```
> restart:with(linalg):
> with(plots):with(liesymm): setup(u,v):
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the name changecoords has been redefined

Warning, the protected name close has been redefined and unprotected

>
```

SPACELIKE ZERO MEAN CURVATURE SURFACES in Lorentz Spaces
ARE "MAXIMAL" SURFACES

MAXLOR.ms
R.M. Kiehn
Updated 5/25/2005 from minlor.mws which was started on Oct 27, 1999
In 1988, at the conference on Geometry and Topology of Submanifolds (World Scientific)
I. Van de Woestijne presented the theory of minimal surfaces in Minkowski 3 space. An immersive map from u,v into a space with a metric of +1,+1,-1. The immersion induces a metric on the 2 surface (u,v) by means of the pullback. From this metric and the shape matrix constructed from this metric, it is possible to construct the mean curvature (trace of shape matrix) and the Gauss curvature (det of shape matrix).
The minimal surface is defined as a surface of zero mean curvature. In Euclidean space with a Euclidean metric, the minimal surface Gauss curvature is negative or zero. However, in Lorentz space (with a Lorentz metric) the curvature can be dominated by the Minkowski (time like) coordinate, or by the spacelike coordinates. This result implies that the spatial features of a minimal surface relative to a Minkowski metric can give some surprises. (They do not look like saddles)
Van de Woestijne classified the Lorentzian minimal surfaces as spacelike (when Q= EE*GG-FF*FF >0) or or timelike, (when Q =EE*GG-FF*FF <0). Q = det(induced metric) The terms in the expression are computed using the Minkowski metric to form the number EE, from the vector E, etc.
For a euclidean metric there is only one minimal surface of revolution - the right catenoid. However, for a Minkowski metric, the are several examples of minimal surfaces of revolution. One looks like a beaded structure along the timelike axis.
The work can be extended to immersions into 4D. Now (according to Sophus Lie) every holomorphic curve generates a minimal surface. I will extend the previous work in 4D minimal surfaces later.
**
Herein some of the examples given by Van de Woestijne are given, where the computations are based upon the induced metric. Then the Minkoiwski space is mapped into the complex euclidean space, and the standard Cartan formulas are used to compute the Shape matrix, and the mean curvature.
The formulas can be used to determine Euclidean (Minsurf) or Lorentzian (Maxsurf) surfaces.

```
>  Minsurf:=proc(R)
       local Yu,Yv,NN,magn,NNU,FFF,DET,EE,FF,GG,Yuu,Yvv,Yuv,b11,b12,b22:

       global GUN,Q,H,K,gun:
        GUN:=array([[1,0,0],[0,1,0],[0,0,1]]);
        Yu:=diff(R,u):
        Yv:=diff(R,v):
        NN:=evalm(simplify(crossprod(Yu,Yv))):
        magn:=(factor(simplify(innerprod(NN,GUN,NN))^(1/2))):
        NNU:=simplify(evalm(NN/magn)):
```

```
      FFF:=transpose(array([Yu,Yv])):
      gun:=simplify(innerprod(transpose(FFF),GUN,(FFF))):DET:=det(gun):
      gun:=evalm(gun):
      EE:=gun[1,1]:
      FF:=gun[1,2]:
      GG:=gun[2,2]:
      Q:=EE*GG-FF*FF:
      Yuu:=diff(Yu,u):b11:=simplify(innerprod(Yuu,GUN,NNU)):
       Yuv:=diff(Yu,v):b12:=simplify(innerprod(Yuv,GUN,NNU)):
       Yvv:=diff(Yv,v):b22:=simplify(innerprod(Yvv,GUN,NNU)):
       H:=factor(simplify(gun[2,2]*b11+gun[1,1]*b22)-2*gun[1,2]*b12)
/(2*det(gun)):
    K:=simplify((b11*b22-b12*b12))/det(gun):print(`Metric` =
evalm(GUN));print(`Immersion R`=R);print(`Induced
metric`=evalm(gun));print(`Metric Det Q`=simplify(Q));print(`Mean
Curvature`=H);print(`Gauss Curvature`=simplify(K));
print(`B11`=b11);print(`B22`=b22);print(`B12`=b12);end:

 MAXsurf:=proc(R)
     local Yu,Yv,NN,magn,NNU,FFF,DET,EE,FF,GG,Yuu,Yvv,Yuv,b11,b12,b22:

     global GUN,Q,H,K,gun:
      GUN:=array([[1,0,0],[0,1,0],[0,0,-1]]);
      Yu:=diff(R,u):
      Yv:=diff(R,v):
      NN:=evalm(simplify(crossprod(Yu,Yv))):
      magn:=(factor(simplify(innerprod(NN,GUN,NN))^(1/2))):
      NNU:=simplify(evalm(NN/magn)):
      FFF:=transpose(array([Yu,Yv])):
      gun:=simplify(innerprod(transpose(FFF),GUN,(FFF))):DET:=det(gun):
      gun:=evalm(gun):
      EE:=gun[1,1]:
      FF:=gun[1,2]:
      GG:=gun[2,2]:
      Q:=EE*GG-FF*FF:
      Yuu:=diff(Yu,u):b11:=simplify(innerprod(Yuu,GUN,NNU)):
       Yuv:=diff(Yu,v):b12:=simplify(innerprod(Yuv,GUN,NNU)):
       Yvv:=diff(Yv,v):b22:=simplify(innerprod(Yvv,GUN,NNU)):
       H:=factor(simplify(gun[2,2]*b11+gun[1,1]*b22)-2*gun[1,2]*b12)
/(2*det(gun)):
     K:=simplify((b11*b22-b12*b12))/det(gun):print(`Metric` =
evalm(GUN));print(`Immersion R`=R);print(`Induced
metric`=evalm(gun));print(`Metric Det Q`=simplify(Q));print(`Mean
Curvature`=H);print(`Gauss Curvature`=simplify(K));
print(`B11`=b11);print(`B22`=b22);print(`B12`=b12);end:
```

EXAMPLES of LORENTZIAN zero mean curvature Surfaces.

