+ u(x, y, z, t) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$+ u(x, y, z, t) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right)$$

$$- v(x, y, z, t)^2 \left(\frac{\partial}{\partial x} w(x, y, z, t)\right)\left(\frac{\partial}{\partial z} u(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$- v(x, y, z, t)^2 \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} u(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right)$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} u(x, y, z, t)\right)\left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} u(x, y, z, t)\right) w(x, y, z, t)^2 \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \phi(x, y, z, t)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} u(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial y} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) u(x, y, z, t)^2$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right)\left(\frac{\partial}{\partial z} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \phi(x, y, z, t)$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$+ v(x, y, z, t)\, u(x, y, z, t) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right)\left(\frac{\partial}{\partial z} u(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$+ w(x, y, z, t)\, u(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y} w(x, y, z, t)\right)$$

$$- u(x, y, z, t) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$- \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial y} u(x, y, z, t)\right) w(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$+ \phi(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) u(x, y, z, t) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right)$$

$$- \phi(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) u(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right)\left(\frac{\partial}{\partial y} w(x, y, z, t)\right)$$

$$+ \phi(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right)$$

$$- \left(\frac{\partial}{\partial x} u(x, y, z, t)\right)\left(\frac{\partial}{\partial y} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \phi(x, y, z, t)^2$$

$$- \phi(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right)$$

$$- \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) u(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y} v(x, y, z, t)\right)$$

$$-\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\mathrm{v}(x,y,z,t)\,\phi(x,y,z,t)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)^2$$
$$-\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$-\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$
$$-\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$
$$+\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$+\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)$$
$$-\mathrm{w}(x,y,z,t)\,\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)$$
$$-\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)$$
$$-\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)$$
$$-\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$-\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)$$
$$+\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)$$
$$+\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)$$
$$+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)$$

$$-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$

$$-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)$$

$$-\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$

$$-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)$$

$$-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$

$$-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$

$$+\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$

$$-\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$

$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2$$

$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$

$$-\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)$$

$$-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\Bigg)\Bigg/$$

$$(\mathrm{u}(x,y,z,t)^2+\mathrm{v}(x,y,z,t)^2+\mathrm{w}(x,y,z,t)^2+\phi(x,y,z,t)^2)^{(5/2)}$$

Note that the classic formula for the third degree adjoint curvature (8th degree in derivatives) of a Monge surface is obtained.

```
> Current:=innerprod(ADJAC,NA4D):Interaction:=innerprod(Current,NA4D);DivJ:=factor
(diverge(Current,[x,y,z,t]));rho:=factor(Current[4]);Jx:=factor(Current[1]);Jy:=
factor(Current[2]);Jz:=factor(Current[3]);Net:=factor(c*Interaction-ADJOINT_CURV
ATURE);
```

$$Interaction := -\Bigg(-\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)^2$$

$$+\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)^2$$

$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\mathrm{v}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)$$

$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$

$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)u(x,y,z,t)$$
$$+ v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,w(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial t}v(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$- v(x,y,z,t)\,u(x,y,z,t)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)u(x,y,z,t)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)\left(\frac{\partial}{\partial t}v(x,y,z,t)\right)u(x,y,z,t)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}v(x,y,z,t)\right)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)w(x,y,z,t)$$

$$+\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)\mathrm{u}(x, y, z, t)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\mathrm{v}(x, y, z, t)\,\phi(x, y, z, t)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\phi(x, y, z, t)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)$$

$$-\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\mathrm{u}(x, y, z, t)\,\phi(x, y, z, t)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\mathrm{u}(x, y, z, t)\,\phi(x, y, z, t)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right)\mathrm{v}(x, y, z, t)\,\phi(x, y, z, t)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)\mathrm{u}(x, y, z, t)\,\phi(x, y, z, t)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)$$

$$-\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)\mathrm{u}(x, y, z, t)$$

$$-\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial x}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)\mathrm{u}(x, y, z, t)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\phi(x, y, z, t)^2$$

$$-\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)\phi(x, y, z, t)^2$$

$$+\mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial t}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)$$

$$+\mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x, y, z, t)\right)\phi(x, y, z, t)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)$$

$$+\mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)$$

$$+\mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)\phi(x, y, z, t)\left(\frac{\partial}{\partial t}\mathrm{v}(x, y, z, t)\right)$$

