```
> restart:
```

(Nonholonomic) Frame fields for Spheres in R4

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```
> with(linalg):with(liesymm):with(difforms):with(plots):
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

Cartan's Repere Mobile will be used to evaluate structural equations,
curvature 2- forms, and torsion 2-forms for two Frame Fields on the sphere S3 in R4.

The First Frame Field to be studied will be constructed from the **Hopf Map** (both left and right versions, and of positive and negative determinant.)

The Hopf map may be considered as map from R4 to R3. Three coordinates (x,y,z) of R3 are given in terms of three functions of the 4 variables (X,Y,Z,S). Three of the (tangent) vectors of the Frame field will be deduced from perfect differentials of the three Hopf map functions. The fourth vector of the Frame Field (the normal field) will be found by constructing the adjoint, N1, to the three perfect differentials. The adjoint field is not the gradient of a scalar function.

The Frame field so constructed defines an A4 space of absolute parallelism.

The range R4 then can be constrained by the equation for a unit (real and imaginary) 3 sphere
(X^2+Y^2+Z^2+S^2)^2 = 1,
which the implies a holonomic constraint exists among the differentials:
XdX+YdY+ZdZ+SdS= 0. The constraint produces a map from S3 to S2.

**

The the second Frame Field to be studied (the Instanton map) will generate the normal field as a perfect differential, and the three remaining (tangent) vectors of the Frame Field will be non-holonomic forms.

**

In summary:

For the Hopf map the three "tangent" vectors which are well defined holonomic differentials, while the "normal" field is anholonomic.

For the Instanton case, the "normal field" is holonomic and is defined as a perfect differential, but the three "tangent" vectors are anholonomic.

```
> setup(X,Y,Z,S):defform(X=0,Y=0,Z=0,S=0,ch=const,B=const,u=0,v=0,w=0,s=0,n=const,e=const,p=const);
> r2:=(x^2+y^2+z^2);dR:=[d(X),d(Y),d(Z),d(S)];scale:=((X^p+Y^p+Z^p+S^p)^(n/p)):
>
```

$$r2 := x^2 + y^2 + z^2$$

$$dR := [\mathrm{d}(X), \mathrm{d}(Y), \mathrm{d}(Z), \mathrm{d}(S)]$$

```
>
```

The Hopf map is defined as a (constrained) projection from R4 {X,Y,Z,S} to R3 {x,y,z},

by means of the functions given below:

(ch is the chiral factor (polarization) and is equal to plus 1 or minus 1.)

```
> x:=(S*Y+ch*X*Z);y:=(Y*Z-ch*S*X);z:=(1/2)*((X^2+Y^2)-(((S)^2+Z^2)));
```

Hopf Map R4 to R3

$$x := S\,Y + ch\,X\,Z$$

$$y := Y\,Z - ch\,S\,X$$

$$z := \frac{1}{2}X^2 + \frac{1}{2}Y^2 - \frac{1}{2}S^2 - \frac{1}{2}Z^2$$

\>

Note that a point on a euclidian 2-sphere (in R3) of radius squared ,
r2 = x^2+y^2+z^2
is related to the points on the euclidean 3 sphere (in R4) of radius to the 4th power.

```
> r2:=factor(subs(ch^2=1,simplify(x^2+y^2+z^2)));R4:=X^2+Y^2+Z^2+S^2;
```

$$r2 := \frac{1}{4}(Z^2 + S^2 + X^2 + Y^2)^2$$

$$R4 := Z^2 + S^2 + X^2 + Y^2$$

The differentials of the three Hopf functions define three independent tangent fields as perfect differentials (gradient fields)

```
> e1:=d(x);e2:=d(y);e3:=d(z);
```

$$e1 := Y\,\mathrm{d}(S) + S\,\mathrm{d}(Y) + ch\,Z\,\mathrm{d}(X) + ch\,X\,\mathrm{d}(Z)$$

$$e2 := Z\,\mathrm{d}(Y) + Y\,\mathrm{d}(Z) - ch\,X\,\mathrm{d}(S) - ch\,S\,\mathrm{d}(X)$$

$$e3 := X\,\mathrm{d}(X) + Y\,\mathrm{d}(Y) - S\,\mathrm{d}(S) - Z\,\mathrm{d}(Z)$$

and can be adjoined to a **1-form normal field** which is orthogonal to each of the tangent fields. The normal field will be proportional to the adjoint of the three given vector fields. It can be constructed by forming the triple exterior product of e1^e2^e3=dx^dy^dz. In this case the common factor is R4. Use the coefficients of dx^dy^dz to form the fourth vector field n1 of the Frame.

```
> `adjoint 3-form`:=wcollect(simplify(subs(ch^2=1,e1&^e2&^e3)));n1:=innerprod([Y,-X,ch*S,-ch*Z],-dR);
```

$$\mathit{adjoint\ 3\text{-}form} := (-X\,Z^2 - X\,Y^2 - X^3 - X\,S^2)\,\&\text{^}(\mathrm{d}(S), \mathrm{d}(X), \mathrm{d}(Z)) + (-Y^2\,ch\,S - ch\,S\,X^2 - S^3\,ch - S\,ch\,Z^2)\,\&\text{^}(\mathrm{d}(S), \mathrm{d}(X), \mathrm{d}(Y)) + (Y^3 + Y\,X^2 + Y\,Z^2 + S^2\,Y)\,\&\text{^}(\mathrm{d}(S), \mathrm{d}(Z), \mathrm{d}(Y)) + (Z\,S^2\,ch + ch\,Z^3 + ch\,X^2\,Z + ch\,Z\,Y^2)\,\&\text{^}(\mathrm{d}(Y), \mathrm{d}(X), \mathrm{d}(Z))$$

$$n1 := -Y\,\mathrm{d}(X) + X\,\mathrm{d}(Y) - ch\,S\,\mathrm{d}(Z) + ch\,Z\,\mathrm{d}(S)$$

\>

The orientation of the adjoint field is arbitrary, so multiply by the factor orientation factor B whcih can takeon value plus or minus one. These fields, to within a factor, lead to an assignment of a global frame field matrix with components proportional to :

```
> HE1:=[ch*Z,S,ch*X,Y];HE2:=[-ch*S,Z,Y,-ch*X];HE3:=[X,Y,-Z,-S];HN1:=evalm(B*[-Y,X,-ch*S,ch*Z]);
```

$$HE1 := [ch\,Z, S, ch\,X, Y]$$

$$HE2 := [-ch\,S, Z, Y, -ch\,X]$$

$$HE3 := [X, Y, -Z, -S]$$

$$HN1 := [-B\,Y, B\,X, -B\,ch\,S, B\,ch\,Z]$$

All four vector fields have zero divergence.
The first three have zero curl.
The HN1 field is NOT integrable as a Pfaffian expression. The associated 1-form has non-zero topological torsion.
Arbitrarily , and for algebraic simplification, each direction field can be divided by the factor Holder factor

(X^p+Y^p+Z^p+S^p)^(n/p). It is of some interest to note that all vectors above have zero divergence with respect to [X,Y,Z,S]. This will not be done in that whch follows.

```
> diverge(HE1,[X,Y,Z,S]);diverge(HE2,[X,Y,Z,S]);diverge(HE3,[X,Y,Z,S]);diverge(evalm(HN1),[X,Y,Z,S]);
```

$$0$$
$$0$$
$$0$$
$$0$$

The Frame matrix is defined as:

```
> FF:=array([[HE1[1],HE1[2],HE1[3],HE1[4]],[HE2[1],HE2[2],HE2[3],HE2[4]],[HE3[1],HE3[2],HE3[3],HE3[4]]
,[HN1[1],HN1[2],HN1[3],HN1[4]]]);DET:=factor(subs(ch^2=1,det(FF)));GG:=simplify(subs(ch^2=1,inver
se(FF)));CHPOLYP:=simplify(factor(subs(ch^2=1,B=1/R4,charpoly(FF,lambda))));CHPOLYM:=simplify
(factor(subs(ch^2=1,B=-1/R4,charpoly(FF,lambda))));
```

FRAME MATRIX FF, INVERSE FRAME GG, ch = chirality factor, B=orientation

\>

$$FF := \begin{bmatrix} ch\,Z & S & ch\,X & Y \\ -ch\,S & Z & Y & -ch\,X \\ X & Y & -Z & -S \\ -B\,Y & B\,X & -B\,ch\,S & B\,ch\,Z \end{bmatrix}$$

$$DET := -B\,(Z^2+S^2+X^2+Y^2)^2$$

$$GG := \begin{bmatrix} \frac{ch\,Z}{Z^2+S^2+X^2+Y^2} & -\frac{ch\,S}{Z^2+S^2+X^2+Y^2} & \frac{X}{Z^2+S^2+X^2+Y^2} & -\frac{Y}{(Z^2+S^2+X^2+Y^2)\,B} \\ \frac{S}{Z^2+S^2+X^2+Y^2} & \frac{Z}{Z^2+S^2+X^2+Y^2} & \frac{Y}{Z^2+S^2+X^2+Y^2} & \frac{X}{(Z^2+S^2+X^2+Y^2)\,B} \\ \frac{ch\,X}{Z^2+S^2+X^2+Y^2} & \frac{Y}{Z^2+S^2+X^2+Y^2} & -\frac{Z}{Z^2+S^2+X^2+Y^2} & -\frac{ch\,S}{(Z^2+S^2+X^2+Y^2)\,B} \\ \frac{Y}{Z^2+S^2+X^2+Y^2} & -\frac{ch\,X}{Z^2+S^2+X^2+Y^2} & -\frac{S}{Z^2+S^2+X^2+Y^2} & \frac{ch\,Z}{(Z^2+S^2+X^2+Y^2)\,B} \end{bmatrix}$$