Example 1: A timelike (Q < 0) Lorentzian surface of revolution about the Z axis.

Specify Immersion:

```
> R:=[cos(v)*cos(u),cos(v)*sin(u),v]:
> MAXsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Immersion\ R = [\cos(v)\cos(u), \cos(v)\sin(u), v]$$

$$Induced\ metric = \begin{bmatrix} \cos(v)^2 & 0 \\ 0 & -\cos(v)^2 \end{bmatrix}$$

$$Metric\ Det\ Q = -\cos(v)^4$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = -\frac{1}{\cos(v)^4}$$

$$B11 = -\mathrm{csgn}(\cos(v)^2)$$

$$B22 = -\mathrm{csgn}(\cos(v)^2)$$

$$B12 = 0$$

```
> R1:=subs(a=1,b=0,e=1,R);R2:=subs(a=1,b=0,e=0,R);R3:=subs(a=1,b=0,e=-1,R);
> subs(e^2=1,K);
```

$$R1 := [\cos(v)\cos(u), \cos(v)\sin(u), v]$$

$$R2 := [\cos(v)\cos(u), \cos(v)\sin(u), v]$$

$$R3 := [\cos(v)\cos(u), \cos(v)\sin(u), v]$$

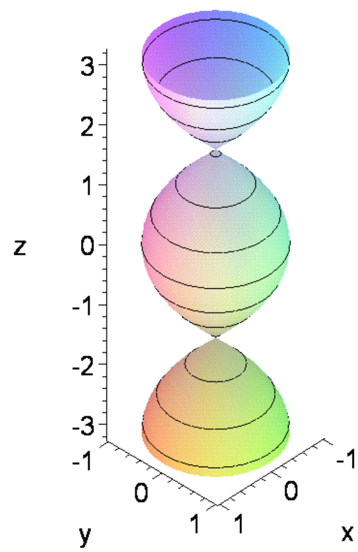
$$-\frac{1}{\cos(v)^4}$$

```
>
> plot3d({R1,R2,R3},u=-1*Pi..Pi,v=-Pi..Pi,orientation=[134,86],numpoints=1000,style=PATCHCONTOUR,title=` 3D Minkowski TIMELIKE zero mean curvature, \n Beaded surface,\n Gauss Curvature = -a^2/(cos(av+b)^4 < 0`,axes=FRAMED,scaling=CONSTRAINED,orientation=[-26,70],labels=[`x`,`y`,`z`]);
```

3D Minkowski TIMELIKE zero mean curvature,

Beaded surface,

Gauss Curvature = -a^2/(cos(av+b)^4 < 0



THe rotation in the display has been limited to Pi.

Example 1E: A Euclidean Ruled Minimal Surface (Q < 0), the Helix about the Z axis.

Specify Immersion:

```
> R:=[1/a*cos(a*v+b)*cos(u),1/a*cos(a*v+b)*sin(u),e*u]:
> Minsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Immersion\ R = \left[\frac{\cos(a\,v+b)\cos(u)}{a}, \frac{\cos(a\,v+b)\sin(u)}{a}, e\,u \right]$$

$$Induced\ metric = \begin{bmatrix} \frac{e^2\,a^2 + \cos(a\,v+b)^2}{a^2} & 0 \\ 0 & 1 - \cos(a\,v+b)^2 \end{bmatrix}$$

$$Metric\ Det\ Q = -\frac{-e^2 a^2 - \cos(a v + b)^2 + \cos(a v + b)^2 e^2 a^2 + \cos(a v + b)^4}{a^2}$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = -\frac{e^2 a^4}{e^4 a^4 + 2\cos(a v + b)^2 e^2 a^2 + \cos(a v + b)^4}$$

$$B11 = 0$$

$$B22 = 0$$

$$B12 = -\frac{(-1 + \cos(a v + b)^2)\, e}{\sqrt{-\frac{-e^2 a^2 - \cos(a v + b)^2 + \cos(a v + b)^2 e^2 a^2 + \cos(a v + b)^4}{a^2}}}$$

```
> R1:=subs(a=1,b=0,e=1,R);R3:=subs(a=1,b=0,e=-1,R);
> factor(subs(e^2=1,K));
```