$$-\mathrm{v}(x, y, z, t)^2\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)$$

$$-\mathrm{v}(x, y, z, t)^2\left(\frac{\partial}{\partial x}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)$$

$$+\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x, y, z, t)\right)\phi(x, y, z, t)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)^2\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)$$

$$+\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x, y, z, t)\right)\phi(x, y, z, t)$$

$$-\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)\phi(x, y, z, t)\left(\frac{\partial}{\partial t}\mathrm{v}(x, y, z, t)\right)$$

$$- v(x,y,z,t)\left(\frac{\partial}{\partial x} v(x,y,z,t)\right)\left(\frac{\partial}{\partial t} w(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z} u(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial y} v(x,y,z,t)\right)\left(\frac{\partial}{\partial z} w(x,y,z,t)\right)\left(\frac{\partial}{\partial t} \phi(x,y,z,t)\right)u(x,y,z,t)^2$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x} w(x,y,z,t)\right)\left(\frac{\partial}{\partial t} v(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z} u(x,y,z,t)\right)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial x} w(x,y,z,t)\right)\left(\frac{\partial}{\partial z} v(x,y,z,t)\right)\left(\frac{\partial}{\partial t} u(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x} v(x,y,z,t)\right)\left(\frac{\partial}{\partial z} u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial t} \phi(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\, u(x,y,z,t)\left(\frac{\partial}{\partial y} w(x,y,z,t)\right)\left(\frac{\partial}{\partial z} u(x,y,z,t)\right)\left(\frac{\partial}{\partial t} \phi(x,y,z,t)\right)$$
$$+ w(x,y,z,t)\, u(x,y,z,t)\left(\frac{\partial}{\partial x} v(x,y,z,t)\right)\left(\frac{\partial}{\partial t} \phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y} w(x,y,z,t)\right)$$
$$- u(x,y,z,t)\left(\frac{\partial}{\partial y} u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial t} v(x,y,z,t)\right)\left(\frac{\partial}{\partial z} \phi(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x} v(x,y,z,t)\right)\left(\frac{\partial}{\partial y} u(x,y,z,t)\right)w(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z} \phi(x,y,z,t)\right)$$
$$+ \phi(x,y,z,t)\left(\frac{\partial}{\partial x} \phi(x,y,z,t)\right)u(x,y,z,t)\left(\frac{\partial}{\partial y} v(x,y,z,t)\right)\left(\frac{\partial}{\partial z} w(x,y,z,t)\right)$$
$$- \phi(x,y,z,t)\left(\frac{\partial}{\partial x} \phi(x,y,z,t)\right)u(x,y,z,t)\left(\frac{\partial}{\partial z} v(x,y,z,t)\right)\left(\frac{\partial}{\partial y} w(x,y,z,t)\right)$$
$$+ \phi(x,y,z,t)\left(\frac{\partial}{\partial x} \phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y} u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial z} v(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x} u(x,y,z,t)\right)\left(\frac{\partial}{\partial y} v(x,y,z,t)\right)\left(\frac{\partial}{\partial z} w(x,y,z,t)\right)\phi(x,y,z,t)^2$$
$$- \phi(x,y,z,t)\left(\frac{\partial}{\partial x} \phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z} u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial y} v(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x} w(x,y,z,t)\right)u(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z} \phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y} v(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x} w(x,y,z,t)\right)\left(\frac{\partial}{\partial y} \phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z} u(x,y,z,t)\right)v(x,y,z,t)\,\phi(x,y,z,t)$$
$$+\left(\frac{\partial}{\partial x} v(x,y,z,t)\right)\left(\frac{\partial}{\partial z} u(x,y,z,t)\right)w(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial y} \phi(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x} u(x,y,z,t)\right)\left(\frac{\partial}{\partial z} v(x,y,z,t)\right)\left(\frac{\partial}{\partial y} w(x,y,z,t)\right)\phi(x,y,z,t)^2$$
$$- w(x,y,z,t)\left(\frac{\partial}{\partial x} u(x,y,z,t)\right)\left(\frac{\partial}{\partial t} w(x,y,z,t)\right)\left(\frac{\partial}{\partial y} v(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+ w(x,y,z,t)\left(\frac{\partial}{\partial x} v(x,y,z,t)\right)\left(\frac{\partial}{\partial y} u(x,y,z,t)\right)\left(\frac{\partial}{\partial t} w(x,y,z,t)\right)\phi(x,y,z,t)$$
$$-\left(\frac{\partial}{\partial y} \phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z} w(x,y,z,t)\right)\left(\frac{\partial}{\partial t} v(x,y,z,t)\right)u(x,y,z,t)^2$$

