```
> restart:
  with(linalg):with(difforms):with(liesymm):with(plots):defforms(X=0
  ,Y=0,Z=0,S=0,a=const,b=const,c=const,e=const,p=const,n=const,rot=c
  onst);
```

Warning, new definition for norm

```
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the names &^, d and wdegree have been redefined

Warning, the protected name close has been redefined and unprotected

Warning, the name changecoords has been redefined
```

$$\mathrm{defforms}(X=0, Y=0, Z=0, S=0, a=const, b=const, c=const, e=const, p=const, n=const, rot=const)$$

THE HOPF MAP and CHIRALITY

R. M. Kiehn July 12,1998 - May 5,1999 - Oct 1 2004

See http://www22.pair.com/csdc/pdf/defects2.pdf
for a tutorial on the Cartan structural equations.

The Hopf Map from the sphere S3 to the sphere S2 gives a representation
for the canonical 1-form of Pfaff dimension 4. It has two different "chiral" representations.
Recall that parity violation does not occur until dimension 4.

The canonical 1-form that generates the Hopf Map is of the classic Zhitomirski form.
A = -YdX + XdY + (or -) (SdZ-ZdS)
.
This form admits three associated gradient fields, which are transversal.to the form
in the sense that each gradient contracted with the direction field of the form yields zero.

These three gradient fields have a non-zero intersection.
The triple of functions that are used form the three gradient fields define the Hopf map.
The Hopf map has two representations with different "chirality".
These two forms are give below explicitly. Each chiral Hopf Map can be
used to compute a basis frame over the 4 dimensional domain mod the origin.
The two chirally different basis frames yield different topological
and geometric structures. The two basis frames are used to compute the
Cartan matrix and the chirally different Cartan equations of structure (see below)

There exist an infinite number of integrating factors to A such that
d(A^dA) = 0. Hence these closed 3-forms of Topological Torsion will produce
contact manifolds of dimension 3. (If d(A^dA) is not zero the manifold
is of dimension 4 and is sympletic.)

Remark:
Weinstein has claimed that the canonical Hopf map is tight. (No Limit cycles). Additional components have to be added to the Hopf 1-form to produce "overtwisted" structures, which are supposed to support limit cycles. (An unfinished objective is to show how the Van der Pohl oscillator fits within this scheme.)
****

In that which follows
Cartan's Repere Mobile will be computed for a representation of the Hopf Map
from S3 in 4D to S2 in 3D.
The Position Vector in 4 space will be presumed to have 4 components R4 = {X,Y,Z,S}.
The position vector in 3 space will be presumed to have 3 components r3 = {x,y,z}.
The origins are presumed to be coincident;
in the sense that the four sphere goes to zero when the 3 sphere goes to zero.

```
> setup(X,Y,Z,S):
```

R4 is the radius of S3,
and beta is an arbitrary scaling function on the Holder type with constant coefficients a,b,c,e.
beta will be used to scale the Toplogical Torsion current.

```
> R4:=((S)^2+X^2+Y^2+Z^2)^(1/2);betaholder:=((e*S^p+a*X^p+b*Z^p+c*Y^p)^(n/p));r3:=(x^2+y^2+z^2)^(1/2);
```

$$R4 := \sqrt{S^2 + X^2 + Y^2 + Z^2}$$

$$betaholder := (e\, S^p + a\, X^p + b\, Z^p + c\, Y^p)^{\left(\frac{n}{p}\right)}$$

$$r3 := \sqrt{x^2 + y^2 + z^2}$$

The Torsion current, A^dA, (in 4D) will have zero 4 divergence when the exponent of the beta divisor is n=2, for any anisotropy (a,b,c,e) and any exponent p.
There are two distinct versions of the Hopf map:
Each version is distinguished by the chirality factor, defined as rot:
rot is either plus or minus 1,
for right or left handed enantiomorphisms.

Consider the map from X,Y,Z,S to x,y,z given by the functions:

```
> x:=2*(S*Y+rot*X*Z);e1:=grad(x,[X,Y,Z,S]);
```

$$x := 2\, S\, Y + 2\, rot\, X\, Z$$

$$e1 := [2\, rot\, Z, 2\, S, 2\, rot\, X, 2\, Y]$$

```
> y:=2*(Y*Z-rot*S*X);e2:=grad(y,[X,Y,Z,S]);
```

$$y := 2\, Y\, Z - 2\, rot\, S\, X$$

$$e2 := [-2\, rot\, S, 2\, Z, 2\, Y, -2\, rot\, X]$$

```
> z:=(X^2+Y^2-(S^2+Z^2));e3:=grad(z,[X,Y,Z,S]);
```

$$z := X^2 + Y^2 - S^2 - Z^2$$

$$e3 := [2\, X, 2\, Y, -2\, Z, -2\, S]$$

Note that the square of the spherical radius in r3 is independent of the rot factor -- a double cover.

```
> r3RH=factor(subs(rot=+1,(x^2+y^2+z^2)));r3LH=factor(subs(rot=-1,(x^2+y^2+z^2)));
```

$$r3RH = (S^2 + X^2 + Y^2 + Z^2)^2$$

$$r3LH = (S^2 + X^2 + Y^2 + Z^2)^2$$

```
>
```

The vector field adjoint to the three gradients, e1,e2,e3 form the components of a Hopf "current", HTJ. THe Hopf current HTJ is the orthogonal compliment to the "normal" field, constructed from the three functions which define a N-3 = 4-3 = 1 dimensional immersed manifold in R4. This current is proportional to the "tangent" vector to the 1D immersed manifold. (This idea of implicit functions producing a normal field - instead of a tangent field - is (in a sense) dual to that concept of a tangent manifold produced by parametrizations).

