```
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=0,Az=0,C=0,Phi=0,phi=0,theta=0,r=0,tau=0,a=const,b=const,c=const,e=const,aa=const,bb=const,cc=const,MM1=0,MM2=0,M=const,ss=0,cc=const,ee=const,Lx=0,Ly=0,Lz=0,vx=const,vy=const,vz=const,omega=const);
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the protected name close has been redefined and unprotected

Warning, the names &^, d and wdegree have been redefined
```

Embedding the Schwarzschild solution in a Basis Frame

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INTRODUCTION

From MapleEP1.mws

Given a connection [C] as a Cartan matrix of 1-forms, the standard formula to produce the
Cartan matrix of curvature 2-forms is:
[OMEGA] = [dC] + [C] ^ [C]
is used to define the curvatures of the space.
*
However, a connection can be computed in several distinct ways.
Given a Frame field, [F], the Right Cartan Connection can be computed from the formula [C] = [G][dF] where
[G] is the inverse of [F]
*
The Frame can be chosen such that the congruent mapping [Ftranspose] [eta] [F] yields some specified [g].
This procedure is easy of [eta] is diagonal.

IN all cases, the Cartan matrix of connection 1-forms can be decomposed into two parts according to the
formula
[C] = [Gamma] - [T]
The standard formula for curvature will be used to yield
[dC] + [C]^[C] ={ [dGamma] + [Gamma]^[Gamma]} + {[dT] + [C]^[T]} + {[Gamma][T] + [T][Gamma] = 0.
For all Frame fields (with det not zero) the far left side vanishes, leaving the result
Christoffel metric curvature = { [dGamma] + [Gamma]^[Gamma]}
inertial curvature = - {[dT] + [C]^[T] + [Gamma][T] + [T][Gamma]}
or finally
[Christoffel metric curvature] = [inertial curvature]
or
[Gravity curvatures] = [inertial curvatures]
*
Thus a principle of equivalence is established without constraints of Absolute Parallelism.
The arguments are based only on the assumption that domain of interest supports a Basis Frame of C2
functions for a global vector space.
*
This example will use the Schwarzshild isotropic metric in spherical coordinates as the example.

The classic coordinate map from spherical to Cartesian Coordinates

is given by the expressions:

```
> x:=r*sin(theta)*cos(phi);y:=r*sin(theta)*sin(phi);z:=r*cos(theta);t:=tau;eta:=ar
  ray([[-1,0,0,0],[0,-1,0,0],[0,0,-1,0],[0,0,0,1]]);
> Sch1BF:=(1+M/(2*r))^2;Sch2BF:=((1-M/(2*r))/(1+M/(2*r)));
> s1:=MM1*d(x);s2:=MM2*d(y);s3:=MM3*d(z);s4:=MM4*d(t);
```

$$x := r\sin(\theta)\cos(\phi)$$

$$y := r\sin(\theta)\sin(\phi)$$

$$z := r\cos(\theta)$$

$$t := \tau$$

$$\eta := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Sch1BF := \left(1 + \frac{\frac{1}{2}M}{r}\right)^2$$

$$Sch2BF := \frac{1 - \frac{1}{2}\frac{M}{r}}{1 + \frac{\frac{1}{2}M}{r}}$$

$$s1 := MM1\,(\sin(\theta)\cos(\phi)\,\mathrm{d}(r) + r\cos(\phi)\cos(\theta)\,\mathrm{d}(\theta) - r\sin(\theta)\sin(\phi)\,\mathrm{d}(\phi))$$

$$s2 := MM2\,(\sin(\theta)\sin(\phi)\,\mathrm{d}(r) + r\sin(\phi)\cos(\theta)\,\mathrm{d}(\theta) + r\sin(\theta)\cos(\phi)\,\mathrm{d}(\phi))$$

$$s3 := MM3\,(\cos(\theta)\,\mathrm{d}(r) - r\sin(\theta)\,\mathrm{d}(\theta))$$

$$s4 := MM4\,\mathrm{d}(\tau)$$

The Coordinate map generates an "unperturbed" Frame field in terms of the Jacobian of the mapping from spherical to Cartesian coordiantes.
The Frame matrix can be perturbed to include the effects of a metric that represents mass and gravitational curvature.
The Frame matrix with a metric perturbation will replace MM1=MM2=MM3 with Sch1BF and MM4 with Sch2BF.
This produces the line element of the isotropic Schwarzschild metric, constructed on the congruent pullback generated by the perturbed basis frame.

```
> IsotropicLineElement:=simpform(simplify(subs(MM1=1*Sch1BF,MM2=1*Sch1BF,MM3=1*Sch
  1BF,MM4=1*Sch2BF,innerprod([s1,s2,s3,s4],eta,[s1,s2,s3,s4]))));
```

$$IsotropicLineElement := \frac{1}{16}(-16\,r^4 + 16\,r^4\cos(\theta)^2 - 32\,r^3 M + 32\,r^3 M\cos(\theta)^2 - 24\,r^2 M^2 + 24\,r^2 M^2\cos(\theta)^2 + 8\,r M^3\cos(\theta)^2 - 8\,r M^3 - M^4 + M^4\cos(\theta)^2)\,\mathrm{d}(\phi)^2 \big/ r^2 - \frac{1}{16}\frac{(16\,r^4 + 32\,r^3 M + 24\,r^2 M^2 + 8\,r M^3 + M^4)\,\mathrm{d}(\theta)^2}{r^2} - \frac{1}{16}\frac{(16\,r^4 + 32\,r^3 M + 24\,r^2 M^2 + 8\,r M^3 + M^4)\,\mathrm{d}(r)^2}{r^4} + \frac{(M^2 - 4\,r M + 4\,r^2)\,\mathrm{d}(\tau)^2}{(2\,r + M)^2}$$

```
> FF11:=getcoeff(d(x)&^d(theta)&^d(phi)&^d(tau)*MM1):FF12:=getcoeff(d(x)&^d(phi)&^d(tau)&^d(r)*MM1):FF13:=getcoeff(d(x)&^d(tau)&^d(r)&^d(theta)*MM1):FF14:=getcoeff(d(x)&^d(r)&^d(theta)&^d(phi)*MM1):
> FF21:=getcoeff(d(y)&^d(theta)&^d(phi)&^d(t)*MM2):FF22:=getcoeff(d(y)&^d(phi)&^d(t)&^d(r)*MM2):FF23:=getcoeff(d(y)&^d(t)&^d(r)&^d(theta)*MM2):FF24:=getcoeff(d(y)&^d(r)&^d(theta)&^d(phi)*MM2):
#Rotation:=simplify((y*d(x)-x*d(y))/(x^2+y^2));
```

The Frame can be modified further by adding rotation terms to dz (with a coefficient aa) and to dt with (coefficients cc). These cases will not be examined herein.

```
> sigmaz:=simpform(wcollect(d(z)+0*aa*(y*d(x)-x*d(y))/(x^2+y^2))):dzz:=sigmaz:sst:=simpform((d(t)+0*cc*(x*d(y)-y*d(x))/(x^2+y^2))):
> FF31:=getcoeff(dzz&^d(theta)&^d(phi)&^d(tau)*MM3):FF32:=getcoeff(dzz&^d(phi)&^d(tau)&^d(r)*MM3):FF33:=factor(simplify(getcoeff(dzz&^d(tau)&^d(r)&^d(theta)))*MM3):FF34:=getcoeff(dzz&^d(r)&^d(theta)&^d(phi)*MM3):
> dtt:=sst:
> FF41:=getcoeff(dtt&^d(theta)&^d(phi)&^d(tau)*MM4):FF42:=simplify(getcoeff(dtt&^d(phi)&^d(tau)&^d(r))*MM4):FF43:=factor(simplify(getcoeff(dtt&^d(tau)&^d(r)&^d(theta)))*MM4):FF44:=getcoeff(dtt&^d(r)&^d(theta)&^d(phi)*MM4):
>

>
> R:=[x,y,z,t]:FF:=simplify(subs(MM1=1*Sch1BF,MM2=1*Sch1BF,MM3=1*Sch1BF,MM4=1*Sch2BF,array([[FF11,FF12,FF13,FF14],[FF21,FF22,FF23,FF24],[FF31,FF32,FF33,FF34],[FF41,FF42,FF43,FF44]]))):`perturbed Basis Frame`:=evalm(FF);print(`Induced metric coefficients from a quadratic congruence using the perturbed Basis Frame (spherical coordiantes)`);
> DZZ:=subs(aa=0,cc=0,M=M,simplify(innerprod(transpose(FF),eta,FF))):g11:=DZZ[1,1];g12:=DZZ[1,2];g13:=simplify(factor(DZZ[1,3]));g14:=DZZ[1,4];g21:=DZZ[2,1];g22:=(factor(subs(DZZ[2,2])));g23:=simplify(factor(DZZ[2,3]));g24:=DZZ[2,4];g31:=simplify(factor(DZZ[3,1]));g32:=simplify(factor(DZZ[3,2]));g33:=simplify(factor(DZZ[3,3]));g34:=simplify(factor(DZZ[3,4]));g41:=DZZ[4,1];g42:=DZZ[4,2];g43:=simplify(factor(DZZ[4,3]));g44:=DZZ[4,4];(simplify(factor(subs(M=M,DZZ[2,2]))-g22));
> print(`For Spherical Coordinates d(r), d(theta), d(phi), d(tau) `);print(`
`);pullbackmetric:=evalm((array([[g11,g12,g13,g14],[g21,subs(g22),g23,g24],[g31,g32,g33,g34],[g41,g42,g43,g44]]))):`Pullback metric of the perturbed Frame`:=evalm(pullbackmetric);
```

perturbed Basis Frame :=

$$\begin{bmatrix} \frac{1}{4}\frac{\sin(\theta)\cos(\phi)(2r+M)^2}{r^2} & \frac{1}{4}\frac{\cos(\phi)\cos(\theta)(2r+M)^2}{r} & -\frac{1}{4}\frac{\sin(\theta)\sin(\phi)(2r+M)^2}{r} & 0 \\ \frac{1}{4}\frac{\sin(\theta)\sin(\phi)(2r+M)^2}{r^2} & \frac{1}{4}\frac{\sin(\phi)\cos(\theta)(2r+M)^2}{r} & \frac{1}{4}\frac{\sin(\theta)\cos(\phi)(2r+M)^2}{r} & 0 \\ \frac{1}{4}\frac{\cos(\theta)(2r+M)^2}{r^2} & -\frac{1}{4}\frac{\sin(\theta)(2r+M)^2}{r} & 0 & 0 \\ 0 & 0 & 0 & \frac{2r-M}{2r+M} \end{bmatrix}$$