$$CHPOLYP := -(ch\,Z^3\lambda^3 - ch\,S^4\lambda^2 - ch\,Z^5\lambda - 2\,Z\,ch\,X^2\lambda\,S^2 - 2\,ch\,Z\,Y^2\lambda\,S^2 - 2\,ch\,Z\,Y^2\lambda\,X^2 - 2\,ch\,Z^3\,Y^2\lambda - ch\,Z\,Y^4\lambda - 2\,Z^3\,ch\,X^2\lambda - Z\,ch\,X^4\lambda + ch\,Z\,\lambda^3 S^2 + ch\,Z\,\lambda^3 X^2 + ch\,Z\,\lambda^3 Y^2 - 2\,ch\,Z^3\lambda\,S^2 - ch\,S^2\lambda^2 Z^2 + Z^2\lambda^2 S^2 + Z^2\lambda^2 X^2 + 2\,Z^2\lambda^2 Y^2 + Y^2\lambda^2 S^2 + Y^2\lambda^2 X^2 + Z^4\lambda^2 + Y^4\lambda^2 - \lambda^4 Z^2 - \lambda^4 S^2 - \lambda^4 X^2 - \lambda^4 Y^2 - ch\,Z\,Y^2\lambda - ch\,S^2 Z\,\lambda - Z\,ch\,X^2\lambda + ch\,Z\,\lambda^3 - ch\,Z^3\lambda + ch\,S^2\lambda^2 - Z^2\lambda^2 - Y^2\lambda^2 - ch\,X^2\lambda^2 + S^4 + X^4 + Y^4 - ch\,S^2\lambda^2 Y^2 + ch\,X^2\lambda^2 Z^2 + ch\,X^2\lambda^2 Y^2 - ch\,S^4 Z\,\lambda + ch\,X^4\lambda^2 + 2\,Z^2 X^2 + 2\,Y^2 S^2 + 2\,S^2 X^2 + 2\,X^2 Y^2 + 2\,Z^2 S^2 + 2\,Y^2 Z^2 + Z^4) \Big/ (Z^2+S^2+X^2+Y^2)$$

$$CHPOLYM := -(ch\,Z^3\lambda^3 - ch\,S^4\lambda^2 - ch\,Z^5\lambda - 2\,Z\,ch\,X^2\lambda\,S^2 - 2\,ch\,Z\,Y^2\lambda\,S^2 - 2\,ch\,Z\,Y^2\lambda\,X^2 - 2\,ch\,Z^3\,Y^2\lambda - ch\,Z\,Y^4\lambda - 2\,Z^3\,ch\,X^2\lambda - Z\,ch\,X^4\lambda + ch\,Z\,\lambda^3 S^2 + ch\,Z\,\lambda^3 X^2 + ch\,Z\,\lambda^3 Y^2 - 2\,ch\,Z^3\lambda\,S^2 - ch\,S^2\lambda^2 Z^2 + Z^2\lambda^2 S^2 + Z^2\lambda^2 X^2 + 2\,Z^2\lambda^2 Y^2 + Y^2\lambda^2 S^2 + Y^2\lambda^2 X^2 + Z^4\lambda^2 + Y^4\lambda^2 - \lambda^4 Z^2 - \lambda^4 S^2 - \lambda^4 X^2 - \lambda^4 Y^2 + ch\,Z\,Y^2\lambda + ch\,S^2 Z\,\lambda + Z\,ch\,X^2\lambda - ch\,Z\,\lambda^3 + ch\,Z^3\lambda - ch\,S^2\lambda^2 + Z^2\lambda^2 + Y^2\lambda^2 + ch\,X^2\lambda^2 - S^4 - X^4 - Y^4 - ch\,S^2\lambda^2 Y^2 + ch\,X^2\lambda^2 Z^2 + ch\,X^2\lambda^2 Y^2 - ch\,S^4 Z\,\lambda + ch\,X^4\lambda^2 - 2\,Z^2 X^2 - 2\,Y^2 S^2 - 2\,S^2 X^2 - 2\,X^2 Y^2 - 2\,Z^2 S^2 - 2\,Y^2 Z^2 - Z^4) \Big/ (Z^2+S^2+X^2+Y^2)$$

The characteristic polynomial depends upon both the chirality factor, ch, and the sign of the determinant B (orientation).

For B=plus one the Frame has a negative determinant (the discrete reflective cases and global abnormalities or catastrophies). For B =minus one, the Frame has a positive determinant which implies perturbations about the identity are possible (local abnormalities) . The orientation is independent from the chirality

The next equation checks to see that the specified frame produces the desired differential structures: [FF]|dR>=|sigma>

```
> sigma:=(innerprod(FF,[d(X),d(Y),d(Z),d(S)]));
```

$$\sigma := [Y\,d(S) + S\,d(Y) + ch\,Z\,d(X) + ch\,X\,d(Z),\ Z\,d(Y) + Y\,d(Z) - ch\,X\,d(S) - ch\,S\,d(X),$$
$$X\,d(X) + Y\,d(Y) - S\,d(S) - Z\,d(Z),\ -B\,Y\,d(X) + B\,X\,d(Y) - B\,ch\,S\,d(Z) + B\,ch\,Z\,d(S)]$$

```
> sigma1:=sigma[1];e1;
> sigma2:=sigma[2];e2;
> sigma3:=sigma[3];e3;
> omega:=factor(sigma[4]);(B*n1);
```

$$\sigma1 := Y\,d(S) + S\,d(Y) + ch\,Z\,d(X) + ch\,X\,d(Z)$$
$$Y\,d(S) + S\,d(Y) + ch\,Z\,d(X) + ch\,X\,d(Z)$$
$$\sigma2 := Z\,d(Y) + Y\,d(Z) - ch\,X\,d(S) - ch\,S\,d(X)$$
$$Z\,d(Y) + Y\,d(Z) - ch\,X\,d(S) - ch\,S\,d(X)$$
$$\sigma3 := X\,d(X) + Y\,d(Y) - S\,d(S) - Z\,d(Z)$$
$$X\,d(X) + Y\,d(Y) - S\,d(S) - Z\,d(Z)$$
$$\omega := B\,(-Y\,d(X) + X\,d(Y) - ch\,S\,d(Z) + ch\,Z\,d(S))$$
$$B\,(-Y\,d(X) + X\,d(Y) - ch\,S\,d(Z) + ch\,Z\,d(S))$$

The vector of induced 1-forms [FF]|dR>=|sigma> can written as the column vector of components [sigma1,sigma2,sigma3,omega] .

The induced 1-forms created by the Frame acting on the differentials on the domain are tested for integrablility. As the Hopf map was defined by a triple of functions for the first three tangent vectors, it is no surprise that the first three induced 1-forms are exact. It is the fourth component of the Frame Field (the normal or adjoint field) that is of interest. The fourth induced 1-form, omega, is not closed and does not obey the Frobenius integrability theorem. Note that the normal field depends upon both the choice of the chirality, ch, and the orientation.

```
> `dsigma1`:=d(sigma[1]);`dsigma2`:=d(sigma[2]);`dsigma3`:=d(sigma[3]);`omega`:=omega;`vorticy of omega`:=factor(d(omega));`topological torsion of omega`:=factor(subs(B^2=1,omega&^d(omega)));`topological parity of omega`:=subs(B^2=1,d(omega)&^d(omega));
```

$$dsigma1 := 0$$
$$dsigma2 := 0$$
$$dsigma3 := 0$$
$$\omega := B\,(-Y\,d(X) + X\,d(Y) - ch\,S\,d(Z) + ch\,Z\,d(S))$$
$$vorticy\ of\ omega := -2\,B\,(ch\,(d(S)\ \&\wedge\ d(Z)) + (d(Y)\ \&\wedge\ d(X)))$$

$$topological\ torsion\ of\ omega :=$$
$$2\,ch\,(\&\wedge(d(X), d(S), d(Z))\,Y - \&\wedge(d(Y), d(S), d(Z))\,X + S\,\&\wedge(d(Z), d(Y), d(X)) - \&\wedge(d(S), d(Y), d(X))\,Z)$$
$$topological\ parity\ of\ omega := 8\,ch\,\&\wedge(d(S), d(Z), d(Y), d(X))$$

It is remarkable that the Topological Torsion and the Topological parity of omega, the non - integrable induced 1-form, does not depend upon the orientation, B, but does depend upon the chirality, ch.

A possible candidate for a metric on X,Y,Z,S would be the symmetric form (which is conformal to the Euclidean metric on R4):

```
> pullbackmetric:=subs(ch^2=1,B^2=1,innerprod(transpose(FF),FF));
```

$$pullbackmetric := \begin{bmatrix} Z^2+S^2+X^2+Y^2 & 0 & 0 & 0 \\ 0 & Z^2+S^2+X^2+Y^2 & 0 & 0 \\ 0 & 0 & Z^2+S^2+X^2+Y^2 & 0 \\ 0 & 0 & 0 & Z^2+S^2+X^2+Y^2 \end{bmatrix}$$

Note that the Frame matrix FF is orthogonal but not orthonormal. It is also conformal in that the normalization is the same for all basis elements. It is independent of the orientation, B and the chirality ch. !!!!

Now Compute the Right Cartan Matrix [CR] and the Left Cartan Matrix. These matrices are negative similarity transforms to within a sign.

```
> cartan:=(subs(ch^2=1,innerprod(GG,d(FF)))):cartanL:=evalm((subs(ch^2=1,innerprod(-d(FF),GG)))):
> CLcong:=(subs(ch^2=1,innerprod(-FF,cartan,GG))):zz:=evalm(FF);inverse(zz):
```

$$zz := \begin{bmatrix} ch\,Z & S & ch\,X & Y \\ -ch\,S & Z & Y & -ch\,X \\ X & Y & -Z & -S \\ -B\,Y & B\,X & -B\,ch\,S & B\,ch\,Z \end{bmatrix}$$

```
> `Should be zero if similarity equivalents`:=wcollect(factor(CLcong[1,2]-cartanL[1,2]));
```

$$Should\ be\ zero\ if\ similarity\ equivalents := -\frac{ch\,X^2\,(ch-1)\,(ch+1)\,Y\,d(X)}{(Z^2+S^2+X^2+Y^2)^2} - \frac{ch\,X\,(ch-1)\,(ch+1)\,Y\,Z\,d(Z)}{(Z^2+S^2+X^2+Y^2)^2} - \frac{ch\,X\,(ch-1)\,(ch+1)\,(-X^2-S^2-Z^2)\,d(Y)}{(Z^2+S^2+X^2+Y^2)^2} - \frac{ch\,X\,(ch-1)\,(ch+1)\,S\,Y\,d(S)}{(Z^2+S^2+X^2+Y^2)^2}$$

\>

The matrix elements of the Right Cartan connection matrix using the matrix methods:

Note that the Right Cartan connection does not depend upon B, but does depend upon ch.

