```
> restart: with (linalg):with(liesymm):with(difforms):
> setup(x,y,z,t):defform(x=0,y=0,z=0,t=0,Vx=0,Vy=0,Vz=0,D1=0,D2=0,D3=0,Ax=0,Ay=
  0,Az=0,C=0,Phi=0,a=const,b=const,c=const,Lx=0,Ly=0,Lz=0);
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for close
Warning, new definition for `&^`
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

Two types of AFFINE Transformations

and two types of geometrical torsion

vs

TOPOLOGICAL TORSION

as subsets of

Projective Transformations

used as Frame Fields or the Repere Mobile

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Introduction: In this article, examples will be given of several Frame Fields imposed upon R4. R4 will be partitioned into a space part {x,y,z} and a time part {t} such that the Frame matrix will be composed of 3 (interior) "tangent vectors and 1 (exterior) normal vector Maple will be used to compute the equations of structure, the curvature 2-forms, the Cartan matrix (sometimes called the spin connection under certain constraints), the Vierbein, and the two types of torsion 2-forms, relative to the partition. The interior curvature induced (if any) is the curvature of the space part. Certain Frame fields induce (possibly) a set of non-zero curvature 2-forms, and two sets of torsion 2-forms. Remarkably, if the functions that make up the Frame Matrix are C2 differentiable, then these sets of two forms can be found by algebraic methods. The details of the matrix approach are summarized at http://www22.pair.com/csdc/pdf/projfram.pdf.

If the Frame matrix of 16 parameters (functions) on R4 is constrained by 1 constraint such that the determinant of the matrix is non-zero, than that matrix is defined as a **projective transformation**, and the domain that satisfies such a constraint is a projective domain. The opposite occurs when the determinant vanishes, for then the domain that satisfies the zero determinant condition is of topological dimension 3 rather than dimension 4. Such domains are singular domains relative to the Projective domain. If the determinant is never zero, then the 4D domain is orientable. The non-orientable domain implies that the determinant goes through zero and takes on negative and positive values. The domain is connected but not orientable. If the determinant is never zero it could be positive on one component and negative on another. But then the two components of the domain are not connectable. In another situation the determinant is positive definite, and the domain is connectable. It is important to determine if the non-zero determinant is on a connect domain or a disconnected domain.

Affine maps are subsets of projective matrices that can be created by the elimination of 3 of the 16 parameters, accomplished by setting to zero three of the 16 elements of the General linear group on 4 dimensions. The resulting group has 13 parameters elements. It is usual to continue to insist that the matrix determinant is non-zero. There exist two realizations of such constraints. One such realization leads to the standard Affine group, which will be renamed herein the **ParticleAFFINE** group (for reasons that will be come obvious below). The second, non-standard, and often ignored realization is designated herein as the **WaveAFFINE** group. The format of these matrices is presented below. Symbolically, the matrices of the two distinct groups of three constraints are transposes of one another, but the matrix groups are different. When both sets of constraints are imposed simultaneously, the Frame Matrices become related to groups of 10 parameters, and these frames are the basis of the group of similarity transformations .

One objective of this article is to show the limitations of a ParticleAFFINE map, and to show how such maps (can) lead to the first set of torsion 2-forms usually discussed in the literature.. A second objective is to show how a 1-form of Action, A, emerges in the general case, and how this 1-form is related to the ParticleAFFINE torsion 2-forms. If this Action 1-form is zero, then the ParticalAFFINE torsion 2-forms vanish. The concept of topological torsion A^dA depends upon whether or not this Action 1-form A is integrable or not. The examples demonstrate the relationship and differences between what has been called affine torsion and Topological Torsion. ParticleAffine maps preserve parallelism.

A similar situation arises in the WaveAFFINE maps, but here the crucial 1-form is designated by Omega. If (Big) Omega vanishes the WaveAFFINE torsion 2 forms vanish (compare: if (little) omega vanishes, the ParticleAFFINE torsion 2-forms vanish).

The (interior) Torsion two forms vanish if the differential of the normal field does not have components along the tangent vectors. If the Frame field is orthonormal, them the WaveAFFINE torsion 2-forms vanish. Unfortunately, this assumption is the ubiquitous assumption of much of the current literature. (A baby gets thrown out with the wash.)

A standard format of a linear affine transformation is represented by a matrix of n x n elements with n-1 constraints (zeros) along the lower row. Such matrices have a group property such that the product of any two affine matrices produces another affine matrix with the zeros preserved.

```
> ParticleAFFINE=array([[a11,a12,a13,Vx],[a21,a22,a23,Vy],[a31,a32,a33,Vz],[0,0,0,1]]);
```

$$ParticleAFFINE = \begin{bmatrix} a11 & a12 & a13 & Vx \\ a21 & a22 & a23 & Vy \\ a31 & a32 & a33 & Vz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There is another group of matrices, with n-1 zeros as constraints, and similar to the affine group, but not identical to it. The two types are distinguished herein by the words ParticleAFFINE and WaveAFFINE groups. The WaveAFFINE group has a representation given by the format:.