$$R1 := [\cos(v)\cos(u), \cos(v)\sin(u), u]$$

$$R3 := [\cos(v)\cos(u), \cos(v)\sin(u), -u]$$

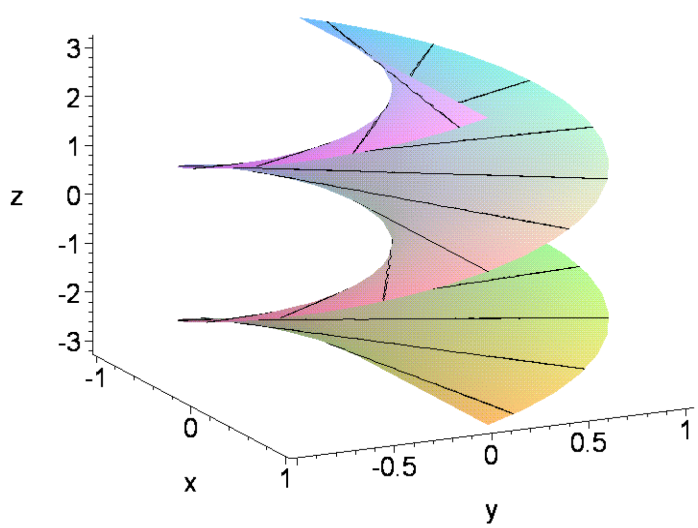
$$-\frac{a^4}{(a^2 + \cos(a v + b)^2)^2}$$

```
>
> plot3d({R1},u=-Pi..Pi,v=-Pi..Pi,orientation=[134,86],numpoints=1000,style=PATCHCONTOUR,title=` 3D Euclidean zero mean curvature, \n Ruled Helix e = +1,\n Gauss Curvature = -a^4/(a^2+cos(a*v+b)^2)^2 < 0`,axes=FRAMED,scaling=UNCONSTRAINED,orientation=[-26,70],labels=[`x`,`y`,`z`]);
plot3d({R3},u=-Pi..Pi,v=-Pi..Pi,orientation=[134,86],numpoints=1000,style=PATCHCONTOUR,title=` 3D Euclidean zero mean curvature, \n Ruled Helix e = -1,\n Gauss Curvature = -a^4/(a^2+cos(a*v+b)^2)^2 < 0`,axes=FRAMED,scaling=UNCONSTRAINED,orientation=[-26,70],labels=[`x`,`y`,`z`]);
```

3D Euclidean zero mean curvature,

Ruled Helix e = +1,

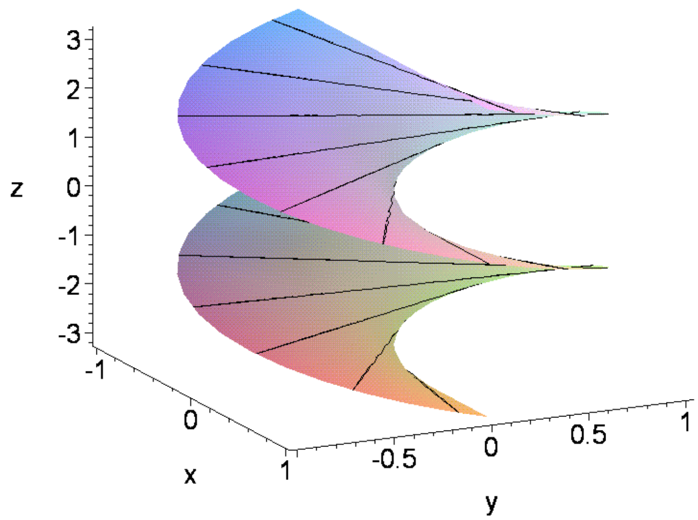
Gauss Curvature = -a^4/(a^2+cos(a*v+b)^2)^2 < 0



3D Euclidean zero mean curvature,

Ruled Helix e = -1,

Gauss Curvature = -a^4/(a^2+cos(a*v+b)^2)^2 < 0



The surface has different polarities in the inbound and outbound directions

Example 1a: A space like (Q > 0) Lorentzian (zero mean curvature) surface of revolution (helicoids) about the z axis.

Specify Immersion:

```
> R:=[1/a*cos(a*v+b)*sin(u),1/a*sin(a*v+b)*sin(u),e*v]:
> MAXsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Immersion\ R = \left[\frac{\cos(a\,v+b)\sin(u)}{a}, \frac{\sin(a\,v+b)\sin(u)}{a}, e\,v\right]$$

$$Induced\ metric = \begin{bmatrix} \frac{\cos(u)^2}{a^2} & 0 \\ 0 & 1-\cos(u)^2-e^2 \end{bmatrix}$$

$$Metric\ Det\ Q = -\frac{\cos(u)^2\,(e^2-1+\cos(u)^2)}{a^2}$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = \frac{e^2\,a^2}{e^4-2\,e^2+2\cos(u)^2\,e^2+1-2\cos(u)^2+\cos(u)^4}$$

$$B11 = 0$$

$$B22 = 0$$

$$B12 = -\frac{\cos(u)^2\, e}{\sqrt{\frac{\cos(u)^2\,(e^2 - 1 + \cos(u)^2)}{a^2}}\; a}$$