\>

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);Gamma14:=wcollect(cartan[1,4]);
```

$$\Gamma 11 := \frac{X\,d(X)}{Z^2+S^2+X^2+Y^2} + \frac{Z\,d(Z)}{Z^2+S^2+X^2+Y^2} + \frac{Y\,d(Y)}{Z^2+S^2+X^2+Y^2} + \frac{S\,d(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 12 := -\frac{Y\,d(X)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,S\,d(Z)}{Z^2+S^2+X^2+Y^2} + \frac{X\,d(Y)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,Z\,d(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 13 := \frac{Z\,d(X)}{Z^2+S^2+X^2+Y^2} - \frac{X\,d(Z)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,S\,d(Y)}{Z^2+S^2+X^2+Y^2} + \frac{Y\,ch\,d(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 14 := \frac{S\,d(X)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,Y\,d(Z)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,Z\,d(Y)}{Z^2+S^2+X^2+Y^2} - \frac{X\,d(S)}{Z^2+S^2+X^2+Y^2}$$

****
****

The Left Cartan matix elements can also be computed by the matrix method (for example)

```
> Gamma11L:=wcollect(cartanL[1,1]);Gamma12L:=wcollect(cartanL[1,2]);Gamma13L:=wcollect(cartanL[1,3]);Gamma14L:=wcollect(cartanL[1,4]);
```

$$Gamma11L := -\frac{X\,d(X)}{Z^2+S^2+X^2+Y^2} - \frac{Z\,d(Z)}{Z^2+S^2+X^2+Y^2} - \frac{Y\,d(Y)}{Z^2+S^2+X^2+Y^2} - \frac{S\,d(S)}{Z^2+S^2+X^2+Y^2}$$

$$Gamma12L := -\frac{ch\,Y\,d(X)}{Z^2+S^2+X^2+Y^2} + \frac{S\,d(Z)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,X\,d(Y)}{Z^2+S^2+X^2+Y^2} - \frac{Z\,d(S)}{Z^2+S^2+X^2+Y^2}$$

$$Gamma13L := \frac{ch\,Z\,d(X)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,X\,d(Z)}{Z^2+S^2+X^2+Y^2} + \frac{S\,d(Y)}{Z^2+S^2+X^2+Y^2} - \frac{Y\,d(S)}{Z^2+S^2+X^2+Y^2}$$

$$Gamma14L := \frac{S\,\mathrm{d}(X)}{(Z^2+S^2+X^2+Y^2)\,B} + \frac{ch\,Y\,\mathrm{d}(Z)}{(Z^2+S^2+X^2+Y^2)\,B} - \frac{ch\,Z\,\mathrm{d}(Y)}{(Z^2+S^2+X^2+Y^2)\,B} - \frac{X\,\mathrm{d}(S)}{(Z^2+S^2+X^2+Y^2)\,B}$$

Note that the Left Cartan connection is sensitive to both B and ch.

****

> **Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(cartan[2,3]);Gamma24:=wcollect(cartan[2,4]);**

$$\Gamma 21 := \frac{Y\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,S\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} - \frac{X\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,Z\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 22 := \frac{X\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{Z\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} + \frac{Y\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} + \frac{S\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 23 := \frac{ch\,S\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} - \frac{Y\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} + \frac{Z\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,X\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 24 := -\frac{ch\,Z\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,X\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} + \frac{S\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} - \frac{Y\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

> **Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(cartan[3,3]);Gamma34:=wcollect(cartan[3,4]);**

$$\Gamma 31 := -\frac{Z\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{X\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,S\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} - \frac{Y\,ch\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 32 := -\frac{ch\,S\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{Y\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} - \frac{Z\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,X\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 33 := \frac{X\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{Z\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} + \frac{Y\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} + \frac{S\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 34 := -\frac{ch\,Y\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} - \frac{S\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,X\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} + \frac{Z\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

> **Gamma41:=(wcollect(cartan[4,1]));Gamma42:=wcollect(cartan[4,2]);Gamma43:=wcollect(cartan[4,3]);Gamma44:=wcollect(cartan[4,4]);**

$$\Gamma 41 := -\frac{S\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{ch\,Y\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,Z\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} + \frac{X\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 42 := \frac{ch\,Z\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,X\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} - \frac{S\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} + \frac{Y\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 43 := \frac{ch\,Y\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{S\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} - \frac{ch\,X\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} - \frac{Z\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

$$\Gamma 44 := \frac{X\,\mathrm{d}(X)}{Z^2+S^2+X^2+Y^2} + \frac{Z\,\mathrm{d}(Z)}{Z^2+S^2+X^2+Y^2} + \frac{Y\,\mathrm{d}(Y)}{Z^2+S^2+X^2+Y^2} + \frac{S\,\mathrm{d}(S)}{Z^2+S^2+X^2+Y^2}$$

All the elements of the right Cartan matrix have a common factor (1/R4). Note that the diagonal matix elements are perfect differentials, and if the R4 space is constrained to a 3 sphere, the diagonal elements vanish.

(These matrix elements are related to dilatations - "the expanding universe".)

>

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

> **dim:=4;coord:=[X,Y,Z,S];**

$$dim := 4$$

$$coord := [X, Y, Z, S]$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k] :=
  (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]. The notation is such that (a,-b,-c,) implies (upper,lower,lower) index.

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for m from 1 to dim do ss
  := ss+(d1GG[a,m,k]*FF[m,b]); CC[a,b,k]:=simplify(-ss) od od od od ;
>
> for a from 1 to dim do for b from 1 to dim do for k from 1 to dim do if CC[a,b,k]=0 then else
  print(`Cartan_RIGHT`(a,-b,-k)=factor(subs(ch^2=1,CC[a,b,k]))) fi od od od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients. (Hopf map)

$$\text{Cartan\_RIGHT}(1,-1,-1)=\frac{X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-1,-2)=\frac{Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-1,-3)=\frac{Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-1,-4)=\frac{S}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-2,-1)=-\frac{Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-2,-2)=\frac{X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-2,-3)=-\frac{ch\,S}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-2,-4)=\frac{ch\,Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-3,-1)=\frac{Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-3,-2)=-\frac{ch\,S}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-3,-3)=-\frac{X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-3,-4)=\frac{ch\,Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-4,-1)=\frac{S}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-4,-2)=\frac{ch\,Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1,-4,-3)=-\frac{ch\,Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -4) = -\frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -1) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -2) = -\frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -3) = \frac{ch\ S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -4) = -\frac{ch\ Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -1) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -2) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -3) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -4) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -1) = \frac{ch\ S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -2) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -3) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -4) = -\frac{ch\ X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -1) = -\frac{ch\ Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -2) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -3) = \frac{ch\ X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -4) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -1) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -2) = \frac{ch\ S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -3) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -4) = -\frac{ch\ Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -1) = -\frac{ch\,S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -2) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -3) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -4) = \frac{ch\,X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -1) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -2) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -3) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -4) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -1) = -\frac{ch\,Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -2) = \frac{ch\,X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -3) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -4) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -1) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -2) = -\frac{ch\,Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -3) = \frac{ch\,Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -4) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -1) = \frac{ch\,Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -2) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -3) = -\frac{ch\,X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -4) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -1) = \frac{ch\,Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -2) = -\frac{ch\,X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -3) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -4) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -1) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -2) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -3) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -4) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

These results agree with matrix method given above.

Next check for Affine Torsion of the Right Cartan matrix using the tensor methods:

Torsion coefficients for the Right Cartan matrix are defined as the difference between Gamma(a,-b,-c)-Gamma(a,-c,-b) times 1/2;

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss := (CC[i,j,k]-CC[i,k,j])/2;
  CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if CCTTS[i,j,k]=0 then else
  print(`RIGHT_AffineTorsion`(i,-k,-j)=simplify(subs(ch^2=1,CCTTS[i,k,j]))) fi od od od ;
```

$$\text{RIGHT\_AffineTorsion}(1, -2, -1) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -3, -1) = 0$$

$$\text{RIGHT\_AffineTorsion}(1, -4, -1) = 0$$

$$\text{RIGHT\_AffineTorsion}(1, -1, -2) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -3, -2) = 0$$

$$\text{RIGHT\_AffineTorsion}(1, -4, -2) = 0$$

$$\text{RIGHT\_AffineTorsion}(1, -1, -3) = 0$$

$$\text{RIGHT\_AffineTorsion}(1, -2, -3) = 0$$

$$\text{RIGHT\_AffineTorsion}(1, -4, -3) = -\frac{ch\,Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -1, -4) = 0$$

$$\text{RIGHT\_AffineTorsion}(1, -2, -4) = 0$$

$$\text{RIGHT\_AffineTorsion}(1, -3, -4) = \frac{ch\,Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -2, -1) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -1, -2) = -\frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -4, -3) = \frac{ch\, X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -3, -4) = -\frac{ch\, X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -2, -1) = -\frac{ch\, S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -3, -1) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -4, -1) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -1, -2) = \frac{ch\, S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -3, -2) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -4, -2) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -1, -3) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -2, -3) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -4, -3) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -1, -4) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -2, -4) = 0$$

$$\text{RIGHT\_AffineTorsion}(3, -3, -4) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -1) = \frac{ch\, Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -1) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -1) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -2) = -\frac{ch\, Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -2) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -2) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -3) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -3) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -3) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -4) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -4) = 0$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -4) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

For the Hopf map and the frame constructed above, it is remarkable that there is a Torsion component that depends on the chirality factor, ch, and another torsion component which does not.!

The Right Affine torsion does not depend upon the orientation, B, but has components that depend upon chirality, ch