Note that there are two distinct Hopf currents (A Left rot = -1 and a Right rot=+1) in R4 that map to the sphere in r3.

```
> HTJRH:=factor(subs(rot=1,d(x)&^d(y)&^d(z)));HTJLH:=factor(subs(rot=-1,d(x)&^d(y)&^d(z)));VOL:=d(X)&^d(Y)&^d(Z)&^d(S);HOPFDIFF:=hook(VOL,[Y,X,S,Z]);
```

$$HTJRH := 8\,(S^2 + X^2 + Y^2 + Z^2)\,(\&\hat{}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(Z))\,Z + \&\hat{}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(S))\,S - \&\hat{}(\mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(S))\,Y - \&\hat{}(\mathrm{d}(X), \mathrm{d}(Z), \mathrm{d}(S))\,X)$$

$$HTJLH := -8\,(S^2 + X^2 + Y^2 + Z^2)\,(\&\hat{}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(Z))\,Z + \&\hat{}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(S))\,S + \&\hat{}(\mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(S))\,Y + \&\hat{}(\mathrm{d}(X), \mathrm{d}(Z), \mathrm{d}(S))\,X)$$

$$VOL := -\&\hat{}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(Z), \mathrm{d}(S))$$

$$HOPFDIFF := -\mathrm{X}(X, Y, Z, S)\,\&\hat{}(\mathrm{d}(X), \mathrm{d}(Z), \mathrm{d}(S)) + \mathrm{Y}(X, Y, Z, S)\,\&\hat{}(\mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(S)) - \mathrm{S}(X, Y, Z, S)\,\&\hat{}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(S)) + \mathrm{Z}(X, Y, Z, S)\,\&\hat{}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(Z))$$

Define the 1-form of Hopf Action AA as

```
> beta:=subs(subs(n=n,a=a,b=b,c=c,p=2,betaholder));
```

$$\beta := (e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}$$

```
> AA:=subs([(-Y)/beta,X/beta,(-rot*S)/beta,rot*Z/beta]);
```

$$AA := \left[-\frac{Y}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}, \frac{X}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}, -\frac{rot\,S}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}, \frac{rot\,Z}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}\right]$$

Prove that the three gradient fields are associated directions.

```
> factor(innerprod(e1,AA));factor(innerprod(e2,AA));factor(innerprod(e3,AA));
```

>

$$-2 \frac{S\, X\, (rot-1)\, (rot+1)}{(e\, S^2 + a\, X^2 + b\, Z^2 + c\, Y^2)^{(1/2\, n)}}$$

$$-2 \frac{X\, Z\, (rot-1)\, (rot+1)}{(e\, S^2 + a\, X^2 + b\, Z^2 + c\, Y^2)^{(1/2\, n)}}$$

$$0$$

So it follows that the for the two chiral choices, rot = plus or minus 1, the three 4D gradients are orthogonal to the 4D Adjoint 1-form, AA. The three gradients are components of exact 1-forms that are dual to the current AA. From the dual point of view, the three vectors e1,e2,e3 are associated fields with respect to the 1-form with components formed from AA.

>

>

TO summarize

The mapping functions of Hopf define three exact differentials dx,dy,dz with values on R4.

There are two distinct Hopf maps depending on the a "chiral" choice.

The object is to define a fourth non-exact differential on R4 to form a basis frame on R4. That special 1-form will be defined has been defined as the Action 1-form : A = {-YdX+XdY +rot(- SdZ+ZdS)}/factor(X,Y,Z,T). The arbitrary factor will scale the 1-form of Action on R4. The components of the exact differentials are orthogonal to the Action. Moreover the position (spherical expansion) vector in R4 is orthogonal to the Action 1-form.

The Hopf map is a quartic equation in the sense that the fourth power of the "length" of the position vector R4 is equal to the square of the length of the position vector, r3. The zeros of the two spheres coincide.

```
> AreaS2:=4*pi*factor(r3^2);
```

$$AreaS2 := 4\,\pi\,(2\, S^2\, Y^2 + 4\, rot^2\, X^2\, Z^2 + 2\, Y^2\, Z^2 + 4\, rot^2\, S^2\, X^2 + X^4 + 2\, X^2\, Y^2 - 2\, X^2\, S^2 - 2\, X^2\, Z^2 + Y^4 + S^4 + 2\, S^2\, Z^2 + Z^4)$$

Redefine non-homeneous coordinates x=X/S, y=Y/S,z=Z/S
AREA:=S^4(4pi){1+x^2+y^2+z^2)^2

The exterior product of the three perfect differentials yields a non-zero volume element except at the origin of R4. and note that { i(A)dX^dY^dZ^dS }factor = dx^dy^dz . Similar to i(Z)dA=d(theta)

Now use the (non-integrable) 1-form Action, and compute its Pfaff sequence, note that the arbitrary factor 1/beta can be adjusted such that the Pfaff dimension of the 1-form is 3, for any anisotropy and any exponent p, as long as n = 2. This means that there exists a hierarchy of conserved Torsion (topological) invariants, depending on p and the anisotropy coefficients.. Note that the Pfaff dimension of the 1-form of Action is 4, if n is not equal to 2.
Hence the 1-form usually defines a symplectic manifold, but when n = 2 , the 1-form defines a contact manifold.
There are two distinct action 1-forms depending upon the chirality. The sum and differences produce 1-forms that are of rank 2, not rank 4.

```
> 
> 
> AA:=[-Y/beta,X/beta,-rot*S/beta,rot*Z/beta];Action:=innerprod(AA,[d(X),d(Y),d(Z),d(S)]);AAA:=[AA[1],AA[2],AA[3]];phi:=-AA[4];
```

$$AA := \left[-\frac{Y}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}, \frac{X}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}, -\frac{rot\,S}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}, \frac{rot\,Z}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}} \right]$$

$$Action := -\frac{Y\,\mathrm{d}(X) - X\,\mathrm{d}(Y) + rot\,S\,\mathrm{d}(Z) - rot\,Z\,\mathrm{d}(S)}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}$$

$$AAA := \left[-\frac{Y}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}, \frac{X}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}, -\frac{rot\,S}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}} \right]$$

