```
> restart:with(linalg):with(plots):
Warning, the protected names norm and trace have been redefined and unprotected

Warning, the name changecoords has been redefined
```

Fresnel Kummer WAVE Surfaces

fresnel.mws R.M.Kiehn 11/15/97 updated 2/25/2005

This program will compute and plot the Fresnel WAVE surfaces for a generalized constitutive matrix. The surfaces are specializations of a Kummer quartic surface, and are related to the Clifford algebra Cl(3,3). Specialized forms have been selected for easy visualization. The formulas can be modified to handle the general case.
The constitutive tensor is a 6 x 6 complex matrix partitioned into 3x3 matrices.
The on-diagonal upper 3x3 matrix is the epsilon matrix.
The epsilon matrix real part describes electric birefringence.
The lower on-diagonal 3x3 matrix is the reciprocal mu matrix.
The real part of the on diagonal reciprocal mu matrix represents magnetic birefringence
The off-diagonal 3x3 matrix has a real part which represents Fresnel-Fizeau effects,
and a complex part Gamma which represents optical activity.
The imaginary part 3x3 on-diagonal matrices represents electric and magnetic Faraday effects.

The fundamental reference for the constitutive matrix is
E.J. Post "The Formal Structure of Electromagnetics" Dover 1997
The sign convention of Post is used below.

```
> constitutive_tensor:=matrix([[-epsilon,Gamma(D)],[Gamma(H),1/mu]])
  ;D=[-epsilon]*(-E)+[Gamma(D)]*B;
> H=[Gamma(H)]*(-E)+[1/mu]*B;
```

$$constitutive\_tensor := \begin{bmatrix} -\varepsilon & \Gamma(D) \\ \Gamma(H) & \dfrac{1}{\mu} \end{bmatrix}$$

$$D = -[-\varepsilon]\, E + [\Gamma(D)]\, B$$

$$H = -[\Gamma(H)]\, E + \left[\frac{1}{\mu}\right] B$$

The Maxwell Faraday and Maxwell Ampere equations become a differntial ideal annihilated by the exterior product of a wave vector, nm1 or nm2.

```
> MF:=nm1*E-B;MA:=nm2*H+D;
```

$$MF := nm1\, E - B$$

$$MA := nm2\, H + D$$

```
>
> MAT:=subs(D=epsilon*E+Gamma(D)*B,H=-Gamma(H)*E+(1/mu)*B,MA);
```

$$MAT := nm2 \left(-\Gamma(H)\, E + \frac{B}{\mu} \right) + \varepsilon\, E + \Gamma(D)\, B$$

```
> MATRIXEQ:=factor(MAT/E);
```

$$MATRIXEQ := -\frac{nm2\, \Gamma(H)\, E\, \mu - nm2\, B - \varepsilon\, E\, \mu - \Gamma(D)\, B\, \mu}{\mu\, E}$$

The MATRIXEQ is to be solved for its eigen values which represent the effective index of refraction in the chosen direction. If
the matrix is Hermitean, then the eigenvalues are real.

Typical Constitutive Matrix entries are shown below. The elements have been scaled such that the "position vector" to a Fresnel surface point in this space has a value equal to the "effective" index of refraction in that direction. The "phase velocity" in that direction is c/n, where c is defined as 1/sqrt(epsilon*mu). The "phase velocity" should be viewed as at the speed of the momentum flux (DxB) in the direction of the position vector.

For the Fresnel Ray surface which defines the propagation speed of the energy flux (ExH) (usually described as the "group" velocity)

See Kiehn, R. M. , Kiehn, G. P., and Roberds, (1991), Parity and time-reversal symmetry breaking, singular solutions and Fresnel surfaces, Phys. Rev A 43, pp. 5165-5671
or http://www22.pair.com/csdc/pdf/timerev.pdf

Note that when the media is dispersive, the energy flux and the momentum flux do not propagate at the same speed, but the product of the group speed times the phase speed is always 1/(epsilon.mu).

In that which follows typical entries have been encoded into the submatrices.

With Maple you can change the entries to anything you want.

The epsilon (permittivity) matrix

```
> eps:=(matrix([[A*epsilon+0*I*fd,0,0],[0,B*epsilon+0*I*fd,0],[0,0,C
  *epsilon+I*fd]]));
```

$$eps := \begin{bmatrix} A\,\varepsilon & 0 & 0 \\ 0 & B\,\varepsilon & 0 \\ 0 & 0 & C\,\varepsilon + I\,fd \end{bmatrix}$$

The real (symetric) part of the epsilon matrix (above) is the permittivity matrix and leads to bifrefringence;
the complex antisymmetric part represents dielectric Faraday effects.

The Gamma matrix (Real and Imaginary Parts)

```
> Gamma(real):=matrix([[0,r,0],[-r,0,0],[0,0,0]]);
```

$$\Gamma(real) := \begin{bmatrix} 0 & r & 0 \\ -r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The real part of the off-diagonal gamma matrix is given above.
The Fresnel-Fizeau and Sagnac effect is contained the real part of the off diagonal gamma matrix
and optical activity in the imaginary part.

```
> Gamma(imag):=matrix([[I*g,I*p,0],[I*p,I*g,0],[0,0,I*g]]);
```

$$\Gamma(imag) := \begin{bmatrix} I\,g & I\,p & 0 \\ I\,p & I\,g & 0 \\ 0 & 0 & I\,g \end{bmatrix}$$

The optical activity is due to the complex part of the off diagonal submatrix above.
Realize that the coefficients used above are representative
and can be adjusted to suit.

```
> Gamma(D):=evalm(Gamma(real)+Gamma(imag));Gamma(A):=evalm(transpose
  (Gamma(real)+Gamma(imag)));
```

$$\Gamma(\mathrm{D}) := \begin{bmatrix} I\,g & r + I\,p & 0 \\ -r + I\,p & I\,g & 0 \\ 0 & 0 & I\,g \end{bmatrix}$$

$$\Gamma(A) := \begin{bmatrix} I\,g & -r + I\,p & 0 \\ r + I\,p & I\,g & 0 \\ 0 & 0 & I\,g \end{bmatrix}$$

The complex conjugate transpose of the Gamma Matrix

```
> Gamma(H):=evalm(transpose(Gamma(real)-transpose(Gamma(imag))));
```

$$\Gamma(H) := \begin{bmatrix} -I\,g & -r - I\,p & 0 \\ r - I\,p & -I\,g & 0 \\ 0 & 0 & -I\,g \end{bmatrix}$$

The Gamma matrix need not be Hermitian. It can be anti-hermitian as well.

The inverse mu matrix (reciprocal permeability matrix)

```
> mu(permeability):=matrix([[a/(mu),I*fm,0],[-I*fm,b/(mu),0],[0,0,c/
  (mu)]]);
```

$$\mu(permeability) := \begin{bmatrix} \frac{a}{\mu} & I\,fm & 0 \\ -I\,fm & \frac{b}{\mu} & 0 \\ 0 & 0 & \frac{c}{\mu} \end{bmatrix}$$

The real (symetric) part of the mu matrix is the magnetic birefringence part; the imaginary antisymmetric part represents magnetic Faraday effects.

```
> mu(inverse):=evalm(inverse(mu(permeability)));
```

$$\mu(inverse) := \begin{bmatrix} \frac{b\,\mu}{a\,b - fm^2\,\mu^2} & \frac{-I\,fm\,\mu^2}{a\,b - fm^2\,\mu^2} & 0 \\ \frac{I\,fm\,\mu^2}{a\,b - fm^2\,\mu^2} & \frac{a\,\mu}{a\,b - fm^2\,\mu^2} & 0 \\ 0 & 0 & \frac{\mu}{c} \end{bmatrix}$$

The matrices are now scaled for ease of computation and plotting.
Values of n = [x.y.z] > 1 imply a phase velocity less than c =1/sqrt(epsilon*mu).
Values of n < 1 imply phase speeds greater than c.

The N matrix (Index of Refraction Operator)

```
> cc:=1/simplify((epsilon)^(1/2)*(mu)^(1/2),sqrt);N:=matrix([[0,z/cc
  ,-y/cc],[-z/cc,0,x/cc],[y/cc,-x/cc,0]]);
```

$$cc := \frac{1}{\sqrt{\varepsilon}\,\sqrt{\mu}}$$

$$N := \begin{bmatrix} 0 & z\sqrt{\varepsilon}\,\sqrt{\mu} & -y\sqrt{\varepsilon}\,\sqrt{\mu} \\ -z\sqrt{\varepsilon}\,\sqrt{\mu} & 0 & x\sqrt{\varepsilon}\,\sqrt{\mu} \\ y\sqrt{\varepsilon}\,\sqrt{\mu} & -x\sqrt{\varepsilon}\,\sqrt{\mu} & 0 \end{bmatrix}$$

The N matrix above is the scaled "index of refraction matrix" which acts like a cross product operator. The N matrix has three components which form the position vector to the Kummer surface. The magnitude of N is the "index" of refraction, n, in the direction of the vector N. The phase speed is then related to 1/n.