$$-\left(\frac{\partial}{\partial y} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) \mathrm{u}(x, y, z, t)^2$$

$$+ \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{v}(x, y, z, t)\right) \phi(x, y, z, t)$$

$$+ \mathrm{v}(x, y, z, t)^2\left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$+ \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{v}(x, y, z, t)\right) \phi(x, y, z, t)$$

$$+ \mathrm{v}(x, y, z, t)^2\left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)$$

$$- \mathrm{w}(x, y, z, t) \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)$$

$$- \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial t} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right)$$

$$- \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{w}(x, y, z, t)\right) \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{v}(x, y, z, t)\right)$$

$$- \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right) \phi(x, y, z, t)$$

$$- \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{v}(x, y, z, t)\right) \phi(x, y, z, t)$$

$$+ \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial t} \mathrm{v}(x, y, z, t)\right)$$

$$+ \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)$$

$$+ \mathrm{v}(x, y, z, t)^2\left(\frac{\partial}{\partial x} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$+ \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{v}(x, y, z, t)\right) \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial z} \mathrm{u}(x, y, z, t)\right)$$

$$+ \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{v}(x, y, z, t)\right) \phi(x, y, z, t)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)$$

$$- \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right) \phi(x, y, z, t)\left(\frac{\partial}{\partial z} \mathrm{v}(x, y, z, t)\right)$$

$$- \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{w}(x, y, z, t)\right) \phi(x, y, z, t)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)$$

$$- \mathrm{v}(x, y, z, t)^2\left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$- \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{v}(x, y, z, t)\right) \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)$$

$$- \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \mathrm{v}(x, y, z, t)\right) \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial z} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$- \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{v}(x, y, z, t)\right) \phi(x, y, z, t)\left(\frac{\partial}{\partial z} \mathrm{u}(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$
$$+\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$
$$-\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2$$
$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2$$
$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\Bigg)\Bigg/$$
$$(\mathrm{u}(x,y,z,t)^2+\mathrm{v}(x,y,z,t)^2+\mathrm{w}(x,y,z,t)^2+\phi(x,y,z,t)^2)^{(5/2)}$$

$$DivJ := 0$$

$$\rho := -\Bigg(\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)$$
$$-\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)$$
$$-\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)$$
$$+\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)$$
$$+\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)$$
$$-\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial x}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)$$

$$+\mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)$$

$$-\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)\phi(x, y, z, t)$$

$$-\mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x, y, z, t)\right)$$

$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x, y, z, t)\right)\phi(x, y, z, t)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)\phi(x, y, z, t)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)\phi(x, y, z, t)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x, y, z, t)\right)\phi(x, y, z, t)$$

$$+\left(\frac{\partial}{\partial x}\phi(x, y, z, t)\right)\mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial z}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x, y, z, t)\right)$$

$$-\left(\frac{\partial}{\partial x}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\phi(x, y, z, t)$$

$$-\left(\frac{\partial}{\partial x}\mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)\Bigg)\Bigg) \Big/$$

$$\left(\mathrm{u}(x, y, z, t)^2+\mathrm{v}(x, y, z, t)^2+\mathrm{w}(x, y, z, t)^2+\phi(x, y, z, t)^2\right)^2$$

$$Jx := \Bigg(-\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)$$

$$+\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial y}\phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x, y, z, t)\right)$$

$$+\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)$$

$$+\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)$$

$$-\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)$$

$$-\mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial y}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial z}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t}\phi(x, y, z, t)\right)$$

$$-\left(\frac{\partial}{\partial y}\mathrm{v}(x, y, z, t)\right)\mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial t}\mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z}\phi(x, y, z, t)\right)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$