$$\phi := -\frac{rot\,Z}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}$$

```
> BBB:=factor(curl(AAA,[X,Y,Z]));EEE:=evalm(-diff(AAA,S)-grad(phi,[X,Y,Z]));EDOTB:=innerprod(EEE,BBB);AdotB:=innerprod(AAA,BBB);
```

$$BBB := \left[\frac{rot\,S\,n\,c\,Y}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}\,(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)} + \frac{X\,n\,b\,Z}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}\,(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)}, \right.$$
$$\frac{Y\,n\,b\,Z}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}\,(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)} - \frac{rot\,S\,n\,a\,X}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}\,(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)}, 2\,\frac{1}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}}$$
$$\left. - \frac{X^2\,n\,a}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}\,(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)} - \frac{Y^2\,n\,c}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}\,(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)} \right]$$

$$EEE := \left[-\frac{Y\,n\,e\,S}{(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)}\,(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)} \right.$$

$$-\frac{rot\,Z\,n\,a\,X}{(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(1/2\,n)}\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)},$$

$$\frac{X\,n\,e\,S}{(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(1/2\,n)}\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)}$$

$$-\frac{rot\,Z\,n\,c\,Y}{(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(1/2\,n)}\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)},\ 2\,\frac{rot}{(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(1/2\,n)}}$$

$$-\frac{rot\,S^2\,n\,e}{(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(1/2\,n)}\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)}$$

$$-\frac{rot\,Z^2\,n\,b}{(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(1/2\,n)}\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)}\Bigg]$$

$$EDOTB := -2\,\frac{(n-2)\,rot}{\left((e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(1/2\,n)}\right)^2}$$

$$AdotB := -2\,\frac{rot\,S}{\left((e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(1/2\,n)}\right)^2}$$

Next compute the 2-form for the field intensities (Maxwell-Faraday equations)

```
> F:=wcollect(simplify(wcollect(factor(subs(wcollect(d(Action)))))))
  ;
```

$$F := (-2\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,e\,S^2-2\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,a\,X^2$$
$$-2\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,c\,Y^2+(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,Y^2\,n\,c$$
$$+(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,X^2\,n\,a-2\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,b\,Z^2)$$
$$(\mathrm{d}(Y)\ \&\hat{}\ \mathrm{d}(X))+$$
$$(-(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,Z\,n\,c\,Y+(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,X\,n\,e\,S)$$
$$(\mathrm{d}(Y)\ \&\hat{}\ \mathrm{d}(S))+$$
$$(-(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,Y\,n\,b\,Z+(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,S\,n\,a\,X)$$
$$(\mathrm{d}(X)\ \&\hat{}\ \mathrm{d}(Z))+(-(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,S^2\,n\,e$$
$$+2\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,c\,Y^2+2\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,e\,S^2$$
$$+2\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,a\,X^2+2\,(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,b\,Z^2$$
$$-(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,Z^2\,n\,b)\,(\mathrm{d}(Z)\ \&\hat{}\ \mathrm{d}(S))+$$
$$((e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,rot\,S\,n\,c\,Y+(e\,S^2+a\,X^2+b\,Z^2+c\,Y^2)^{(-1/2\,n-1)}\,X\,n\,b\,Z)$$
$$(\mathrm{d}(Y)\ \&\hat{}\ \mathrm{d}(Z))+$$

$$(-(e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(-1/2\,n-1)}\,rot\,Z\,n\,a\,X - (e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(-1/2\,n-1)}\,Y\,n\,e\,S)$$
$$(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))$$

Next compute the Topological Torsion 3-form. Note that the components of this 3-form are proportional to the position vector R4 contracted with the 4D volume element, dX^dY^dZ^dS.

```
> H:=factor(Action&^d(Action));
```

$$H := -2\,(Z\,\&\wedge(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(S)) - S\,\&\wedge(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(Z)) + Y\,\&\wedge(\mathrm{d}(X), \mathrm{d}(Z), \mathrm{d}(S))$$
$$- X\,\&\wedge(\mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(S)))\,rot\,/\,(((e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)})^2)$$

The components of the 3-form H are orthogonal to the components of the 1-form of Action (transversal). The components of the three form H are a current in the direction of the "radial expansion" position vector in R4. T **he sign of H is dependent upon the chirality. Inbound versus outbound.**

Finally compute the Topological Parity 4-form

```
> K:=factor(d(Action)&^d(Action));
```

$$K := 4\,\frac{\&\wedge(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(Z), \mathrm{d}(S))\,(n-2)\,rot}{((e\,S^2 + a\,X^2 + b\,Z^2 + c\,Y^2)^{(1/2\,n)})^2}$$

This result is extraordinary, for it shows that the sign of K depends on the chirality factor, rot, which can be plus or minus one. The integer n determines the homogeneity of the form, and the coefficients a,b,c,e, determine the isotropy - or lack thereof. For arbitrary evolution, K is like a Liapunov function. When K is positive, then the evolution is divergent exponentially. When K is negative, the evolution is convergent exponentially. The remarkable feature is that the sign is determined by the homogeneity conditions (the choice of n) and the chirality (rot).

Typically n=1 and p = 2, with a,b,c,e all equal to 1. Then rot = -1 is stable and rot = +1 is unstable!

Note that when n = 2, the 4-form K vanishes, such that the Pffaf dimension of the Action is 3, not 4.
The 1-form then defines a contact manifold, for which there exists a unique extremal vector, such that the closed integrals of the Action 1-form and the Torsion 3 -form are deformation invariants. The result is valid for any anisotropy denominator and any value of a,b,c,e and any value of p, subject to n=2

The effect of 4D radial expansions is null for n=2. The Lie derivative of the forms of the Pfaff sequence generated by A,dA,A^dA,F^F are zero in the direction of H (the torsion vector is the radial expansion vector in 4D when n=2 any p ,etc.)