The elements of the constitutive matrix have been scaled (below) for algebraic reduction purposes.

```
> fd:=fda*epsilon;p:=pa*epsilon^(1/2)/(mu)^(1/2);g:=ga*epsilon^(1/2)
  /(mu)^(1/2);r:=ra*epsilon^(1/2)/(mu)^(1/2);fm:=fma/mu;
```

$$fd := fda\,\varepsilon$$

$$p := \frac{pa\sqrt{\varepsilon}}{\sqrt{\mu}}$$

$$g := \frac{ga\sqrt{\varepsilon}}{\sqrt{\mu}}$$

$$r := \frac{ra\sqrt{\varepsilon}}{\sqrt{\mu}}$$

$$fm := \frac{fma}{\mu}$$

fd = dielectric faraday, fm =magnetic faraday, r = axion, p = optical activity, g = chirality

Now compute the various matrices whose determinant create the Kummer equation.

```
> eps:=(matrix([[A*epsilon,I*fd,0],[-I*fd,B*epsilon,0],[0,0,C*epsilo
  n]]));
> M:=innerprod(N,mu(permeability)):MM:=innerprod(M,N);
```

The Fresnel Hamiltonian Matrix HHH

$$eps := \begin{bmatrix} A\,\varepsilon & I\,fda\,\varepsilon & 0 \\ -I\,fda\,\varepsilon & B\,\varepsilon & 0 \\ 0 & 0 & C\,\varepsilon \end{bmatrix}$$

$$MM := \begin{bmatrix} -z^2\,\varepsilon\,b - y^2\,\varepsilon\,c & -I\,z^2\,\varepsilon\,fma + y\,\varepsilon\,c\,x & I\,z\,\varepsilon\,fma\,y + z\,\varepsilon\,b\,x \\ I\,z^2\,\varepsilon\,fma + y\,\varepsilon\,c\,x & -z^2\,\varepsilon\,a - x^2\,\varepsilon\,c & z\,\varepsilon\,a\,y - I\,z\,\varepsilon\,fma\,x \\ -\varepsilon\,(I\,y\,fma - x\,b)\,z & \varepsilon\,(y\,a + I\,x\,fma)\,z & -\varepsilon\,y^2\,a - \varepsilon\,x^2\,b \end{bmatrix}$$

```
> HHH:=evalm(eps)+evalm(innerprod(Gamma(D),N)-innerprod(N,Gamma(H)))
  +evalm(MM);
```

$$HHH := \begin{bmatrix} A\,\varepsilon & I\,fda\,\varepsilon & 0 \\ -I\,fda\,\varepsilon & B\,\varepsilon & 0 \\ 0 & 0 & C\,\varepsilon \end{bmatrix}$$

$$+ \begin{bmatrix} -\varepsilon\,(ra + I\,pa)\,z + \varepsilon\,(-ra + I\,pa)\,z\,, 2\,I\,ga\,\varepsilon\,z\,, -2\,I\,ga\,\varepsilon\,y + \varepsilon\,x\,ra + I\,\varepsilon\,x\,pa \\ \text{-}2\,I\,ga\,\varepsilon\,z\,, -\varepsilon\,(ra + I\,pa)\,z + \varepsilon\,(-ra + I\,pa)\,z\,, \varepsilon\,y\,ra - I\,\varepsilon\,y\,pa + 2\,I\,ga\,\varepsilon\,x \\ 2\,I\,ga\,\varepsilon\,y + \varepsilon\,x\,ra - I\,\varepsilon\,x\,pa\,, \quad -2\,I\,ga\,\varepsilon\,x + \varepsilon\,y\,ra + I\,\varepsilon\,y\,pa\,, \quad 0 \end{bmatrix}$$

$$+ \begin{bmatrix} -z^2\,\varepsilon\,b - y^2\,\varepsilon\,c & -I\,z^2\,\varepsilon\,fma + y\,\varepsilon\,c\,x & I\,z\,\varepsilon\,fma\,y + z\,\varepsilon\,b\,x \\ I\,z^2\,\varepsilon\,fma + y\,\varepsilon\,c\,x & -z^2\,\varepsilon\,a - x^2\,\varepsilon\,c & z\,\varepsilon\,a\,y - I\,z\,\varepsilon\,fma\,x \\ -\varepsilon\,(I\,y\,fma - x\,b)\,z & \varepsilon\,(y\,a + I\,x\,fma)\,z & -\varepsilon\,y^2\,a - \varepsilon\,x^2\,b \end{bmatrix}$$

The Kummer Fresnel WAVE Quartic Polynomial

```
> HAMILTONIAN:=factor(det(HHH)/epsilon^3);
```

$$\begin{aligned} HAMILTONIAN := {} & -8\,z\,ra\,y\,pa\,ga\,x + 2\,z^3\,ra\,a\,C + y^4\,c\,ra^2 + 4\,z^2\,ra^2\,C - A\,y^2\,pa^2 - A\,y^2\,ra^2 \\ & + z^2\,b\,y^2\,pa^2 + z^2\,b\,B\,y^2\,a + 2\,z^3\,b\,ra\,C + z^4\,b\,a\,C + 4\,z^2\,ra\,y\,pa\,fma\,x + 2\,z\,ra^3\,y^2 + y^4\,c\,pa^2 \\ & - 4\,fda\,ga\,z\,C + 4\,ga\,z^3\,fma\,C + 4\,ga\,y\,x\,pa\,B - 2\,x\,pa\,z\,fma\,y\,B - z^2\,fma^2\,y^2\,B + 2\,fda\,x\,pa\,z\,a\,y \\ & + 2\,fda\,z\,fma\,y^2\,ra + 2\,fda\,z^2\,fma\,C + 2\,z\,b\,x\,fda\,y\,pa + 2\,y^2\,c\,x^2\,ra^2 - 2\,y^2\,c\,x^2\,pa^2 \\ & - 4\,fda\,ga\,y^2\,ra + fda^2\,x^2\,b + 2\,x^2\,ra^3\,z + fda^2\,y^2\,a - x^2\,pa^2\,B - 4\,ga^2\,y^2\,B + x^4\,pa^2\,c + x^2\,ra^2\,z^2\,a \\ & + 4\,x\,ra\,fda\,y\,pa - 4\,x^2\,ra\,fda\,ga + 2\,x^2\,ra\,fda\,z\,fma + A\,x^4\,c\,b - A\,z^2\,fma^2\,x^2 + 4\,A\,ga\,x^2\,z\,fma \\ & - 2\,A\,y\,pa\,z\,fma\,x + A\,B\,C - A\,z^2\,a\,C + A\,z^2\,a\,x^2\,b - A\,x^2\,c\,C + 2\,A\,z\,ra\,x^2\,b + A\,x^2\,c\,y^2\,a \\ & - A\,B\,y^2\,a - A\,B\,x^2\,b - z^4\,fma^2\,C - 4\,ga^2\,z^2\,C - x^2\,ra^2\,B + x^4\,ra^2\,c + 4\,A\,y\,pa\,ga\,x - 2\,z\,ra\,B\,C \\ & + 2\,z\,ra\,x^2\,c\,C + 2\,z\,ra\,y^2\,pa^2 - 2\,A\,z\,ra\,C + x^2\,pa^2\,z^2\,a + 2\,x^2\,pa^2\,z\,ra + y^2\,c\,z^2\,a\,C + z^2\,b\,y^2\,ra^2 \\ & - y^2\,c\,B\,C + y^4\,c\,B\,a + y^2\,c\,B\,x^2\,b - z^2\,b\,B\,C + z^2\,b\,x^2\,c\,C + 2\,z\,ra\,B\,y^2\,a - 4\,A\,ga^2\,x^2 \\ & + 2\,y^2\,c\,z\,ra\,C + 4\,ga\,y^2\,z\,fma\,B - fda^2\,C \end{aligned}$$

For specific examples choose values for the matrix elements. A,B,C are numeric factors times epsilon. a,b,c are numeric factors times mu. The Optical Activity part is scaled by the impedance of "free space" or sqrt(epsilon/mu). The algebraic

method presented permits symbolic factorization. The dielectric Faraday fd is scaled by epsilon. The magnetic faraday is scaled by mu. The example below is for a chiral Optical Activity coefficient gamma of sqrt(2)/2. Note that the phase velocity of one of the polarization states can be faster than the speed of light and the other is slower.!!!
Select the effects to be studied algebraically by elimating all effects but one or two of the scalars,
fma,fda,ra,pa,ga
Example 1 studies optical activity algebraically (chirality term?), by setting all factors to zero, except ga:
Example 2 studies optical activity algebraically, by setting all factors to zero, except pa:
Example 3 studies optical activity combined with Faraday rotation algebraically, by setting all factors to zero, except ga and pa:

Example 1 Optical Activity - diagonal - chiral,

```
>
```

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMILTONIAN;
```

$$HAMRED := -8\, z\, ra\, y\, pa\, ga\, x + 2\, z^3\, ra\, a\, C + y^4\, c\, ra^2 + 4\, z^2\, ra^2\, C - A\, y^2\, pa^2 - A\, y^2\, ra^2 + z^2\, b\, y^2\, pa^2 + z^2\, b\, B\, y^2\, a + 2\, z^3\, b\, ra\, C + z^4\, b\, a\, C + 4\, z^2\, ra\, y\, pa\, fma\, x + 2\, z\, ra^3\, y^2 + y^4\, c\, pa^2 - 4\, fda\, ga\, z\, C + 4\, ga\, z^3\, fma\, C + 4\, ga\, y\, x\, pa\, B - 2\, x\, pa\, z\, fma\, y\, B - z^2\, fma^2\, y^2\, B + 2\, fda\, x\, pa\, z\, a\, y + 2\, fda\, z\, fma\, y^2\, ra + 2\, fda\, z^2\, fma\, C + 2\, z\, b\, x\, fda\, y\, pa + 2\, y^2\, c\, x^2\, ra^2 - 2\, y^2\, c\, x^2\, pa^2 - 4\, fda\, ga\, y^2\, ra + fda^2\, x^2\, b + 2\, x^2\, ra^3\, z + fda^2\, y^2\, a - x^2\, pa^2\, B - 4\, ga^2\, y^2\, B + x^4\, pa^2\, c + x^2\, ra^2\, z^2\, a + 4\, x\, ra\, fda\, y\, pa - 4\, x^2\, ra\, fda\, ga + 2\, x^2\, ra\, fda\, z\, fma + A\, x^4\, c\, b - A\, z^2\, fma^2\, x^2 + 4\, A\, ga\, x^2\, z\, fma - 2\, A\, y\, pa\, z\, fma\, x + A\, B\, C - A\, z^2\, a\, C + A\, z^2\, a\, x^2\, b - A\, x^2\, c\, C + 2\, A\, z\, ra\, x^2\, b + A\, x^2\, c\, y^2\, a - A\, B\, y^2\, a - A\, B\, x^2\, b - z^4\, fma^2\, C - 4\, ga^2\, z^2\, C - x^2\, ra^2\, B + x^4\, ra^2\, c + 4\, A\, y\, pa\, ga\, x - 2\, z\, ra\, B\, C + 2\, z\, ra\, x^2\, c\, C + 2\, z\, ra\, y^2\, pa^2 - 2\, A\, z\, ra\, C + x^2\, pa^2\, z^2\, a + 2\, x^2\, pa^2\, z\, ra + y^2\, c\, z^2\, a\, C + z^2\, b\, y^2\, ra^2 - y^2\, c\, B\, C + y^4\, c\, B\, a + y^2\, c\, B\, x^2\, b - z^2\, b\, B\, C + z^2\, b\, x^2\, c\, C + 2\, z\, ra\, B\, y^2\, a - 4\, A\, ga^2\, x^2 + 2\, y^2\, c\, z\, ra\, C + 4\, ga\, y^2\, z\, fma\, B - fda^2\, C$$

Set anisotropic coefficients:

```
> HAMA:=subs(A=1,B=1,C=1,a=1,b=1,c=1,HAMRED):eval(%):HAMA:=evalc(HAMA);
>
```

$$HAMA := 1 - 4\, y\, pa\, z\, fma\, x - 8\, z\, ra\, y\, pa\, ga\, x + 4\, z^2\, ra\, y\, pa\, fma\, x + 2\, z\, ra^3\, y^2 - y^2\, ra^2 + 4\, ga\, x^2\, z\, fma - x^2\, pa^2 - 2\, z^2 - 2\, y^2 - 2\, x^2 + x^4 + 4\, fda\, x\, pa\, z\, y + x^2\, pa^2\, z^2 + 2\, fda\, z\, fma\, y^2\, ra + x^2\, ra^2\, z^2 - 4\, fda\, ga\, y^2\, ra + 2\, x^2\, ra^3\, z + 4\, x\, ra\, fda\, y\, pa - 4\, x^2\, ra\, fda\, ga + 2\, x^2\, ra\, fda\, z\, fma + 2\, z\, ra\, y^2\, pa^2 + 2\, x^2\, pa^2\, z\, ra + 8\, y\, pa\, ga\, x - z^2\, fma^2\, y^2 - 4\, ga^2\, x^2 - y^2\, pa^2 + z^4 + y^4 - z^2\, fma^2\, x^2 + y^4\, pa^2 + 2\, x^2\, y^2 + x^4\, pa^2 + x^4\, ra^2 + z^2\, y^2\, pa^2 + z^2\, y^2\, ra^2 + 4\, z\, ra\, x^2 + 2\, y^2\, x^2\, ra^2 - 2\, y^2\, x^2\, pa^2 + 4\, z\, ra\, y^2 + 4\, z^2\, ra^2 + 4\, z^3\, ra + fda^2\, x^2 + 2\, z^2\, y^2 + 4\, ga\, y^2\, z\, fma - 4\, fda\, ga\, z + 4\, ga\, z^3\, fma + 2\, fda\, z^2\, fma - 4\, ga^2\, y^2 - 4\, z\, ra - x^2\, ra^2 + y^4\, ra^2 - z^4\, fma^2 + fda^2\, y^2 - 4\, ga^2\, z^2 - fda^2 + 2\, z^2\, x^2$$

Set the numeric values for the coefficients

```
> HAM:=subs(fda=0,ra=0,pa=0,ga=0.3,fma=0,HAMA);
```

$$HAM := 1. - 2.36\, z^2 - 2.36\, y^2 - 2.36\, x^2 + x^4 + z^4 + y^4 + 2\, x^2\, y^2 + 2\, z^2\, y^2 + 2\, z^2\, x^2$$

Coefficients were chosen to display optical activity

```
> KUMM:=factor(HAM);LORENTZ:=(x^2+y^2+z^2)^2-1;KUMMR:=x^2+y^2+z^2-1;
```

$$KUMM := 1. - 2.36\, z^2 - 2.36\, y^2 - 2.36\, x^2 + x^4 + z^4 + y^4 + 2.\, x^2\, y^2 + 2.\, z^2\, y^2 + 2.\, z^2\, x^2$$

$$LORENTZ := (x^2 + y^2 + z^2)^2 - 1$$

$$KUMMR := x^2 + y^2 + z^2 - 1$$

```
>
```

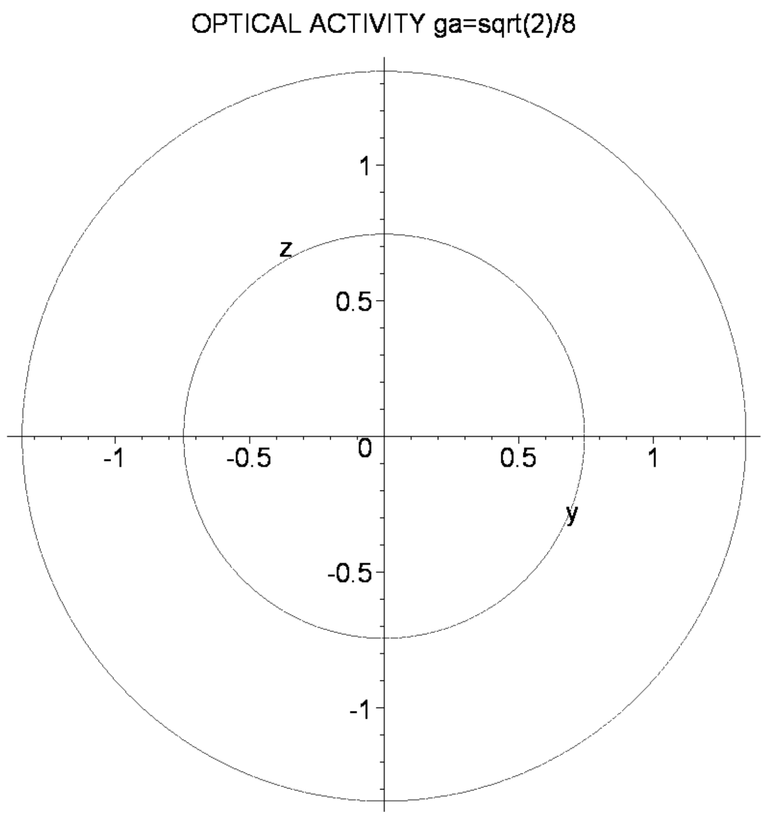
```
> KUMMX:=factor(subs(x=0,KUMM));
```

$$KUMMX := 1. - 2.36\, z^2 - 2.36\, y^2 + z^4 + y^4 + 2.\, z^2\, y^2$$