Induced metric coefficients from a quadratic congruence using the perturbed Basis Frame (spherical coordiantes)

$$g11 := -\frac{1}{16}\frac{(2r+M)^4}{r^4}$$

$$g12 := 0$$

$$g13 := 0$$

$$g14 := 0$$

$$g21 := 0$$

$$g22 := -\frac{1}{16}\frac{(2r+M)^4}{r^2}$$

$$g23 := 0$$

$$g24 := 0$$

$$g31 := 0$$

$$g32 := 0$$

$$g33 := \frac{1}{16}\frac{(-1+\cos(\theta)^2)(2r+M)^4}{r^2}$$

$$g34 := 0$$

$$g41 := 0$$

$$g42 := 0$$

$$g43 := 0$$

$$g44 := \frac{(2r-M)^2}{(2r+M)^2}$$

$$0$$

For Spherical Coordinates d(r), d(theta), d(phi), d(tau)

Pullback metric of the perturbed Frame :=

$$\begin{bmatrix} -\frac{1}{16}\frac{(2r+M)^4}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{1}{16}\frac{(2r+M)^4}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{16}\frac{(-1+\cos(\theta)^2)(2r+M)^4}{r^2} & 0 \\ 0 & 0 & 0 & \frac{(2r-M)^2}{(2r+M)^2} \end{bmatrix}$$

>

Properties of the vector of 1-forms |sigma>

```
> dim:=4;coord:=[r,theta,phi,tau];GG:=simplify(inverse(FF)):dGG:=d(evalm(GG)):DETFF:=factor(simplify(det(FF)));DETGG:=factor(simplify(det(GG)));DETMETRIC:=det(pullbackmetric);print(`  `);print(`1-form sigmas projected by Frame Matrix  [F][dy> = [sigma>`);print(`
`);SIGMAFF:=simplify(innerprod(evalm(FF),[d(r),d(theta),d(phi),d(tau)])):SIGMAGG:=simplify(innerprod(GG,[d(x),d(y),d(z),d(t)])):sigma1:=simpform(simplify(factor(SIGMAFF[1])));sigma2:=simpform(simplify(factor(SIGMAFF[2])));sigma3:=simpform(simplify(factor(SIGMAFF[3])));sigma4:=simpform(simplify(factor(SIGMAFF[4])));
>
> print(`
`);dsigma1:=simpform(d(sigma1));dsigma2:=d(sigma2);dsigma3:=simpform(d(sigma3));dsigma4:=d(sigma4);print(`  `);
> TopTor1:=sigma1&^dsigma1;TopTor2:=sigma2&^dsigma2;TopTor3:=factor(sigma3&^dsigma3);TopTor4:=sigma4&^dsigma4;
```

$$dim := 4$$

$$coord := [r, \theta, \phi, \tau]$$

$$DETFF := \frac{1}{64} \frac{\sin(\theta)\,(2r+M)^5\,(2r-M)}{r^4}$$

$$DETGG := 64 \frac{r^4}{\sin(\theta)\,(2r+M)^5\,(2r-M)}$$

$$DETMETRIC := \frac{1}{4096} \frac{(2r+M)^{10}\,(-1+\cos(\theta)^2)\,(2r-M)^2}{r^8}$$

1-form sigmas projected by Frame Matrix [F][dy> = [sigma>

$$\sigma 1 := \frac{1}{4} \frac{(2r+M)^2 \sin(\theta)\cos(\phi)\, d(r)}{r^2} + \frac{\frac{1}{4}(2r+M)^2 \cos(\phi)\cos(\theta)\, d(\theta)}{r} - \frac{1}{4} \frac{(2r+M)^2 \sin(\theta)\sin(\phi)\, d(\phi)}{r}$$

$$\sigma 2 := \frac{1}{4} \frac{(2r+M)^2 \sin(\theta)\sin(\phi)\, d(r)}{r^2} + \frac{\frac{1}{4}(2r+M)^2 \sin(\phi)\cos(\theta)\, d(\theta)}{r} + \frac{\frac{1}{4}(2r+M)^2 \sin(\theta)\cos(\phi)\, d(\phi)}{r}$$

$$\sigma 3 := \frac{1}{4} \frac{(2r+M)^2 \cos(\theta)\, d(r)}{r^2} - \frac{1}{4} \frac{(2r+M)^2 \sin(\theta)\, d(\theta)}{r}$$

$$\sigma 4 := \frac{(2r-M)\, d(\tau)}{2r+M}$$

$$dsigma1 := -\frac{1}{2} \frac{\sin(\theta)\sin(\phi)\,(2r+M)\,M\,(d(\phi)\ \&\hat{}\ d(r))}{r^2} + \frac{\frac{1}{2}\cos(\phi)\cos(\theta)\,(2r+M)\,M\,(d(\theta)\ \&\hat{}\ d(r))}{r^2}$$

$$dsigma2 := \frac{1}{2} \frac{\sin(\theta)\cos(\phi)\,(2r+M)\,M\,(d(\phi)\ \&\hat{}\ d(r))}{r^2} + \frac{\frac{1}{2}\sin(\phi)\cos(\theta)\,(2r+M)\,M\,(d(\theta)\ \&\hat{}\ d(r))}{r^2}$$

$$dsigma3 := -\frac{1}{2} \frac{\sin(\theta)\,(2r+M)\,M\,(d(\theta)\ \&\hat{}\ d(r))}{r^2}$$

$$dsigma4 := 4\,\frac{M\,(\mathrm{d}(r)\,\&\wedge\,\mathrm{d}(\tau))}{(2\,r+M)^2}$$

$$TopTor1 := 0$$
$$TopTor2 := 0$$
$$TopTor3 := 0$$
$$TopTor4 := 0$$

\>

\>

With pure mass perturbation, all the 1-forms [F][dy> = |sigma> admit integrating factors, as topological torsion is zero for each 1-form.
(A new Basis FRame can be constructed such that the affine torsion of the new frame is zero)

Not done herein

```
>
> AAA:=array([[0,0,0,d(r)],[0,0,0,d(theta)],[0,0,0,d(phi)],[0,0,0,d(tau)]]):
>
> Cartan_Connection:=simpform(simplify(innerprod(-d(evalm(GG)),evalm(FF)))):`Carta
  n Connection Matrix of 1-forms [C] constucted form the Basis Frame
  [F]`=evalm(Cartan_Connection);Congruent_Quadratic_metric:=evalm(pullbackmetric):
  `Congruent Metric (quadratic form) based on Pullback by the basis frame
  [F]`:=evalm(Congruent_Quadratic_metric);CCCC:=simpform(simplify(factor(evalm(Car
  tan_Connection)))):AT:=CCCC&^evalm(AAA):ATr:=simpform(simplify(AT[1,4])):ATtheta
  :=simpform(AT[2,4]):ATphi:=simpform(AT[3,4]):ATtau:=simpform(AT[4,4]):VATC:=arra
  y([[ATr],[ATtheta],[ATphi],[ATtau]]):`Vector of Affine Torsion 2-forms based on
  [C]`:=evalm(VATC);
```

Cartan Connection Matrix of 1-forms [C] constucted form the Basis Frame [F] =

$$\begin{bmatrix} -2\,\frac{M\,\mathrm{d}(r)}{r\,(2\,r+M)} & -r\,\mathrm{d}(\theta) & \mathrm{d}(\phi)\,r\,(-1+\cos(\theta)^2) & 0 \\ \frac{\mathrm{d}(\theta)}{r} & \frac{\mathrm{d}(r)\,(2\,r-M)}{r\,(2\,r+M)} & -\mathrm{d}(\phi)\cos(\theta)\sin(\theta) & 0 \\ \frac{\mathrm{d}(\phi)}{r} & \frac{\mathrm{d}(\phi)\cos(\theta)}{\sin(\theta)} & \frac{\mathrm{d}(r)\,(2\,r-M)}{r\,(2\,r+M)}+\frac{\cos(\theta)\,\mathrm{d}(\theta)}{\sin(\theta)} & 0 \\ 0 & 0 & 0 & 4\,\frac{M\,\mathrm{d}(r)}{4\,r^2-M^2} \end{bmatrix}$$

Congruent Metric (quadratic form) based on Pullback by the basis frame [F] :=

$$\begin{bmatrix} -\frac{1}{16}\frac{(2r+M)^4}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{1}{16}\frac{(2r+M)^4}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{16}\frac{(-1+\cos(\theta)^2)(2r+M)^4}{r^2} & 0 \\ 0 & 0 & 0 & \frac{(2r-M)^2}{(2r+M)^2} \end{bmatrix}$$

$$\textit{Vector of Affine Torsion 2-forms based on [C]} := \begin{bmatrix} 0 \\ 2\frac{(d(\theta)\ \&\hat{}\ d(r))\,M}{r\,(2r+M)} \\ 2\frac{(d(\phi)\ \&\hat{}\ d(r))\,M}{r\,(2r+M)} \\ 4\frac{M\,(d(r)\ \&\hat{}\ d(\tau))}{4r^2-M^2} \end{bmatrix}$$

```
>
> TOR_AF:=simpform(TopTor1+TopTor2+TopTor3+TopTor4);P2:=simpform(d(TOR_AF));
```

$$TOR\_AF := 0$$

$$P2 := 0$$

```
> G:=simplify(evalm(VATC));F:=array([[dsigma1],[dsigma2],[dsigma3],[dsigma4]]);G[1
,1];FG:=dsigma1&^G[1,1]+dsigma2&^G[2,1]+dsigma3&^G[3,1]+dsigma4&^G[4,1];J:=simpl
ify(d(G));AJ:=sigma1&^J[1,1]+sigma2&^J[2,1]+sigma3&^J[3,1]+sigma4&^J[4,1];
```

$$G := \begin{bmatrix} 0 \\ 2\frac{(d(\theta)\ \&\hat{}\ d(r))\,M}{r\,(2r+M)} \\ 2\frac{(d(\phi)\ \&\hat{}\ d(r))\,M}{r\,(2r+M)} \\ 4\frac{M\,(d(r)\ \&\hat{}\ d(\tau))}{4r^2-M^2} \end{bmatrix}$$