```
> WaveAFFINE=array([[a11,a12,a13,0],[a21,a22,a23,0],[a31,a32,a33,0],[Ax,Ay,Az,Phi]]);
```

$$WaveAFFINE = \begin{bmatrix} a11 & a12 & a13 & 0 \\ a21 & a22 & a23 & 0 \\ a31 & a32 & a33 & 0 \\ Ax & Ay & Az & \Phi \end{bmatrix}$$

A simple translation ParticleAFFINE matrix operates on the differential position vector to yield a Vierbein collection of 1 forms:

```
> TransAFFINE:=array([[1,0,0,Vx],[0,1,0,Vy],[0,0,1,Vz],[0,0,0,1]]);PVierbein:=innerprod(TransAFFINE,[dx,dy,dz,dt]);DETTR:=det(TransAFFINE):
```

$$TransAFFINE := \begin{bmatrix} 1 & 0 & 0 & Vx \\ 0 & 1 & 0 & Vy \\ 0 & 0 & 1 & Vz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$PVierbein := [dx + Vx\,dt, dy + Vy\,dt, dz + Vz\,dt, dt]$$

This formual is the classic formula of Sophomore kinematics where it is learned that the "absolute velocity of a point p" is equal to the "velocity of p with respect to center of mass" plus the "relative velocity" of the center of mass to the "absolute" origin. The Translational Affine matrix transforms the differential position vector in the direction of the velocity vector, and leaves the time coordinate alone. Physicists call this a Galilean transformation.

Consider a simplification of the WaveAFFINE matrix defined as the PhaseAFFINE matrix:

```
> PhaseAFFINE:=array([[1,0,0,0],[0,1,0,0],[0,0,1,0],[Ax,Ay,Az,Phi]]);WVierbein:
  =innerprod(PhaseAFFINE,[dx,dy,dz,dt]);
```

$$PhaseAFFINE := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ Ax & Ay & Az & \Phi \end{bmatrix}$$

$$WVierbein := [dx, dy, dz, Ax\,dx + Ay\,dy + Az\,dz + \Phi\,dt]$$

The PhaseAFFINE matrix preserves the spatial parts of the differential position vector, but changes the time component into a differential 1-form of Action

```
> Action:=WVierbein[4];
```

$$Action := Ax\,dx + Ay\,dy + Az\,dz + \Phi\,dt$$

A more general matrix can be represented by the General Linear matrix in 4D

```
> GL:=array([[a11,a12,a13,-Vx],[a21,a22,a23,-Vy],[a31,a32,a33,-Vz],[Ax,Ay,Az,Ph
  i]]);factor(det(GL)):
```

$$GL := \begin{bmatrix} a11 & a12 & a13 & -Vx \\ a21 & a22 & a23 & -Vy \\ a31 & a32 & a33 & -Vz \\ Ax & Ay & Az & \Phi \end{bmatrix}$$

Any of these matrices (which ultimately will be viewed as matrices of functions) can be used as a Projective basis frame over the domain where the inverse does not vanish. Hence a projective constraint is that the determinant of the Frame Field must not be zero. Hence the fabric of space time constrained by some equivalence class of Projective Frame fields, contains a compliment hypersurface that may have many components. If the hypersurfaces are considered to be boundaries, then space time is segmented in projective parts which are not connected. Such questions are related to whether or not the determinant of the Frame matrix is harmonic (satisfies Laplaces equation).

In that which follows, it will be presumed that the the inverse of the Frame Field will be of a simpified or constrained GL format. The cases of a particle Affine restriction will be compared to the Wave Affine restriction. The differential geometry components of the torsion and curvature two forms will be evaluted for both cases. The method clearly demonstrates the differences between the two types of torsion 2-forms (a fact often ignored in the literature).

Note that in the simple examples that follow, the Torsion most often is related to accelerations of one form or another.

```
>
```

THE INVERSE FRAME FFINVD and the Vierbein.

Example 1: Fixed expansion, variable rotation about the z axis, variable velocity along the z axis (Particle Affine)

```
> FFINV:=array([[D2,Lz,0,0],[-Lz,D2,0,0],[0,0,-D1,-Vz],[0,0,0,-Phi]]);
```

$$FFINV := \begin{bmatrix} D2 & Lz & 0 & 0 \\ -Lz & D2 & 0 & 0 \\ 0 & 0 & -D1 & -Vz \\ 0 & 0 & 0 & -\Phi \end{bmatrix}$$

```
> Z:=innerprod(FFINV,[d(x),d(y),d(z),d(t)]):sigma1:=Z[1];sigma2:=Z[2];sigma3:=Z
  [3];omega:=(Z[4]);
```

The Vierbein 1-forms.

$$\sigma 1 := D2\, d(x) + Lz\, d(y)$$

$$\sigma 2 := -Lz\, d(x) + D2\, d(y)$$

$$\sigma 3 := -D1\, d(z) - Vz\, d(t)$$

$$\omega := -\Phi\, d(t)$$