$$-\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)$$

$$-\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)$$

$$-\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)$$

$$-\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)$$

$$-\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$

$$-\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)$$

$$-\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)$$

$$+\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$

$$-\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\Bigg) \Bigg/$$

$$(\mathrm{u}(x,y,z,t)^2+\mathrm{v}(x,y,z,t)^2+\mathrm{w}(x,y,z,t)^2+\phi(x,y,z,t)^2)^2$$

$$Jy := \Bigg(\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)$$

$$+\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)$$

$$+\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$

$$-\mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$

$$
\begin{aligned}
&- v(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z} u(x, y, z, t)\right)\left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \\
&- v(x, y, z, t) \left(\frac{\partial}{\partial x} u(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) \\
&- \left(\frac{\partial}{\partial x} w(x, y, z, t)\right)\left(\frac{\partial}{\partial z} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) u(x, y, z, t) \\
&+ \left(\frac{\partial}{\partial x} u(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \\
&+ \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} w(x, y, z, t)\right)\left(\frac{\partial}{\partial z} v(x, y, z, t)\right) u(x, y, z, t) \\
&+ \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) u(x, y, z, t) \\
&- \left(\frac{\partial}{\partial x} u(x, y, z, t)\right)\left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \\
&+ \left(\frac{\partial}{\partial x} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) u(x, y, z, t) \\
&- \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} v(x, y, z, t)\right) u(x, y, z, t) \\
&+ \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) \\
&- \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} w(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) u(x, y, z, t) \\
&- \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) \\
&- \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) \\
&+ \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) \\
&+ \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) \\
&+ \left(\frac{\partial}{\partial x} w(x, y, z, t)\right)\left(\frac{\partial}{\partial z} v(x, y, z, t)\right)\left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \phi(x, y, z, t) \\
&- \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z} v(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \\
&+ \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} u(x, y, z, t)\right) w(x, y, z, t) \\
&- \left(\frac{\partial}{\partial x} v(x, y, z, t)\right)\left(\frac{\partial}{\partial z} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \phi(x, y, z, t) \\
&- \left(\frac{\partial}{\partial x} w(x, y, z, t)\right)\left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right)\Bigg) \Bigg/ \\
&(u(x, y, z, t)^2 + v(x, y, z, t)^2 + w(x, y, z, t)^2 + \phi(x, y, z, t)^2)^2
\end{aligned}
$$

$$
\begin{aligned}
Jz := \Bigg(& \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right) \\
& + \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right) \\
& - \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right) \\
& - \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right) \\
& - \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right) \\
& + \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right) \\
& + \left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t) \\
& - \left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right) \\
& + \left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right) \\
& - \mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right) \\
& + \left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right) \\
& + \left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t) \\
& + \left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t) \\
& - \left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t) \\
& - \left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t) \\
& + \left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right) \\
& + \mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right) \\
& + \left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t) \\
& + \left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right) \\
& - \left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t) \\
& - \left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)
\end{aligned}
$$

$$- \left(\frac{\partial}{\partial x} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial t} \mathrm{v}(x, y, z, t)\right)$$

$$- \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)$$

$$- \left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{v}(x, y, z, t)\right) \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)\Bigg)\Bigg) \Big/$$

$$\left(\mathrm{u}(x, y, z, t)^2+\mathrm{v}(x, y, z, t)^2+\mathrm{w}(x, y, z, t)^2+\phi(x, y, z, t)^2\right)^2$$

$$Net := \Bigg((-1+c)\Bigg(\left(\frac{\partial}{\partial x} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right) \phi(x, y, z, t)^2$$

$$- \left(\frac{\partial}{\partial x} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{w}(x, y, z, t)\right) \phi(x, y, z, t)^2$$

$$+ \left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right) \mathrm{v}(x, y, z, t)\, \phi(x, y, z, t)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right) \phi(x, y, z, t)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial z} \mathrm{v}(x, y, z, t)\right)$$

$$- \mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) \mathrm{u}(x, y, z, t)$$

$$- \mathrm{v}(x, y, z, t)\, \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)$$

$$- \mathrm{v}(x, y, z, t)\, \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right)$$

$$- \mathrm{v}(x, y, z, t)\, \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$+ \mathrm{v}(x, y, z, t)\, \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)$$

$$+ \mathrm{v}(x, y, z, t)\, \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right)$$

$$+ \mathrm{v}(x, y, z, t)\, \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)$$

$$+ \mathrm{v}(x, y, z, t)\, \phi(x, y, z, t)\left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{w}(x, y, z, t)\right)$$