****

Now compute the CARTAN LEFT CONNECTION

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[a,j,k] :=
```

```
simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrix product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do  ss:=0;for m to dim do ss :=
  ss+FF[i,m]*(d1GG[m,j,k]); DD[i,j,k]:=simplify(ss) od od od od ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if DD[i,j,k]=0 then else
  print(`Cartan_LEFT`(i,-j,-k)=simplify(subs(ch^2=1,DD[i,j,k]))) fi od od od ;
```

$$\text{Cartan\_LEFT}(1,-1,-1) = -\frac{X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-1,-2) = -\frac{Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-1,-3) = -\frac{Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-1,-4) = -\frac{S}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-2,-1) = -\frac{Y\,ch}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-2,-2) = \frac{ch\,X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-2,-3) = \frac{S}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-2,-4) = -\frac{Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-3,-1) = \frac{ch\,Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-3,-2) = \frac{S}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-3,-3) = -\frac{ch\,X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-3,-4) = -\frac{Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(1,-4,-1) = \frac{S}{(Z^2+S^2+X^2+Y^2)\,B}$$

$$\text{Cartan\_LEFT}(1,-4,-2) = -\frac{ch\,Z}{(Z^2+S^2+X^2+Y^2)\,B}$$

$$\text{Cartan\_LEFT}(1,-4,-3) = \frac{Y\,ch}{(Z^2+S^2+X^2+Y^2)\,B}$$

$$\text{Cartan\_LEFT}(1,-4,-4) = -\frac{X}{(Z^2+S^2+X^2+Y^2)\,B}$$

$$\text{Cartan\_LEFT}(2,-1,-1) = \frac{Y\,ch}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(2,-1,-2) = -\frac{ch\,X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Cartan\_LEFT}(2, -1, -3) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -1, -4) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -2, -1) = -\frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -2, -2) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -2, -3) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -2, -4) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -3, -1) = -\frac{ch\ S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -3, -2) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -3, -3) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -3, -4) = \frac{ch\ X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(2, -4, -1) = \frac{Z}{(Z^2 + S^2 + X^2 + Y^2)\ B}$$

$$\text{Cartan\_LEFT}(2, -4, -2) = \frac{ch\ S}{(Z^2 + S^2 + X^2 + Y^2)\ B}$$

$$\text{Cartan\_LEFT}(2, -4, -3) = -\frac{X}{(Z^2 + S^2 + X^2 + Y^2)\ B}$$

$$\text{Cartan\_LEFT}(2, -4, -4) = -\frac{Y\ ch}{(Z^2 + S^2 + X^2 + Y^2)\ B}$$

$$\text{Cartan\_LEFT}(3, -1, -1) = -\frac{ch\ Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -1, -2) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -1, -3) = \frac{ch\ X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -1, -4) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -2, -1) = \frac{ch\ S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -2, -2) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -2, -3) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -2, -4) = -\frac{ch\,X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -3, -1) = -\frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -3, -2) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -3, -3) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -3, -4) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(3, -4, -1) = \frac{Y}{(Z^2 + S^2 + X^2 + Y^2)\,B}$$

$$\text{Cartan\_LEFT}(3, -4, -2) = -\frac{X}{(Z^2 + S^2 + X^2 + Y^2)\,B}$$

$$\text{Cartan\_LEFT}(3, -4, -3) = -\frac{ch\,S}{(Z^2 + S^2 + X^2 + Y^2)\,B}$$

$$\text{Cartan\_LEFT}(3, -4, -4) = \frac{ch\,Z}{(Z^2 + S^2 + X^2 + Y^2)\,B}$$

$$\text{Cartan\_LEFT}(4, -1, -1) = -\frac{B\,S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -1, -2) = \frac{B\,ch\,Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -1, -3) = -\frac{Y\,B\,ch}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -1, -4) = \frac{B\,X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -2, -1) = -\frac{B\,Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -2, -2) = -\frac{B\,ch\,S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -2, -3) = \frac{B\,X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -2, -4) = \frac{Y\,B\,ch}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -3, -1) = -\frac{B\,Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -3, -2) = \frac{B\,X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -3, -3) = \frac{B\,ch\,S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -3, -4) = -\frac{B\,ch\,Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -4, -1) = -\frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -4, -2) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -4, -3) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Cartan\_LEFT}(4, -4, -4) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

The components of f the **LEFT CARTAN Connection** appear above. Note that they are not the same as the components of the Right Cartan matrix, moreover they **depend upon the choice of orientation, B, and chirality ch.**

Check for asymmetry (LEFT Torsion) defined as {Cartan_LEFT(a,-b,-c) - Cartan_LEFT(a.-c,-b)} times 1/2.

```
> for j from 1 to \dim do for i from 1 to dim do for k from 1 to dim do  ss := (DD[i,j,k]-DD[i,k,j])/2;
  TTS[i,j,k]:=simplify(ss) od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if TTS[i,j,k]=0 then else
  print(`LEFT_Torsion`(i,-k,-j)=simplify(subs(ch^2=1,TTS[i,k,j]))) fi od od od ;
```

$$\text{LEFT\_Torsion}(1, -2, -1) = -\frac{1}{2}\frac{Y(ch-1)}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{LEFT\_Torsion}(1, -3, -1) = \frac{1}{2}\frac{Z(ch+1)}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{LEFT\_Torsion}(1, -4, -1) = \frac{1}{2}\frac{(B+1)S}{(Z^2 + S^2 + X^2 + Y^2)B}$$

$$\text{LEFT\_Torsion}(1, -1, -2) = \frac{1}{2}\frac{Y(ch-1)}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{LEFT\_Torsion}(1, -3, -2) = 0$$

$$\text{LEFT\_Torsion}(1, -4, -2) = \frac{1}{2}\frac{(B-ch)Z}{(Z^2 + S^2 + X^2 + Y^2)B}$$

$$\text{LEFT\_Torsion}(1, -1, -3) = -\frac{1}{2}\frac{Z(ch+1)}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{LEFT\_Torsion}(1, -2, -3) = 0$$

$$\text{LEFT\_Torsion}(1, -4, -3) = \frac{1}{2}\frac{(ch+B)Y}{(Z^2 + S^2 + X^2 + Y^2)B}$$

$$\text{LEFT\_Torsion}(1, -1, -4) = -\frac{1}{2}\frac{(B+1)S}{(Z^2 + S^2 + X^2 + Y^2)B}$$

$$\text{LEFT\_Torsion}(1, -2, -4) = -\frac{1}{2}\frac{(B-ch)Z}{(Z^2 + S^2 + X^2 + Y^2)B}$$

$$\text{LEFT\_Torsion}(1, -3, -4) = -\frac{1}{2}\frac{(ch+B)Y}{(Z^2 + S^2 + X^2 + Y^2)B}$$

$$\text{LEFT\_Torsion}(2, -2, -1) = \frac{1}{2} \frac{X(ch-1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(2, -3, -1) = -\frac{1}{2} \frac{S(ch-1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(2, -4, -1) = -\frac{1}{2} \frac{(B-1)Z}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(2, -1, -2) = -\frac{1}{2} \frac{X(ch-1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(2, -3, -2) = \frac{Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(2, -4, -2) = \frac{1}{2} \frac{(ch+B)S}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(2, -1, -3) = \frac{1}{2} \frac{S(ch-1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(2, -2, -3) = -\frac{Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(2, -4, -3) = -\frac{1}{2} \frac{(1+B\,ch)X}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(2, -1, -4) = \frac{1}{2} \frac{(B-1)Z}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(2, -2, -4) = -\frac{1}{2} \frac{(ch+B)S}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(2, -3, -4) = \frac{1}{2} \frac{(1+B\,ch)X}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(3, -2, -1) = \frac{1}{2} \frac{S(ch+1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(3, -3, -1) = -\frac{1}{2} \frac{X(ch+1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(3, -4, -1) = -\frac{1}{2} \frac{Y(B-1)}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(3, -1, -2) = -\frac{1}{2} \frac{S(ch+1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(3, -3, -2) = -\frac{Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(3, -4, -2) = \frac{1}{2} \frac{X(B\,ch-1)}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(3, -1, -3) = \frac{1}{2} \frac{X(ch+1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(3, -2, -3) = \frac{Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(3, -4, -3) = \frac{1}{2} \frac{S(B-ch)}{(Z^2+S^2+X^2+Y^2)B}$$

$$\text{LEFT\_Torsion}(3, -1, -4) = \frac{1}{2} \frac{Y (B-1)}{(Z^2+S^2+X^2+Y^2) B}$$

$$\text{LEFT\_Torsion}(3, -2, -4) = -\frac{1}{2} \frac{X (B\,ch-1)}{(Z^2+S^2+X^2+Y^2) B}$$

$$\text{LEFT\_Torsion}(3, -3, -4) = -\frac{1}{2} \frac{S (B-ch)}{(Z^2+S^2+X^2+Y^2) B}$$

$$\text{LEFT\_Torsion}(4, -2, -1) = -\frac{1}{2} \frac{B Z (ch+1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -3, -1) = \frac{1}{2} \frac{B Y (ch-1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -4, -1) = -\frac{1}{2} \frac{X (B+1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -1, -2) = \frac{1}{2} \frac{B Z (ch+1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -3, -2) = 0$$

$$\text{LEFT\_Torsion}(4, -4, -2) = -\frac{1}{2} \frac{Y (1+B\,ch)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -1, -3) = -\frac{1}{2} \frac{B Y (ch-1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -2, -3) = 0$$

$$\text{LEFT\_Torsion}(4, -4, -3) = \frac{1}{2} \frac{Z (B\,ch-1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -1, -4) = \frac{1}{2} \frac{X (B+1)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -2, -4) = \frac{1}{2} \frac{Y (1+B\,ch)}{Z^2+S^2+X^2+Y^2}$$

$$\text{LEFT\_Torsion}(4, -3, -4) = -\frac{1}{2} \frac{Z (B\,ch-1)}{Z^2+S^2+X^2+Y^2}$$

The Right and the Left Cartan matrices are negative similarity transforms, **but the LEFT AND RIGHT torsion terms appear to be diffferent, and the left "torsion" depends upon the both the chirality factor, ch, and the orientation, B !!!!!.**

****

Next the Christoffel symbols will be computed for the **subsumed pullback metric** on the initial state. The pullback mnetric is conformal to the indentity matrix.

Christoffel Connection coefficients from the induced metric

It is assumed that the "metric" is the pull back metric given below, which is conformal.

```
> metric:=evalm( pullbackmetric);
```

$$metric := \begin{bmatrix} Z^2+S^2+X^2+Y^2 & 0 & 0 & 0 \\ 0 & Z^2+S^2+X^2+Y^2 & 0 & 0 \\ 0 & 0 & Z^2+S^2+X^2+Y^2 & 0 \\ 0 & 0 & 0 & Z^2+S^2+X^2+Y^2 \end{bmatrix}$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1gun[i,j,k] :=
  (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if d1gun[i,j,k]=0 then  else
  print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k] := 0  od od od; for i from
  1 to dim do for j from 1 to dim do for k from 1 to dim do C1S[i,j,k] :=
  1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0; for m to dim do ss :=
  ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] := simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if C2S[i,j,k]=0 then  else
  print(`Christoffel_Gamma2`(i,-j,-k)=C2S[i,j,k]) fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

$$\text{Christoffel\_Gamma2}(1,-1,-1) = \frac{X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Christoffel\_Gamma2}(1,-1,-2) = \frac{Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{Christoffel\_Gamma2}(1,-1,-3) = \frac{Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Christoffel\_Gamma2}(1,-1,-4) = \frac{S}{Z^2+S^2+X^2+Y^2}$$

$$\text{Christoffel\_Gamma2}(1,-2,-1) = \frac{Y}{Z^2+S^2+X^2+Y^2}$$

$$\text{Christoffel\_Gamma2}(1,-2,-2) = -\frac{X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Christoffel\_Gamma2}(1,-3,-1) = \frac{Z}{Z^2+S^2+X^2+Y^2}$$

$$\text{Christoffel\_Gamma2}(1,-3,-3) = -\frac{X}{Z^2+S^2+X^2+Y^2}$$

$$\text{Christoffel\_Gamma2}(1, -4, -1) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(1, -4, -4) = -\frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -1, -1) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -1, -2) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -1) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -2) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -3) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -4) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -3, -2) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -3, -3) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -4, -2) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(2, -4, -4) = -\frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -1, -1) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -1, -3) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -2, -2) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -2, -3) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -1) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -2) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -3) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -4) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -4, -3) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(3, -4, -4) = -\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -1, -1) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -1, -4) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -2, -2) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -2, -4) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -3, -3) = -\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -3, -4) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -1) = \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -2) = \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -3) = \frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -4) = \frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

If no entries appear above the Christoffel symbols on the domain space vanish

Note that the Christoffel Symbols for the Conformal metric are not zero, and are not the same as the Right or Left Cartan Connection matrices. More over, the **Christoffel symbols built upon the metric defined above are independent from the choice of chirality and orientation.** The metric is symmetric and has a positive definite determinant.

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) := ChristoffelGamma(ijk) + T(ijk)

Using this *definition for the Rotation coefficients,* Compute the T(i,j,k):

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss := (CC[i,j,k]-C2S[i,j,k]);
  SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if C2S[i,j,k]=0 and CC[i,j,k]=0 then
                  else print(`T`(i,-j,-k)=simplify(subs(ch^2=1,SHIPTR[i,j,k]))) fi od od od ;
```