$$F := \begin{bmatrix} -\frac{1}{2}\frac{\sin(\theta)\sin(\phi)(2r+M)\,M\,(d(\phi)\ \&\hat{}\ d(r))}{r^2} + \frac{\frac{1}{2}\cos(\phi)\cos(\theta)(2r+M)\,M\,(d(\theta)\ \&\hat{}\ d(r))}{r^2} \\ \frac{1}{2}\frac{\sin(\theta)\cos(\phi)(2r+M)\,M\,(d(\phi)\ \&\hat{}\ d(r))}{r^2} + \frac{\frac{1}{2}\sin(\phi)\cos(\theta)(2r+M)\,M\,(d(\theta)\ \&\hat{}\ d(r))}{r^2} \\ -\frac{1}{2}\frac{\sin(\theta)(2r+M)\,M\,(d(\theta)\ \&\hat{}\ d(r))}{r^2} \\ 4\frac{M\,(d(r)\ \&\hat{}\ d(\tau))}{(2r+M)^2} \end{bmatrix}$$

$$0$$

$$FG := 0$$

$$J := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AJ := 0$$

```
> SPIN_AG:=simplify(simpform(sigma1&^G[1,1])+simpform(sigma2&^G[2,1])+simpform(sigma3&^G[3,1])+simpform(sigma4&^G[4,1]));
> P1:=d(SPIN_AG);
```

$$SPIN\_AG := \frac{1}{2}\frac{\sin(\theta)\,(2\,r+M)\,M\,(\cos(\phi)\,\&\hat{}(d(\phi), d(\theta), d(r)) - \&\hat{}(d(\theta), d(\phi), d(r)))}{r^2}$$

$$P1 := 0$$

```
>
```

It is also possible to come to the same conclusions using tensor methods.

The right Cartan Connection is based on the perturbed Frame: [C] = [G][dF]= - [dG][F]

First compute the differentials of the inverse matrix [GG]

```
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do d1GG[i,j,k] := (diff(GG[i,j],coord[k])) od od od:
```

Compute the elements of the matrix product of - d[G][F]= G[dF]

```
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do  ss:=0;for m from 1 to dim do ss := ss+(d1GG[a,m,k]*FF[m,b]); C2C[a,b,k]:=simplify(-ss) od od od od ;
```

Right Cartan connection coefficients by tensor methods are displayed below:

```
>
> for b from 1 to dim do for a from 1 to dim do for k from 1 to dim do if C2C[a,b,k]=0 then else print(`Cartan_RIGHT`(a,b,k)=factor(C2C[a,b,k])) fi od od od ;
```

$$\text{Cartan\_RIGHT}(1, 1, 1) = -2\frac{M}{r\,(2\,r+M)}$$

$$\text{Cartan\_RIGHT}(2, 1, 2) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(3, 1, 3) = \frac{1}{r}$$

$$\text{Cartan\_RIGHT}(1, 2, 2) = -r$$

$$\text{Cartan\_RIGHT}(2, 2, 1) = \frac{2\,r-M}{r\,(2\,r+M)}$$

$$\text{Cartan\_RIGHT}(3, 2, 3) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\text{Cartan\_RIGHT}(1, 3, 3) = r\,(\cos(\theta)-1)\,(\cos(\theta)+1)$$

$$\text{Cartan\_RIGHT}(2, 3, 3) = -\cos(\theta)\sin(\theta)$$

$$\text{Cartan\_RIGHT}(3, 3, 1) = \frac{2\,r-M}{r\,(2\,r+M)}$$

$$\text{Cartan\_RIGHT}(3, 3, 2) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\text{Cartan\_RIGHT}(4, 4, 1) = 4\frac{M}{(2\,r+M)\,(2\,r-M)}$$

```
>
```

Next check for Affine Torsion using the tensor methods:

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (C2C[i,j,k]-C2C[i,k,j])/2; CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  CCTTS[i,j,k]=0 then else print(`[C]_AffineTorsion`(i,k,j)=factor(CCTTS[i,k,j]))
  fi od od od ;
```

$$\text{[C]\_AffineTorsion}(2,2,1) = -\frac{M}{r\,(2\,r+M)}$$

$$\text{[C]\_AffineTorsion}(2,1,2) = \frac{M}{r\,(2\,r+M)}$$

$$\text{[C]\_AffineTorsion}(3,3,1) = -\frac{M}{r\,(2\,r+M)}$$

$$\text{[C]\_AffineTorsion}(3,1,3) = \frac{M}{r\,(2\,r+M)}$$

$$\text{[C]\_AffineTorsion}(4,4,1) = 2\,\frac{M}{(2\,r+M)\,(2\,r-M)}$$

$$\text{[C]\_AffineTorsion}(4,1,4) = -2\,\frac{M}{(2\,r+M)\,(2\,r-M)}$$

IF NO ENTRIES APPEAR ABOVE, THE AFFINE TORSION IS ZERO

Compare the matrix elements of the right Cartan matrix of connection 1-forms
computed by Matrix Methods and Tensor Methods.

```
> CGamma11:=C2C[1,1,1]*d(r)+C2C[1,1,2]*d(theta)+C2C[1,1,3]*d(phi)+C2C[1,1,4]*d(tau
  ):
> CGamma12:=C2C[1,2,1]*d(r)+C2C[1,2,2]*d(theta)+C2C[1,2,3]*d(phi)+C2C[1,2,4]*d(tau
  ):
> CGamma13:=C2C[1,3,1]*d(r)+C2C[1,3,2]*d(theta)+C2C[1,3,3]*d(phi)+C2C[1,3,4]*d(tau
  ):
> CGamma14:=C2C[1,4,1]*d(r)+C2C[1,4,2]*d(theta)+C2C[1,4,3]*d(phi)+C2C[1,4,4]*d(tau
  ):
> CGamma21:=C2C[2,1,1]*d(r)+C2C[2,1,2]*d(theta)+C2C[2,1,3]*d(phi)+C2C[2,1,4]*d(tau
  ):
> CGamma22:=C2C[2,2,1]*d(r)+C2C[2,2,2]*d(theta)+C2C[2,2,3]*d(phi)+C2C[2,2,4]*d(tau
  ):
> CGamma23:=C2C[2,3,1]*d(r)+C2C[2,3,2]*d(theta)+C2C[2,3,3]*d(phi)+C2C[2,3,4]*d(tau
  ):
> CGamma24:=C2C[2,4,1]*d(r)+C2C[2,4,2]*d(theta)+C2C[2,4,3]*d(phi)+C2C[2,4,4]*d(tau
  ):
> CGamma31:=C2C[3,1,1]*d(r)+C2C[3,1,2]*d(theta)+C2C[3,1,3]*d(phi)+C2C[3,1,4]*d(tau
```

```
):
> CGamma32:=C2C[3,2,1]*d(r)+C2C[3,2,2]*d(theta)+C2C[3,2,3]*d(phi)+C2C[3,2,4]*d(tau
):
> CGamma33:=C2C[3,3,1]*d(r)+C2C[3,3,2]*d(theta)+C2C[3,3,3]*d(phi)+C2C[3,3,4]*d(tau
):
> CGamma34:=C2C[3,4,1]*d(r)+C2C[3,4,2]*d(theta)+C2C[3,4,3]*d(phi)+C2C[3,4,4]*d(tau
):
> CGamma41:=C2C[4,1,1]*d(r)+C2C[4,1,2]*d(theta)+C2C[4,1,3]*d(phi)+C2C[4,1,4]*d(tau
):
> CGamma42:=C2C[4,2,1]*d(r)+C2C[4,2,2]*d(theta)+C2C[4,2,3]*d(phi)+C2C[4,2,4]*d(tau
):
> CGamma43:=C2C[4,3,1]*d(r)+C2C[4,3,2]*d(theta)+C2C[4,3,3]*d(phi)+C2C[4,3,4]*d(tau
):
> CGamma44:=C2C[4,4,1]*d(r)+C2C[4,4,2]*d(theta)+C2C[4,4,3]*d(phi)+C2C[4,4,4]*d(tau
):
> CartanC:=array([[CGamma11,CGamma12,CGamma13,CGamma14],[CGamma21,CGamma22,CGamma2
3,CGamma24],[CGamma31,CGamma32,CGamma33,CGamma34],[CGamma41,CGamma42,CGamma43,CG
amma44]]):print(Cartan_Connection_by_tensor_methods);evalm(CartanC);print(Cartan
_Connection_by_matrix_methods);simpform(factor(Cartan_Connection));
> CHECKSUM:=simpform(evalm(CartanC-Cartan_Connection));
>
```

$$Cartan\_Connection\_by\_tensor\_methods$$

$$\begin{bmatrix} -2\frac{M\,d(r)}{r(2r+M)} & -r\,d(\theta) & d(\phi)\,r\,(-1+\cos(\theta)^2) & 0 \\ \frac{d(\theta)}{r} & \frac{d(r)(2r-M)}{r(2r+M)} & -d(\phi)\cos(\theta)\sin(\theta) & 0 \\ \frac{d(\phi)}{r} & \frac{d(\phi)\cos(\theta)}{\sin(\theta)} & \frac{d(r)(2r-M)}{r(2r+M)}+\frac{\cos(\theta)\,d(\theta)}{\sin(\theta)} & 0 \\ 0 & 0 & 0 & 4\frac{M\,d(r)}{4r^2-M^2} \end{bmatrix}$$

$$Cartan\_Connection\_by\_matrix\_methods$$

$$\begin{bmatrix} -2\frac{M\,d(r)}{r(2r+M)} & -r\,d(\theta) & d(\phi)\,r\,(-1+\cos(\theta)^2) & 0 \\ \frac{d(\theta)}{r} & \frac{d(r)(2r-M)}{r(2r+M)} & -d(\phi)\cos(\theta)\sin(\theta) & 0 \\ \frac{d(\phi)}{r} & \frac{d(\phi)\cos(\theta)}{\sin(\theta)} & \frac{d(r)(2r-M)}{r(2r+M)}+\frac{\cos(\theta)\,d(\theta)}{\sin(\theta)} & 0 \\ 0 & 0 & 0 & 4\frac{M\,d(r)}{4r^2-M^2} \end{bmatrix}$$

$$CHECKSUM := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Cartan matrix of curvature 2-forms based upon d[C] + [C]^[C]

first compute the [C]^[C} terms using matrix methods