$$- \mathrm{v}(x, y, z, t)\, \phi(x, y, z, t)\left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial y} \mathrm{w}(x, y, z, t)\right)$$

$$- \mathrm{v}(x, y, z, t)\left(\frac{\partial}{\partial x} \mathrm{u}(x, y, z, t)\right) \mathrm{w}(x, y, z, t)\left(\frac{\partial}{\partial t} \mathrm{v}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$- \mathrm{v}(x, y, z, t)\, \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{w}(x, y, z, t)\right)$$

$$- \mathrm{v}(x, y, z, t)\, \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$+ \mathrm{v}(x, y, z, t)\, \mathrm{u}(x, y, z, t)\left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)\left(\frac{\partial}{\partial z} \mathrm{u}(x, y, z, t)\right)\left(\frac{\partial}{\partial t} \mathrm{w}(x, y, z, t)\right)$$

$$
\begin{aligned}
&- \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\\
&+ \mathrm{v}(x,y,z,t)\,\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\\
&+ \mathrm{v}(x,y,z,t)\,\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\\
&+ \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\\
&+ \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\\
&+ \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\\
&- \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\\
&+ \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\\
&- \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\\
&- \left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\mathrm{v}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\\
&- \left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\\
&+ \left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\\
&- \left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\\
&- \left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{v}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\\
&- \left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\,\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\\
&+ \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\\
&+ \mathrm{v}(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\\
&- \left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)^2\\
&+ \left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)^2\\
&- \mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\\
&- \mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)
\end{aligned}
$$

$$- u(x, y, z, t) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$- u(x, y, z, t) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right)$$

$$+ v(x, y, z, t)^2 \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$+ v(x, y, z, t)^2 \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right)$$

$$- \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) w(x, y, z, t)^2 \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \phi(x, y, z, t)$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right)$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right)$$

$$- \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) u(x, y, z, t)^2$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \phi(x, y, z, t) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right)$$

$$+ v(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \phi(x, y, z, t)$$

$$- v(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$- v(x, y, z, t)\, u(x, y, z, t) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$

$$- w(x, y, z, t)\, u(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right)$$

$$+ u(x, y, z, t) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) w(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$- \phi(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) u(x, y, z, t) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right)$$

$$+ \phi(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) u(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right)$$

$$- \phi(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right)$$

$$+ \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \phi(x, y, z, t)^2$$

$$+ \phi(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) w(x, y, z, t) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right)$$
$$+ \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) u(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right)$$
$$+ \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) v(x, y, z, t)\, \phi(x, y, z, t)$$
$$- \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) w(x, y, z, t)\, \phi(x, y, z, t) \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right)$$
$$- \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} v(x, y, z, t)\right) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right) \phi(x, y, z, t)^2$$
$$+ w(x, y, z, t) \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \phi(x, y, z, t)$$
$$- w(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \phi(x, y, z, t)$$
$$+ \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) u(x, y, z, t)^2$$
$$+ \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right) u(x, y, z, t)^2$$
$$- w(x, y, z, t) \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \phi(x, y, z, t)$$
$$- v(x, y, z, t)^2 \left(\frac{\partial}{\partial x} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right)$$
$$- w(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \phi(x, y, z, t)$$
$$- v(x, y, z, t)^2 \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial z} u(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right)$$
$$+ w(x, y, z, t)\, u(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right)$$
$$+ w(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) u(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right)$$
$$+ w(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) u(x, y, z, t) \left(\frac{\partial}{\partial t} \phi(x, y, z, t)\right) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right)$$
$$+ w(x, y, z, t) \left(\frac{\partial}{\partial x} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \left(\frac{\partial}{\partial y} w(x, y, z, t)\right) \phi(x, y, z, t)$$
$$+ w(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial y} u(x, y, z, t)\right) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right) \phi(x, y, z, t)$$
$$- w(x, y, z, t) \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial y} \phi(x, y, z, t)\right) u(x, y, z, t) \left(\frac{\partial}{\partial t} v(x, y, z, t)\right)$$
$$- w(x, y, z, t) \left(\frac{\partial}{\partial x} \phi(x, y, z, t)\right) u(x, y, z, t) \left(\frac{\partial}{\partial y} v(x, y, z, t)\right) \left(\frac{\partial}{\partial t} w(x, y, z, t)\right)$$
$$- v(x, y, z, t)^2 \left(\frac{\partial}{\partial x} w(x, y, z, t)\right) \left(\frac{\partial}{\partial t} u(x, y, z, t)\right) \left(\frac{\partial}{\partial z} \phi(x, y, z, t)\right)$$