THE HOPF MAP BASIS FRAME.

Cartan's methods of the repere mobile are applied to the two chiral Hopf basis frames.
The different chiral maps produce different results!

For algebraic clarity the constants will be chosen as:

(If you have Maple, you can change these values and Maple will recompute the formulas)

```
> ### WARNING: `Adjoint` might conflict with Maple's meaning of that name
n:=0;p:=2;a:=1;b:=1;c:=1;e:=1;AA1:=evalm(Gamma*AA);AA:=evalm(AA);Adjoint:=AA;
```

$$n := 0$$
$$p := 2$$
$$a := 1$$
$$b := 1$$
$$c := 1$$
$$e := 1$$
$$AA1 := [-\Gamma\, Y, \Gamma\, X, -\Gamma\, rot\, S, \Gamma\, rot\, Z]$$
$$AA := [-Y, X, -rot\, S, rot\, Z]$$
$$Adjoint := AA$$

The Basis Frame on R4 then is constructed from the three gradients and the adjoint field.

```
> E1:=[rot*Z,S,rot*X,Y];INNERE1A:=innerprod(E1,AA);
```

$$E1 := [rot\, Z, S, rot\, X, Y]$$
$$INNERE1A := S\, X - rot^2\, X\, S$$

```
> E2:=[-rot*S,Z,Y,-rot*X];INNERE2A:=innerprod(E2,AA);
```

$$E2 := [-rot\, S, Z, Y, -rot\, X]$$
$$INNERE2A := Z\, X - rot^2\, Z\, X$$

```
> E3:=[-X,-Y,Z,S];INNERE3A:=innerprod(E3,AA);
```

$$E3 := [-X, -Y, Z, S]$$
$$INNERE3A := 0$$

```
> FT:=array([[E1[1],E1[2],E1[3],E1[4]],[E2[1],E2[2],E2[3],E2[4]],[E3[1],E3[2],E3[3],E3[4]],[AA[1],AA[2],AA[3],AA[4]]]);factor(subs(rot^2=1,det(FT)));FTR:=subs(rot=1,rot^2=1,evalm(FT));FTL:=subs(rot=-1,rot^2=1,evalm(FT));
```

$$FT := \begin{bmatrix} rot\, Z & S & rot\, X & Y \\ -rot\, S & Z & Y & -rot\, X \\ -X & -Y & Z & S \\ -Y & X & -rot\, S & rot\, Z \end{bmatrix}$$

$$(S^2 + X^2 + Y^2 + Z^2)^2$$

$$FTR := \begin{bmatrix} Z & S & X & Y \\ -S & Z & Y & -X \\ -X & -Y & Z & S \\ -Y & X & -S & Z \end{bmatrix}$$

$$FTL := \begin{bmatrix} -Z & S & -X & Y \\ S & Z & Y & X \\ -X & -Y & Z & S \\ -Y & X & S & -Z \end{bmatrix}$$

Form at the point R4 the matrix array of E1, E2, E3, Action, to create Cartan's Repere Mobile. Orthogonality is preserved for either choice of the chirality. There are two Frame matrices FTR and FTL,

The extraordinary result is that the different chiralities produce very different structures. The minimum polynomial for the LH case (rot=-1) is a quartic as a product of squares. The trace of the frame matrix vanishes! On the otherhand, the RH (rot=+1) case yields a cubic term (as the trace is nonzero) The eigenvalues of the LH case are equal and opposite pairs with only two distinct magnitudes. The eigenvalues of the RH case can all be distinct.

Note as the determinant of the Frame is not zero (except at the excluded origin), the Frame matrix is global.

```
> DET:=factor(subs(rot^2=1,det(FTR)));FTR1:=trace(FTR);FTL1:=trace(FTL);FTR3:=trace(adjoint(FTR));FTL3:=trace(adjoint(FTL));
```

$$DET := (S^2+X^2+Y^2+Z^2)^2$$

$$FTR1 := 4\,Z$$

$$FTL1 := 0$$

$$FTR3 := 4\,Z\,Y^2+4\,S^2\,Z+4\,Z^3+4\,X^2\,Z$$

$$FTL3 := 0$$

```
>
> EVFTR:=eigenvalues(FTR);EVFTL:=eigenvalues(FTL);
```

$$EVFTR := Z+\sqrt{-S^2-X^2-Y^2},\, Z-\sqrt{-S^2-X^2-Y^2},\, Z+\sqrt{-S^2-X^2-Y^2},\, Z-\sqrt{-S^2-X^2-Y^2}$$

$$EVFTL := \sqrt{Z^2-Y^2+S^2+X^2+2\sqrt{-Y^2\,Z^2-S^2\,Y^2-X^2\,Y^2}},$$
$$-\sqrt{Z^2-Y^2+S^2+X^2+2\sqrt{-Y^2\,Z^2-S^2\,Y^2-X^2\,Y^2}},$$
$$\sqrt{Z^2-Y^2+S^2+X^2-2\sqrt{-Y^2\,Z^2-S^2\,Y^2-X^2\,Y^2}},$$
$$-\sqrt{Z^2-Y^2+S^2+X^2-2\sqrt{-Y^2\,Z^2-S^2\,Y^2-X^2\,Y^2}}$$

The determinant never vanishes except at the origin of R4.
Next create the induced (from the basis frame) metric defined as:

```
>
> GUN:=subs(rot^2=1,innerprod(FT,transpose(FT)));
```

$$GUN := \begin{bmatrix} S^2+X^2+Y^2+Z^2 & 0 & 0 & 0 \\ 0 & S^2+X^2+Y^2+Z^2 & 0 & 0 \\ 0 & 0 & S^2+X^2+Y^2+Z^2 & 0 \\ 0 & 0 & 0 & S^2+X^2+Y^2+Z^2 \end{bmatrix}$$

The induced Metric is conformal and independent of chiralilty.

```
>
> FF:=transpose(FT):
> FFINV:=subs(rot^2=1,inverse(FF));
```