```
>
```

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/8`);
```

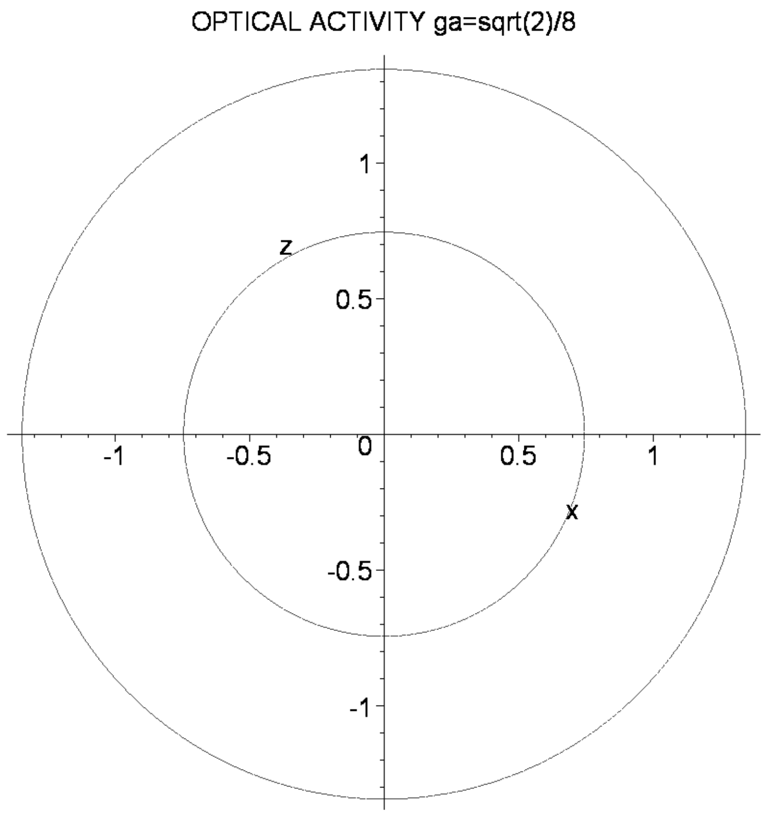


```
> KUMMY:=subs(y=0,KUMM);
```

$$KUMMY := 1. - 2.36\, z^2 - 2.36\, x^2 + x^4 + z^4 + 2.\, z^2\, x^2$$

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/8`);
```

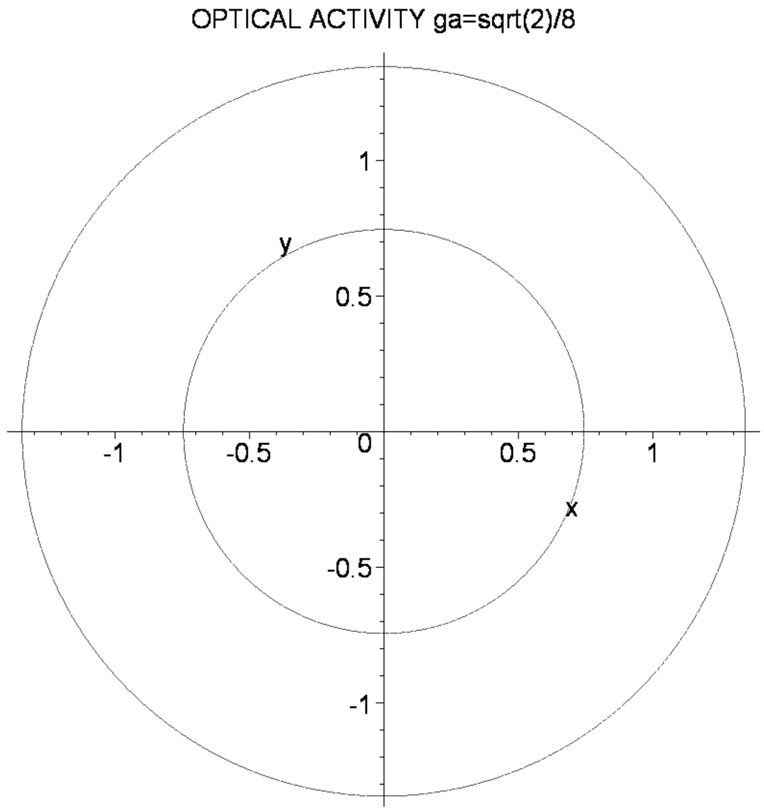


```
> KUMMZ:=subs(z=0,KUMM);
```

$$KUMMZ := 1. - 2.36\, y^2 - 2.36\, x^2 + x^4 + y^4 + 2.\, x^2\, y^2$$

Z=0 section of Fresnel Kummer Wave Vector Surface

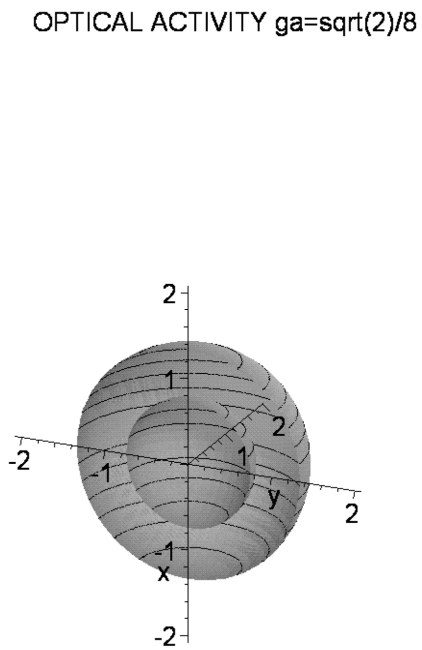
```
> implicitplot(KUMMZ,x=-2.0..2.0,y=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/8`);
```



Optical Activity gives Two concentric spheres

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM,x=-2..2,y=-0..2,z=-2..2,shading=XYZ,lightmodel
  =light3,axes=NORMAL,style=PATCHCONTOUR,scaling=CONSTRAINED,title=`
  OPTICAL ACTIVITY ga=sqrt(2)/8`
  ,numpoints=50000,orientation=[-66,69]);
```



Example 2 Optical Activity - chiral off-diagonal,

>

<u>Reduced Fresnel Kummer quartic polynomial</u>

```
> HAMRED:=HAMILTONIAN;
```

$$
\begin{aligned}
HAMRED := & -4\, A\, ga^2\, x^2 - A\, y^2\, pa^2 - A\, y^2\, ra^2 + 4\, z^2\, ra^2\, C + 2\, z\, ra^3\, y^2 + y^4\, c\, ra^2 - x^2\, ra^2\, B \\
& + A\, B\, C + x^4\, ra^2\, c - 2\, z\, ra\, B\, C - 4\, ga^2\, z^2\, C + 2\, fda\, z\, fma\, y^2\, ra - 2\, y^2\, c\, x^2\, pa^2 - 4\, x^2\, ra\, fda\, ga \\
& + 4\, x\, ra\, fda\, y\, pa - A\, B\, y^2\, a - A\, B\, x^2\, b - 2\, A\, z\, ra\, C + 2\, A\, z\, ra\, x^2\, b - A\, z^2\, a\, C + A\, z^2\, a\, x^2\, b \\
& - A\, x^2\, c\, C + A\, x^2\, c\, y^2\, a + A\, x^4\, c\, b - A\, z^2\, fma^2\, x^2 - 4\, fda\, ga\, z\, C - z^2\, fma^2\, y^2\, B + 2\, x^2\, pa^2\, z\, ra \\
& + x^2\, pa^2\, z^2\, a - 2\, x\, pa\, z\, fma\, y\, B + 2\, z\, b\, x\, fda\, y\, pa + 2\, x^2\, ra\, fda\, z\, fma + x^2\, ra^2\, z^2\, a + 2\, y^2\, c\, x^2\, ra^2 \\
& + z^2\, b\, y^2\, pa^2 + z^2\, b\, y^2\, ra^2 - y^2\, c\, B\, C + y^4\, c\, B\, a + y^2\, c\, B\, x^2\, b + 2\, y^2\, c\, z\, ra\, C + y^2\, c\, z^2\, a\, C \\
& + 2\, z^3\, ra\, a\, C + 2\, z\, ra\, x^2\, c\, C - 8\, z\, ra\, y\, pa\, ga\, x + 4\, z^2\, ra\, y\, pa\, fma\, x + 2\, z\, ra\, y^2\, pa^2 - z^2\, b\, B\, C \\
& + z^2\, b\, B\, y^2\, a + 2\, z^3\, b\, ra\, C + z^4\, b\, a\, C + z^2\, b\, x^2\, c\, C + fda^2\, y^2\, a + fda^2\, x^2\, b + y^4\, c\, pa^2 - z^4\, fma^2\, C \\
& - 4\, ga^2\, y^2\, B + x^4\, pa^2\, c - x^2\, pa^2\, B + 4\, ga\, y\, x\, pa\, B + 4\, ga\, y^2\, z\, fma\, B + 4\, ga\, z^3\, fma\, C \\
& + 2\, fda\, x\, pa\, z\, a\, y + 2\, fda\, z^2\, fma\, C + 2\, x^2\, ra^3\, z + 2\, z\, ra\, B\, y^2\, a + 4\, A\, ga\, x^2\, z\, fma + 4\, A\, y\, pa\, ga\, x \\
& - 4\, fda\, ga\, y^2\, ra - 2\, A\, y\, pa\, z\, fma\, x - fda^2\, C
\end{aligned}
$$