$$
\begin{aligned}
&-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\\
&-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\\
&+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\\
&+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\\
&+\mathrm{v}(x,y,z,t)^2\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\\
&+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\\
&+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\\
&+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\\
&-\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2\\
&-\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2\\
&+\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)^2\\
&+\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\\
&-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\\
&+\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\phi(x,y,z,t)\right)\\
&-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial t}\mathrm{v}(x,y,z,t)\right)\\
&+\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial t}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)^2\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)))\Big/\\
&(\mathrm{u}(x,y,z,t)^2+\mathrm{v}(x,y,z,t)^2+\mathrm{w}(x,y,z,t)^2+\phi(x,y,z,t)^2)^{(5/2)}
\end{aligned}
$$

It is now apparent that the interaction between the potentials and the conserved current is equal to the ADJOINT curvature of the 4D Monge surface.

Moreover, the charge density associated with the the induced charge current is not zero and is equal to the Adjoint curvature. This implies that the charge density is cubic in the curvatures of the implicit surface.

For the Monge surface, unique in time, if the signature of the Holder norm is changed to +++-, then the Interaction energy density is the negative of the Adjoint curvature. Only the denominators change in the expressions.

```
>
>
> FF:=wcollect(d(AA)):
> Helicity:=factor(AA&^FF);
```

$$Helicity := \&\hat{}(d(x), d(z), d(t))\, w(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right) + \&\hat{}(d(x), d(z), d(t))\, w(x,y,z,t)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)$$
$$- \&\hat{}(d(x), d(z), d(t))\, u(x,y,z,t)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right) - \&\hat{}(d(x), d(z), d(t))\, u(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$+ \&\hat{}(d(x), d(z), d(t))\, \phi(x,y,z,t)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right) - \&\hat{}(d(x), d(z), d(t))\, \phi(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)$$
$$- \&\hat{}(d(x), d(y), d(z))\, u(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right) + \&\hat{}(d(x), d(y), d(z))\, u(x,y,z,t)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$
$$+ \&\hat{}(d(x), d(y), d(z))\, v(x,y,z,t)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right) - \&\hat{}(d(x), d(y), d(z))\, v(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)$$
$$+ \&\hat{}(d(x), d(y), d(z))\, w(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right) - \&\hat{}(d(x), d(y), d(z))\, w(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)$$
$$- \&\hat{}(d(x), d(y), d(t))\, u(x,y,z,t)\left(\frac{\partial}{\partial t}v(x,y,z,t)\right) - \&\hat{}(d(x), d(y), d(t))\, u(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)$$
$$+ \&\hat{}(d(x), d(y), d(t))\, v(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right) + \&\hat{}(d(x), d(y), d(t))\, v(x,y,z,t)\left(\frac{\partial}{\partial t}u(x,y,z,t)\right)$$
$$- \&\hat{}(d(x), d(y), d(t))\, \phi(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right) + \&\hat{}(d(x), d(y), d(t))\, \phi(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)$$
$$+ \&\hat{}(d(y), d(z), d(t))\, w(x,y,z,t)\left(\frac{\partial}{\partial t}v(x,y,z,t)\right) + \&\hat{}(d(y), d(z), d(t))\, w(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)$$
$$- \&\hat{}(d(y), d(z), d(t))\, v(x,y,z,t)\left(\frac{\partial}{\partial t}w(x,y,z,t)\right) - \&\hat{}(d(y), d(z), d(t))\, v(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$+ \&\hat{}(d(y), d(z), d(t))\, \phi(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right) - \&\hat{}(d(y), d(z), d(t))\, \phi(x,y,z,t)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$

```
>
> B:=curl([A[1],A[2],A[3]],[x,y,z]):B1:=factor(B[1]);B2:=factor(B[2]);B3:=factor(B
[3]);
>
```