$FFINV :=$

$$\left[\frac{rot\,Z^3+Z\,S^2\,rot+rot\,Z\,Y^2+rot\,X^2\,Z}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}, \frac{S}{S^2+X^2+Y^2+Z^2}, \frac{rot\,X\,Z^2+rot\,S^2\,X+rot\,X^3+rot\,X\,Y^2}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}, \frac{Y\,(S^2+X^2+Y^2+Z^2)}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}\right]$$

$$\left[-\frac{S\,rot\,Z^2+S^3\,rot+rot\,X^2\,S+Y^2\,rot\,S}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}, \frac{Z}{S^2+X^2+Y^2+Z^2}, \frac{Y\,(S^2+X^2+Y^2+Z^2)}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}, -\frac{rot\,X\,Z^2+rot\,S^2\,X+rot\,X^3+rot\,X\,Y^2}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}\right]$$

$$\left[-\frac{X}{S^2+X^2+Y^2+Z^2}, -\frac{Y}{S^2+X^2+Y^2+Z^2}, \frac{Z}{S^2+X^2+Y^2+Z^2}, \frac{S}{S^2+X^2+Y^2+Z^2}\right]$$

$$\left[-\frac{Y\,(S^2+X^2+Y^2+Z^2)}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}, \frac{X}{S^2+X^2+Y^2+Z^2}, -\frac{S\,rot\,Z^2+S^3\,rot+rot\,X^2\,S+Y^2\,rot\,S}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}, \frac{rot\,Z^3+Z\,S^2\,rot+rot\,Z\,Y^2+rot\,X^2\,Z}{Z^4+2\,S^2\,Z^2+2\,Y^2\,Z^2+2\,X^2\,Z^2+S^4+2\,S^2\,Y^2+2\,X^2\,S^2+X^4+2\,X^2\,Y^2+Y^4}\right]$$

```
> innerprod(FF,FFINV):
> DR:=subs(rot^2=1,innerprod(FFINV,[d(X),d(Y),d(Z),d(S)])):sigma1:=w
  collect(factor(wcollect((DR[1]))));sigma2:=wcollect(factor(wcollec
  t((DR[2]))));sigma3:=factor(wcollect((DR[3])));omega:=factor(wcoll
  ect((DR[4])));dsigma1:=wcollect(factor(simpform(d(sigma1))));dsigm
  a2:=d(sigma2);dsigma3:=d(sigma3);domega:=wcollect(factor(simpform(
```

```
d(omega))));
```

Now compute the component of the position vector in 4D with respect to the basis frame.
Note that the small omega term is not zero, hence affine torsion can exist.
In fact note that small omega is precisely the adjoint field, to within a factor.
In Gauss Weingarten theory, omega is zero.

$$\sigma 1 := \frac{S\,\mathrm{d}(Y)}{S^2+X^2+Y^2+Z^2} + \frac{rot\,Z\,\mathrm{d}(X)}{S^2+X^2+Y^2+Z^2} + \frac{rot\,X\,\mathrm{d}(Z)}{S^2+X^2+Y^2+Z^2} + \frac{Y\,\mathrm{d}(S)}{S^2+X^2+Y^2+Z^2}$$

$$\sigma 2 := \frac{Z\,\mathrm{d}(Y)}{S^2+X^2+Y^2+Z^2} - \frac{rot\,S\,\mathrm{d}(X)}{S^2+X^2+Y^2+Z^2} + \frac{Y\,\mathrm{d}(Z)}{S^2+X^2+Y^2+Z^2} - \frac{rot\,X\,\mathrm{d}(S)}{S^2+X^2+Y^2+Z^2}$$

$$\sigma 3 := \frac{-X\,\mathrm{d}(X) - Y\,\mathrm{d}(Y) + Z\,\mathrm{d}(Z) + S\,\mathrm{d}(S)}{S^2+X^2+Y^2+Z^2}$$

$$\omega := \frac{rot\,Z\,\mathrm{d}(S) - rot\,S\,\mathrm{d}(Z) + X\,\mathrm{d}(Y) - Y\,\mathrm{d}(X)}{S^2+X^2+Y^2+Z^2}$$

$$dsigma1 := 2\,\frac{(S\,X - rot\,Z\,Y)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(X))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{2\,(S^2-Y^2)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$+ \frac{2\,(-rot\,X^2 + rot\,Z^2)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{2\,(rot\,S\,X - Y\,Z)\,(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$+ \frac{2\,(Z\,S - rot\,X\,Y)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{2\,(Z\,rot\,S - X\,Y)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

$$dsigma2 := 2\,\frac{Z\,X\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(X))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{(-S^2-X^2-Y^2+Z^2)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$+ \frac{2\,S\,Z\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{2\,rot\,S\,Y\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(X))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{2\,rot\,Z\,S\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$+ \frac{rot\,(-S^2+X^2+Y^2+Z^2)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{(S^2+X^2-Y^2+Z^2)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$- \frac{2\,Y\,X\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{2\,Y\,S\,(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{2\,rot\,X\,Y\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$- \frac{rot\,(S^2-X^2+Y^2+Z^2)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{2\,rot\,X\,Z\,(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

$$dsigma3 := -4\,\frac{X\,Z\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{4\,S\,X\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{4\,Y\,Z\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$- \frac{4\,Y\,S\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

$$domega := -2\,\frac{(S^2+Z^2)\,(d(Y)\ \&\hat{}\ d(X))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{2\,(-S\,X+rot\,Z\,Y)\,(d(Y)\ \&\hat{}\ d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$- \frac{2\,(-rot\,S\,X+Y\,Z)\,(d(X)\ \&\hat{}\ d(Z))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{2\,(-rot\,X^2-rot\,Y^2)\,(d(Z)\ \&\hat{}\ d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$- \frac{2\,(-Z\,X-Y\,rot\,S)\,(d(Y)\ \&\hat{}\ d(Z))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{2\,(S\,Y+rot\,X\,Z)\,(d(X)\ \&\hat{}\ d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

The forms sigma1 sigma2 and sigma3 are of Pfaff dimension 2. The form omega is of Pfaff dimension 4 unless the exponent n = 2, any p, any signature.