T(ijk) index (1,-1,-1)

$$T(1, -1, -1) = 0$$
$$T(1, -1, -2) = 0$$
$$T(1, -1, -3) = 0$$
$$T(1, -1, -4) = 0$$
$$T(1, -2, -1) = -2 \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(1, -2, -2) = 2 \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(1, -2, -3) = -\frac{ch\, S}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(1, -2, -4) = \frac{ch\, Z}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(1, -3, -1) = 0$$
$$T(1, -3, -2) = -\frac{ch\, S}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(1, -3, -3) = 0$$
$$T(1, -3, -4) = \frac{ch\, Y}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(1, -4, -1) = 0$$
$$T(1, -4, -2) = \frac{ch\, Z}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(1, -4, -3) = -\frac{ch\, Y}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(1, -4, -4) = 0$$
$$T(2, -1, -1) = 2 \frac{Y}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(2, -1, -2) = -2 \frac{X}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(2, -1, -3) = \frac{ch\, S}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(2, -1, -4) = -\frac{ch\, Z}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(2, -2, -1) = 0$$
$$T(2, -2, -2) = 0$$
$$T(2, -2, -3) = 0$$
$$T(2, -2, -4) = 0$$
$$T(2, -3, -1) = \frac{ch\, S}{Z^2 + S^2 + X^2 + Y^2}$$
$$T(2, -3, -2) = 0$$

$$T(2, -3, -3) = 0$$

$$T(2, -3, -4) = -\frac{ch\, X}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(2, -4, -1) = -\frac{ch\, Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(2, -4, -2) = 0$$

$$T(2, -4, -3) = \frac{ch\, X}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(2, -4, -4) = 0$$

$$T(3, -1, -1) = 0$$

$$T(3, -1, -2) = \frac{ch\, S}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(3, -1, -3) = 0$$

$$T(3, -1, -4) = -\frac{ch\, Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(3, -2, -1) = -\frac{ch\, S}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(3, -2, -2) = 0$$

$$T(3, -2, -3) = 0$$

$$T(3, -2, -4) = \frac{ch\, X}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(3, -3, -1) = 0$$

$$T(3, -3, -2) = 0$$

$$T(3, -3, -3) = 0$$

$$T(3, -3, -4) = 0$$

$$T(3, -4, -1) = -\frac{ch\, Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(3, -4, -2) = \frac{ch\, X}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(3, -4, -3) = -2\,\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(3, -4, -4) = 2\,\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -1, -1) = 0$$

$$T(4, -1, -2) = -\frac{ch\, Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -1, -3) = \frac{ch\, Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -1, -4) = 0$$

$$T(4, -2, -1) = \frac{ch\, Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -2, -2) = 0$$

$$T(4, -2, -3) = -\frac{ch\, X}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -2, -4) = 0$$

$$T(4, -3, -1) = \frac{ch\, Y}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -3, -2) = -\frac{ch\, X}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -3, -3) = 2\frac{S}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -3, -4) = -2\frac{Z}{Z^2 + S^2 + X^2 + Y^2}$$

$$T(4, -4, -1) = 0$$

$$T(4, -4, -2) = 0$$

$$T(4, -4, -3) = 0$$

$$T(4, -4, -4) = 0$$

The Rotation matrices also depend upon the chirality factor.

NOW RESTART FOR THE INSTANTON CASE

The objective will be to build another orthogonal frame where three of the induced 1-forms are not integable, and only the fourth is a perfect differential. The FRAME so constructed will have the SAME (pullback) metric !!!

```
> restart:
```

```
> with(linalg):with(liesymm):with(difforms):with(plots):defform(X=0,Y=0,Z=0,S=0,ch=const,B=const,u=0,v=0,w=0,s=0);
```

```
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

The idea is to define the normal field as a gradient field of the function (X^2+Y^2+Z^2+S^2)/2, and then to find three other vector fields that annihilate the gradient field. A particulat choice is the "Instanton Map" for which each of the 3 fields is not integrable.

The choice produces a global frame field matrix with components proportional to :

```
> HE3:=[-Y,+X,S,-Z];HE2:=[-ch*Z,-S,X,ch*Y];HE1:=[ch*S,-Z,ch*Y,-X];HN1:=evalm(B*[X,Y,ch*Z,ch*S]);
```

$$HE3 := [-Y, X, S, -Z]$$

$$HE2 := [-ch\, Z, -S, X, ch\, Y]$$

$$HE1 := [ch\, S, -Z, ch\, Y, -X]$$

$$HN1 := [B\, X, B\, Y, B\, ch\, Z, B\, ch\, S]$$

```
> FF:=array([[HE1[1],HE1[2],HE1[3],HE1[4]],[HE2[1],HE2[2],HE2[3],HE2[4]],[HE3[1],HE3[2],HE3[3],HE3[4]],[HN1[1],HN1[2],HN1[3],HN1[4]]]);DET:=factor(subs(ch^2=1,det(FF)));GG:=simplify(subs(ch^2=1,inver
```

se(FF))):

$$FF := \begin{bmatrix} ch\,S & -Z & ch\,Y & -X \\ -ch\,Z & -S & X & ch\,Y \\ -Y & X & S & -Z \\ B\,X & B\,Y & B\,ch\,Z & B\,ch\,S \end{bmatrix}$$

$$DET := -B\,(S^2+Z^2+Y^2+X^2)^2$$

The Instanton Frame is orthogonal -- but not orthonormal **The Pullback metric is the same as for the Hopf map above.**

> pullbackmetric:=subs(ch^2=1,B^2=1,innerprod(transpose(FF),FF));

$$pullbackmetric := \begin{bmatrix} S^2+Z^2+Y^2+X^2 & 0 & 0 & 0 \\ 0 & S^2+Z^2+Y^2+X^2 & 0 & 0 \\ 0 & 0 & S^2+Z^2+Y^2+X^2 & 0 \\ 0 & 0 & 0 & S^2+Z^2+Y^2+X^2 \end{bmatrix}$$

> CHPOLYP:=simplify(factor(subs(ch^2=1,B=1,charpoly(FF,lambda))));CHPOLYM:=simplify(factor(subs(ch^2=1,B=-1,charpoly(FF,lambda))));

FRAME MATRIX FF, INVERSE FRAME GG, ch = chirality factor, B=orientation

>

$$CHPOLYP := -(-\lambda^2+S^2+Y^2+Z^2+X^2)\,(Y^2-2\,S\,ch\,\lambda+S^2+Z^2+X^2+\lambda^2)$$

$$CHPOLYM := 2\,ch\,Y^2\,\lambda^2+2\,S^2\,Z^2-2\,S^2\,\lambda^2-2\,X^2\,\lambda^2+\lambda^4-2\,ch\,Z^2\,\lambda^2+S^4+Z^4+Y^4+X^4+2\,X^2\,S^2+2\,S^2\,Y^2 + 2\,X^2\,Z^2+2\,Z^2\,Y^2+2\,Y^2\,X^2$$

The characteristic polynomial depends upon both the chirality factor, ch, and the sign of the determinant B (orientation).

For B=plus one the Frame has a negative determinant (the discrete reflective cases and global abnormalities or catastrophies).

For B =minus one, the Frame has a positive determinant which implies perturbations about the identity are possible (local abnormalities) .

The orientation is independent from the chirality

Under the interchange S<=>Z, the B=plus 1 CHPOLYP of the Hopf map and the Instanton map are the same.

Under the interchange S<=>Y and X<=> Z the B=-1 CHPOLYL of the Hopf map and the Instanton map are the same.

The next equation checks to see that the specified frame produces the desired differential structures: [FF]|dR>=|sigma>

> sigma:=evalm(innerprod(FF,[d(X),d(Y),d(Z),d(S)])):

> sigma1:=sigma[1];e1;

> sigma2:=sigma[2];e2;

> sigma3:=sigma[3];e3;omega:=sigma[4];

$$\sigma 1 := ch\,S\,d(X)-Z\,d(Y)+ch\,Y\,d(Z)-X\,d(S)$$

$$e1$$

$$\sigma 2 := -ch\,Z\,d(X)-S\,d(Y)+X\,d(Z)+ch\,Y\,d(S)$$

$$e2$$

$$\sigma 3 := -Y\,d(X)+X\,d(Y)+S\,d(Z)-Z\,d(S)$$

$$e3$$

$$\omega := B\,X\,d(X)+B\,Y\,d(Y)+B\,ch\,Z\,d(Z)+B\,ch\,S\,d(S)$$

The vector of induced 1-forms [FF]|dR>=|sigma> can written as the column vector of components [sigma1,sigma2,sigma3,omega] .

The induced 1-forms created by the Frame acting on the differentials on the domain are tested for integrablility. In the Instanton Map, the first three induced 1-forms are not integrable. The fourth component of the Frame Field is exact.