```
>
> C2CxC2C:=simpform(evalm(CartanC)&^evalm(CartanC));
```

```
> dC2C:=simpform(d(evalm(CartanC))):
>
> CartanCurvature:=simpform(simplify(evalm(evalm(dC2C)+evalm(C2CxC2C)))):`Cartan
  Curvature matrix of 2-forms form d[C]+[C]^[C}`:=evalm(CartanCurvature);
```

$C2CxC2C :=$

$$\begin{bmatrix} 0, & d(r)\ \&\hat{}\ d(\theta), & \frac{(\sin(\theta)^2+1-\cos(\theta)^2)\, r\cos(\theta)\,(d(\theta)\ \&\hat{}\ d(\phi))}{\sin(\theta)} + (1-\cos(\theta)^2)\,(d(r)\ \&\hat{}\ d(\phi)), & 0 \\ -\frac{d(\theta)\ \&\hat{}\ d(r)}{r^2}, & 0, & (-1+2\cos(\theta)^2)\,(d(\theta)\ \&\hat{}\ d(\phi)), & 0 \\ -\frac{d(\phi)\ \&\hat{}\ d(r)}{r^2}, & -\frac{(\sin(\theta)^2+\cos(\theta)^2)\,(d(\phi)\ \&\hat{}\ d(\theta))}{\sin(\theta)^2}, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{bmatrix}$$

$$\text{Cartan Curvature matrix of 2-forms form } d[C]+[C]\hat{}[C\} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix of curvature 2-forms (based upon d[C]+[C]^[C]) is the zero matrix
as it should be for the right Cartan connection (also it has a zero trace).

\>

****

NEXT, obtain the Christoffel Connection coefficients from the induced perturbed pullback metric using tensor formulas.

```
> metric:=evalm(pullbackmetric):
> metricinverse:=inverse(metric):dim:=4:coord:=[r,theta,phi,tau]:
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do
  d1gun[i,j,k]  := (diff(metric[i,j],coord[k])) od od od:
> #for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  d1gun[i,j,k]=0 then  else print(`dgun`(i,j,k)=d1gun[i,j,k]) fi od od od;
> for i from 1 to dim do for j from i to dim do for k from 1 to dim do C1S[i,j,k]
  := 0  od od od; for i from 1 to dim do for j from 1 to dim do for k from 1 to
  dim do C1S[i,j,k]  := 1/2*d1gun[i,k,j]+1/2*d1gun[j,k,i]-1/2*d1gun[i,j,k]  od od
  od;
>  for k from 1 to dim do for i from 1 to dim do for j from 1 to dim do ss := 0;
  for m to dim do ss := ss+metricinverse[k,m]*C1S[i,j,m] od; C2S[k,i,j] :=
  simplify(factor(ss),trig) od od od;
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  C2S[i,j,k]=0 then  else print(`Christoffel_Gamma of the second kind
  `(i,j,k)=simplify(subs(-1+cos(theta)^2=-sin(theta)^2,aa^2=0,(C2S[i,j,k])))) fi
  od od od;
```

The non zero Christoffel Connection coefficients 2nd kind for the

perturbed frame and perturbed metric, on the initial space (domain) are:

Gamma2 (i,j,k) index (1,-1,-1)

$$\text{Christoffel\_Gamma of the second kind}(1,1,1) = -2\frac{M}{r(2r+M)}$$

$$\text{Christoffel\_Gamma of the second kind}(1,2,2) = -\frac{(2r-M)r}{2r+M}$$

$$\text{Christoffel\_Gamma of the second kind}(1,3,3) = \frac{(2r-M)(-1+\cos(\theta)^2)r}{2r+M}$$

$$\text{Christoffel\_Gamma of the second kind}(1,4,4) = 64\frac{r^4(2r-M)M}{(2r+M)^7}$$

$$\text{Christoffel\_Gamma of the second kind}(2,1,2) = \frac{2r-M}{r(2r+M)}$$

$$\text{Christoffel\_Gamma of the second kind}(2,2,1) = \frac{2r-M}{r(2r+M)}$$

$$\text{Christoffel\_Gamma of the second kind}(2,3,3) = -\cos(\theta)\sin(\theta)$$

$$\text{Christoffel\_Gamma of the second kind}(3,1,3) = \frac{2r-M}{r(2r+M)}$$

$$\text{Christoffel\_Gamma of the second kind}(3,2,3) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\text{Christoffel\_Gamma of the second kind}(3,3,1) = \frac{2r-M}{r(2r+M)}$$

$$\text{Christoffel\_Gamma of the second kind}(3,3,2) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\text{Christoffel\_Gamma of the second kind}(4,1,4) = 4\frac{M}{4r^2-M^2}$$

$$\text{Christoffel\_Gamma of the second kind}(4,4,1) = 4\frac{M}{4r^2-M^2}$$

```
>
```

Now compute the matrix elements of the matrix of connection 1-forms based upon the Christoffel connection

```
> Gamma11:=C2S[1,1,1]*d(r)+C2S[1,1,2]*d(theta)+C2S[1,1,3]*d(phi)+C2S[1,1,4]*d(tau)
  :
> Gamma12:=C2S[1,2,1]*d(r)+C2S[1,2,2]*d(theta)+C2S[1,2,3]*d(phi)+C2S[1,2,4]*d(tau)
  :
> Gamma13:=C2S[1,3,1]*d(r)+C2S[1,3,2]*d(theta)+C2S[1,3,3]*d(phi)+C2S[1,3,4]*d(tau)
  :
> Gamma14:=C2S[1,4,1]*d(r)+C2S[1,4,2]*d(theta)+C2S[1,4,3]*d(phi)+C2S[1,4,4]*d(tau)
  :
> Gamma21:=C2S[2,1,1]*d(r)+C2S[2,1,2]*d(theta)+C2S[2,1,3]*d(phi)+C2S[2,1,4]*d(tau)
  :
> Gamma22:=C2S[2,2,1]*d(r)+C2S[2,2,2]*d(theta)+C2S[2,2,3]*d(phi)+C2S[2,2,4]*d(tau)
  :
> Gamma23:=C2S[2,3,1]*d(r)+C2S[2,3,2]*d(theta)+C2S[2,3,3]*d(phi)+C2S[2,3,4]*d(tau)
```

```
  :
> Gamma24:=C2S[2,4,1]*d(r)+C2S[2,4,2]*d(theta)+C2S[2,4,3]*d(phi)+C2S[2,4,4]*d(tau)
  :
> Gamma31:=C2S[3,1,1]*d(r)+C2S[3,1,2]*d(theta)+C2S[3,1,3]*d(phi)+C2S[3,1,4]*d(tau)
  :
> Gamma32:=C2S[3,2,1]*d(r)+C2S[3,2,2]*d(theta)+C2S[3,2,3]*d(phi)+C2S[3,2,4]*d(tau)
  :
> Gamma33:=C2S[3,3,1]*d(r)+C2S[3,3,2]*d(theta)+C2S[3,3,3]*d(phi)+C2S[3,3,4]*d(tau)
  :
> Gamma34:=C2S[3,4,1]*d(r)+C2S[3,4,2]*d(theta)+C2S[3,4,3]*d(phi)+C2S[3,4,4]*d(tau)
  :
>
> Gamma41:=C2S[4,1,1]*d(r)+C2S[4,1,2]*d(theta)+C2S[4,1,3]*d(phi)+C2S[4,1,4]*d(tau)
  :
> Gamma42:=C2S[4,2,1]*d(r)+C2S[4,2,2]*d(theta)+C2S[4,2,3]*d(phi)+C2S[4,2,4]*d(tau)
  :
> Gamma43:=C2S[4,3,1]*d(r)+C2S[4,3,2]*d(theta)+C2S[4,3,3]*d(phi)+C2S[4,3,4]*d(tau)
  :
> Gamma44:=C2S[4,4,1]*d(r)+C2S[4,4,2]*d(theta)+C2S[4,4,3]*d(phi)+C2S[4,4,4]*d(tau)
  :
> Christ:=array([[Gamma11,Gamma12,Gamma13,Gamma14],[Gamma21,Gamma22,Gamma23,Gamma2
  4],[Gamma31,Gamma32,Gamma33,Gamma34],[Gamma41,Gamma42,Gamma43,Gamma44]]):`
  Christoffel Connection Matrix [Gamma] of 1-forms `:=eval(Christ);
> `Compare to Cartan Connection Matrix [C] of
  1-forms`:=evalm(Cartan_Connection);TT:=simplify(evalm(evalm(Cartan_Connection)-e
  valm(Christ))):evalm(C2CC2C):
>
> `Compare to Residue Connection Matrix [T] = [C]-[Gamma] of 1-forms`:=evalm(TT);
```

Christoffel Connection Matrix [Gamma] of 1-forms :=

$$\begin{bmatrix} -2\frac{M\,d(r)}{r\,(2\,r+M)} & -\frac{(2\,r-M)\,r\,d(\theta)}{2\,r+M} & \frac{(2\,r-M)\,(-1+\cos(\theta)^2)\,r\,d(\phi)}{2\,r+M} & 64\frac{r^4\,(2\,r-M)\,M\,d(\tau)}{(2\,r+M)^7} \\ \frac{(2\,r-M)\,d(\theta)}{r\,(2\,r+M)} & \frac{d(r)\,(2\,r-M)}{r\,(2\,r+M)} & -d(\phi)\cos(\theta)\sin(\theta) & 0 \\ \frac{(2\,r-M)\,d(\phi)}{r\,(2\,r+M)} & -\frac{\cos(\theta)\sin(\theta)\,d(\phi)}{-1+\cos(\theta)^2} & \frac{d(r)\,(2\,r-M)}{r\,(2\,r+M)}-\frac{\cos(\theta)\sin(\theta)\,d(\theta)}{-1+\cos(\theta)^2} & 0 \\ 4\frac{M\,d(\tau)}{(2\,r-M)\,(2\,r+M)} & 0 & 0 & 4\frac{M\,d(r)}{(2\,r-M)\,(2\,r+M)} \end{bmatrix}$$

Compare to Cartan Connection Matrix [C] of 1-forms :=

$$\begin{bmatrix} -2\frac{M\,d(r)}{r\,(2\,r+M)} & -r\,d(\theta) & d(\phi)\,r\,(-1+\cos(\theta)^2) & 0 \\ \frac{d(\theta)}{r} & \frac{d(r)\,(2\,r-M)}{r\,(2\,r+M)} & -d(\phi)\cos(\theta)\sin(\theta) & 0 \\ \frac{d(\phi)}{r} & \frac{d(\phi)\cos(\theta)}{\sin(\theta)} & \frac{d(r)\,(2\,r-M)}{r\,(2\,r+M)}+\frac{\cos(\theta)\,d(\theta)}{\sin(\theta)} & 0 \\ 0 & 0 & 0 & 4\frac{M\,d(r)}{4\,r^2-M^2} \end{bmatrix}$$

Compare to Residue Connection Matrix [T] = [C]-[Gamma] of 1-forms :=

$$\begin{bmatrix} 0 & -2\frac{r\,\mathrm{d}(\theta)\,M}{2\,r+M} & 2\frac{\mathrm{d}(\phi)\,r\,(-1+\cos(\theta)^2)\,M}{2\,r+M} & -64\frac{r^4\,(2\,r-M)\,M\,\mathrm{d}(\tau)}{(2\,r+M)^7} \\ 2\frac{\mathrm{d}(\theta)\,M}{r\,(2\,r+M)} & 0 & 0 & 0 \\ 2\frac{\mathrm{d}(\phi)\,M}{r\,(2\,r+M)} & 0 & 0 & 0 \\ -4\frac{M\,\mathrm{d}(\tau)}{4\,r^2-M^2} & 0 & 0 & 0 \end{bmatrix}$$