```
> Vol4:=wcollect(simplify(sigma1&^sigma2&^sigma3&^Z[4])):rho:=factor(getcoeff(V
  ol4));
```

The density (determinant)

$$\rho := \Phi\, (D2^2 + Lz^2)\, D1$$

The determinant cannot go to zero for the projective domain. The zero sets of the density function determine a hypersurface. IF the hypersurface is harmonic then it can be a boundary.

There is an induced metric on R4

```
> FF:=inverse(FFINV):Gun:=subs(innerprod(transpose(FF),FF));
```

$$Gun := \begin{bmatrix} \frac{1}{D2^2 + Lz^2} & 0 & 0 & 0 \\ 0 & \frac{1}{D2^2 + Lz^2} & 0 & 0 \\ 0 & 0 & \frac{1}{D1^2} & -\frac{Vz}{D1^2\, \Phi} \\ 0 & 0 & -\frac{Vz}{D1^2\, \Phi} & \frac{Vz^2 + D1^2}{D1^2\, \Phi^2} \end{bmatrix}$$

From the Frame Field use the standard methods to compute the
Cartan Matrix of connection 1-forms.
See http://www.uh.edu/~rkiehn/pdf/projfram.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,
  2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(F
  F[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]):
> cartan:=(evalm(FFINV&*dFF)):
```

The Interior (space-space) Connection 1 forms

```
> Gamma11:=factor(wcollect(cartan[1,1]));Gamma21:=factor(wcollect(cartan[2,1]))
  ;Gamma31:=factor(wcollect(cartan[3,1]));
```

$$\Gamma 11 := -\frac{D2\, d(D2) + Lz\, d(Lz)}{D2^2 + Lz^2}$$

$$\Gamma 21 := \frac{-Lz\, d(D2) + D2\, d(Lz)}{D2^2 + Lz^2}$$

$$\Gamma 31 := 0$$

```
> Gamma12:=factor(wcollect(cartan[1,2]));Gamma22:=factor(wcollect(cartan[2,2]))
  ;Gamma32:=factor(wcollect(cartan[3,2]));
```

$$\Gamma 12 := -\frac{-Lz\, d(D2) + D2\, d(Lz)}{D2^2 + Lz^2}$$

$$\Gamma22 := -\frac{\mathrm{D2}\ \mathrm{d}(\mathrm{D2}) + Lz\ \mathrm{d}(Lz)}{\mathrm{D2}^2 + Lz^2}$$

$$\Gamma32 := 0$$

```
> Gamma13:=factor(wcollect(cartan[1,3]));Gamma23:=factor(wcollect(cartan[2,3]));Gamma33:=factor(wcollect(cartan[3,3]));
```

$$\Gamma13 := 0$$

$$\Gamma23 := 0$$

$$\Gamma33 := -\frac{\mathrm{d}(\mathrm{D1})}{\mathrm{D1}}$$

```
>
```

The "space-time" connection 1-forms are:

```
> hh1:=factor(wcollect(cartan[4,1]));hh2:=factor(wcollect(cartan[4,2]));hh3:=factor(wcollect(cartan[4,3]));
>
```

$$hh1 := 0$$

$$hh2 := 0$$

$$hh3 := 0$$

As all of the hh terms are zero, there will be no curvature induced on the 3-space.

The "time-space connection" 1-forms are

```
> gg1:=factor(wcollect(factor(wcollect(cartan[1,4]))));gg2:=factor(wcollect(cartan[2,4]));gg3:=factor(wcollect(cartan[3,4]));
```

$$gg1 := 0$$

$$gg2 := 0$$

$$gg3 := \frac{-\mathrm{d}(Vz)\ \mathrm{D1} + Vz\ \mathrm{d}(\mathrm{D1})}{\mathrm{D1}\ \Phi}$$

The abnormality (time-time) connection 1-form

```
> Omega:=wcollect(subs(A=a,simplify(wcollect(cartan[4,4]))));
```

$$\Omega := -\frac{\mathrm{d}(\Phi)}{\Phi}$$

```
> L:=factor(wcollect(hh1&^sigma1+hh2&^sigma2+hh3&^sigma3));
```

$$L := 0$$

```
> S:=(wcollect(factor(hh1&^gg1+hh2&^gg2+hh3&^gg3)));
```

$$S := 0$$

There are in general two sets of torsion two forms.

1. Particle AFFINE (**PA**) torsion 2-forms which depend upon the product of little omega (the timelike part of the Vierbein) and the (time-space) connection components, little gamma.

2. WaveAFFINE (**WA**) torsion 2-forms which depend upon Big Omega (the time-time connection component or abnormality) and again the (time-space) connection components, little gamma.

If the time-space connection 1-forms vanish, neither form of torsion exists.