```
> `vorticity sigma1`:=d(sigma[1]);`vorticity sigma2`:=d(sigma[2]);`vorticity sigma3`:=d(sigma[3]);`omega`:=omega;`vorticy of  omega`:=factor(d(omega));`topological torsion of sigma1`:=factor(subs(B^2=1,sigma[1]&^d(sigma[1])));`topological parity of sigma1`:=subs(B^2=1,d(sigma[1])&^d(sigma[1]));`topological torsion of sigma2`:=factor(subs(B^2=1,sigma[2]&^d(sigma[2])));`topological parity of sigma2`:=subs(B^2=1,d(sigma[2])&^d(sigma[2]));`topological torsion of sigma3`:=factor(subs(B^2=1,sigma[3]&^d(sigma[3])));`topological parity of sigma3`:=subs(B^2=1,d(sigma[3])&^d(sigma[3]));
```

$$\text{vorticity sigma1} := (-1 - ch)\,(d(Z)\,\&\wedge\,d(Y)) + (ch + 1)\,(d(S)\,\&\wedge\,d(X))$$

$$\text{vorticity sigma2} := (-1 - ch)\,(d(S)\,\&\wedge\,d(Y)) + (-1 - ch)\,(d(Z)\,\&\wedge\,d(X))$$

$$\text{vorticity sigma3} := -2\,(d(Y)\,\&\wedge\,d(X)) + 2\,(d(S)\,\&\wedge\,d(Z))$$

$$\omega := B\,X\,d(X) + B\,Y\,d(Y) + B\,ch\,Z\,d(Z) + B\,ch\,S\,d(S)$$

$$\text{vorticy of omega} := 0$$

$$\text{topological torsion of sigma1} := -(ch + 1)\,(-ch\,Y\,\&\wedge(d(Z), d(S), d(X)) + ch\,S\,\&\wedge(d(X), d(Z), d(Y)) - X\,\&\wedge(d(S), d(Z), d(Y)) + Z\,\&\wedge(d(Y), d(S), d(X)))$$

$$\text{topological parity of sigma1} := -2\,(ch + 1)^2\,\&\wedge(d(Z), d(Y), d(S), d(X))$$

$$\text{topological torsion of sigma2} := -(ch + 1)\,(ch\,Y\,\&\wedge(d(S), d(Z), d(X)) - ch\,Z\,\&\wedge(d(X), d(S), d(Y)) - S\,\&\wedge(d(Y), d(Z), d(X)) + X\,\&\wedge(d(Z), d(S), d(Y)))$$

$$\text{topological parity of sigma2} := 2\,(ch + 1)^2\,\&\wedge(d(S), d(Y), d(Z), d(X))$$

$$\text{topological torsion of sigma3} := -2\,Y\,\&\wedge(d(X), d(S), d(Z)) + 2\,X\,\&\wedge(d(Y), d(S), d(Z)) - 2\,S\,\&\wedge(d(Z), d(Y), d(X)) + 2\,Z\,\&\wedge(d(S), d(Y), d(X))$$

$$\text{topological parity of sigma3} := -8\,\&\wedge(d(Y), d(X), d(S), d(Z))$$

Itis remarkable that the Topological Torsion and the Topological parity of the non - integrable induced 1-forms, does not depend upon the orientation, B, but does depend upon the chirality, ch.

IT is also remarkable that the Cholce of ch = minus 1 reduces the complexity of the system to where the Topological Torsion and the Topological parity of of the sigma1 and sigma2 are zero!!! One form of polarization behaves differently from the other form of polarization. !!!!

This result implies that these forms are integrable to within an integrating factor (for only 1 choice of chirality - polarization)

IS this some how connected to left handed neutrinos ???

A possible candidate for a metric on X,Y,Z,S would be the symmetric form (which is conformal to the Euclidean metric on R4.)

```
> pullbackmetric:=subs(ch^2=1,B^2=1,innerprod(transpose(FF),FF));
```

$$\text{pullbackmetric} := \begin{bmatrix} S^2 + Z^2 + Y^2 + X^2 & 0 & 0 & 0 \\ 0 & S^2 + Z^2 + Y^2 + X^2 & 0 & 0 \\ 0 & 0 & S^2 + Z^2 + Y^2 + X^2 & 0 \\ 0 & 0 & 0 & S^2 + Z^2 + Y^2 + X^2 \end{bmatrix}$$

Note that the Frame matrix FF is orthogonal but not orthonormal. It is also conformal in that the normalization is the same for all basis elements. It is independent of the orientation, B and the chirality ch. !!!! The Pullbackmetric is the same for the Instanton map and the Hopf map !!!

This shows that the frame matrix and the connection carry much more information than does the metric.

Now Compute the Right Cartan Matrix [CR] and the Left Cartan Matrix. These matrices are negative similarity transforms to within a sign.

```
> cartan:=(subs(ch^2=1,innerprod(GG,d(FF)))):cartanL:=evalm((subs(ch^2=1,innerprod(-d(FF),GG)))):
> CLcong:=(subs(ch^2=1,innerprod(-FF,cartan,GG))):zz:=evalm(FF);inverse(zz):
```

$$zz := \begin{bmatrix} ch\,S & -Z & ch\,Y & -X \\ -ch\,Z & -S & X & ch\,Y \\ -Y & X & S & -Z \\ B\,X & B\,Y & B\,ch\,Z & B\,ch\,S \end{bmatrix}$$

```
> `Should be zero if similarity equivalents`:=wcollect(factor(CLcong[1,2]-cartanL[1,2]));
```

$$Should\ be\ zero\ if\ similarity\ equivalents := -\frac{ch\,Y\,(ch-1)\,(ch+1)\,(-S^2-Y^2-Z^2)\,\mathrm{d}(X)}{(S^2+Z^2+Y^2+X^2)^2} - \frac{ch\,Y^2\,(ch-1)\,(ch+1)\,X\,\mathrm{d}(Y)}{(S^2+Z^2+Y^2+X^2)^2} - \frac{ch\,Y\,(ch-1)\,(ch+1)\,X\,Z\,\mathrm{d}(Z)}{(S^2+Z^2+Y^2+X^2)^2} - \frac{ch\,Y\,(ch-1)\,(ch+1)\,X\,S\,\mathrm{d}(S)}{(S^2+Z^2+Y^2+X^2)^2}$$

>

The matrix elements of the Right Cartan connection matrix using the matrix methods:
Note that the Right Cartan connection does not depend upon B, but does depend upon ch.
These connection matrices are **NOT** the same as the connection matrices for the Hopf map.

>

```
> Gamma11:=wcollect(cartan[1,1]);Gamma12:=wcollect(cartan[1,2]);Gamma13:=wcollect(cartan[1,3]);Gamma14:=wcollect(cartan[1,4]);
```

$$\Gamma 11 := \frac{X\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{S\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 12 := -\frac{Y\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{X\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,S\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,Z\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 13 := -\frac{ch\,Z\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{S\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{X\,ch\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} - \frac{Y\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 14 := -\frac{ch\,S\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{Z\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{X\,ch\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

****
****

The Left Cartan matix elements can also be computed by the matrix method (for example)

```
> Gamma11L:=wcollect(cartanL[1,1]);Gamma12L:=wcollect(cartanL[1,2]);Gamma13L:=wcollect(cartanL[1,3]);Gamma14L:=wcollect(cartanL[1,4]);
```

$$Gamma11L := -\frac{X\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{Y\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{Z\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} - \frac{S\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$Gamma12L := \frac{ch\,Y\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{X\,ch\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{S\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$Gamma13L := -\frac{Z\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,S\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{X\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,Y\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$Gamma14L := \frac{ch\,S\,\mathrm{d}(X)}{(S^2+Z^2+Y^2+X^2)\,B} - \frac{Z\,\mathrm{d}(Y)}{(S^2+Z^2+Y^2+X^2)\,B} + \frac{Y\,\mathrm{d}(Z)}{(S^2+Z^2+Y^2+X^2)\,B} - \frac{X\,ch\,\mathrm{d}(S)}{(S^2+Z^2+Y^2+X^2)\,B}$$

Note that the Left Cartan connection is sensitive to both B and ch.

****
****

```
> Gamma21:=wcollect(cartan[2,1]);Gamma22:=wcollect(cartan[2,2]);Gamma23:=wcollect(cartan[2,3]);G
```

```
amma24:=wcollect(cartan[2,4]);
```

$$\Gamma 21 := \frac{Y\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{X\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,S\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,Z\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 22 := \frac{X\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{S\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 23 := -\frac{S\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,Z\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,Y\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{X\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 24 := \frac{Z\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,S\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{X\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,Y\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

```
> Gamma31:=wcollect(cartan[3,1]);Gamma32:=wcollect(cartan[3,2]);Gamma33:=wcollect(cartan[3,3]);G
amma34:=wcollect(cartan[3,4]);
```

$$\Gamma 31 := \frac{ch\,Z\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{S\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{X\,ch\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 32 := \frac{S\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,Z\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,Y\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} - \frac{X\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 33 := \frac{X\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{S\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 34 := -\frac{ch\,Y\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{X\,ch\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{S\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

```
> Gamma41:=(wcollect(cartan[4,1]));Gamma42:=wcollect(cartan[4,2]);Gamma43:=wcollect(cartan[4,3]);
Gamma44:=wcollect(cartan[4,4]);
```

$$\Gamma 41 := \frac{ch\,S\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} - \frac{Y\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} - \frac{X\,ch\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 42 := -\frac{Z\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{ch\,S\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{X\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} - \frac{ch\,Y\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 43 := \frac{ch\,Y\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} - \frac{X\,ch\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{S\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} - \frac{Z\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

$$\Gamma 44 := \frac{X\,\mathrm{d}(X)}{S^2+Z^2+Y^2+X^2} + \frac{Y\,\mathrm{d}(Y)}{S^2+Z^2+Y^2+X^2} + \frac{Z\,\mathrm{d}(Z)}{S^2+Z^2+Y^2+X^2} + \frac{S\,\mathrm{d}(S)}{S^2+Z^2+Y^2+X^2}$$

All the elements of the right Cartan matrix have a common factor (1/R4). Note that the diagonal matix elements are perfect differentials, and if the R4 space is constrained to a 3 sphere, the diagonal elements vanish. (These matrix elements are related to dilatations.)