```
> GxT:=simpform(evalm(evalm(TT&^Christ)+evalm(Christ&^TT))):`Interaction 2-forms = [Gamma]^[T]+[T]^[Gamma]`:=evalm(GxT);GxTxGxT:=GxT&^GxT;
>
```

Interaction 2-forms = [Gamma]^[T]+[T]^[Gamma] :=

$$\left[0\,,-2\frac{(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(r))\,M}{2\,r+M}\,,4\frac{r\cos(\theta)\sin(\theta)\,(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(\phi))\,M}{2\,r+M}+\frac{2\,M\,(-1+\cos(\theta)^2)\,(\mathrm{d}(\phi)\ \&\wedge\ \mathrm{d}(r))}{2\,r+M}\,,\right.$$
$$\left.-128\frac{M^2\,r^3\,(4\,r-M)\,(\mathrm{d}(\tau)\ \&\wedge\ \mathrm{d}(r))}{(2\,r+M)^8}\right]$$

$$\left[-2\frac{M\,(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(r))}{r^2\,(2\,r+M)}\,,0\,,4\frac{M\,(-1+\cos(\theta)^2)\,(2\,r-M)\,(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(\phi))}{(2\,r+M)^2}\,,\right.$$
$$\left.-64\frac{M\,r^3\,(2\,r-M)\,(-3\,M+2\,r)\,(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(\tau))}{(2\,r+M)^8}\right]$$

$$\left[-2\frac{M\,(\mathrm{d}(\phi)\ \&\wedge\ \mathrm{d}(r))}{r^2\,(2\,r+M)}\,,-4\frac{M\,(2\,r-M)\,(\mathrm{d}(\phi)\ \&\wedge\ \mathrm{d}(\theta))}{(2\,r+M)^2}\,,0\,,-64\frac{M\,r^3\,(2\,r-M)\,(-3\,M+2\,r)\,(\mathrm{d}(\phi)\ \&\wedge\ \mathrm{d}(\tau))}{(2\,r+M)^8}\right]$$

$$\left[8\frac{M^2\,(4\,r-M)\,(\mathrm{d}(\tau)\ \&\wedge\ \mathrm{d}(r))}{(4\,r^2-M^2)\,r\,(2\,r+M)\,(2\,r-M)}\,,4\frac{r\,M\,(-3\,M+2\,r)\,(\mathrm{d}(\tau)\ \&\wedge\ \mathrm{d}(\theta))}{(2\,r-M)\,(2\,r+M)^2}\,,\right.$$
$$\left.-4\frac{r\,M\,(-1+\cos(\theta)^2)\,(-3\,M+2\,r)\,(\mathrm{d}(\tau)\ \&\wedge\ \mathrm{d}(\phi))}{(2\,r-M)\,(2\,r+M)^2}\,,0\right]$$

$$GxTxGxT := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 32\frac{M^3\,(4\,r-M)\cos(\theta)\sin(\theta)\ \&\wedge(\mathrm{d}(\tau),\mathrm{d}(r),\mathrm{d}(\theta),\mathrm{d}(\phi))}{(4\,r^2-M^2)\,(2\,r+M)^2\,(2\,r-M)} & 0 \end{bmatrix}$$

```
> GxG:=simpform(Christ&^Christ);TxT:=simpform(TT&^TT);CCGT:=simpform(evalm(TxT+GxG+GxT));CCGT&^CCGT;simpform(simplify(evalm(C2CxC2C-CCGT)));
```

GxG :=

$$\left[0\,,\frac{(2\,r-M)\,(\mathrm{d}(r)\ \&\wedge\ \mathrm{d}(\theta))}{2\,r+M}\,,\right.$$
$$2\frac{(2\,r-M)\,r\cos(\theta)\sin(\theta)\,(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(\phi))}{2\,r+M}-\frac{(-1+\cos(\theta)^2)\,(2\,r-M)\,(\mathrm{d}(r)\ \&\wedge\ \mathrm{d}(\phi))}{2\,r+M}\,,$$
$$\left.-128\frac{M^2\,r^3\,(4\,r-M)\,(\mathrm{d}(r)\ \&\wedge\ \mathrm{d}(\tau))}{(2\,r+M)^8}\right]$$

$$\left[-\frac{(2\,r-M)\,(\mathrm{d}(\theta)\ \&\wedge\ \mathrm{d}(r))}{(2\,r+M)\,r^2}\,,0\,,-(-M^2+2\,M^2\cos(\theta)^2-M^2\cos(\theta)^4+4\,r\,M-8\,r\,M\cos(\theta)^2+4\,r\,M\cos(\theta)^4\right.$$
$$-4\,r^2+8\,r^2\cos(\theta)^2-4\,r^2\cos(\theta)^4+4\cos(\theta)^2\sin(\theta)^2\,r^2+4\cos(\theta)^2\sin(\theta)^2\,r\,M+\cos(\theta)^2\sin(\theta)^2\,M^2)$$

$$(d(\theta) \mathbin{\&\wedge} d(\phi)) \Big/ ((2r+M)^2(-1+\cos(\theta)^2)),\ 64\,\frac{(2r-M)^2 r^3 M\,(d(\theta) \mathbin{\&\wedge} d(\tau))}{(2r+M)^8}\Bigg]$$

$$\Bigg[-\frac{(2r-M)\,(d(\phi) \mathbin{\&\wedge} d(r))}{(2r+M)\,r^2},\ -(M^2-2M^2\cos(\theta)^2+M^2\cos(\theta)^4-4rM+8rM\cos(\theta)^2-4rM\cos(\theta)^4+4r^2$$
$$-8r^2\cos(\theta)^2+4r^2\cos(\theta)^4+4\cos(\theta)^2\sin(\theta)^2r^2+4\cos(\theta)^2\sin(\theta)^2rM+\cos(\theta)^2\sin(\theta)^2M^2)\,(d(\phi) \mathbin{\&\wedge} d(\theta))$$
$$\Big/ ((-1+\cos(\theta)^2)^2(2r+M)^2),\ 0,\ 64\,\frac{(2r-M)^2 r^3 M\,(d(\phi) \mathbin{\&\wedge} d(\tau))}{(2r+M)^8}\Bigg]$$

$$\left[-8\,\frac{M^2(4r-M)\,(d(\tau) \mathbin{\&\wedge} d(r))}{(2r-M)^2(2r+M)^2 r},\ -4\,\frac{rM\,(d(\tau) \mathbin{\&\wedge} d(\theta))}{(2r+M)^2},\ 4\,\frac{rM(-1+\cos(\theta)^2)\,(d(\tau) \mathbin{\&\wedge} d(\phi))}{(2r+M)^2},\ 0\right]$$

$TxT :=$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4\,\frac{M^2(-1+\cos(\theta)^2)\,(d(\theta) \mathbin{\&\wedge} d(\phi))}{(2r+M)^2} & -128\,\frac{M^2r^3(2r-M)\,(d(\theta) \mathbin{\&\wedge} d(\tau))}{(2r+M)^8} \\ 0 & -4\,\frac{M^2\,(d(\phi) \mathbin{\&\wedge} d(\theta))}{(2r+M)^2} & 0 & -128\,\frac{M^2r^3(2r-M)\,(d(\phi) \mathbin{\&\wedge} d(\tau))}{(2r+M)^8} \\ 0 & 8\,\frac{M^2 r\,(d(\tau) \mathbin{\&\wedge} d(\theta))}{(4r^2-M^2)(2r+M)} & -8\,\frac{M^2(-1+\cos(\theta)^2)\,r\,(d(\tau) \mathbin{\&\wedge} d(\phi))}{(4r^2-M^2)(2r+M)} & 0 \end{bmatrix}$$

$$CCGT := \begin{bmatrix} 0,\ d(r) \mathbin{\&\wedge} d(\theta),\ 2r\cos(\theta)\sin(\theta)\,(d(\theta) \mathbin{\&\wedge} d(\phi)) + (1-\cos(\theta)^2)\,(d(r) \mathbin{\&\wedge} d(\phi)),\ 0 \\ -\frac{d(\theta) \mathbin{\&\wedge} d(r)}{r^2},\ 0,\ -\frac{(2\cos(\theta)^2-\cos(\theta)^4+\cos(\theta)^2\sin(\theta)^2-1)\,(d(\theta) \mathbin{\&\wedge} d(\phi))}{-1+\cos(\theta)^2},\ 0 \\ -\frac{d(\phi) \mathbin{\&\wedge} d(r)}{r^2},\ -\frac{(-2\cos(\theta)^2+\cos(\theta)^4+\cos(\theta)^2\sin(\theta)^2+1)\,(d(\phi) \mathbin{\&\wedge} d(\theta))}{(-1+\cos(\theta)^2)^2},\ 0,\ 0 \\ 0,\ 0,\ 0,\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
>
> dChrist:=simpform(simplify(evalm(d(Christ)))):
> ChristxChrist:=simpform(simplify(Christ&^Christ)):
> CHCURV:=(simpform(evalm(evalm(dChrist)+evalm(ChristxChrist)))):`PHI -
  Christoffel Matrix of Curvature 2 forms`:=evalm(CHCURV);
  PHIxPHI:=evalm(CHCURV)&^evalm(CHCURV);
```