The forms sigma and omega are the 1-forms relative to the basis frame FF
In the example, the 1-forms sigma1 sigma2 and sigma3 are proportional to exact differentials of the mapping functions defined by the Hopf map giving x,y,z in terms of X,Y,Z,T
The in-exact 1-form of Action is proportional to the 1-form small omega.
For a parametrized surface, small omega vanishes, and such subspaces have Zero Torsion of the Affine type.
Such is not the case for the implicit manifolds. (Recall the three functions define a 1 D manifold , a curve, in 4D space.

From the Frame matrix, now use the standard methods to compute the Cartan Matrix of connection 1-forms.

See http://www.uh.edu/~rkiehn/pdf/defects2.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]):
> cartan:=(evalm(FFINV&*dFF)):
```

Evaluate each component of the connection coefficients on transverse subspace of E1,E2,E3.

```
> Gamma11:=subs(rot^2=1,factor(wcollect(cartan[1,1])));
```

$$\Gamma 11 := \frac{Z\,d(Z)+Y\,d(Y)+X\,d(X)+S\,d(S)}{S^2+X^2+Y^2+Z^2}$$

```
> Gamma12:=subs(rot^2=1,factor(wcollect(cartan[1,2])));
```

$$\Gamma 12 := -\frac{Z\,d(S)+Y\,rot\,d(X)-S\,d(Z)-rot\,X\,d(Y)}{S^2+X^2+Y^2+Z^2}$$

```
> Gamma13:=subs(rot^2=1,factor(wcollect(cartan[1,3])));
```

$$\Gamma 13 := -\frac{rot\,d(X)\,Z+S\,d(Y)-d(S)\,Y-rot\,X\,d(Z)}{S^2+X^2+Y^2+Z^2}$$

```
> Gamma21:=factor(wcollect(cartan[2,1]));
```

$$\Gamma 21 := \frac{Z\,d(S)+Y\,rot\,d(X)-rot^2\,S\,d(Z)-rot\,X\,d(Y)}{S^2+X^2+Y^2+Z^2}$$

```
> Gamma22:=subs(rot^2=1,factor(wcollect(cartan[2,2])));
```

$$\Gamma 22 := \frac{Z\,\mathrm{d}(Z) + Y\,\mathrm{d}(Y) + X\,\mathrm{d}(X) + S\,\mathrm{d}(S)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> Gamma23:=subs(rot^2=1,factor(wcollect(cartan[2,3])));
```

$$\Gamma 23 := -\frac{\mathrm{d}(Y)\,Z + rot\,\mathrm{d}(S)\,X - Y\,\mathrm{d}(Z) - rot\,S\,\mathrm{d}(X)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> Gamma31:=factor(wcollect(cartan[3,1]));
```

$$\Gamma 31 := \frac{rot\,\mathrm{d}(X)\,Z + S\,\mathrm{d}(Y) - \mathrm{d}(S)\,Y - rot\,X\,\mathrm{d}(Z)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> Gamma32:=subs(rot^2=1,factor(wcollect(cartan[3,2])));
```

$$\Gamma 32 := \frac{\mathrm{d}(Y)\,Z + rot\,\mathrm{d}(S)\,X - Y\,\mathrm{d}(Z) - rot\,S\,\mathrm{d}(X)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> Gamma33:=subs(rot^2=1,factor(wcollect(cartan[3,3])));
```

$$\Gamma 33 := \frac{Z\,\mathrm{d}(Z) + Y\,\mathrm{d}(Y) + X\,\mathrm{d}(X) + S\,\mathrm{d}(S)}{S^2 + X^2 + Y^2 + Z^2}$$

The "Space-S" components are:

```
> hh1:=subs(rot^2=1,factor(wcollect(cartan[4,1])));
>
```

$$hh1 := \frac{rot\,Z\,\mathrm{d}(Y) - Y\,rot\,\mathrm{d}(Z) - S\,\mathrm{d}(X) + X\,\mathrm{d}(S)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> gg1:=subs(rot^2=1,factor(wcollect(factor(wcollect(cartan[1,4])))))
;
```

$$gg1 := -\frac{rot\,Z\,\mathrm{d}(Y) - Y\,rot\,\mathrm{d}(Z) - S\,\mathrm{d}(X) + X\,\mathrm{d}(S)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> hh2:=subs(rot^2=1,factor(wcollect(cartan[4,2])));
```

$$hh2 := -\frac{Z\,\mathrm{d}(X) - Y\,rot\,\mathrm{d}(S) - \mathrm{d}(Z)\,X + rot\,S\,\mathrm{d}(Y)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> gg2:=subs(rot^2=1,factor(wcollect(cartan[2,4])));
```

$$gg2 := \frac{Z\,\mathrm{d}(X) - Y\,rot\,\mathrm{d}(S) - \mathrm{d}(Z)\,X + rot\,S\,\mathrm{d}(Y)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> hh3:=subs(rot^2=1,factor(wcollect(cartan[4,3])));
```

$$hh3 := \frac{rot\,Z\,\mathrm{d}(S) - X\,\mathrm{d}(Y) + Y\,\mathrm{d}(X) - rot\,S\,\mathrm{d}(Z)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> gg3:=factor(wcollect(cartan[3,4]));
```

$$gg3 := -\frac{rot\,Z\,\mathrm{d}(S) - X\,\mathrm{d}(Y) + Y\,\mathrm{d}(X) - rot\,S\,\mathrm{d}(Z)}{S^2 + X^2 + Y^2 + Z^2}$$

```
> Omega:=subs(rot^2=1,simplify(wcollect(cartan[4,4])));
```

$$\Omega := \frac{Z\,\mathrm{d}(Z) + Y\,\mathrm{d}(Y) + X\,\mathrm{d}(X) + S\,\mathrm{d}(S)}{S^2 + X^2 + Y^2 + Z^2}$$

Note that the Big Omega term is a perfect differential, and is zero only when the argument R4 is a constant. That is off the sphere R4 = constant the Omega term does not vanish. Hence if the radius of the sphere is expanding, then R4 is not constant and one has a dilatation. (The source of dilatons?)