Set anisotropic coefficients:

```
> HAMA:=subs(A=1,B=1,C=1,a=1,b=1,c=1,HAMRED):eval(%):HAMA:=evalc(HAMA);
>
```

$$HAMA := 1 + x^2\, ra^2\, z^2 - y^2\, ra^2 + z^4 - 2\, z^2 - 4\, y\, pa\, z\, fma\, x + 2\, y^2\, x^2\, ra^2 - 2\, y^2\, x^2\, pa^2 + x^4 + fda^2\, y^2 + 4\, z\, ra\, x^2 + y^4 + z^2\, y^2\, ra^2 + z^2\, y^2\, pa^2 - 2\, y^2 + x^4\, pa^2 + y^4\, pa^2 + 2\, x^2\, y^2 + x^4\, ra^2 + y^4\, ra^2 + 2\, z^2\, y^2 + fda^2\, x^2 + 2\, z^2\, x^2 + 4\, ga\, y^2\, z\, fma + 4\, z^3\, ra - z^4\, fma^2 - 4\, ga^2\, z^2 + 4\, z^2\, ra^2 - 4\, z\, ra - x^2\, pa^2 - 4\, ga^2\, y^2 + 8\, y\, pa\, ga\, x + 4\, ga\, x^2\, z\, fma - x^2\, ra^2 - y^2\, pa^2 - fda^2 - 2\, x^2 - 4\, ga^2\, x^2 - z^2\, fma^2\, y^2 - z^2\, fma^2\, x^2 - 4\, fda\, ga\, z + 2\, fda\, z^2\, fma + 2\, z\, ra^3\, y^2 + 4\, ga\, z^3\, fma + 2\, fda\, z\, fma\, y^2\, ra + 4\, z\, ra\, y^2 + 4\, fda\, x\, pa\, z\, y - 4\, x^2\, ra\, fda\, ga + 4\, x\, ra\, fda\, y\, pa + 2\, x^2\, pa^2\, z\, ra + 2\, x^2\, ra\, fda\, z\, fma - 8\, z\, ra\, y\, pa\, ga\, x + 4\, z^2\, ra\, y\, pa\, fma\, x + 2\, z\, ra\, y^2\, pa^2 + 2\, x^2\, ra^3\, z - 4\, fda\, ga\, y^2\, ra + x^2\, pa^2\, z^2$$

Set anisotropic coefficients:

```
> HAMA:=subs(A=1,B=1,C=1,a=1,b=1,c=1,HAMRED):eval(%):HAMA:=evalc(HAMA);
>
```

$$HAMA := 1 + x^2\, ra^2\, z^2 - y^2\, ra^2 + z^4 - 2\, z^2 - 4\, y\, pa\, z\, fma\, x + 2\, y^2\, x^2\, ra^2 - 2\, y^2\, x^2\, pa^2 + x^4 + fda^2\, y^2 + 4\, z\, ra\, x^2 + y^4 + z^2\, y^2\, ra^2 + z^2\, y^2\, pa^2 - 2\, y^2 + x^4\, pa^2 + y^4\, pa^2 + 2\, x^2\, y^2 + x^4\, ra^2 + y^4\, ra^2 + 2\, z^2\, y^2 + fda^2\, x^2 + 2\, z^2\, x^2 + 4\, ga\, y^2\, z\, fma + 4\, z^3\, ra - z^4\, fma^2 - 4\, ga^2\, z^2 + 4\, z^2\, ra^2 - 4\, z\, ra - x^2\, pa^2 - 4\, ga^2\, y^2 + 8\, y\, pa\, ga\, x + 4\, ga\, x^2\, z\, fma - x^2\, ra^2 - y^2\, pa^2 - fda^2 - 2\, x^2 - 4\, ga^2\, x^2 - z^2\, fma^2\, y^2 - z^2\, fma^2\, x^2 - 4\, fda\, ga\, z + 2\, fda\, z^2\, fma + 2\, z\, ra^3\, y^2 + 4\, ga\, z^3\, fma + 2\, fda\, z\, fma\, y^2\, ra + 4\, z\, ra\, y^2 + 4\, fda\, x\, pa\, z\, y - 4\, x^2\, ra\, fda\, ga + 4\, x\, ra\, fda\, y\, pa + 2\, x^2\, pa^2\, z\, ra + 2\, x^2\, ra\, fda\, z\, fma - 8\, z\, ra\, y\, pa\, ga\, x + 4\, z^2\, ra\, y\, pa\, fma\, x + 2\, z\, ra\, y^2\, pa^2 + 2\, x^2\, ra^3\, z - 4\, fda\, ga\, y^2\, ra + x^2\, pa^2\, z^2$$

Set the numeric values for the coefficients

```
> HAM:=subs(fda=0,ra=0,ga=0,pa=2,fma=0,HAMA);
```

$$HAM := 1 + z^4 - 2\, z^2 - 6\, x^2\, y^2 + 5\, x^4 + 5\, y^4 + 6\, z^2\, y^2 - 6\, y^2 + 6\, z^2\, x^2 - 6\, x^2$$

Coefficients were chosen to display optical activity

```
> KUMM:=factor(HAM);KUMMSQ:=expand(KUMM);LORENTZ:=(x^2+y^2+z^2)^2-1;
  Difference:=factor(KUMM-LORENTZ);
```

$$KUMM := 1 + z^4 - 2\, z^2 - 6\, x^2\, y^2 + 5\, x^4 + 5\, y^4 + 6\, z^2\, y^2 - 6\, y^2 + 6\, z^2\, x^2 - 6\, x^2$$

$$KUMMSQ := 1 + z^4 - 2\, z^2 - 6\, x^2\, y^2 + 5\, x^4 + 5\, y^4 + 6\, z^2\, y^2 - 6\, y^2 + 6\, z^2\, x^2 - 6\, x^2$$

$$LORENTZ := (x^2 + y^2 + z^2)^2 - 1$$

$$Difference := 2 - 2\, z^2 - 8\, x^2\, y^2 + 4\, x^4 + 4\, y^4 + 4\, z^2\, y^2 - 6\, y^2 + 4\, z^2\, x^2 - 6\, x^2$$