PHI - Christoffel Matrix of Curvature 2 forms :=

$$\Bigg[0,\ -4\,\frac{rM\,(d(r) \mathbin{\&\wedge} d(\theta))}{(2r+M)^2},\ 4\,\frac{rM(-1+\cos(\theta)^2)\,(d(r) \mathbin{\&\wedge} d(\phi))}{(2r+M)^2},$$
$$-128\,\frac{r^3M(M^2-4rM+4r^2)\,(d(r) \mathbin{\&\wedge} d(\tau))}{(2r+M)^8}\Bigg]$$

$$\left[4\,\frac{M\,(d(r) \mathbin{\&\wedge} d(\theta))}{r(2r+M)^2},\ 0,\ -8\,\frac{rM(-1+\cos(\theta)^2)\,(d(\theta) \mathbin{\&\wedge} d(\phi))}{(2r+M)^2},\ 64\,\frac{(2r-M)^2r^3M\,(d(\theta) \mathbin{\&\wedge} d(\tau))}{(2r+M)^8}\right]$$

$$\left[4\,\frac{M\,(\mathrm{d}(r)\ \&\hat{}\ \mathrm{d}(\phi))}{r\,(2\,r+M)^2},\,-8\,\frac{(\mathrm{d}(\theta)\ \&\hat{}\ \mathrm{d}(\phi))\,r\,M}{(2\,r+M)^2},\,0\,,\,64\,\frac{(2\,r-M)^2\,r^3\,M\,(\mathrm{d}(\phi)\ \&\hat{}\ \mathrm{d}(\tau))}{(2\,r+M)^8}\right]$$

$$\left[-8\,\frac{M\,(\mathrm{d}(r)\ \&\hat{}\ \mathrm{d}(\tau))}{r\,(2\,r+M)^2},\,-4\,\frac{r\,M\,(\mathrm{d}(\tau)\ \&\hat{}\ \mathrm{d}(\theta))}{(2\,r+M)^2},\,4\,\frac{r\,M\,(-1+\cos(\theta)^2)\,(\mathrm{d}(\tau)\ \&\hat{}\ \mathrm{d}(\phi))}{(2\,r+M)^2},\,0\right]$$

$$PHIxPHI := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

\>

\>

\>

```
> dTT:=simpform(simplify(evalm(d(TT)))):
> TTxTT:=simpform(simplify(TT&^TT)):
> TTCURV:=(simpform(evalm(evalm(dTT)+evalm(TTxTT)))):`Residue Matrix of Curvature
  2 forms built on T`:=evalm(TTCURV);TTCURVxTTCURV:=TTCURV&^TTCURV;
```

Residue Matrix of Curvature 2 forms built on T :=

$$\left[0\,,\,-2\,\frac{M^2\,(\mathrm{d}(r)\ \&\hat{}\ \mathrm{d}(\theta))}{(2\,r+M)^2},\,-4\,\frac{r\cos(\theta)\sin(\theta)\,(\mathrm{d}(\theta)\ \&\hat{}\ \mathrm{d}(\phi))\,M}{2\,r+M}+\frac{2\,M^2\,(-1+\cos(\theta)^2)\,(\mathrm{d}(r)\ \&\hat{}\ \mathrm{d}(\phi))}{(2\,r+M)^2},\right.$$
$$\left.256\,\frac{r^3\,M\,(2\,r^2+M^2-4\,r\,M)\,(\mathrm{d}(r)\ \&\hat{}\ \mathrm{d}(\tau))}{(2\,r+M)^8}\right]$$

$$\left[-2\,\frac{M\,(4\,r+M)\,(\mathrm{d}(r)\ \&\hat{}\ \mathrm{d}(\theta))}{r^2\,(2\,r+M)^2},\,0\,,\,4\,\frac{M^2\,(-1+\cos(\theta)^2)\,(\mathrm{d}(\theta)\ \&\hat{}\ \mathrm{d}(\phi))}{(2\,r+M)^2},\right.$$
$$\left.-128\,\frac{M^2\,r^3\,(2\,r-M)\,(\mathrm{d}(\theta)\ \&\hat{}\ \mathrm{d}(\tau))}{(2\,r+M)^8}\right]$$

$$\left[-2\,\frac{M\,(4\,r+M)\,(\mathrm{d}(r)\ \&\hat{}\ \mathrm{d}(\phi))}{r^2\,(2\,r+M)^2},\,-4\,\frac{M^2\,(\mathrm{d}(\phi)\ \&\hat{}\ \mathrm{d}(\theta))}{(2\,r+M)^2},\,0\,,\,-128\,\frac{M^2\,r^3\,(2\,r-M)\,(\mathrm{d}(\phi)\ \&\hat{}\ \mathrm{d}(\tau))}{(2\,r+M)^8}\right]$$

$$\left[32\,\frac{M\,r\,(\mathrm{d}(r)\ \&\hat{}\ \mathrm{d}(\tau))}{(4\,r^2-M^2)^2},\,8\,\frac{M^2\,r\,(\mathrm{d}(\tau)\ \&\hat{}\ \mathrm{d}(\theta))}{(4\,r^2-M^2)\,(2\,r+M)},\,-8\,\frac{M^2\,(-1+\cos(\theta)^2)\,r\,(\mathrm{d}(\tau)\ \&\hat{}\ \mathrm{d}(\phi))}{(4\,r^2-M^2)\,(2\,r+M)},\,0\right]$$

$$TTCURVxTTCURV := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -128\,\frac{M^2\,r^2\cos(\theta)\sin(\theta)\ \&\hat{}(\mathrm{d}(r),\,\mathrm{d}(\tau),\,\mathrm{d}(\theta),\,\mathrm{d}(\phi))}{(4\,r^2-M^2)^2\,(2\,r+M)} & 0 \end{bmatrix}$$

```
> `CheckSUM for 2-forms  {[d{Gamma]+[Gamma]^[Gamma]}  +
  {[Gamma]^[T]+[T]^[Gamma]}  +  {d[T]
  +[T]^[T]}`:=simpform(evalm(evalm(TTCURV)+evalm(GxT)+evalm(CHCURV)));
```

*CheckSUM for 2-forms {[d{Gamma]+[Gamma]^[Gamma]} + {[Gamma]^[T]+[T]^[Gamma]} + {d[T] +[T]^[T *
]} :=
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the sum of the Riemann curvature (2-forms) based on d{Gamma) +Gamma^Gamma,

and the residue curvature 1-forms based on d(T)+T^T plus the interaction 2-forms Gamma^T+T^Gamma indeed balance to zero as they should.

>

>

The Trace of the Metric curvature is Zero !
The Trace of the Cartan Curvature is Zero!
(Hence the Trace of the Inertial terms must be Zero.

NEXT Compute the CURRENT and Interaction 3-Forms

```
> J1CH:=simpform(evalm(CHCURV)&^evalm(Christ)):
> J2CH:=simpform(evalm(Christ)&^evalm(CHCURV)):`Positve Christoffel Current 3-form .. J1CH `:=evalm(J1CH);`Negative Christoffel Current 3-form .. J2CH `:=evalm(J2CH);
> NETJCH:=simpform(subs(M=M,evalm(evalm(J1CH)-evalm(J2CH))));
>
```

Positve Christoffel Current 3-form .. J1CH :=

$$\left[0, 0, 8\frac{r M \cos(\theta)\sin(\theta) \&\hat{}(d(r), d(\theta), d(\phi))}{(2r+M)^2}, 0\right]$$

$$\left[0, 0, -4\frac{M(-1+\cos(\theta)^2)(2r-M) \&\hat{}(d(r), d(\theta), d(\phi))}{(2r+M)^3}, 512\frac{(2r-M) r^3 M^2 \&\hat{}(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 4\frac{M(2r-M) \&\hat{}(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0, 512\frac{(2r-M) r^3 M^2 \&\hat{}(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 4\frac{M(2r-M) \&\hat{}(d(r), d(\tau), d(\theta))}{(2r+M)^3}, 8\frac{r M \cos(\theta)\sin(\theta) \&\hat{}(d(\tau), d(\theta), d(\phi))}{(2r+M)^2} - \frac{4M(-2r+M+2r\cos(\theta)^2 - M\cos(\theta)^2) \&\hat{}(d(r), d(\tau), d(\phi))}{(2r+M)^3}, 0\right]$$

Negative Christoffel Current 3-form .. J2CH :=

$$[0, 0, 0, 0]$$

$$\left[0, 0, 12\frac{M(-1+\cos(\theta)^2)(2r-M) \&\hat{}(d(\theta), d(r), d(\phi))}{(2r+M)^3}, -192\frac{(2r-M) r^2 M (M^2-4rM+4r^2) \&\hat{}(d(\theta), d(r), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, -12\frac{M(2r-M)\,\&\hat{}(d(\phi), d(r), d(\theta))}{(2r+M)^3}, 0, -192\frac{(2r-M)r^2 M(M^2-4rM+4r^2)\,\&\hat{}(d(\phi), d(r), d(\tau))}{(2r+M)^9}\right]$$

$$[0, 0, 0, 0]$$

NETJCH :=

$$\left[0, 0, 8\frac{r M\cos(\theta)\sin(\theta)\,\&\hat{}(d(r), d(\theta), d(\phi))}{(2r+M)^2}, 0\right]$$

$$\left[0, 0, 8\frac{M(-1+\cos(\theta)^2)(2r-M)\,\&\hat{}(d(r), d(\theta), d(\phi))}{(2r+M)^3}, -64\frac{(2r-M)r^2 M(-20rM+3M^2+12r^2)\,\&\hat{}(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, -8\frac{M(2r-M)\,\&\hat{}(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0, -64\frac{(2r-M)r^2 M(-20rM+3M^2+12r^2)\,\&\hat{}(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 4\frac{M(2r-M)\,\&\hat{}(d(r), d(\tau), d(\theta))}{(2r+M)^3}, 8\frac{r M\cos(\theta)\sin(\theta)\,\&\hat{}(d(\tau), d(\theta), d(\phi))}{(2r+M)^2} - \frac{4M(-2r+M+2r\cos(\theta)^2-M\cos(\theta)^2)\,\&\hat{}(d(r), d(\tau), d(\phi))}{(2r+M)^3}, 0\right]$$

```
> J1T:=simpform(evalm(TTCURV)&^evalm(TT)):`J1T  - Positive Current due to
  [T]`:=evalm(J1T);
> J2T:=simpform(evalm(TT)&^evalm(TTCURV)):`J2T  - Negative Current due to
  [T]`:=evalm(J1I);NETJT:=simpform(subs(evalm(evalm(J1T)-evalm(J2T))));NETCH:=simp
  form(subs(evalm(evalm(J1CH)-evalm(J2CH))));
```