```
> K:=simplify(subs(e^2=1,K));
```

$$K := \frac{a^2}{\cos(u)^4}$$

```
> R1:=subs(a=1,e=1,b=0,R);R2:=subs(a=1,e=-1,b=0,R);
```

$$R1 := [\cos(v)\sin(u), \sin(v)\sin(u), v]$$

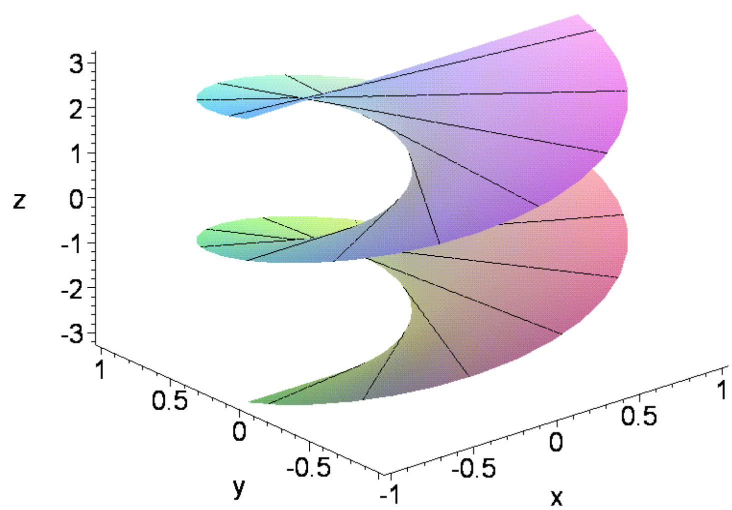
$$R2 := [\cos(v)\sin(u), \sin(v)\sin(u), -v]$$

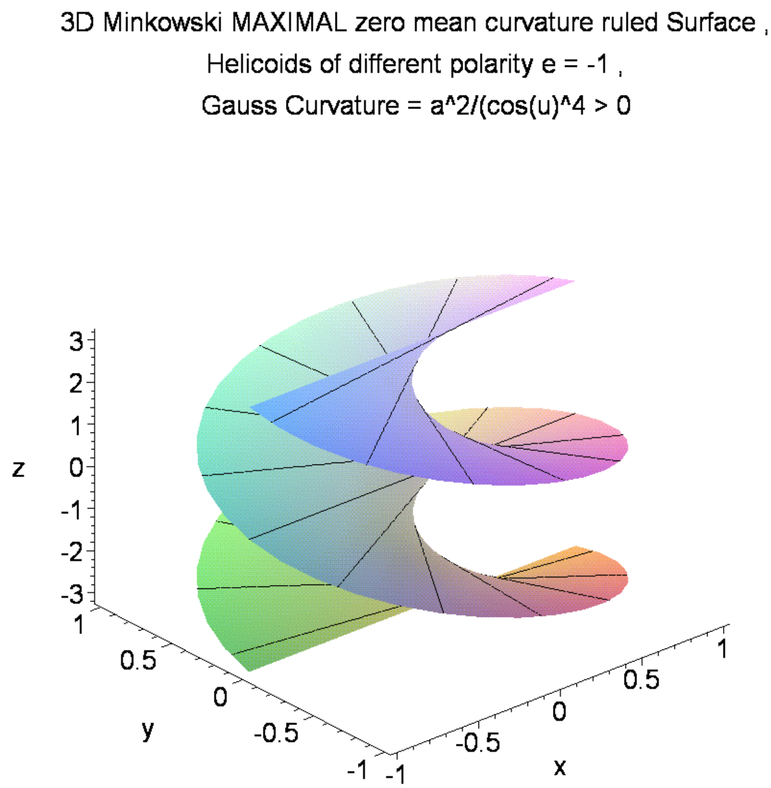
```
> plot3d({R1},u=-Pi/2..Pi/2,v=-Pi..Pi,orientation=[134,86],numpoints=1000,style=PA
TCHCONTOUR,title=` 3D Minkowski MAXIMAL zero mean curvature ruled Surface ,\n
Helicoids of different polarity e = +1,\n Gauss Curvature = a^2/(cos(u)^4 >
0`,axes=FRAMED,orientation=[-130,60],labels=[`x`,`y`,`z`]);plot3d({R2},u=-Pi/2..
Pi/2,v=-Pi..Pi,orientation=[134,86],numpoints=1000,style=PATCHCONTOUR,title=` 3D
Minkowski MAXIMAL zero mean curvature ruled Surface ,\n Helicoids of different
polarity e = -1 ,\n Gauss Curvature = a^2/(cos(u)^4 >
0`,axes=FRAMED,orientation=[-130,60],labels=[`x`,`y`,`z`]);
```

3D Minkowski MAXIMAL zero mean curvature ruled Surface ,

Helicoids of different polarity e = +1,

Gauss Curvature = a^2/(cos(u)^4 > 0





This surface has the visual appearance of a helix. There is a metric singularity at u = 0.

Example 2: A Lorentzian MAXIMAL (Space like Q > 0 zero mean curvature) surface of revolution about the z axis.

Specify Immersion:

```
> R:=[1/a*sinh(a*v+b)*cos(u),-1/a*sinh(a*v+b)*sin(u),(a*v+b)/a]:
>
> MAXsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Immersion\ R = \left[\frac{\sinh(a\,v+b)\cos(u)}{a}, -\frac{\sinh(a\,v+b)\sin(u)}{a}, \frac{a\,v+b}{a}\right]$$

$$Induced\ metric = \begin{bmatrix} \frac{\cosh(a\,v+b)^2-1}{a^2} & 0 \\ 0 & \cosh(a\,v+b)^2-1 \end{bmatrix}$$

$$Metric\ Det\ Q = \frac{(\cosh(a\,v+b)^2-1)^2}{a^2}$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = \frac{a^2}{(\cosh(a\,v+b)^2-1)^2}$$

$$B11 = \frac{\cosh(a\,v+b)^2 - 1}{a^2 \sqrt{-\frac{(\cosh(a\,v+b)-1)^2\,(\cosh(a\,v+b)+1)^2}{a^2}}}$$

$$B22 = -\frac{\cosh(a\,v+b)^2 - 1}{\sqrt{-\frac{(\cosh(a\,v+b)-1)^2\,(\cosh(a\,v+b)+1)^2}{a^2}}}$$

$$B12 = 0$$

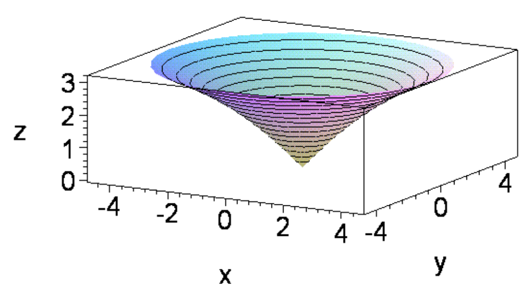
```
> plot3d(subs(a=1/2,b=0,R),u=-Pi..Pi,v=-0..Pi,orientation=[134,86],numpoints=1000,
  style=PATCHCONTOUR,title=`3D Minkowski Maximal Catenoid Surface \n with Conical
  Singularity \n FALACO SOLITON \n Gauss Curvature = +a^2/(cosh(a*v+b)^2-1)^2 >
  0`,axes=BOXED,orientation=[-62,73],scaling=CONSTRAINED,labels=[`x`,`y`,`z`]);
```

3D Minkowski Maximal Catenoid Surface

with Conical Singularity

FALACO SOLITON

Gauss Curvature = +a^2/(cosh(a*v+b)^2-1)^2 > 0



If I could figure a way to let the v axis be a circle, then this surface looks like the Falaco solitons.