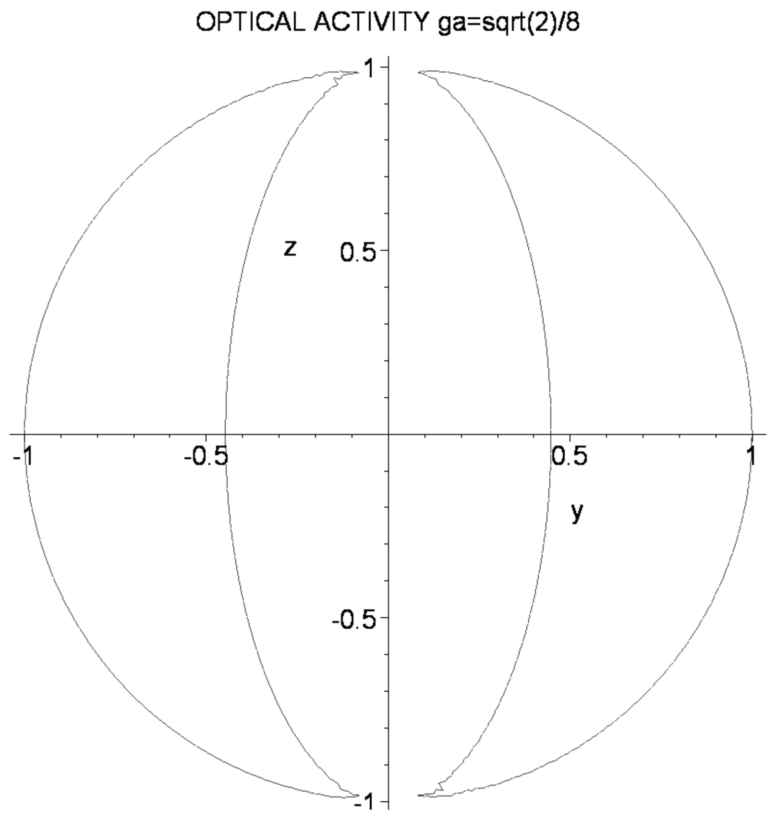
>

```
> KUMMX:=subs(x=0,KUMMSQ);
```

$$KUMMX := 1 + z^4 - 2\,z^2 + 5\,y^4 + 6\,z^2\,y^2 - 6\,y^2$$

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/8`);
```

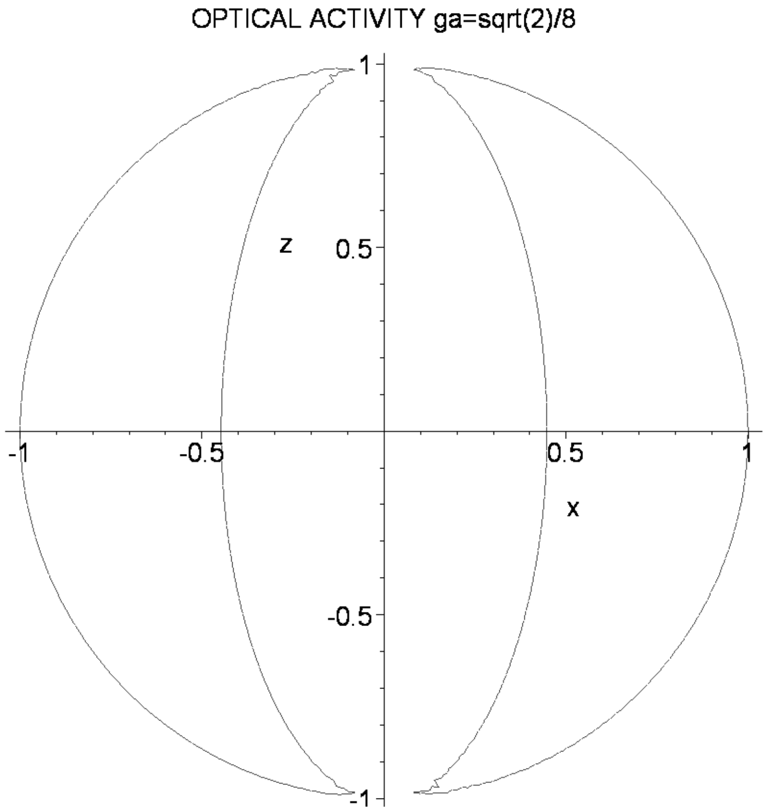


```
> KUMMY:=subs(y=.0,KUMM);
```

$$KUMMY := 1. + z^4 - 2\,z^2 + 5\,x^4 + 6\,z^2\,x^2 - 6\,x^2$$

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/8`);
```

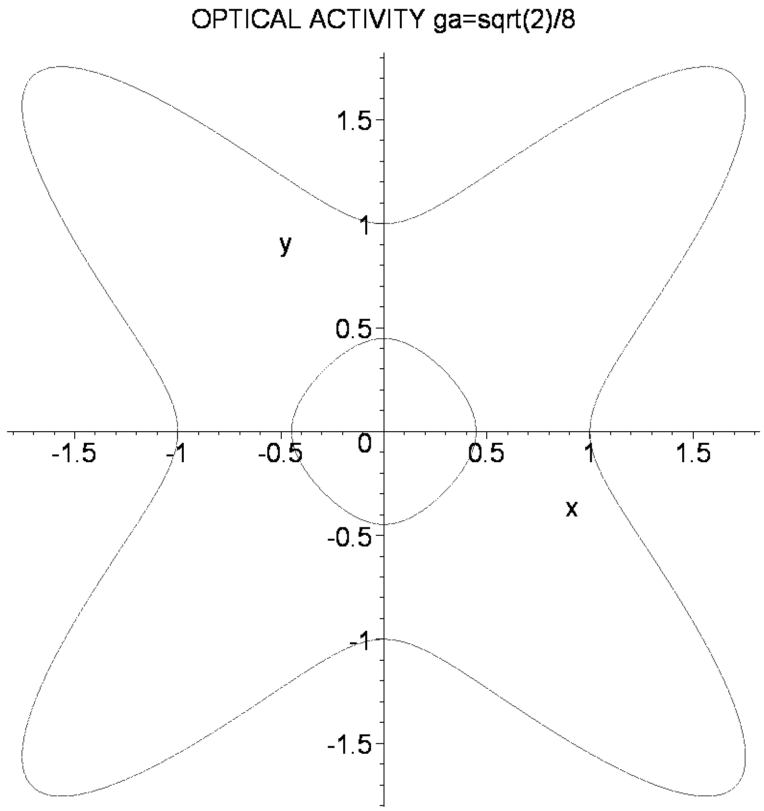


```
> KUMMZ:=subs(z=.0,KUMM);
```

$$KUMMZ := 1. - 6\,x^2\,y^2 + 5\,x^4 + 5\,y^4 - 6\,y^2 - 6\,x^2$$

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ=0,x=-2.0..2.0,y=-2..2,numpoints =
  50000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/8`);
```

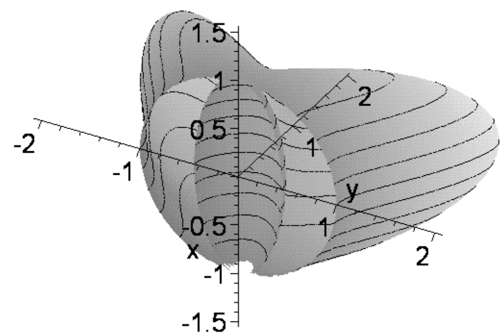


Optical Activity gives Two concentric spheres

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM=0,x=-2..2,y=-0..2,z=-1.5..1.5,shading=XYZ,axes
  =NORMAL,style=PATCHCONTOUR,scaling=CONSTRAINED,numpoints=100000,or
  ientation=[-61,59],title=`OPTICAL ACTIVITY pa=1`);
>
>
```

OPTICAL ACTIVITY pa=1



>

>

>

Example 3 diagonal Optical Activity + Faraday,

>

<u>Reduced Fresnel Kummer quartic polynomial</u>

```
> HAMRED:=HAMILTONIAN;
```

$$HAMRED := -8\, z\, ra\, y\, pa\, ga\, x + 2\, z^3\, ra\, a\, C + y^4\, c\, ra^2 + 4\, z^2\, ra^2\, C - A\, y^2\, pa^2 - A\, y^2\, ra^2 + z^2\, b\, y^2\, pa^2 + z^2\, b\, B\, y^2\, a + 2\, z^3\, b\, ra\, C + z^4\, b\, a\, C + 4\, z^2\, ra\, y\, pa\, fma\, x + 2\, z\, ra^3\, y^2 + y^4\, c\, pa^2 - 4\, fda\, ga\, z\, C + 4\, ga\, z^3\, fma\, C + 4\, ga\, y\, x\, pa\, B - 2\, x\, pa\, z\, fma\, y\, B - z^2\, fma^2\, y^2\, B + 2\, fda\, x\, pa\, z\, a\, y + 2\, fda\, z\, fma\, y^2\, ra + 2\, fda\, z^2\, fma\, C + 2\, z\, b\, x\, fda\, y\, pa + 2\, y^2\, c\, x^2\, ra^2 - 2\, y^2\, c\, x^2\, pa^2 - 4\, fda\, ga\, y^2\, ra + fda^2\, x^2\, b + 2\, x^2\, ra^3\, z + fda^2\, y^2\, a - x^2\, pa^2\, B - 4\, ga^2\, y^2\, B + x^4\, pa^2\, c + x^2\, ra^2\, z^2\, a$$

$$+ 4 x\, ra\, fda\, y\, pa - 4 x^2 ra\, fda\, ga + 2 x^2 ra\, fda\, z\, fma + A x^4 c\, b - A z^2 fma^2 x^2 + 4 A\, ga\, x^2 z\, fma - 2 A y\, pa\, z\, fma\, x + A B C - A z^2 a C + A z^2 a x^2 b - A x^2 c C + 2 A z\, ra\, x^2 b + A x^2 c y^2 a - A B y^2 a - A B x^2 b - z^4 fma^2 C - 4 ga^2 z^2 C - x^2 ra^2 B + x^4 ra^2 c + 4 A y\, pa\, ga\, x - 2 z\, ra\, B C + 2 z\, ra\, x^2 c C + 2 z\, ra\, y^2 pa^2 - 2 A z\, ra\, C + x^2 pa^2 z^2 a + 2 x^2 pa^2 z\, ra + y^2 c z^2 a C + z^2 b y^2 ra^2 - y^2 c B C + y^4 c B a + y^2 c B x^2 b - z^2 b B C + z^2 b x^2 c C + 2 z\, ra\, B y^2 a - 4 A\, ga^2 x^2 + 2 y^2 c z\, ra\, C + 4 ga\, y^2 z\, fma\, B - fda^2 C$$

Set anisotropic coefficients:

```
> HAMA:=subs(A=1,B=1,C=1,a=1,b=1,c=1,HAMRED):eval(%):HAMA:=evalc(HAMA);
>
```

$$HAMA := 1 - y^2 ra^2 - y^2 pa^2 + x^4 ra^2 + x^4 pa^2 + y^4 pa^2 + fda^2 x^2 + fda^2 y^2 + 4 z^3 ra - 4 ga^2 z^2 - z^4 fma^2 + 4 z^2 ra^2 - 4 z\, ra - x^2 ra^2 - 4 ga^2 y^2 - x^2 pa^2 - 4 ga^2 x^2 - z^2 fma^2 x^2 - 8 z\, ra\, y\, pa\, ga\, x + 4 z^2 ra\, y\, pa\, fma\, x + 2 z\, ra^3 y^2 + 2 fda\, z^2 fma + 4 ga\, z^3 fma - 4 fda\, ga\, z - z^2 fma^2 y^2 + 4 z\, ra\, x^2 + z^2 y^2 pa^2 + x^2 pa^2 z^2 + x^2 ra^2 z^2 - 2 y^2 x^2 pa^2 + 2 y^2 x^2 ra^2 + z^2 y^2 ra^2 - fda^2 + 4 ga\, y^2 z\, fma + 4 z\, ra\, y^2 + 4 fda\, x\, pa\, z\, y + 8 y\, pa\, ga\, x - 2 z^2 - 2 y^2 - 2 x^2 + y^4 ra^2 + 4 ga\, x^2 z\, fma - 4 y\, pa\, z\, fma\, x + x^4 + 2 fda\, z\, fma\, y^2 ra - 4 fda\, ga\, y^2 ra + 2 x^2 ra^3 z + 4 x\, ra\, fda\, y\, pa - 4 x^2 ra\, fda\, ga + 2 x^2 ra\, fda\, z\, fma + 2 z\, ra\, y^2 pa^2 + 2 x^2 pa^2 z\, ra + z^4 + y^4 + 2 x^2 y^2 + 2 z^2 y^2 + 2 z^2 x^2$$