J1T - Positive Current due to [T] :=

$$[0, 0, 0, 0]$$

$$\left[0, 0, -4\frac{M^2(4r+M)(-1+\cos(\theta)^2)\,\&\hat{}(d(r), d(\theta), d(\phi))}{r(2r+M)^3}, 128\frac{M^2(4r+M)r^2(2r-M)\,\&\hat{}(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 4\frac{M^2(4r+M)\,\&\hat{}(d(r), d(\phi), d(\theta))}{r(2r+M)^3}, 0, 128\frac{M^2(4r+M)r^2(2r-M)\,\&\hat{}(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, -64\frac{M^2 r^2\,\&\hat{}(d(r), d(\tau), d(\theta))}{(4r^2-M^2)^2(2r+M)}, 64\frac{M^2 r^2(-1+\cos(\theta)^2)\,\&\hat{}(d(r), d(\tau), d(\phi))}{(4r^2-M^2)^2(2r+M)}, 0\right]$$

J2T - Negative Current due to [T] := *J1I*

NETJT :=

$$[0, 0, 0, 0]$$

$$\left[0, 0, -16\frac{M^2(-1+\cos(\theta)^2)\,\&\hat{}(d(r), d(\theta), d(\phi))}{(2r+M)^3}, 128\frac{r^2 M^2(16r^2-18rM+3M^2)\,\&\hat{}(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 16\frac{M^2\,\&\hat{}(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0, 128\frac{r^2 M^2(16r^2-18rM+3M^2)\,\&\hat{}(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, -8\frac{M^2(8r^2-2rM+M^2)\,\&\hat{}(d(r), d(\tau), d(\theta))}{(2r+M)^2(2r-M)(4r^2-M^2)},\right.$$

$$-16\,\frac{M^2\, r\cos(\theta)\sin(\theta)\ \&\hat{}(d(\tau), d(\theta), d(\phi))}{(2r+M)^2\,(2r-M)} + \frac{8\,(8r^2-2rM+M^2)\,M^2\,(-1+\cos(\theta)^2)\ \&\hat{}(d(r), d(\tau), d(\phi))}{(2r+M)^2\,(2r-M)\,(4r^2-M^2)},\ 0\Bigg]$$

NETCH :=

$$\left[0,\ 0,\ 8\,\frac{r M\cos(\theta)\sin(\theta)\ \&\hat{}(d(r), d(\theta), d(\phi))}{(2r+M)^2},\ 0\right]$$

$$\Bigg[0,\ 0,\ 8\,\frac{M\,(-1+\cos(\theta)^2)\,(2r-M)\ \&\hat{}(d(r), d(\theta), d(\phi))}{(2r+M)^3},$$

$$-64\,\frac{(2r-M)\,r^2 M\,(-20rM+3M^2+12r^2)\ \&\hat{}(d(r), d(\theta), d(\tau))}{(2r+M)^9}\Bigg]$$

$$\left[0,\ -8\,\frac{M\,(2r-M)\ \&\hat{}(d(r), d(\phi), d(\theta))}{(2r+M)^3},\ 0,\ -64\,\frac{(2r-M)\,r^2 M\,(-20rM+3M^2+12r^2)\ \&\hat{}(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right.$$

$$\Bigg]$$

$$\Bigg[0,\ 4\,\frac{M\,(2r-M)\ \&\hat{}(d(r), d(\tau), d(\theta))}{(2r+M)^3},$$

$$8\,\frac{r M\cos(\theta)\sin(\theta)\ \&\hat{}(d(\tau), d(\theta), d(\phi))}{(2r+M)^2} - \frac{4M\,(-2r+M+2r\cos(\theta)^2-M\cos(\theta)^2)\ \&\hat{}(d(r), d(\tau), d(\phi))}{(2r+M)^3},\ 0\Bigg]$$

```
> 
> J1T:=simpform(evalm(TTCURV)&^evalm(TT));
> J2T:=simpform(evalm(TT)&^evalm(TTCURV));NETJT:=simpform(subs(evalm(evalm(J1T)-evalm(J2T))));NETCH:=simpform(subs(evalm(evalm(J1CH)-evalm(J2CH))));
> 
```

J1T :=

$$[0,\ 0,\ 0,\ 0]$$

$$\Bigg[0,\ 0,\ -4\,\frac{M^2\,(4r+M)\,(-1+\cos(\theta)^2)\ \&\hat{}(d(r), d(\theta), d(\phi))}{r\,(2r+M)^3},$$

$$128\,\frac{M^2\,(4r+M)\,r^2\,(2r-M)\ \&\hat{}(d(r), d(\theta), d(\tau))}{(2r+M)^9}\Bigg]$$

$$\left[0,\ 4\,\frac{M^2\,(4r+M)\ \&\hat{}(d(r), d(\phi), d(\theta))}{r\,(2r+M)^3},\ 0,\ 128\,\frac{M^2\,(4r+M)\,r^2\,(2r-M)\ \&\hat{}(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0,\ -64\,\frac{M^2 r^2\ \&\hat{}(d(r), d(\tau), d(\theta))}{(4r^2-M^2)^2\,(2r+M)},\ 64\,\frac{M^2 r^2\,(-1+\cos(\theta)^2)\ \&\hat{}(d(r), d(\tau), d(\phi))}{(4r^2-M^2)^2\,(2r+M)},\ 0\right]$$

J2T :=

$$[0,\ 0,\ 0,\ 0]$$

$$\left[0,\ 0,\ 4\,\frac{M^3\,(-1+\cos(\theta)^2)\ \&\hat{}(d(\theta), d(r), d(\phi))}{r\,(2r+M)^3},\ 512\,\frac{M^2 r^2\,(2r^2+M^2-4rM)\ \&\hat{}(d(\theta), d(r), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0,\ -4\,\frac{M^3\ \&\hat{}(d(\phi), d(r), d(\theta))}{(2r+M)^3\, r},\ 0,\ 512\,\frac{M^2 r^2\,(2r^2+M^2-4rM)\ \&\hat{}(d(\phi), d(r), d(\tau))}{(2r+M)^9}\right]$$

$$\Bigg[0,\ 8\,\frac{M^3\ \&\hat{}(d(\tau), d(r), d(\theta))}{(4r^2-M^2)\,(2r+M)^2},$$

$$16\,\frac{M^2\, r\cos(\theta)\sin(\theta)\ \&\hat{}(d(\tau), d(\theta), d(\phi))}{(2r+M)^2\,(2r-M)} - \frac{8M^3\,(-1+\cos(\theta)^2)\ \&\hat{}(d(\tau), d(r), d(\phi))}{(4r^2-M^2)\,(2r+M)^2},\ 0\Bigg]$$

NETJT :=

$$[0, 0, 0, 0]$$

$$\left[0, 0, -16\frac{M^2(-1+\cos(\theta)^2)\ \&\verb|^|(d(r), d(\theta), d(\phi))}{(2r+M)^3}, 128\frac{r^2 M^2(16r^2-18rM+3M^2)\ \&\verb|^|(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 16\frac{M^2\ \&\verb|^|(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0, 128\frac{r^2 M^2(16r^2-18rM+3M^2)\ \&\verb|^|(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, -8\frac{M^2(8r^2-2rM+M^2)\ \&\verb|^|(d(r), d(\tau), d(\theta))}{(2r+M)^2(2r-M)(4r^2-M^2)},\right.$$
$$\left.-16\frac{M^2 r\cos(\theta)\sin(\theta)\ \&\verb|^|(d(\tau), d(\theta), d(\phi))}{(2r+M)^2(2r-M)}+\frac{8(8r^2-2rM+M^2)M^2(-1+\cos(\theta)^2)\ \&\verb|^|(d(r), d(\tau), d(\phi))}{(2r+M)^2(2r-M)(4r^2-M^2)}, 0\right]$$

NETCH :=

$$\left[0, 0, 8\frac{rM\cos(\theta)\sin(\theta)\ \&\verb|^|(d(r), d(\theta), d(\phi))}{(2r+M)^2}, 0\right]$$

$$\left[0, 0, 8\frac{M(-1+\cos(\theta)^2)(2r-M)\ \&\verb|^|(d(r), d(\theta), d(\phi))}{(2r+M)^3},\right.$$
$$\left.-64\frac{(2r-M)r^2 M(-20rM+3M^2+12r^2)\ \&\verb|^|(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, -8\frac{M(2r-M)\ \&\verb|^|(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0, -64\frac{(2r-M)r^2 M(-20rM+3M^2+12r^2)\ \&\verb|^|(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right.$$
$$\left.\right]$$

$$\left[0, 4\frac{M(2r-M)\ \&\verb|^|(d(r), d(\tau), d(\theta))}{(2r+M)^3},\right.$$
$$\left.8\frac{rM\cos(\theta)\sin(\theta)\ \&\verb|^|(d(\tau), d(\theta), d(\phi))}{(2r+M)^2}-\frac{4M(-2r+M+2r\cos(\theta)^2-M\cos(\theta)^2)\ \&\verb|^|(d(r), d(\tau), d(\phi))}{(2r+M)^3}, 0\right]$$

```
> NETCHT:=simpform(evalm(evalm(NETCH)+evalm(NETJT)));
```