See http://www.uh.edu/~rkiehn/pdf/projfram.pdf

PA TORSION 2-forms

```
> Sigma1:=simplify(subs(wcollect(factor(omega&^gg1))));
  Sigma2:=factor(simplify(wcollect(factor(omega&^gg2))));Sigma3:=simplify(omega&^gg3);
```

$$\Sigma1 := 0$$

$$\Sigma 2 := 0$$

$$\Sigma 3 := \frac{D1\,(d(t)\ \&\wedge\ d(Vz)) - Vz\,(d(t)\ \&\wedge\ d(D1))}{D1}$$

WA TORSION 2-forms

```
> Phi1:=wcollect(subs(factor(Omega&^gg1)));Phi2:=wcollect(factor(Omega&^gg2));Phi3:=wcollect(factor(Omega&^gg3));
```

$$\Phi 1 := 0$$

$$\Phi 2 := 0$$

$$\Phi 3 := \frac{d(\Phi)\ \&\wedge\ d(Vz)}{\Phi^2} - \frac{Vz\,(d(\Phi)\ \&\wedge\ d(D1))}{\Phi^2\,D1}$$

Next compute the matrix of curvature 2-forms on the x,y,z subspace

Curvature 2-forms

```
> Theta:=array([[gg1&^hh1,gg1&^hh2,gg1&^hh3],[gg2&^hh1,gg2&^hh2,gg2&^hh3],[gg3&^hh1,gg3&^hh2,(gg3&^hh3)]]);
>
```

$$\Theta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

DISCUSSION:

The projective frame field does not induce any curvature on the interior set {x,y,z},
if the exterior derivative of the abnormality is zero, the WA torsion is zero.
If the abnormality is not a constant, and if accelerations exist in either the expansion or in the Velocity, then the WA torsion need not be zero.
The PA torsion depends upon the non-uniformity of the Acceleration, d(Vz), and the non-uniformity of the expansion, d(G1). If d(Vz)=Az d(t), and if d(G1)=B1 d(t), then the PA Torsion is zero.
Note that the angular motion Lz does not enter into either of the torsion coeffients.

Example 2: Constant rotation, variable velocity (Particle Affine)

```
> FFINV:=array([[1,c,-b,-Vx],[-c,1,a,-Vy],[b,-a,1,-Vz],[0,0,0,Phi]]);
```

$$FFINV := \begin{bmatrix} 1 & c & -b & -Vx \\ -c & 1 & a & -Vy \\ b & -a & 1 & -Vz \\ 0 & 0 & 0 & \Phi \end{bmatrix}$$

```
> Z:=innerprod(FFINV,[d(x),d(y),d(z),d(t)]):sigma1:=Z[1];sigma2:=Z[2];sigma3:=Z[3];omega:=(Z[4]);
```

The Vierbein 1-forms.

$$\sigma 1 := d(x) + c\,d(y) - b\,d(z) - Vx\,d(t)$$

$$\sigma 2 := -c\,d(x) + d(y) + a\,d(z) - Vy\,d(t)$$

$$\sigma 3 := b\,d(x) - a\,d(y) + d(z) - Vz\,d(t)$$

$$\omega := \Phi\,d(t)$$

```
> Vol4:=wcollect(simplify(sigma1&^sigma2&^sigma3&^Z[4])):rho:=factor(getcoeff(Vol4));
```

The density (determinant)

$$\rho := \Phi\,(1 + c^2 + a^2 + b^2)$$

The determinant cannot go to zero for the projective domain. The zero sets of the density function determine a hypersurface. IF the hypersurface is harmonic then it can be a boundary.

There is an induced metric on R4

```
> FF:=inverse(FFINV):Gun:=subs(innerprod(transpose(FF),FF));
```

$$Gun := \left[\left[\frac{1+a^2}{1+c^2+a^2+b^2}, \frac{b\,a}{1+c^2+a^2+b^2}, \frac{c\,a}{1+c^2+a^2+b^2}, \frac{Vx\,a^2+Vz\,c\,a+a\,b\,Vy+Vx}{\Phi\,(1+c^2+a^2+b^2)}\right] \left[\frac{b\,a}{1+c^2+a^2+b^2}, \frac{1+b^2}{1+c^2+a^2+b^2}, \frac{b\,c}{1+c^2+a^2+b^2}, \frac{b\,Vx\,a+Vz\,b\,c+b^2\,Vy+Vy}{\Phi\,(1+c^2+a^2+b^2)}\right] \left[\frac{c\,a}{1+c^2+a^2+b^2}, \frac{b\,c}{1+c^2+a^2+b^2}, \frac{1+c^2}{1+c^2+a^2+b^2}, \frac{a\,Vx\,c+b\,c\,Vy+Vz\,c^2+Vz}{\Phi\,(1+c^2+a^2+b^2)}\right] \left[\frac{Vx\,a^2+Vz\,c\,a+a\,b\,Vy+Vx}{\Phi\,(1+c^2+a^2+b^2)}, \frac{b\,Vx\,a+Vz\,b\,c+b^2\,Vy+Vy}{\Phi\,(1+c^2+a^2+b^2)}, \frac{a\,Vx\,c+b\,c\,Vy+Vz\,c^2+Vz}{\Phi\,(1+c^2+a^2+b^2)}, \frac{a^2+Vx^2\,a^2+2\,Vx\,a\,b\,Vy+2\,Vz\,c\,a\,Vx+b^2+c^2+b^2\,Vy^2+Vx^2+Vz^2+1+Vy^2+Vz^2\,c^2+2\,Vz\,b\,c\,Vy}{(1+c^2+a^2+b^2)\,\Phi^2}\right]\right]$$