Example 2E: A Euclidean Minimal (Q < 0) surface of revolution (Catenoid) about the z axis.

Specify Immersion:

```
> R:=[1/a*cosh(a*v+b)*cos(u),-1/a*cosh(a*v+b)*sin(u),v]:
>
> Minsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Immersion\ R = \left[\frac{\cosh(a\,v+b)\cos(u)}{a}, -\frac{\cosh(a\,v+b)\sin(u)}{a}, v\right]$$

$$Induced\ metric = \begin{bmatrix} \frac{\cosh(a\,v+b)^2}{a^2} & 0 \\ 0 & \cosh(a\,v+b)^2 \end{bmatrix}$$

$$Metric\ Det\ Q = \frac{\cosh(a\,v+b)^4}{a^2}$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = -\frac{a^2}{\cosh(a\,v+b)^4}$$

$$B11 = \frac{\cosh(a\,v+b)^2}{a^2\sqrt{\frac{\cosh(a\,v+b)^4}{a^2}}}$$

$$B22 = -\frac{\cosh(a\,v+b)^2}{\sqrt{\frac{\cosh(a\,v+b)^4}{a^2}}}$$

$$B12 = 0$$

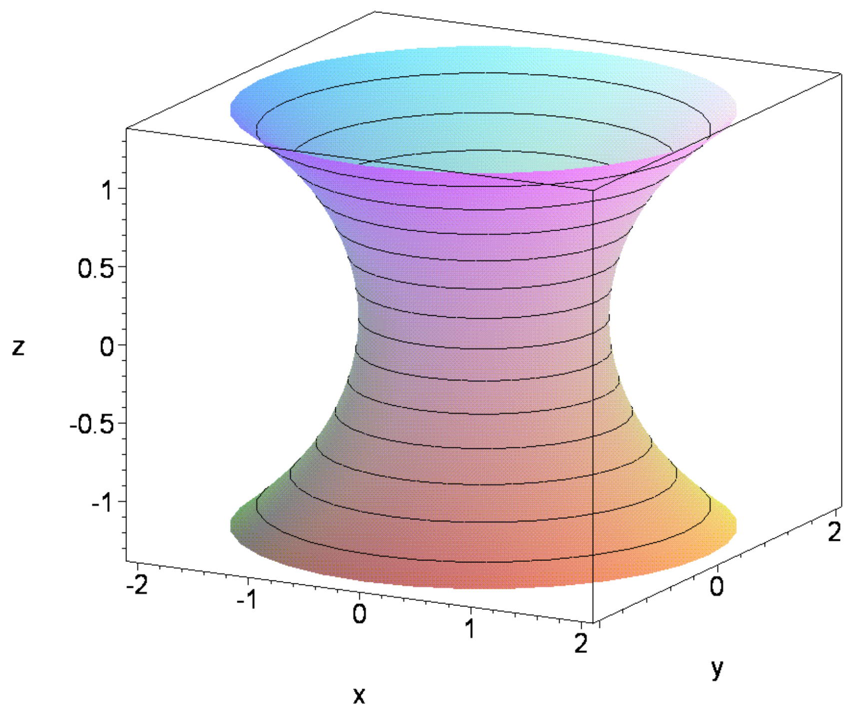
```
> plot3d(subs(a=1,b=0,R),u=-Pi..Pi,v=-1.33..1.33,orientation=[134,86],numpoints=1000,style=PATCHCONTOUR,title=`Distorted 3D Euclidean Minimal Catenoid Surface \n WHEELER WORM HOLE \n Possible Model of a Falaco pair \n Gauss Curvature = -a^2/(cosh(a*v+b)^4 < 0`,axes=BOXED,orientation=[-62,73],scaling=UNCONSTRAINED,labels=[`x`,`y`,`z`],titlefont=[COURIER,BOLD,12]);
```

Distorted 3D Euclidean Minimal Catenoid Surface

WHEELER WORM HOLE

Possible Model of a Falaco pair

Gauss Curvature = -a^2/(cosh(a*v+b)^4 < 0



If I could figure a way to let the v axis be a circle, then this surface looks like the Falaco solitons.

Example 2a: A Lorentzian MAXIMAL (Space like Q > 0, mean curvature = 0) surface of revolution about the z axis.

Specify Immersion:

```
> R:=[1/a*cosh(a*v+b)*cos(u),1/a*cosh(a*v+b)*sin(u),e*u]:
>
> MAXsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Immersion\ R = \left[\frac{\cosh(a\,v+b)\cos(u)}{a}, \frac{\cosh(a\,v+b)\sin(u)}{a}, e\,u \right]$$

$$Induced\ metric = \begin{bmatrix} \frac{-e^2 a^2 + \cosh(a v + b)^2}{a^2} & 0 \\ 0 & \cosh(a v + b)^2 - 1 \end{bmatrix}$$

$$Metric\ Det\ Q = \frac{(\cosh(a v + b)^2 - 1)(-e^2 a^2 + \cosh(a v + b)^2)}{a^2}$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = \frac{e^2 a^4}{(-e^2 a^2 + \cosh(a v + b)^2)^2}$$

$$B11 = 0$$

$$B22 = 0$$

$$B12 = \frac{(\cosh(a v + b)^2 - 1) e}{\sqrt{-\frac{(\cosh(a v + b)^2 - 1)(-e^2 a^2 + \cosh(a v + b)^2)}{a^2}}}$$

```
> R1:=subs(a=.5,e=1,b=0,R);R2:=subs(a=.5,e=0,b=0,R);R3:=subs(a=.5,e=-1,b=0,R);
>
```