NETCHT :=

$$\left[0, 0, 8\frac{rM\cos(\theta)\sin(\theta)\ \&\verb|^|(d(r), d(\theta), d(\phi))}{(2r+M)^2}, 0\right]$$

$$\left[0, 0, 8\frac{M(-1+\cos(\theta)^2)(-3M+2r)\ \&\verb|^|(d(r), d(\theta), d(\phi))}{(2r+M)^3},\right.$$
$$\left.-64\frac{r^2 M(-84r^2 M+62rM^2+24r^3-9M^3)\ \&\verb|^|(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, -8\frac{M(-3M+2r)\ \&\verb|^|(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0,\right.$$
$$\left.-64\frac{r^2 M(-84r^2 M+62rM^2+24r^3-9M^3)\ \&\verb|^|(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 4\frac{(8r^3-28r^2 M+10rM^2-3M^3)M\ \&\verb|^|(d(r), d(\tau), d(\theta))}{(2r+M)^2(2r-M)(4r^2-M^2)},\right.$$
$$8\frac{r\cos(\theta)\sin(\theta)M(-3M+2r)\ \&\verb|^|(d(\tau), d(\theta), d(\phi))}{(2r-M)(2r+M)^2}-4$$
$$(-8r^3+8r^3\cos(\theta)^2+28r^2 M-28r^2\cos(\theta)^2 M-10rM^2+10r\cos(\theta)^2 M^2+3M^3-3M^3\cos(\theta)^2)M$$

$\&\wedge(d(r), d(\tau), d(\phi)) \Big/ ((2r+M)^2 (2r-M)(4r^2-M^2)), 0\Big]$

```
> J1Int:=simpform(d(evalm(evalm(TT)&^evalm(Christ))));J2Int:=simpform(d(evalm(eval
m(Christ)&^evalm(TT))));NETInt:=simpform(evalm(J1Int+J2Int));
```

J1Int :=

$$\left[0, 0, -8\frac{\cos(\theta)\sin(\theta) M(-M+r)\,\&\wedge(d(r), d(\theta), d(\phi))}{(2r+M)^2}, 0\right]$$

$$\left[0, 0, -4\frac{M(-1+\cos(\theta)^2)(-3M+2r)\,\&\wedge(d(r), d(\theta), d(\phi))}{(2r+M)^3},\right.$$
$$\left.-128\frac{r^2 M^2 (16r^2-18rM+3M^2)\,\&\wedge(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 4\frac{M(-3M+2r)\,\&\wedge(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0, -128\frac{r^2 M^2 (16r^2-18rM+3M^2)\,\&\wedge(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, -4\frac{M(2r-M)\,\&\wedge(d(r), d(\tau), d(\theta))}{(2r+M)^3},\right.$$
$$\left.8\frac{rM\cos(\theta)\sin(\theta)\,\&\wedge(d(\theta), d(\tau), d(\phi))}{(2r+M)^2} + \frac{4M(-1+\cos(\theta)^2)(2r-M)\,\&\wedge(d(r), d(\tau), d(\phi))}{(2r+M)^3}, 0\right]$$

J2Int :=

$$\left[0, 0, 8\frac{M^2\cos(\theta)\sin(\theta)\,\&\wedge(d(\theta), d(r), d(\phi))}{(2r+M)^2}, 0\right]$$

$$\left[0, 0, -4\frac{M(-1+\cos(\theta)^2)(-3M+2r)\,\&\wedge(d(r), d(\theta), d(\phi))}{(2r+M)^3},\right.$$
$$\left.64\frac{(2r-M) r^2 M(-20rM+3M^2+12r^2)\,\&\wedge(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 4\frac{M(-3M+2r)\,\&\wedge(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0, 64\frac{(2r-M) r^2 M(-20rM+3M^2+12r^2)\,\&\wedge(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 8\frac{M^2(8r^2-2rM+M^2)\,\&\wedge(d(r), d(\tau), d(\theta))}{(2r-M)^2(2r+M)^3},\right.$$
$$\left.-16\frac{M^2 r\cos(\theta)\sin(\theta)\,\&\wedge(d(\theta), d(\tau), d(\phi))}{(2r-M)(2r+M)^2} - \frac{8(-1+\cos(\theta)^2) M^2(8r^2-2rM+M^2)\,\&\wedge(d(r), d(\tau), d(\phi))}{(2r-M)^2(2r+M)^3}, 0\right]$$

NETInt :=

$$\left[0, 0, -8\frac{rM\cos(\theta)\sin(\theta)\,\&\wedge(d(r), d(\theta), d(\phi))}{(2r+M)^2}, 0\right]$$

$$\left[0, 0, -8\frac{M(-1+\cos(\theta)^2)(-3M+2r)\,\&\wedge(d(r), d(\theta), d(\phi))}{(2r+M)^3},\right.$$
$$\left.64\frac{r^2 M(-84r^2M+62rM^2+24r^3-9M^3)\,\&\wedge(d(r), d(\theta), d(\tau))}{(2r+M)^9}\right]$$

$$\left[0, 8\frac{M(-3M+2r)\,\&\wedge(d(r), d(\phi), d(\theta))}{(2r+M)^3}, 0, 64\frac{r^2 M(-84r^2M+62rM^2+24r^3-9M^3)\,\&\wedge(d(r), d(\phi), d(\tau))}{(2r+M)^9}\right.$$

$$\left[0, -4\frac{M(8r^3-28r^2M+10rM^2-3M^3)\ \&\text{^}(d(r), d(\tau), d(\theta))}{(2r-M)^2(2r+M)^3},\right.$$

$$8\frac{rM\cos(\theta)\sin(\theta)(-3M+2r)\ \&\text{^}(d(\theta), d(\tau), d(\phi))}{(2r-M)(2r+M)^2}$$

$$\left.+\frac{4M(-1+\cos(\theta)^2)(8r^3-28r^2M+10rM^2-3M^3)\ \&\text{^}(d(r), d(\tau), d(\phi))}{(2r-M)^2(2r+M)^3}, 0\right]$$

```
> `CheckSUM of the
  3-forms`:=simpform(evalm(evalm(NETInt)+evalm(NETJT)+evalm(NETCH)));
```

$$\mathit{CheckSUM\ of\ the\ 3\text{-}forms} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the Three forms balance as they should.

```
>
```

Compute the T(i,j,k): These coefficients - if not zero - indicate the effects of the pertubations on the metric and Basis. If there is no difference between the Christoffel symbols and the Cartan connection symbols, then the T(i,j,k) are zero.

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (C2C[i,j,k]-C2S[i,k,j]); SHIPTR[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  SHIPTR[i,j,k]=0 then else print(`Residue [T]
  connection`(i,k,j)=factor(SHIPTR[i,k,j])) fi od od od ;
>
```

$$\text{Residue [T] connection}(1, 2, 2) = -2\frac{rM}{2r+M}$$

$$\text{Residue [T] connection}(1, 3, 3) = 2\frac{r(\cos(\theta)-1)(\cos(\theta)+1)M}{2r+M}$$

$$\text{Residue [T] connection}(1, 4, 4) = -64\frac{r^4(2r-M)M}{(2r+M)^7}$$

$$\text{Residue [T] connection}(2, 2, 1) = 0$$

$$\text{Residue [T] connection}(3, 3, 1) = 0$$

$$\text{Residue [T] connection}(3, 3, 2) = \frac{\cos(\theta)(-1+\cos(\theta)^2+\sin(\theta)^2)}{\sin(\theta)(\cos(\theta)-1)(\cos(\theta)+1)}$$

$$\text{Residue [T] connection}(3, 2, 3) = \frac{\cos(\theta)\,(-1 + \cos(\theta)^2 + \sin(\theta)^2)}{\sin(\theta)\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)}$$

$$\text{Residue [T] connection}(4, 4, 1) = 0$$

$$\text{Residue [T] connection}(4, 1, 4) = -4\,\frac{M}{(2\,r - M)\,(2\,r + M)}$$

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (SHIPTR[i,j,k]-SHIPTR[i,k,j])/2; TTTOR[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  TTTOR[i,j,k]=0 then else print(`[T] AffineTorsion`(i,k,j)=factor(TTTOR[i,k,j]))
  fi od od od ;
```

$$\text{[T] AffineTorsion}(2, 2, 1) = -\frac{M}{r\,(2\,r + M)}$$

$$\text{[T] AffineTorsion}(2, 1, 2) = \frac{M}{r\,(2\,r + M)}$$

$$\text{[T] AffineTorsion}(3, 3, 1) = -\frac{M}{r\,(2\,r + M)}$$

$$\text{[T] AffineTorsion}(3, 1, 3) = \frac{M}{r\,(2\,r + M)}$$

$$\text{[T] AffineTorsion}(4, 4, 1) = 2\,\frac{M}{(2\,r - M)\,(2\,r + M)}$$

$$\text{[T] AffineTorsion}(4, 1, 4) = -2\,\frac{M}{(2\,r - M)\,(2\,r + M)}$$

```
> for j from 1 to dim do for i from 1 to dim do for k from 1 to dim do  ss :=
  (C2C[i,j,k]-C2C[i,k,j])/2; CCTTS[i,j,k]:=ss od od od  ;
>
> for i from 1 to dim do for j from 1 to dim do for k from 1 to dim do if
  CCTTS[i,j,k]=0 then else print(`[C]_AffineTorsion`(i,k,j)=factor(CCTTS[i,k,j]))
  fi od od od ;
```

$$\text{[C]\_AffineTorsion}(2, 2, 1) = -\frac{M}{r\,(2\,r + M)}$$

$$\text{[C]\_AffineTorsion}(2, 1, 2) = \frac{M}{r\,(2\,r + M)}$$

$$\text{[C]\_AffineTorsion}(3, 3, 1) = -\frac{M}{r\,(2\,r + M)}$$

$$\text{[C]\_AffineTorsion}(3, 1, 3) = \frac{M}{r\,(2\,r + M)}$$

$$\text{[C]\_AffineTorsion}(4, 4, 1) = 2\,\frac{M}{(2\,r - M)\,(2\,r + M)}$$

$$\text{[C]\_AffineTorsion}(4, 1, 4) = -2\,\frac{M}{(2\,r - M)\,(2\,r + M)}$$

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Affine Torsion of [C] = Affine Torsion of [T]
Affine Torsion of [Gamma] = 0.
The Frame Field pertubation, that allows the Metric field with non zero curvatures to be generated from a quadratic congruence,
produces Affine torsion. However, as the 1-forms |sigma> are integrable, another algebraic choice of a Frame field can make the Affine Torsion disappear. Irreducible Affine torsion occurs only when the 1-forms |sigma> are NOT integrable.
Then the Topological Torsion at least one of the 1-forms |sigma> is not zero, and at least one of the 1-forms |sigma> is of Pfaff dimension 3 or higher. To me, this immediately implies that the system being modeled by the Basis Frame [F] is not in thermodynamic equilibrium.

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