From the Frame Field use the standard methods to compute the
Cartan Matrix of connection 1-forms.
See http://www.uh.edu/~rkiehn/pdf/defects2.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]):
> cartan:=(evalm(FFINV&*dFF)):
```

The Interior (space-space) Connection 1 forms

```
> Gamma11:=factor(wcollect(cartan[1,1]));Gamma21:=factor(wcollect(cartan[2,1]));Gamma31:=factor(wcollect(cartan[3,1]));
```

$$\Gamma11 := 0$$
$$\Gamma21 := 0$$
$$\Gamma31 := 0$$

```
> Gamma12:=factor(wcollect(cartan[1,2]));Gamma22:=factor(wcollect(cartan[2,2]));Gamma32:=factor(wcollect(cartan[3,2]));
```

$$\Gamma12 := 0$$
$$\Gamma22 := 0$$
$$\Gamma32 := 0$$

```
> Gamma13:=factor(wcollect(cartan[1,3]));Gamma23:=factor(wcollect(cartan[2,3]));Gamma33:=factor(wcollect(cartan[3,3]));
```

$$\Gamma13 := 0$$
$$\Gamma23 := 0$$
$$\Gamma33 := 0$$

```
>
```

The "space-time" connection 1-forms are:

```
> hh1:=factor(wcollect(cartan[4,1]));hh2:=factor(wcollect(cartan[4,2]));hh3:=factor(wcollect(cartan[4,3]));
>
```

$$hh1 := 0$$
$$hh2 := 0$$
$$hh3 := 0$$

The "time-space connection" 1-forms are

```
> gg1:=factor(wcollect(factor(wcollect(cartan[1,4]))));gg2:=factor(wcollect(cartan[2,4]));gg3:=factor(wcollect(cartan[3,4]));
```

$$gg1 := \frac{d(Vx)}{\Phi}$$

$$gg2 := \frac{d(Vy)}{\Phi}$$

$$gg3 := \frac{d(Vz)}{\Phi}$$

The abnormality (time-time) connection 1-form

```
> Omega:=wcollect(subs(A=a,simplify(wcollect(cartan[4,4]))));
```

$$\Omega := -\frac{d(\Phi)}{\Phi}$$

```
> L:=factor(wcollect(hh1&^sigma1+hh2&^sigma2+hh3&^sigma3));
```

$$L := 0$$

```
> S:=(wcollect(factor(hh1&^gg1+hh2&^gg2+hh3&^gg3)));
```

$$S := 0$$

There are in general two sets of torsion two forms.

1. Particle AFFINE (**PA**) torsion 2-forms which depend upon the product of little omega (the timelike part of the Vierbein) and the (time-space) connection components, little gamma.

2. WaveAFFINE (**WA**) torsion 2-forms which depend upon Big Omega (the time-time connection component or abnormality) and again the (time-space) connection components, little gamma.

If the time-space connection 1-forms vanish, neither form of torsion exists.

See http://www.uh.edu/~rkiehn/pdf/projfram.pdf

PA TORSION 2-forms

```
> Sigma1:=simplify(subs(Lx=a,Ly=b,Lz=c,wcollect(factor(omega&^gg1))));
  Sigma2:=factor(simplify(wcollect(factor(omega&^gg2))));Sigma3:=simplify(omega
  &^gg3);
```

$$\Sigma 1 := d(t) \&\wedge d(Vx)$$

$$\Sigma 2 := d(t) \&\wedge d(Vy)$$

$$\Sigma 3 := d(t) \&\wedge d(Vz)$$

WA TORSION 2-forms

```
> Phi1:=simplify(wcollect(subs(Lx=a,Ly=b,Lz=c,factor(Omega&^gg1))));Phi2:=wcoll
  ect(factor(Omega&^gg2));Phi3:=wcollect(factor(Omega&^gg3));
```

$$\Phi 1 := -\frac{d(\Phi) \&\wedge d(Vx)}{\Phi^2}$$

$$\Phi 2 := -\frac{d(\Phi) \&\wedge d(Vy)}{\Phi^2}$$

$$\Phi 3 := -\frac{d(\Phi) \&\wedge d(Vz)}{\Phi^2}$$

Next compute the matrix of curvature 2-forms on the x,y,z subspace

Curvature 2-forms