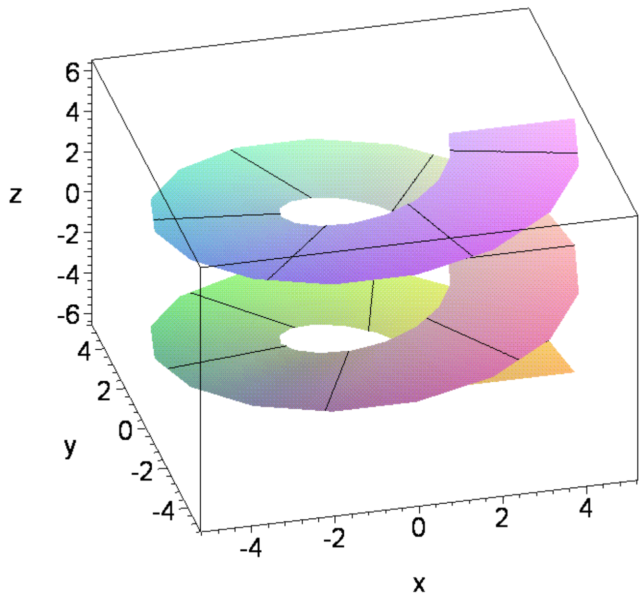
$$R1 := [2.000000000 \cosh(.5\ v) \cos(u), 2.000000000 \cosh(.5\ v) \sin(u), u]$$

$$R2 := [2.000000000 \cosh(.5\ v) \cos(u), 2.000000000 \cosh(.5\ v) \sin(u), 0]$$

$$R3 := [2.000000000 \cosh(.5\ v) \cos(u), 2.000000000 \cosh(.5\ v) \sin(u), -u]$$

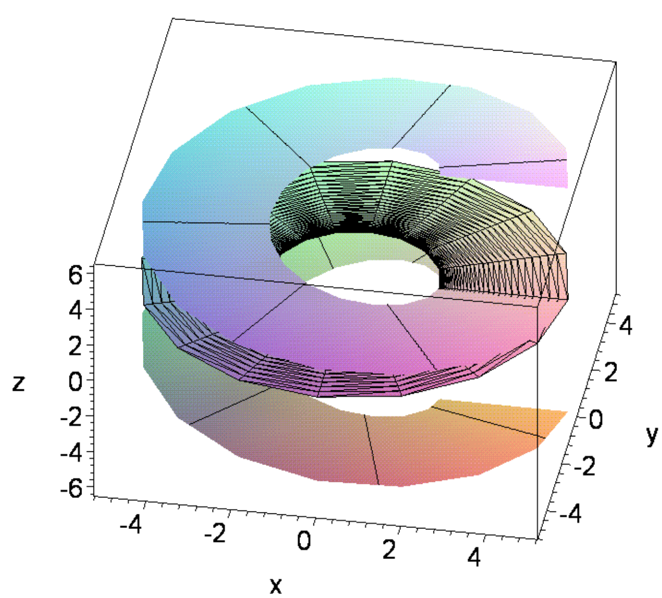
```
> plot3d({R1},u=-2*Pi..2*Pi,v=-0*Pi..Pi,orientation=[134,86],numpoints=1000,style=
PATCHCONTOUR,title=`3D Minkowski Maximal Surface with Cylindrical Singularity \n
Gauss Curvature = 1/(-a^2+cosh(a*v+b)^2)^2*a^4 > 0 ,\n Positive z
outbound`,axes=BOXED,orientation=[-80,43],scaling=UNCONSTRAINED,labels=[`x`,`y`,
`z`]);

>
```

3D Minkowski Maximal Surface with Cylindrical Singularity
Gauss Curvature = 1/(-a^2+cosh(a*v+b)^2)^2*a^4 > 0 ,
Positive z outbound




```
> plot3d({R2,R3},u=-2*Pi..2*Pi,v=-0*Pi..Pi,orientation=[134,86],numpoints=1000,style=PATCHCONTOUR,title=`3D Minkowski Maximal Surface with Cylindrical Singularity \n Gauss Curvature = 1/(-a^2+cosh(a*v+b)^2)^2*a^4 > 0,\n negative z inbound`,axes=BOXED,orientation=[-80,43],scaling=UNCONSTRAINED,labels=[`x`,`y`,`z`]);
```

3D Minkowski Maximal Surface with Cylindrical Singularity
Gauss Curvature = 1/(-a^2+cosh(a*v+b)^2)^2*a^4 > 0,
negative z inbound



This is a ruled surface of Helices which rap aroun the cylinder of radius 1/a.   The immersion does not admit a conical singularity. The Helices are polarization dependent in terms of motion along positive of negative Z dependent in terms of Z.

>

Example 3: A Lorentzian (Space like Q > 0, mean curvature = 0) surface about the y axis.