```
>
```

Now the components of the right Cartan matrix will be computed by the tensor method, as a check

```
> dim:=4;coord:=[X,Y,Z,S];
```

$$dim := 4$$

$$coord := [X, Y, Z, S]$$

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k] :=
(diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]. The notation is such that (a,-b,-c,) implies (upper,lower,lower) index.

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for m from 1 to dim do ss
:= ss+(d1GG[a,m,k]*FF[m,b]); CC[a,b,k]:=simplify(-ss) od od od od ;
>
```

```
> for a from 1 to dim do for b from 1 to dim do for k from 1 to dim do if CC[a,b,k]=0 then else
  print(`Cartan_RIGHT`(a,-b,-k)=factor(subs(ch^2=1,CC[a,b,k]))) fi od od od ;
```

THE non zero CARTAN RIGHT CONNECTION coefficients. (Instanton map)

$$\text{Cartan\_RIGHT}(1, -1, -1) = \frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -2) = \frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -3) = \frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -1, -4) = \frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -1) = -\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -2) = \frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -3) = -\frac{ch\, S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -2, -4) = \frac{ch\, Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -1) = -\frac{ch\, Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -2) = \frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -3) = \frac{X\, ch}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -3, -4) = -\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -1) = -\frac{ch\, S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -2) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -3) = \frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(1, -4, -4) = \frac{X\, ch}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -1) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -2) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -3) = \frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -1, -4) = -\frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -2, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -1) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -2) = -\frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -3) = \frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -3, -4) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -1) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -2) = -\frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -3) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(2, -4, -4) = \frac{ch\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -1) = \frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -2) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -3) = -\frac{X\, ch}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -1, -4) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -1) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -2) = \frac{ch\ Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -3) = -\frac{ch\ Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -2, -4) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -3, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -1) = -\frac{ch\ Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -2) = \frac{X\ ch}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -3) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(3, -4, -4) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -1) = \frac{ch\ S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -2) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -3) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -1, -4) = -\frac{X\ ch}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -1) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -2) = \frac{ch\ S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -3) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -2, -4) = -\frac{ch\ Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -1) = \frac{ch\ Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -2) = -\frac{X\ ch}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -3) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -3, -4) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_RIGHT}(4, -4, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

These results agree with matrix method given above.

Next check for Affine Torsion of the Right Cartan matrix using the tensor methods: Torsion coefficients for the Right Cartan matrix are defined as the difference between Gamma(a,-b,-c)-Gamma(a,-c,-b) times 1/2;

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss := (CC[i,j,k]-CC[i,k,j])/2;
  CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if CCTTS[i,j,k]=0 then else
  print(`RIGHT_AffineTorsion`(i,-k,-j)=simplify(subs(ch^2=1,CCTTS[i,k,j]))) fi od od od ;
```

$$\text{RIGHT\_AffineTorsion}(1, -2, -1) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -3, -1) = -\frac{1}{2}\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -4, -1) = -\frac{1}{2}\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -1, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -3, -2) = \frac{1}{2}\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -4, -2) = -\frac{1}{2}\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -1, -3) = \frac{1}{2}\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -2, -3) = -\frac{1}{2}\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -4, -3) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -1, -4) = \frac{1}{2}\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -2, -4) = \frac{1}{2}\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(1, -3, -4) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -2, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -3, -1) = -\frac{1}{2}\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -4, -1) = \frac{1}{2}\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -1, -2) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -3, -2) = -\frac{1}{2}\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -4, -2) = -\frac{1}{2}\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -1, -3) = \frac{1}{2}\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -2, -3) = \frac{1}{2}\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -4, -3) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -1, -4) = -\frac{1}{2}\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -2, -4) = \frac{1}{2}\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(2, -3, -4) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -2, -1) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -3, -1) = \frac{1}{2}\frac{X\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -4, -1) = -\frac{1}{2}\frac{Y\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -1, -2) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -3, -2) = \frac{1}{2}\frac{Y\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -4, -2) = \frac{1}{2}\frac{X\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -1, -3) = -\frac{1}{2}\frac{X\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -2, -3) = -\frac{1}{2}\frac{Y\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -4, -3) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -1, -4) = \frac{1}{2}\frac{Y(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -2, -4) = -\frac{1}{2}\frac{X(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(3, -3, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -1) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -1) = \frac{1}{2}\frac{Y(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -1) = \frac{1}{2}\frac{X(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -2) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -2) = -\frac{1}{2}\frac{X(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -2) = \frac{1}{2}\frac{Y(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -3) = -\frac{1}{2}\frac{Y(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -3) = \frac{1}{2}\frac{X(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -4, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -1, -4) = -\frac{1}{2}\frac{X(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -2, -4) = -\frac{1}{2}\frac{Y(ch+1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{RIGHT\_AffineTorsion}(4, -3, -4) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

For the Hopf map and the frame constructed above, it is remarkable that there is a Torsion component that depends on the chirality factor, ch, and another torsion component which does not.!

The Right Affine torsion does not depend upon the orientation, B, but has components that depend upon chirality, ch

****

Now compute the CARTAN LEFT CONNECTION

```
> for a from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[a,j,k] :=
  simplify(diff(GG[a,j],coord[k])) od od od:
```

Compute the elements of the matrix product of [F]d[G]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do  ss:=0;for m to dim do ss :=
```

```
ss+FF[i,m]*(d1GG[m,j,k]); DD[i,j,k]:=simplify(ss) od od od ;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if DD[i,j,k]=0 then else
print(`Cartan_LEFT`(i,-j,-k)=simplify(subs(ch^2=1,DD[i,j,k]))) fi od od od ;
```

$$\text{Cartan\_LEFT}(1,-1,-1) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-1,-2) = -\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-1,-3) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-1,-4) = -\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-2,-1) = \frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-2,-2) = -\frac{X\,ch}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-2,-3) = -\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-2,-4) = \frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-3,-1) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-3,-2) = -\frac{ch\,S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-3,-3) = \frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-3,-4) = \frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(1,-4,-1) = \frac{ch\,S}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{Cartan\_LEFT}(1,-4,-2) = -\frac{Z}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{Cartan\_LEFT}(1,-4,-3) = \frac{Y}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{Cartan\_LEFT}(1,-4,-4) = -\frac{X\,ch}{(S^2+Z^2+Y^2+X^2)\,B}$$

$$\text{Cartan\_LEFT}(2,-1,-1) = -\frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-1,-2) = \frac{X\,ch}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-1,-3) = \frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-1,-4) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-2,-1) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-2,-2) = -\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-2,-3) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-2,-4) = -\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-3,-1) = -\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-3,-2) = \frac{ch\, Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-3,-3) = -\frac{ch\, Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-3,-4) = \frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(2,-4,-1) = -\frac{ch\, Z}{(S^2+Z^2+Y^2+X^2)\, B}$$

$$\text{Cartan\_LEFT}(2,-4,-2) = -\frac{S}{(S^2+Z^2+Y^2+X^2)\, B}$$

$$\text{Cartan\_LEFT}(2,-4,-3) = \frac{X\, ch}{(S^2+Z^2+Y^2+X^2)\, B}$$

$$\text{Cartan\_LEFT}(2,-4,-4) = \frac{Y}{(S^2+Z^2+Y^2+X^2)\, B}$$

$$\text{Cartan\_LEFT}(3,-1,-1) = \frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-1,-2) = \frac{ch\, S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-1,-3) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-1,-4) = -\frac{ch\, Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-2,-1) = \frac{S}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-2,-2) = -\frac{ch\, Z}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-2,-3) = \frac{ch\, Y}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3,-2,-4) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Cartan\_LEFT}(3, -3, -1) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -3, -2) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -3, -3) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -3, -4) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(3, -4, -1) = -\frac{Y}{(S^2 + Z^2 + Y^2 + X^2)\, B}$$

$$\text{Cartan\_LEFT}(3, -4, -2) = \frac{X}{(S^2 + Z^2 + Y^2 + X^2)\, B}$$

$$\text{Cartan\_LEFT}(3, -4, -3) = \frac{ch\, S}{(S^2 + Z^2 + Y^2 + X^2)\, B}$$

$$\text{Cartan\_LEFT}(3, -4, -4) = -\frac{ch\, Z}{(S^2 + Z^2 + Y^2 + X^2)\, B}$$

$$\text{Cartan\_LEFT}(4, -1, -1) = -\frac{B\, ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -1, -2) = \frac{B\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -1, -3) = -\frac{B\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -1, -4) = \frac{B\, X\, ch}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -2, -1) = \frac{B\, ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -2, -2) = \frac{B\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -2, -3) = -\frac{B\, X\, ch}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -2, -4) = -\frac{B\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -3, -1) = \frac{B\, Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -3, -2) = -\frac{B\, X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -3, -3) = -\frac{B\, ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -3, -4) = \frac{B\, ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -4, -1) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -4, -2) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -4, -3) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Cartan\_LEFT}(4, -4, -4) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

The components of f the LEFT CARTAN Connection appear above. Note that they are not the same as the components of the Right Cartan matrix, moreover they depend upon the choice of orientation, B, and chirality ch.

Check for asymmetry (LEFT Torsion) defined as {Cartan_LEFT(a,-b,-c) - Cartan_LEFT(a.-c,-b)} times 1/2.

```
> for j from 1 to \dim do for i from 1 to dim do for k from 1 to dim do  ss := (DD[i,j,k]-DD[i,k,j])/2;
  TTS[i,j,k]:=simplify(ss) od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if TTS[i,j,k]=0 then else
  print(`LEFT_Torsion`(i,-k,-j)=simplify(subs(ch^2=1,TTS[i,k,j]))) fi od od od ;
```

$$\text{LEFT\_Torsion}(1, -1, -1) = TTS_{1,1,1}$$
$$\text{LEFT\_Torsion}(1, -2, -1) = TTS_{1,2,1}$$
$$\text{LEFT\_Torsion}(1, -3, -1) = TTS_{1,3,1}$$
$$\text{LEFT\_Torsion}(1, -4, -1) = TTS_{1,4,1}$$
$$\text{LEFT\_Torsion}(1, -1, -2) = TTS_{1,1,2}$$
$$\text{LEFT\_Torsion}(1, -2, -2) = TTS_{1,2,2}$$
$$\text{LEFT\_Torsion}(1, -3, -2) = TTS_{1,3,2}$$
$$\text{LEFT\_Torsion}(1, -4, -2) = TTS_{1,4,2}$$
$$\text{LEFT\_Torsion}(1, -1, -3) = TTS_{1,1,3}$$
$$\text{LEFT\_Torsion}(1, -2, -3) = TTS_{1,2,3}$$
$$\text{LEFT\_Torsion}(1, -3, -3) = TTS_{1,3,3}$$
$$\text{LEFT\_Torsion}(1, -4, -3) = TTS_{1,4,3}$$
$$\text{LEFT\_Torsion}(1, -1, -4) = TTS_{1,1,4}$$
$$\text{LEFT\_Torsion}(1, -2, -4) = TTS_{1,2,4}$$
$$\text{LEFT\_Torsion}(1, -3, -4) = TTS_{1,3,4}$$
$$\text{LEFT\_Torsion}(1, -4, -4) = TTS_{1,4,4}$$
$$\text{LEFT\_Torsion}(2, -1, -1) = TTS_{2,1,1}$$
$$\text{LEFT\_Torsion}(2, -2, -1) = TTS_{2,2,1}$$
$$\text{LEFT\_Torsion}(2, -3, -1) = TTS_{2,3,1}$$
$$\text{LEFT\_Torsion}(2, -4, -1) = TTS_{2,4,1}$$
$$\text{LEFT\_Torsion}(2, -1, -2) = TTS_{2,1,2}$$
$$\text{LEFT\_Torsion}(2, -2, -2) = TTS_{2,2,2}$$
$$\text{LEFT\_Torsion}(2, -3, -2) = TTS_{2,3,2}$$
$$\text{LEFT\_Torsion}(2, -4, -2) = TTS_{2,4,2}$$