Set anisotropic coefficients:

```
> HAMA:=subs(A=1,B=1,C=1,a=1,b=1,c=1,HAMRED):eval(%):HAMA:=evalc(HAMA);
>
```

$$HAMA := 1 - y^2 ra^2 - y^2 pa^2 + x^4 ra^2 + x^4 pa^2 + y^4 pa^2 + fda^2 x^2 + fda^2 y^2 + 4 z^3 ra - 4 ga^2 z^2 - z^4 fma^2 + 4 z^2 ra^2 - 4 z\, ra - x^2 ra^2 - 4 ga^2 y^2 - x^2 pa^2 - 4 ga^2 x^2 - z^2 fma^2 x^2 - 8 z\, ra\, y\, pa\, ga\, x + 4 z^2 ra\, y\, pa\, fma\, x + 2 z\, ra^3 y^2 + 2 fda\, z^2 fma + 4 ga\, z^3 fma - 4 fda\, ga\, z - z^2 fma^2 y^2 + 4 z\, ra\, x^2 + z^2 y^2 pa^2 + x^2 pa^2 z^2 + x^2 ra^2 z^2 - 2 y^2 x^2 pa^2 + 2 y^2 x^2 ra^2 + z^2 y^2 ra^2 - fda^2 + 4 ga\, y^2 z\, fma + 4 z\, ra\, y^2 + 4 fda\, x\, pa\, z\, y + 8 y\, pa\, ga\, x - 2 z^2 - 2 y^2 - 2 x^2 + y^4 ra^2 + 4 ga\, x^2 z\, fma - 4 y\, pa\, z\, fma\, x + x^4 + 2 fda\, z\, fma\, y^2 ra - 4 fda\, ga\, y^2 ra + 2 x^2 ra^3 z + 4 x\, ra\, fda\, y\, pa - 4 x^2 ra\, fda\, ga + 2 x^2 ra\, fda\, z\, fma + 2 z\, ra\, y^2 pa^2 + 2 x^2 pa^2 z\, ra + z^4 + y^4 + 2 x^2 y^2 + 2 z^2 y^2 + 2 z^2 x^2$$

Set the numeric values for the coefficients

```
> HAM:=subs(fda=1,ra=1,ga=.0,pa=0,fma=.0,HAMA);HAMtest:=subs(fda=0,r
  a=0,ga=.3,pa=0,fma=.3,HAMA);
```

$$HAM := 6\,x^2\,z + 6\,y^2\,z + 4\,z^3 + 2\,z^2 - 2\,y^2 - 2\,x^2 - 4\,z + 2\,x^4 + z^4 + 2\,y^4 + 4\,x^2\,y^2 + 3\,z^2\,y^2 + 3\,z^2\,x^2$$

$$HAMtest := .36\,z^3 + .36\,y^2\,z + .36\,x^2\,z - 2.36\,z^2 - 2.36\,y^2 - 2.36\,x^2 + x^4 + 1. + 1.91\,z^2\,y^2 + .91\,z^4 + y^4 + 1.91\,z^2\,x^2 + 2\,x^2\,y^2$$

Coefficients were chosen to display optical activity mixed with Faraday

```
> KUMM:=factor(HAM);KUMMSQ:=expand(KUMM);LORENTZ:=(x^2+y^2+z^2)^2-1;
  Difference:=factor(KUMM-LORENTZ);
```

$$KUMM := (2\,y^2 + 2\,z + z^2 - 2 + 2\,x^2)\,(y^2 + 2\,z + z^2 + x^2)$$

$$KUMMSQ := 6\,x^2\,z + 6\,y^2\,z + 4\,z^3 + 2\,z^2 - 2\,y^2 - 2\,x^2 - 4\,z + 2\,x^4 + z^4 + 2\,y^4 + 4\,x^2\,y^2 + 3\,z^2\,y^2 + 3\,z^2\,x^2$$

$$LORENTZ := (x^2 + y^2 + z^2)^2 - 1$$

$$Difference := 6\,x^2\,z + 6\,y^2\,z + 4\,z^3 + 2\,z^2 - 2\,y^2 - 2\,x^2 - 4\,z + x^4 + y^4 + 2\,x^2\,y^2 + z^2\,y^2 + z^2\,x^2 + 1$$

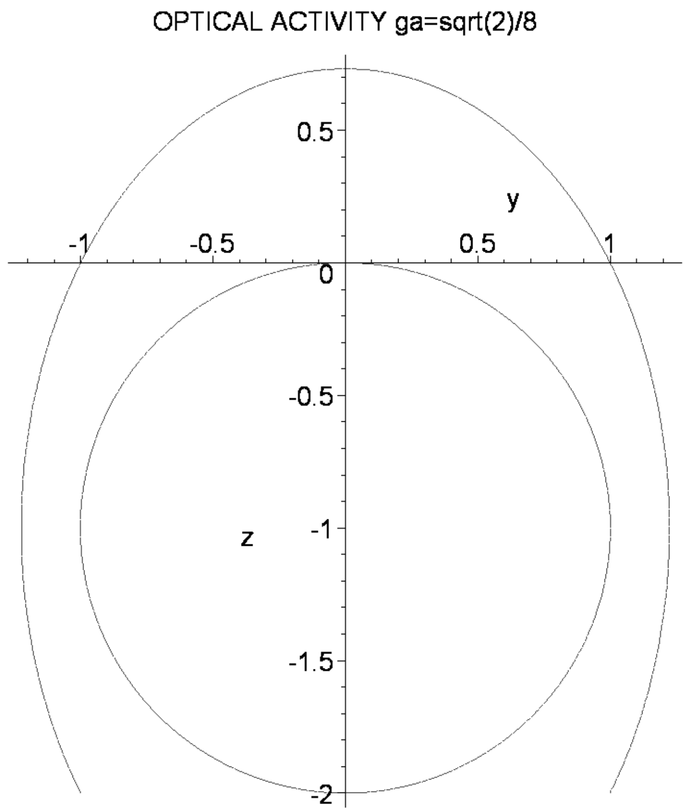
```
>
```

```
> KUMMX:=subs(x=0,KUMMSQ);
```

$$KUMMX := 6\,y^2\,z + 4\,z^3 + 2\,z^2 - 2\,y^2 - 4\,z + z^4 + 2\,y^4 + 3\,z^2\,y^2$$

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/8`);
```

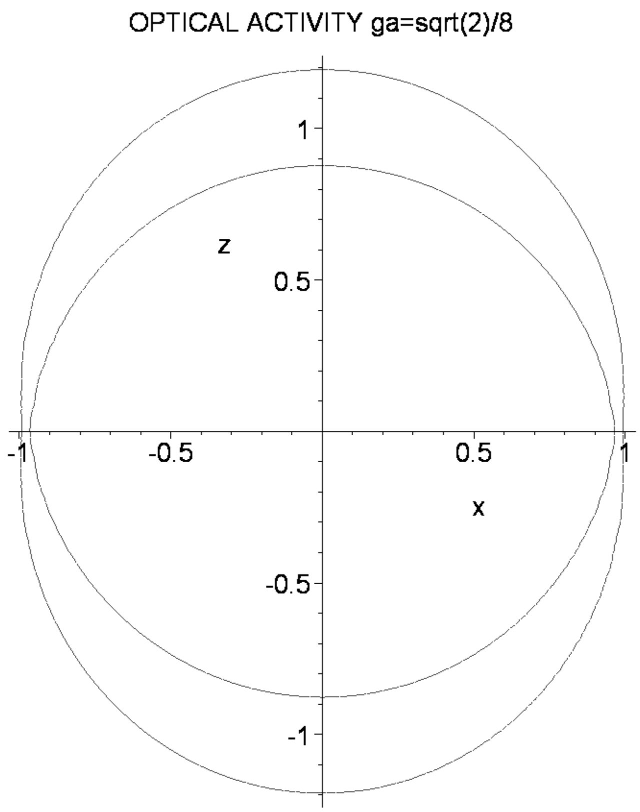