There are in general two sets of torsion two forms. The affine two forms, big Sigma, which depend upon the product of little omega and the connection components, little gamma.

The second set of torsion 2-forms is related to Big Omega and the connection components, little gamma. See http://www.uh.edu/~rkiehn/pdf/defects2.pdf

AFFINE TORSION 2-forms due to translations |Sigma>

```
> Sigma1:=subs(rot^2=1,(wcollect(factor(omega&^gg1))));Sigma2:=subs(rot^2=1,wcollect(factor(omega&^gg2)));Sigma3:=subs(rot^2=1,omega&^gg3);
```

$$\Sigma 1 := \frac{(S\,X - rot\,Z\,Y)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(X))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{(Z^2 - X^2)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{(S^2\,rot - rot\,Y^2)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{(rot\,S\,X - Y\,Z)\,(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{(-Z\,S + rot\,X\,Y)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{(-Z\,rot\,S + X\,Y)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

$$\Sigma 2 := -\frac{(-Z\,X - Y\,rot\,S)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(X))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{(Z\,S + rot\,X\,Y)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{(-Z\,rot\,S - X\,Y)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{(-S\,Y - rot\,X\,Z)\,(\mathrm{d}(Z)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{(-S^2 + X^2)\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{(rot\,Z^2 - rot\,Y^2)\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

$$\Sigma 3 := 2\,\frac{rot\,X\,S\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{2\,rot\,Z\,X\,(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2} - \frac{2\,rot\,S\,Y\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(Z))}{(S^2+X^2+Y^2+Z^2)^2} + \frac{2\,rot\,Z\,Y\,(\mathrm{d}(X)\ \&\wedge\ \mathrm{d}(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

```
> GSIGMA1:=subs(rot^2=1,wcollect(Gamma11&^sigma1+Gamma12&^sigma2+Gamma13&^sigma3));diff2forms:=subs(rot^2=1,wcollect(factor(GSIGMA1-Sigma1)));
```

$$GSIGMA1 := \left(-\frac{S\,X}{(S^2+X^2+Y^2+Z^2)^2} + \frac{rot\,Z\,Y}{(S^2+X^2+Y^2+Z^2)^2}\right)(\mathrm{d}(Y)\ \&\wedge\ \mathrm{d}(X)) +$$

$$\left(-2\frac{S^2}{(S^2+X^2+Y^2+Z^2)^2}+\frac{2\,Y^2}{(S^2+X^2+Y^2+Z^2)^2}+\frac{Z^2}{(S^2+X^2+Y^2+Z^2)^2}-\frac{X^2}{(S^2+X^2+Y^2+Z^2)^2}\right)$$
$$(d(Y)\ \&\wedge\ d(S))+$$
$$\left(\frac{rot\,S^2}{(S^2+X^2+Y^2+Z^2)^2}-\frac{rot\,Y^2}{(S^2+X^2+Y^2+Z^2)^2}-\frac{2\,rot\,Z^2}{(S^2+X^2+Y^2+Z^2)^2}+\frac{2\,rot\,X^2}{(S^2+X^2+Y^2+Z^2)^2}\right)$$
$$(d(X)\ \&\wedge\ d(Z))+\left(\frac{Y\,Z}{(S^2+X^2+Y^2+Z^2)^2}-\frac{rot\,X\,S}{(S^2+X^2+Y^2+Z^2)^2}\right)(d(Z)\ \&\wedge\ d(S))$$
$$+\left(-3\frac{S\,Z}{(S^2+X^2+Y^2+Z^2)^2}+\frac{3\,rot\,X\,Y}{(S^2+X^2+Y^2+Z^2)^2}\right)(d(Y)\ \&\wedge\ d(Z))$$
$$+\left(-3\frac{rot\,Z\,S}{(S^2+X^2+Y^2+Z^2)^2}+\frac{3\,Y\,X}{(S^2+X^2+Y^2+Z^2)^2}\right)(d(X)\ \&\wedge\ d(S))$$

$$diff2forms := -2\frac{(S\,X-rot\,Z\,Y)\,(d(Y)\ \&\wedge\ d(X))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{2\,(S^2-Y^2)\,(d(Y)\ \&\wedge\ d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$-\frac{2\,(-rot\,X^2+rot\,Z^2)\,(d(X)\ \&\wedge\ d(Z))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{2\,(rot\,S\,X-Y\,Z)\,(d(Z)\ \&\wedge\ d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$-\frac{2\,(Z\,S-rot\,X\,Y)\,(d(Y)\ \&\wedge\ d(Z))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{2\,(Z\,rot\,S-X\,Y)\,(d(X)\ \&\wedge\ d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