```
> Theta:=array([[gg1&^hh1,gg1&^hh2,gg1&^hh3],[gg2&^hh1,gg2&^hh2,gg2&^hh3],[gg3&
  ^hh1,gg3&^hh2,(gg3&^hh3)]]);
>
```

$$\Theta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

>

DISCUSSION:

The projective frame field does not induce any curvature on the interior set {x,y,z},
Both sets of torsion coefficients depend upon the accererations. (d(Vx) etc.
If the Accelerations are kinematic and depend upon d(t) only, then the PA torsion coefficients are zero.
If the Accerelations are kinematic and the Abnormality is spatially dependent, then the WA torsion coefficients are not zero.

>

Example 3: Variable expansion, variable rotation about the z axis, variable Wave vector a (wave Affine)

```
> FFINV:=array([[D2,Lz,0,0],[-Lz,D2,0,0],[0,0,-D1,0],[Ax,Ay,Az,-Phi]]);
```

$$FFINV := \begin{bmatrix} D2 & Lz & 0 & 0 \\ -Lz & D2 & 0 & 0 \\ 0 & 0 & -D1 & 0 \\ Ax & Ay & Az & -\Phi \end{bmatrix}$$

```
> Z:=innerprod(FFINV,[d(x),d(y),d(z),d(t)]):sigma1:=Z[1];sigma2:=Z[2];sigma3:=Z[3];omega:=(Z[4]);d(omega);
```

The Vierbein 1-forms.

$$\sigma1 := D2\ d(x) + Lz\ d(y)$$

$$\sigma2 := -Lz\ d(x) + D2\ d(y)$$

$$\sigma3 := -D1\ d(z)$$

$$\omega := Ax\ d(x) + Ay\ d(y) + Az\ d(z) - \Phi\ d(t)$$

$$(d(Ax)\ \&\wedge\ d(x)) + (d(Ay)\ \&\wedge\ d(y)) + (d(Az)\ \&\wedge\ d(z)) - (d(\Phi)\ \&\wedge\ d(t))$$

```
> Vol4:=wcollect(simplify(sigma1&^sigma2&^sigma3&^Z[4])):rho:=factor(getcoeff(Vol4));
```

The density (determinant)

$$\rho := \Phi\,(D2^2 + Lz^2)\,D1$$

The determinant cannot go to zero for the projective domain. The zero sets of the density function determine a hypersurface. IF the hypersurface is harmonic then it can be a boundary.

There is an induced metric on R4

```
> FF:=inverse(FFINV):Gun:=subs(innerprod(transpose(FF),FF));
```

$$Gun := \left[\frac{D2^2\,\Phi^2 + Lz^2\,\Phi^2 + Lz^2\,Ay^2 + 2\,Lz\,Ay\,Ax\,D2 + Ax^2\,D2^2}{\Phi^2\,(D2^2 + Lz^2)^2}, \frac{(Lz\,Ay + Ax\,D2)\,(D2\,Ay - Ax\,Lz)}{\Phi^2\,(D2^2 + Lz^2)^2}, -\frac{(Lz\,Ay + Ax\,D2)\,Az}{\Phi^2\,(D2^2 + Lz^2)\,D1}, -\frac{Lz\,Ay + Ax\,D2}{\Phi^2\,(D2^2 + Lz^2)}\right]$$

$$\left[\frac{(Lz\,Ay + Ax\,D2)\,(D2\,Ay - Ax\,Lz)}{\Phi^2\,(D2^2 + Lz^2)^2}, \frac{Lz^2\,\Phi^2 + D2^2\,\Phi^2 + D2^2\,Ay^2 - 2\,Lz\,Ay\,Ax\,D2 + Lz^2\,Ax^2}{\Phi^2\,(D2^2 + Lz^2)^2}, -\frac{(D2\,Ay - Ax\,Lz)\,Az}{\Phi^2\,(D2^2 + Lz^2)\,D1}, -\frac{D2\,Ay - Ax\,Lz}{\Phi^2\,(D2^2 + Lz^2)}\right]$$

$$\left[-\frac{(Lz\,Ay + Ax\,D2)\,Az}{\Phi^2\,(D2^2 + Lz^2)\,D1}, -\frac{(D2\,Ay - Ax\,Lz)\,Az}{\Phi^2\,(D2^2 + Lz^2)\,D1}, \frac{\Phi^2 + Az^2}{D1^2\,\Phi^2}, \frac{Az}{D1\,\Phi^2}\right]$$

$$\left[-\frac{Lz\,Ay + Ax\,D2}{\Phi^2\,(D2^2 + Lz^2)}, -\frac{D2\,Ay - Ax\,Lz}{\Phi^2\,(D2^2 + Lz^2)}, \frac{Az}{D1\,\Phi^2}, \frac{1}{\Phi^2}\right]$$

From the Frame Field use the standard methods to compute the
Cartan Matrix of connection 1-forms.
See http://www.uh.edu/~rkiehn/pdf/projfram.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]):
> cartan:=(evalm(FFINV&*dFF)):
```