```
> KUMMY:=subs(y=.0,KUMM);
```

$$KUMMY := 1.000000000 - 2.090000000\, x^2 - 2.000000000\, z^2 + 1.090000000\, x^4 + 2.000000000\, z^2\, x^2 + .9100000001\, z^4$$

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-2.0..2.0,z=-2..2,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/8`);
```

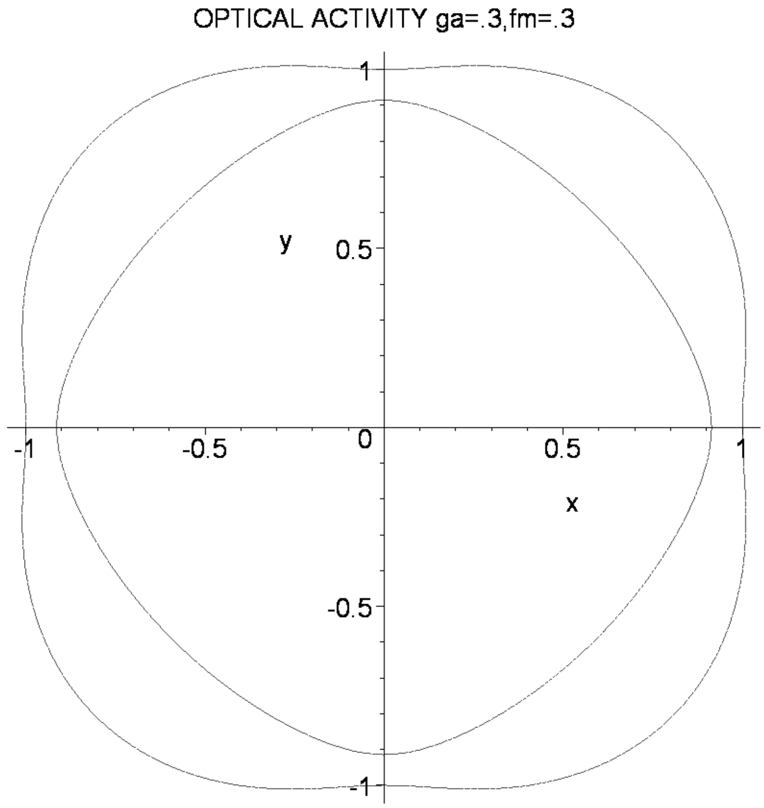


```
> KUMMZ:=subs(z=.0,KUMM);
```

$$KUMMZ := .9100000001 - 2.000000000\,x^2 - 2.000000000\,y^2 + 1.090000000\,x^4 + 1.090000000\,y^4 + 1.820000000\,x^2\,y^2$$

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ=0,x=-2.0..2.0,y=-2..2,numpoints =
  50000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=.3,fm=.3`);
```

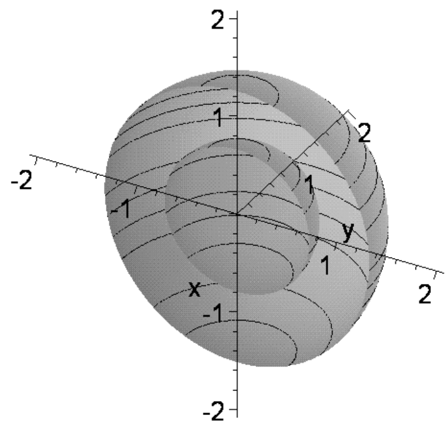


Optical Activity gives Two concentric spheres

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM=0,x=-2..2,y=-0..2,z=-2..2,shading=XYZ,axes=NOR
  MAL,style=PATCHCONTOUR,scaling=CONSTRAINED,numpoints=100000,orient
  ation=[-61,59],title=`OPTICAL ACTIVITY pa=.3, fm=.3`);
>
>
```

OPTICAL ACTIVITY pa=.3, fm=.3



>

Example 3 Optical Activity combined with magnetic Faraday,

>

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMIL;
```

HAMRED := HAMIL

Set anisotropic coefficients:

```
> HAMA:=subs(A=1,B=1,C=1,a=1,b=1,c=1,HAMRED):eval(%):HAMA:=factor(HA
  MA);
>
```

HAMA := HAMIL

Set the numeric values for the coefficients

```
> HAM:=subs(fda=0,ra=0,pa=0,ga=.3,fma=1,HAMA);eval(%);
```

$$HAM := HAMIL$$

$$HAMIL$$

Coefficients were chosen to display chiral vacuum effect.

```
> KUMM:=factor(HAM);LORENTZ:=(x^2+y^2+z^2)^2-1;Difference:=factor(KU
  MM-LORENTZ);
```

$$KUMM := HAMIL$$

$$LORENTZ := (x^2 + y^2 + z^2)^2 - 1$$

$$Difference := HAMIL - x^4 - 2\, x^2\, y^2 - 2\, z^2\, x^2 - y^4 - 2\, z^2\, y^2 - z^4 + 1$$

```
>
```

Now for an x=0 section to get more detail.

```
> KUMMX:=subs(x=0,KUMM);
```

$$KUMMX := HAMIL$$

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-4.0..4.0,z=-4..4,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/4 \n
  with FARADAY fa=sqrt(2)/4`);

>
Error, (in implicitplot) could not evaluate expression
```

```
> KUMMY:=subs(y=0,KUMM);
```

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-4.0..4.0,z=-4..4,numpoints =
  15000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/4 \n
  with FARADAY fa=sqrt(2)/4`);

>
```

```
> KUMMZ:=factor(subs(z=0,KUMM));
```

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ=0,x=-2.0..2.0,y=-2..2,numpoints =
  5000,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/4 \n
  with FARADAY fa=sqrt(2)/4`);
```

```
>
```

It would appear that the propagation velocity exceeds c for the inner sphere?

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM=0,x=-2..2,y=-0..2,z=-2..2,shading=XYZ,lightmodel=light3,axes=NORMAL,style=PATCHCONTOUR,scaling=CONSTRAINED,title=`OPTICAL ACTIVITY ga=sqrt(2)/4 \n with FARADAY fa=sqrt(2)/4 \n 4 Distinct wave
  speeds`,numpoints=50000,orientation=[-69,62]);R4:=subs(y=0,x=0,KUMM):R4x:=subs(subs(z=x,R4)):ZaxisRoots:=allvalues(RootOf(R4x=0));RootOf(R4x=0);
> evalf(allvalues(RootOf(R4x=0)));
>
>
>
>
>
```

Example 4 Anisotropic Ferrite,

```
> HAMIL:=subs(fda=0,pa=0,ra=0,HAMILTONIAN);eval(%);
```

Reduced Fresnel Kummer quartic polynomial

```
> HAMRED:=HAMIL;
```

Set anisotropic coefficients:

```
> HAMA:=subs(A=1,B=1,C=1,a=5/4,b=5/4,c=1/2,HAMRED):eval(%):HAMA:=factor(HAMA);
>
```

Set the numeric values for the coefficients

```
> HAM:=subs(fda=0,ra=0,pa=0,ga=0*2^(1/2)/4,fma=2^(1/2)/4,HAMA);eval(%);
```

Coefficients were chosen to display chiral vacuum effect.

```
> KUMM:=factor(HAM);LORENTZ:=(x^2+y^2+z^2)^2-1;Difference:=factor(KUMM-LORENTZ);
>
```

Now for an x=0 section to get more detail.

```
> KUMMX:=subs(x=0,KUMM);
```

X=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMX=0,y=-4.0..4.0,z=-4..4,numpoints =
  15000,scaling=CONSTRAINED,title=`Ferrite with a=b=5/4,c=1/2
  fa=sqrt(2)/4`);

>
```

```
> KUMMY:=subs(y=0,KUMM);
```

Y=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMY=0,x=-4.0..4.0,z=-4..4,numpoints =
  15000,scaling=CONSTRAINED,title=`Ferrite with a=b=5/4,c=1/2
  fa=sqrt(2)/4`);

>
```

```
> KUMMZ:=factor(subs(z=0,KUMM));
```

Z=0 section of Fresnel Kummer Wave Vector Surface

```
> implicitplot(KUMMZ=0,x=-2.0..2.0,y=-2..2,numpoints =
  5000,scaling=CONSTRAINED,title=`Ferrite with a=b=5/4,c=1/2
  fa=sqrt(2)/4`);
```

>

It would appear that the propagation velocity exceeds c for the inner sphere?

3-D Plot of Fresnel Wave Vector Surface

```
> implicitplot3d(KUMM=0,x=-2..2,y=-0..2,z=-2..2,shading=XYZ,lightmod
el=light3,axes=NORMAL,style=PATCHCONTOUR,scaling=CONSTRAINED,title
=`Ferrite with a=b=5/4,c=1/2
fa=sqrt(2)/4`,numpoints=50000,orientation=[-69,62]);R4:=subs(y=0,x
=0,KUMM):R4x:=subs(subs(z=x,R4)):ZaxisRoots:=allvalues(RootOf(R4x=
0));
>
```


>

>

>