$LEFT\_Torsion(2, -1, -3) = TTS_{2,1,3}$
$LEFT\_Torsion(2, -2, -3) = TTS_{2,2,3}$
$LEFT\_Torsion(2, -3, -3) = TTS_{2,3,3}$
$LEFT\_Torsion(2, -4, -3) = TTS_{2,4,3}$
$LEFT\_Torsion(2, -1, -4) = TTS_{2,1,4}$
$LEFT\_Torsion(2, -2, -4) = TTS_{2,2,4}$
$LEFT\_Torsion(2, -3, -4) = TTS_{2,3,4}$
$LEFT\_Torsion(2, -4, -4) = TTS_{2,4,4}$
$LEFT\_Torsion(3, -1, -1) = TTS_{3,1,1}$
$LEFT\_Torsion(3, -2, -1) = TTS_{3,2,1}$
$LEFT\_Torsion(3, -3, -1) = TTS_{3,3,1}$
$LEFT\_Torsion(3, -4, -1) = TTS_{3,4,1}$
$LEFT\_Torsion(3, -1, -2) = TTS_{3,1,2}$
$LEFT\_Torsion(3, -2, -2) = TTS_{3,2,2}$
$LEFT\_Torsion(3, -3, -2) = TTS_{3,3,2}$
$LEFT\_Torsion(3, -4, -2) = TTS_{3,4,2}$
$LEFT\_Torsion(3, -1, -3) = TTS_{3,1,3}$
$LEFT\_Torsion(3, -2, -3) = TTS_{3,2,3}$
$LEFT\_Torsion(3, -3, -3) = TTS_{3,3,3}$
$LEFT\_Torsion(3, -4, -3) = TTS_{3,4,3}$
$LEFT\_Torsion(3, -1, -4) = TTS_{3,1,4}$
$LEFT\_Torsion(3, -2, -4) = TTS_{3,2,4}$
$LEFT\_Torsion(3, -3, -4) = TTS_{3,3,4}$
$LEFT\_Torsion(3, -4, -4) = TTS_{3,4,4}$
$LEFT\_Torsion(4, -1, -1) = TTS_{4,1,1}$
$LEFT\_Torsion(4, -2, -1) = TTS_{4,2,1}$
$LEFT\_Torsion(4, -3, -1) = TTS_{4,3,1}$
$LEFT\_Torsion(4, -4, -1) = TTS_{4,4,1}$
$LEFT\_Torsion(4, -1, -2) = TTS_{4,1,2}$
$LEFT\_Torsion(4, -2, -2) = TTS_{4,2,2}$
$LEFT\_Torsion(4, -3, -2) = TTS_{4,3,2}$
$LEFT\_Torsion(4, -4, -2) = TTS_{4,4,2}$
$LEFT\_Torsion(4, -1, -3) = TTS_{4,1,3}$
$LEFT\_Torsion(4, -2, -3) = TTS_{4,2,3}$
$LEFT\_Torsion(4, -3, -3) = TTS_{4,3,3}$
$LEFT\_Torsion(4, -4, -3) = TTS_{4,4,3}$
$LEFT\_Torsion(4, -1, -4) = TTS_{4,1,4}$
$LEFT\_Torsion(4, -2, -4) = TTS_{4,2,4}$
$LEFT\_Torsion(4, -3, -4) = TTS_{4,3,4}$
$LEFT\_Torsion(4, -4, -4) = TTS_{4,4,4}$

The Right and the Left Cartan matrices are negative similarity transforms, **but the LEFT AND RIGHT torsion terms appear to be diffferent, and the left "torsion" depends upon the both the chirality factor, ch, and the oreintation, B !!!!!.**

****

Next the Christoffel symbols will be computed for the **subsumed pullback metric** on the initial state. The pullback mnetric is conformal to the indentity matrix.

Christoffel Connection coefficients from the induced metric

It is assumed that the "metric" is the pull back metric given below, which is conformal.

```
> metric:=evalm( pullbackmetric);
```

$$metric := \begin{bmatrix} S^2+Z^2+Y^2+X^2 & 0 & 0 & 0 \\ 0 & S^2+Z^2+Y^2+X^2 & 0 & 0 \\ 0 & 0 & S^2+Z^2+Y^2+X^2 & 0 \\ 0 & 0 & 0 & S^2+Z^2+Y^2+X^2 \end{bmatrix}$$

```
> metricinverse:=inverse(metric):
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1gun[i,j,k] :=
  (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if d1gun[i,j,k]=0 then  else
  print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k] := 0  od od od; for i from
  1 to dim do for j from 1 to dim do for k from 1 to dim do C1S[i,j,k] :=
  1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0; for m to dim do ss :=
  ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] := simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if C2S[i,j,k]=0 then  else
  print(`Christoffel_Gamma2`(i,-j,-k)=C2S[i,j,k]) fi od od od;
```

The non zero Christoffel Connection coefficients 2nd kind on the initial space (domain)

Gamma2(i,j,k) index (1,-1,-1)

$$\text{Christoffel\_Gamma2}(1,-1,-1) = \frac{X}{S^2+Z^2+Y^2+X^2}$$

$$\text{Christoffel\_Gamma2}(1, -1, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(1, -1, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(1, -1, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(1, -2, -1) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(1, -2, -2) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(1, -3, -1) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(1, -3, -3) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(1, -4, -1) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(1, -4, -4) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -1, -1) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -1, -2) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -2, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -3, -2) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -3, -3) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -4, -2) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(2, -4, -4) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -1, -1) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -1, -3) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -2, -2) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -2, -3) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -3, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -4, -3) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(3, -4, -4) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -1, -1) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -1, -4) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -2, -2) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -2, -4) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -3, -3) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -3, -4) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -1) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -2) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -3) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$\text{Christoffel\_Gamma2}(4, -4, -4) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

If no entries appear above the Christoffel symbols on the domain space vanish

Note that the Christoffel Symbols for the Conformal metric are not zero, but are not the same as the Right or Left Cartan Connection matrices. More over, the Christoffel symbols built upon the metric defined above are independent from the choice of chirality and orientation. The metric is symmetric and has a positive definite determinant.

The Right Cartan matrix is often defined as the sum of Christoffel Symbols and Rotation coefficients, T(i,j,k)

CartanRight(ijk) := ChristoffelGamma(ijk) + T(ijk)

Using this **definition for the Rotation coefficients,** Compute the T(i,j,k):

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do ss:=0; ss := (CC[i,j,k]-C2S[i,j,k]);
  SHIPTR[i,j,k]:=simplify(ss)  od od od   ;
>
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if C2S[i,j,k]=0 and CC[i,j,k]=0 then
  else print(`T`(i,-j,-k)=simplify(subs(ch^2=1,SHIPTR[i,j,k]))) fi od od od ;
```

T(ijk) index (1,-1,-1)

$$T(1, -1, -1) = 0$$

$$T(1, -1, -2) = 0$$

$$T(1, -1, -3) = 0$$

$$T(1, -1, -4) = 0$$

$$T(1, -2, -1) = -2 \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -2, -2) = 2 \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -2, -3) = -\frac{ch\, S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -2, -4) = \frac{ch\, Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -3, -1) = -\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -3, -2) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -3, -3) = \frac{X\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -3, -4) = -\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -4, -1) = -\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1, -4, -2) = -\frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-4,-3) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(1,-4,-4) = \frac{X\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-1,-1) = 2\,\frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-1,-2) = -2\,\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-1,-3) = \frac{ch\,S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-1,-4) = -\frac{ch\,Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-2,-1) = 0$$

$$T(2,-2,-2) = 0$$

$$T(2,-2,-3) = 0$$

$$T(2,-2,-4) = 0$$

$$T(2,-3,-1) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-3,-2) = -\frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-3,-3) = \frac{Y\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-3,-4) = \frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-4,-1) = \frac{Z}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-4,-2) = -\frac{S\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-4,-3) = -\frac{X}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(2,-4,-4) = \frac{Y\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3,-1,-1) = \frac{Z\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3,-1,-2) = -\frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3,-1,-3) = -\frac{X\,(ch + 1)}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3,-1,-4) = \frac{Y}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3,-2,-1) = \frac{S}{S^2 + Z^2 + Y^2 + X^2}$$

$$T(3, -2, -2) = \frac{Z\,(ch+1)}{S^2+Z^2+Y^2+X^2}$$

$$T(3, -2, -3) = -\frac{Y\,(ch+1)}{S^2+Z^2+Y^2+X^2}$$

$$T(3, -2, -4) = -\frac{X}{S^2+Z^2+Y^2+X^2}$$

$$T(3, -3, -1) = 0$$

$$T(3, -3, -2) = 0$$

$$T(3, -3, -3) = 0$$

$$T(3, -3, -4) = 0$$

$$T(3, -4, -1) = -\frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$T(3, -4, -2) = \frac{X\,ch}{S^2+Z^2+Y^2+X^2}$$

$$T(3, -4, -3) = -2\,\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$T(3, -4, -4) = 2\,\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -1, -1) = \frac{S\,(ch+1)}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -1, -2) = \frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -1, -3) = -\frac{Y}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -1, -4) = -\frac{X\,(ch+1)}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -2, -1) = -\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -2, -2) = \frac{S\,(ch+1)}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -2, -3) = \frac{X}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -2, -4) = -\frac{Y\,(ch+1)}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -3, -1) = \frac{ch\,Y}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -3, -2) = -\frac{X\,ch}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -3, -3) = 2\,\frac{S}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -3, -4) = -2\,\frac{Z}{S^2+Z^2+Y^2+X^2}$$

$$T(4, -4, -1) = 0$$
$$T(4, -4, -2) = 0$$
$$T(4, -4, -3) = 0$$
$$T(4, -4, -4) = 0$$

The Rotation matrices also depend upon the chirality factor.

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