The Interior (space-space) Connection 1 forms

```
> Gamma11:=factor(wcollect(cartan[1,1]));Gamma21:=factor(wcollect(cartan[2,1]));Gamma31:=factor(wcollect(cartan[3,1]));
```

$$\Gamma 11 := -\frac{D2\,d(D2) + Lz\,d(Lz)}{D2^2 + Lz^2}$$

$$\Gamma 21 := \frac{-Lz\,d(D2) + D2\,d(Lz)}{D2^2 + Lz^2}$$

$$\Gamma 31 := 0$$

```
> Gamma12:=factor(wcollect(cartan[1,2]));Gamma22:=factor(wcollect(cartan[2,2]));Gamma32:=factor(wcollect(cartan[3,2]));
```

$$\Gamma 12 := -\frac{d(Lz)\,D1\,D2 - Lz\,D1\,d(D2) - Lx\,Lz\,d(Lx) + d(Lz)\,Lx^2}{D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2}$$

$$\Gamma 22 := -\frac{D2\,D1\,d(D2) + Lz\,D1\,d(Lz) + Lx^2\,d(D2)}{D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2}$$

$$\Gamma 32 := -\frac{Lz\,(Lx\,d(D1) - D1\,d(Lx))}{D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2}$$

```
> Gamma13:=factor(wcollect(cartan[1,3]));Gamma23:=factor(wcollect(cartan[2,3]));Gamma33:=factor(wcollect(cartan[3,3]));
```

$$\Gamma 13 := \frac{d(Lx)\,D2^2 - Lx\,D2\,d(D2) - Lx\,Lz\,d(Lz) + d(Lx)\,Lz^2}{D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2}$$

$$\Gamma 23 := \frac{Lx\,(-Lz\,d(D2) + D2\,d(Lz))}{D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2}$$

$$\Gamma 33 := -\frac{D2^2\,d(D1) + Lz^2\,d(D1) + Lx\,D2\,d(Lx)}{D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2}$$

```
>
```

The "space-time" connection 1-forms are:

```
> hh1:=factor(wcollect(cartan[4,1]));hh2:=factor(wcollect(cartan[4,2]));hh3:=factor(wcollect(cartan[4,3]));
>
```

$$hh1 := \frac{D1\,d(\Phi)\,Lz\,Ay + d(\Phi)\,Lx\,D2\,Az + D2\,D1\,d(\Phi)\,Ax - d(Ay)\,Lz\,D1\,\Phi - D2\,Lx\,d(Az)\,\Phi - D2\,d(Ax)\,D1\,\Phi}{(D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2)\,\Phi}$$

$$hh2 := (D2\,D1\,d(\Phi)\,Ay - d(\Phi)\,Lx\,Lz\,Az + d(\Phi)\,Lx^2\,Ay - D1\,d(\Phi)\,Ax\,Lz - D2\,d(Ay)\,D1\,\Phi - d(Ay)\,\Phi\,Lx^2 + Lx\,Lz\,d(Az)\,\Phi + d(Ax)\,Lz\,D1\,\Phi) \Big/ ((D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2)\,\Phi)$$

$$hh3 := -(Az\,d(\Phi)\,D2^2 + Az\,d(\Phi)\,Lz^2 - d(\Phi)\,Lz\,Lx\,Ay - d(\Phi)\,Ax\,Lx\,D2 + Lx\,Lz\,d(Ay)\,\Phi - d(Az)\,D2^2\,\Phi - d(Az)\,Lz^2\,\Phi + D2\,Lx\,d(Ax)\,\Phi) \Big/ ((D2^2\,D1 + Lz^2\,D1 + Lx^2\,D2)\,\Phi)$$

The "time-space connection" 1-forms are

```
> gg1:=factor(wcollect(factor(wcollect(cartan[1,4]))));gg2:=factor(wcollect(cartan[2,4]));gg3:=factor(wcollect(cartan[3,4]));
```

$$gg1 := 0$$

$$gg2 := 0$$

$$gg3 := 0$$

The abnormality (time-time) connection 1-form

```
> Omega:=wcollect(subs(A=a,simplify(wcollect(cartan[4,4]))));
```

$$\Omega := -\frac{d(\Phi)}{\Phi}$$

```
> L:=factor(wcollect(hh1&^sigma1+hh2&^sigma2+hh3&^sigma3));d(omega);
```

$$L := (Az\,(d(\Phi)\ \&\text{^}\ d(z)) - \Phi\,(d(Ay)\ \&\text{^}\ d(y)) - \Phi\,(d(Ax)\ \&\text{^}\ d(x)) - \Phi\,(d(Az)\ \&\text{^}\ d(z)) + Ay\,(d(\Phi)\ \&\text{^}\ d(y)) + Ax\,(d(\Phi)\ \&\text{^}\ d(x)))\,/\,\Phi$$

$$(d(Ax)\ \&\text{^}\ d(x)) + (d(Ay)\ \&\text{^}\ d(y)) + (d(Az)\ \&\text{^}\ d(z)) - (d(\Phi)\ \&\text{^}\ d(t))$$

```
> S:=(wcollect(factor(hh1&^gg1+hh2&^gg2+hh3&^gg3)));
```

$$S := 0$$

There are in general two sets of torsion two forms.