Specify Immersion:

```
> R:=[1/a*cos(a*v+b)*sinh(u),v,1/a*cos(a*v+b)*cosh(u)]:
> MAXsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Immersion\ R = \left[\frac{\cos(a\,v+b)\sinh(u)}{a}, v, \frac{\cos(a\,v+b)\cosh(u)}{a} \right]$$

$$Induced\ metric = \begin{bmatrix} \frac{\cos(a\,v+b)^2}{a^2} & 0 \\ 0 & \cos(a\,v+b)^2 \end{bmatrix}$$

$$Metric\ Det\ Q = \frac{\cos(a\,v+b)^4}{a^2}$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = \frac{(\cos(a\,v+b)^2 + 4\cosh(u)^4 - 4\cosh(u)^2)\,a^2}{\cos(a\,v+b)^6}$$

$$B11 = -\frac{\cos(a\,v+b)^2\,(2\cosh(u)^2 - 1)}{a^2\sqrt{-\frac{\cos(a\,v+b)^4}{a^2}}}$$

$$B22 = \frac{\cos(a\,v+b)^2\,(2\cosh(u)^2 - 1)}{\sqrt{-\frac{\cos(a\,v+b)^4}{a^2}}}$$

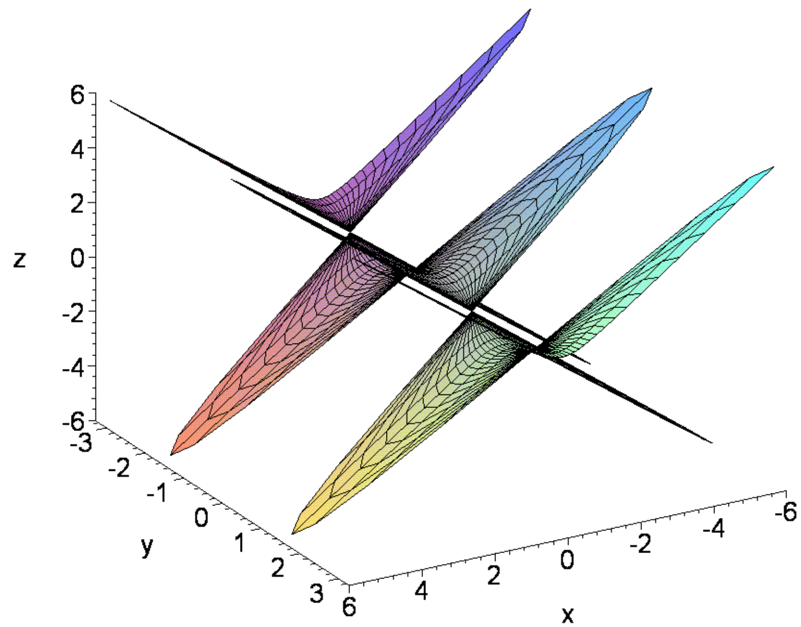
$$B12 = 2\,\frac{\sin(a\,v+b)\cosh(u)\cos(a\,v+b)\sinh(u)}{\sqrt{-\frac{\cos(a\,v+b)^4}{a^2}}\,a}$$

```
> simplify(K);
```

$$\frac{(\cos(a\,v+b)^2 + 4\cosh(u)^4 - 4\cosh(u)^2)\,a^2}{\cos(a\,v+b)^6}$$

```
>
> plot3d(subs(a=2,b=0,R),u=-Pi..Pi,v=-Pi..Pi,orientation=[134,86],numpoints=1000,style=PATCH,title=`Minkowski Enneper #2 Surface  Q > 0`,axes=FRAMED,scaling=UNCONSTRAINED,orientation=[60,60],labels=[`x`,`y`,`z`]);
```

Minkowski Enneper #2 Surface Q > 0



Example 3a: A Maximal (Space like Q > 0, mean curvature = 0) surface of revolution about the z axis.

Specify Immersion:

```
> R:=[1/a*cos(a*v+b)*cosh(u),1/a*sin(a*v+b)*cosh(u),v]:
> MAXsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Immersion\ R = \left[\frac{\cos(a\,v+b)\cosh(u)}{a}, \frac{\sin(a\,v+b)\cosh(u)}{a}, v \right]$$

$$Induced\ metric = \begin{bmatrix} \frac{\cosh(u)^2-1}{a^2} & 0 \\ 0 & \cosh(u)^2-1 \end{bmatrix}$$

$$Metric\ Det\ Q = \frac{(\cosh(u)^2-1)^2}{a^2}$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = \frac{a^2}{(\cosh(u)^2-1)^2}$$

$$B11 = 0$$

$$B22 = 0$$

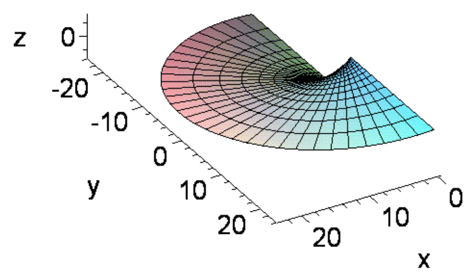
$$B12 = -\frac{\cosh(u)^2-1}{\sqrt{-\frac{(\cosh(u)-1)^2(\cosh(u)+1)^2}{a^2}}\,a}$$

>

```
> plot3d(subs(a=.5,b=0,R),u=-1*Pi..1*Pi,v=-Pi..Pi,orientation=[134,86],numpoints=1
  000,style=PATCH,title=`Minkowski Helix #2 Surface  Q >
  0`,axes=FRAMED,scaling=CONSTRAINED,orientation=[60,60],labels=[`x`,`y`,`z`]);
```

>

Minkowski Helix #2 Surface Q > 0



>

Example 3b: A maximal (Space like Q > 0) surface of revolution about the y axis.