The two methods of computation give the same result

```
> E11:=innerprod(AA,[d(X),d(Y),d(Z),d(S)]);
```

$$E11 := rot\,Z\,d(S)-rot\,S\,d(Z)+X\,d(Y)-Y\,d(X)$$

```
> d(E11)&^E11;
```

Now compute the second set of torsion 2-forms associated with rotations. |Phi>

$$-2\,rot\,Z\,\&\wedge(d(Y), d(X), d(S))+2\,rot\,S\,\&\wedge(d(Y), d(X), d(Z))+2\,rot\,X\,\&\wedge(d(Y), d(Z), d(S))$$
$$-2\,rot\,Y\,\&\wedge(d(X), d(Z), d(S))$$

```
> Phi1:=wcollect(factor(Omega&^gg1));Phi2:=wcollect(factor(Omega&^gg
  2));Phi3:=wcollect(factor(Omega&^gg3));
```

$$\Phi1 := \frac{(S\,Y+rot\,X\,Z)\,(d(Y)\ \&\wedge\ d(X))}{(S^2+X^2+Y^2+Z^2)^2}+\frac{(Z\,rot\,S-X\,Y)\,(d(Y)\ \&\wedge\ d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$+\frac{(-Z\,S+rot\,X\,Y)\,(d(X)\ \&\wedge\ d(Z))}{(S^2+X^2+Y^2+Z^2)^2}+\frac{(-Z\,X-Y\,rot\,S)\,(d(Z)\ \&\wedge\ d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

$$+\frac{(rot\,Z^2+rot\,Y^2)\,(d(Y)\,\&\wedge\,d(Z))}{(S^2+X^2+Y^2+Z^2)^2}+\frac{(-S^2-X^2)\,(d(X)\,\&\wedge\,d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

$$\Phi2:=-\frac{(rot\,S\,X-Y\,Z)\,(d(Y)\,\&\wedge\,d(X))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{(S^2\,rot+rot\,Y^2)\,(d(Y)\,\&\wedge\,d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$-\frac{(Z^2+X^2)\,(d(X)\,\&\wedge\,d(Z))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{(-S\,X+rot\,Z\,Y)\,(d(Z)\,\&\wedge\,d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$-\frac{(Z\,rot\,S+X\,Y)\,(d(Y)\,\&\wedge\,d(Z))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{(Z\,S+rot\,X\,Y)\,(d(X)\,\&\wedge\,d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

$$\Phi3:=-\frac{(X^2+Y^2)\,(d(Y)\,\&\wedge\,d(X))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{(rot\,Z\,Y+S\,X)\,(d(Y)\,\&\wedge\,d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$-\frac{(-rot\,S\,X-Y\,Z)\,(d(X)\,\&\wedge\,d(Z))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{(rot\,Z^2+S^2\,rot)\,(d(Z)\,\&\wedge\,d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$
$$-\frac{(-Y\,rot\,S+Z\,X)\,(d(Y)\,\&\wedge\,d(Z))}{(S^2+X^2+Y^2+Z^2)^2}-\frac{(-S\,Y+rot\,X\,Z)\,(d(X)\,\&\wedge\,d(S))}{(S^2+X^2+Y^2+Z^2)^2}$$

```
> factor(Phi1&^Sigma1+Phi2&^Sigma2+Phi3&^Sigma3);
```

$$0$$

```
> factor(wcollect(d(Sigma1)));0
```

$$8\,(Y\,rot\,S+Z\,X)\,(Z\,\&\wedge(d(Y),d(X),d(S))-S\,\&\wedge(d(Y),d(X),d(Z))+Y\,\&\wedge(d(X),d(Z),d(S))$$
$$-X\,\&\wedge(d(Y),d(Z),d(S)))\,/\,(S^2+X^2+Y^2+Z^2)^3$$

```
> VOL4:=d(X)&^d(Y)&^d(Z)&^d(S);
```

$$VOL4:=-\&\wedge(d(Y),d(X),d(Z),d(S))$$

```
> hook(VOL4,E3);
```

$$\mathrm{Y}(X,Y,Z,S)\,\&\wedge(d(X),d(Z),d(S))-\mathrm{X}(X,Y,Z,S)\,\&\wedge(d(Y),d(Z),d(S))$$
$$-\mathrm{Z}(X,Y,Z,S)\,\&\wedge(d(Y),d(X),d(S))+\mathrm{S}(X,Y,Z,S)\,\&\wedge(d(Y),d(X),d(Z))$$

Both of the two forms are chiral dependent.
The two vector two forms of different torsion types are not independent

```
> SS3:=factor(Sigma3*(S^2+X^2+Y^2+Z^2)^2);
```

$$SS3:=-2$$
$$rot\,(-X\,S\,(d(Y)\,\&\wedge\,d(Z))+Z\,X\,(d(Y)\,\&\wedge\,d(S))+S\,Y\,(d(X)\,\&\wedge\,d(Z))-Z\,Y\,(d(X)\,\&\wedge\,d(S)))$$

```
> PP3:=wcollect(factor(Phi3*(S^2+X^2+Y^2+Z^2)^2));factor(PP3&^PP3);
```

$$PP3:=(-X^2-Y^2)\,(d(Y)\,\&\wedge\,d(X))+(-rot\,Z\,Y-S\,X)\,(d(Y)\,\&\wedge\,d(S))$$
$$+(rot\,S\,X+Y\,Z)\,(d(X)\,\&\wedge\,d(Z))+(-S^2\,rot-rot\,Z^2)\,(d(Z)\,\&\wedge\,d(S))$$
$$+(Y\,rot\,S-Z\,X)\,(d(Y)\,\&\wedge\,d(Z))+(-rot\,X\,Z+S\,Y)\,(d(X)\,\&\wedge\,d(S))$$

$$0$$

```
> wcollect(subs(rot=1,factor(PP3+SS3)));FFF:=simplify(subs(rot=1,get
  coeff(PP3&^SS3)));factor(PP3&^SS3);
```

$$(-X^2 - Y^2)\,(d(Y)\ \&\hat{}\ d(X)) + (-Y Z - S X - 2 Z X)\,(d(Y)\ \&\hat{}\ d(S)) + (-2 S Y + S X + Y Z)\,(d(X)\ \&\hat{}\ d(Z)) + (-S^2 - Z^2)\,(d(Z)\ \&\hat{}\ d(S)) + (2 S X + S Y - Z X)\,(d(Y)\ \&\hat{}\ d(Z)) + (2 Y Z - Z X + S Y)\,(d(X)\ \&\hat{}\ d(S))$$

$$FFF := 0$$

$$0$$

The two torsion two forms have no intersection.

```
>
>
>
```