$$B1 := \left(\frac{\partial}{\partial y}w(x,y,z,t)\right) - \left(\frac{\partial}{\partial z}v(x,y,z,t)\right)$$
$$B2 := \left(\frac{\partial}{\partial z}u(x,y,z,t)\right) - \left(\frac{\partial}{\partial x}w(x,y,z,t)\right)$$
$$B3 := \left(\frac{\partial}{\partial x}v(x,y,z,t)\right) - \left(\frac{\partial}{\partial y}u(x,y,z,t)\right)$$

```
>
> Pot:=eval(A[4]);EP:=grad(Pot,[x,y,z]):evalm(EP);E1:=-factor(EP[1]-diff(Pot[1],t)
);E2:=-factor(EP[2]-diff(Pot[2],t));E3:=-factor(EP[3]-diff(Pot[3],t));
```

$$Pot := -\phi(x,y,z,t)$$

$$\left[-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right), -\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right), -\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\right]$$

$$E1 := \left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right) + \left(\frac{\partial}{\partial t}(-\phi(x,y,z,t))_1\right)$$

$$E2 := \left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right) + \left(\frac{\partial}{\partial t}(-\phi(x,y,z,t))_2\right)$$

$$E3 := \left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right) + \left(\frac{\partial}{\partial t}(-\phi(x,y,z,t))_3\right)$$

```
> EXA:=crossprod([E1,E2,E3],[A[1],A[2],A[3]]):TOP_TORSION1:=factor(EXA[1]+B1*Pot);
  Helicity:=innerprod([A[1],A[2],A[3]],B);
```

$$TOP\_TORSION1 := w(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right) + w(x,y,z,t)\left(\frac{\partial}{\partial t}(-\phi(x,y,z,t))_2\right) - v(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right) - v(x,y,z,t)\left(\frac{\partial}{\partial t}(-\phi(x,y,z,t))_3\right) - \phi(x,y,z,t)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right) + \phi(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)$$

$$Helicity := u(x,y,z,t)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right) - u(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right) + v(x,y,z,t)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right) - v(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right) + w(x,y,z,t)\left(\frac{\partial}{\partial x}v(x,y,z,t)\right) - w(x,y,z,t)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)$$

```
>
> DIVE:=factor(diverge([E1,E2,E3],[x,y,z]));rho;
```

$$DIVE := \left(\frac{\partial^2}{\partial x^2}\phi(x,y,z,t)\right) + \left(\frac{\partial^2}{\partial x\,\partial t}(-\phi(x,y,z,t))_1\right) + \left(\frac{\partial^2}{\partial y^2}\phi(x,y,z,t)\right) + \left(\frac{\partial^2}{\partial y\,\partial t}(-\phi(x,y,z,t))_2\right) + \left(\frac{\partial^2}{\partial z^2}\phi(x,y,z,t)\right) + \left(\frac{\partial^2}{\partial z\,\partial t}(-\phi(x,y,z,t))_3\right)$$

$$-\Bigg(v(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}w(x,y,z,t)\right)$$
$$+ v(x,y,z,t)\left(\frac{\partial}{\partial x}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}w(x,y,z,t)\right)$$
$$- v(x,y,z,t)\left(\frac{\partial}{\partial x}w(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$- \left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}u(x,y,z,t)\right)w(x,y,z,t)\left(\frac{\partial}{\partial z}v(x,y,z,t)\right)$$

$$-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)$$
$$+\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$-\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)$$
$$+\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$+\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\mathrm{u}(x,y,z,t)\left(\frac{\partial}{\partial z}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{w}(x,y,z,t)\right)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{w}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\left(\frac{\partial}{\partial y}\mathrm{v}(x,y,z,t)\right)\phi(x,y,z,t)$$
$$-\left(\frac{\partial}{\partial x}\mathrm{v}(x,y,z,t)\right)\left(\frac{\partial}{\partial z}\mathrm{u}(x,y,z,t)\right)\mathrm{w}(x,y,z,t)\left(\frac{\partial}{\partial y}\phi(x,y,z,t)\right)\Bigg)\Bigg)\Bigg/$$
$$\left(\mathrm{u}(x,y,z,t)^2+\mathrm{v}(x,y,z,t)^2+\mathrm{w}(x,y,z,t)^2+\phi(x,y,z,t)^2\right)^2$$

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Div E is not proportional to the charge density. !!!
This implies that the constitutive relation between F and G is not linear.