Specify Immersion:

```
> R:=[1/a*cos(a*v+b)*sinh(u),1/a*sin(a*v+b)*sinh(u),v]:
> Minsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Immersion\ R = \left[\frac{\cos(a\,v+b)\sinh(u)}{a}, \frac{\sin(a\,v+b)\sinh(u)}{a}, v \right]$$

$$Induced\ metric = \begin{bmatrix} \frac{\cosh(u)^2}{a^2} & 0 \\ 0 & \cosh(u)^2 \end{bmatrix}$$

$$Metric\ Det\ Q = \frac{\cosh(u)^4}{a^2}$$

$$Mean\ Curvature = 0$$

$$Gauss\ Curvature = -\frac{a^2}{\cosh(u)^4}$$

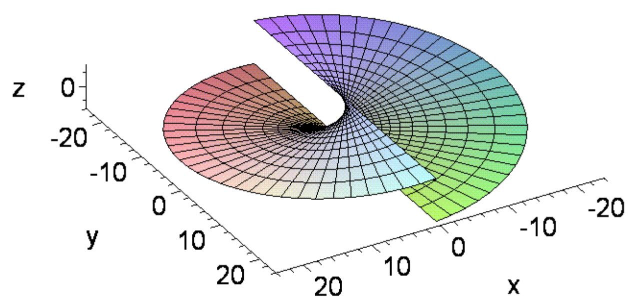
$$B11 = 0$$

$$B22 = 0$$

$$B12 = -\frac{\cosh(u)^2}{\sqrt{\frac{\cosh(u)^4}{a^2}}\, a}$$

```
>
> plot3d(subs(a=.5,b=0,R),u=-1*Pi..1*Pi,v=-Pi..Pi,orientation=[134,86],numpoints=1
  000,style=PATCH,title=`Minkowski Helix #2 Surface  Q >
  0`,axes=FRAMED,scaling=CONSTRAINED,orientation=[60,60],labels=[`x`,`y`,`z`]);
```

Minkowski Helix #2 Surface Q > 0




```
>
```

Example 4: A Lorentz (Light-like Q < 0 zero mean curvature) surface about the y axis.

Specify Immersion:

```
> R:=[-a*v^3-u^2*v+v+b,2*u*v,-a*v^3-u^2*v-v+b]:
> MAXsurf(R):
```

$$Metric = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Immersion\ R = [-a\, v^3 - u^2\, v + v + b,\ 2\, u\, v,\ -a\, v^3 - u^2\, v - v + b]$$

$$Induced\ metric = \begin{bmatrix} 4\, v^2 & 0 \\ 0 & -12\, a\, v^2 \end{bmatrix}$$

$$Metric\ Det\ Q = -48\, a\, v^4$$

$$Mean\ Curvature = 0$$

$$\textit{Gauss Curvature} = -\frac{1}{36}\,\frac{27\,a^3\,v^6 - 27\,a^2\,v^4\,u^2 + 9\,a\,v^2\,u^4 - u^6 - 2\,u^4 + 6\,a\,v^2\,u^2 - u^2}{a^2\,v^6}$$

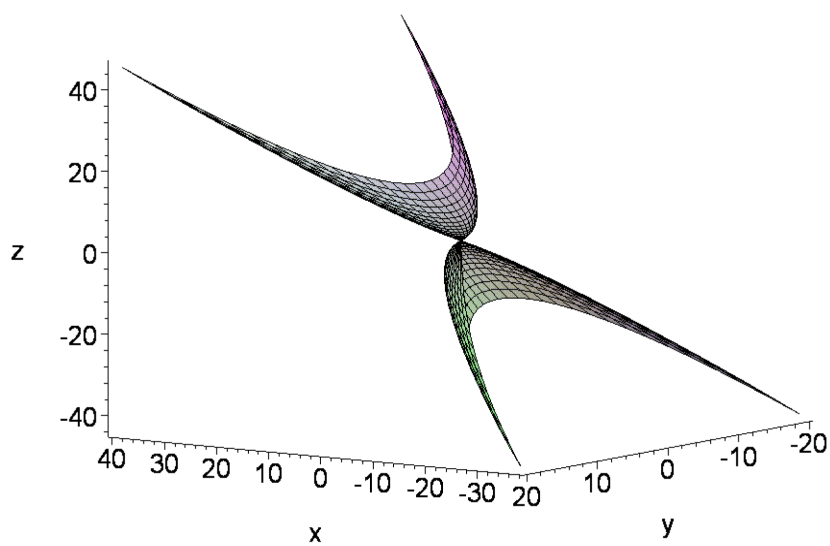
$$B11 = \frac{2}{3}\,\frac{v^2\,\sqrt{3}\,(3\,a\,v^2 - u^2)}{\sqrt{a\,v^4}}$$

$$B22 = 2\,\frac{a\,v^2\,\sqrt{3}\,(3\,a\,v^2 - u^2)}{\sqrt{a\,v^4}}$$

$$B12 = \frac{2}{3}\,\frac{u\,\sqrt{3}\,v\,(-u^2 + 3\,a\,v^2 - 1)}{\sqrt{a\,v^4}}$$

```
>
>
>
> plot3d(subs(a=1/3,b=1,R),u=-Pi..Pi,v=-Pi..Pi,orientation=[134,86],numpoints=1000
,style=PATCH,title=`Minkowski Enneper #2 Conjugate Surface  Q >
0`,axes=FRAMED,scaling=UNCONSTRAINED,orientation=[125,80],labels=[`x`,`y`,`z`]);
```

Minkowski Enneper #2 Conjugate Surface Q > 0




```
>
>
>
>
```