1. Particle AFFINE (**PA**) torsion 2-forms which depend upon the product of little omega (the timelike part of the Vierbein) and the (time-space) connection components, little gamma.

2. WaveAFFINE (**WA**) torsion 2-forms which depend upon Big Omega (the time-time connection component or abnormality) and again the (time-space) connection components, little gamma.

If the time-space connection 1-forms vanish, neither form of torsion exists.

See http://www.uh.edu/~rkiehn/pdf/projfram.pdf

PA TORSION 2-forms

```
> Sigma1:=simplify(subs(wcollect(factor(omega&^gg1))));
  Sigma2:=factor(simplify(wcollect(factor(omega&^gg2))));Sigma3:=simplify(omega&^gg3);
```

$$\Sigma 1 := 0$$

$$\Sigma 2 := 0$$

$$\Sigma 3 := 0$$

WA TORSION 2-forms

```
> Phi1:=wcollect(subs(factor(Omega&^gg1)));Phi2:=wcollect(factor(Omega&^gg2));Phi3:=wcollect(factor(Omega&^gg3));
```

$$\Phi 1 := 0$$

$$\Phi 2 := 0$$

$$\Phi 3 := 0$$

Next compute the matrix of curvature 2-forms on the x,y,z subspace

Curvature 2-forms

```
> Theta:=array([[gg1&^hh1,gg1&^hh2,gg1&^hh3],[gg2&^hh1,gg2&^hh2,gg2&^hh3],[gg3&^hh1,gg3&^hh2,(gg3&^hh3)]]);
>
```

$$\Theta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
>
```

DISCUSSION:

The projective frame field does not induce any curvature on the interior set {x,y,z},

if the exterior derivative of the abnormality is zero, the WA torsion is zero.
If the abnormality is not a constant, and if accelerations exist in either the expansion or in the Velocity, then the WA torsion need not be zero.
The PA torsion depends upon the non-uniformity of the Acceleration, d(Vz), and the non-uniformity of the expansion, d(G1). If d(Vz)=Az d(t), and if d(G1)=B1 d(t), then the PA Torsion is zero.
Note that the angular motion Lz does not enter into either of the torsion coeffients.

>

>

>

>

Example 4: Variable expansion, variable rotation about the z axis, variable Wave vector a (wave Affine)

```
> FFINV:=array([[D2,Lz,0,0],[-Lz,D2,0,0],[0,0,-D1,0],[Ax,Ay,Az,-Phi]]);
```

$$FFINV := \begin{bmatrix} D2 & Lz & 0 & 0 \\ -Lz & D2 & 0 & 0 \\ 0 & 0 & -D1 & 0 \\ Ax & Ay & Az & -\Phi \end{bmatrix}$$

```
> Z:=innerprod(FFINV,[d(x),d(y),d(z),d(t)]):sigma1:=Z[1];sigma2:=Z[2];sigma3:=Z[3];omega:=(Z[4]);d(omega);
```

The Vierbein 1-forms.

$$\sigma1 := D2\ d(x) + Lz\ d(y)$$

$$\sigma2 := -Lz\ d(x) + D2\ d(y)$$

$$\sigma3 := -D1\ d(z)$$

$$\omega := Ax\ d(x) + Ay\ d(y) + Az\ d(z) - \Phi\ d(t)$$

$$(d(Ax)\ \&\wedge\ d(x)) + (d(Ay)\ \&\wedge\ d(y)) + (d(Az)\ \&\wedge\ d(z)) - (d(\Phi)\ \&\wedge\ d(t))$$

```
> Vol4:=wcollect(simplify(sigma1&^sigma2&^sigma3&^Z[4])):rho:=factor(getcoeff(Vol4));
```

The density (determinant)

$$\rho := \Phi\ (D2^2 + Lz^2)\ D1$$

The determinant cannot go to zero for the projective domain. The zero sets of the density function determine a hypersurface. IF the hypersurface is harmonic then it can be a boundary.

There is an induced metric on R4

```
> FF:=inverse(FFINV):Gun:=subs(innerprod(transpose(FF),FF));
```

$Gun :=$

$$\left[\frac{D2^2\,\Phi^2 + Lz^2\,\Phi^2 + Lz^2\,Ay^2 + 2\,Lz\,Ay\,Ax\,D2 + Ax^2\,D2^2}{\Phi^2\,(D2^2+Lz^2)^2}, \frac{(Lz\,Ay + Ax\,D2)\,(D2\,Ay - Ax\,Lz)}{\Phi^2\,(D2^2+Lz^2)^2},\right.$$
$$\left. -\frac{(Lz\,Ay + Ax\,D2)\,Az}{\Phi^2\,(D2^2+Lz^2)\,D1}, -\frac{Lz\,Ay + Ax\,D2}{\Phi^2\,(D2^2+Lz^2)}\right]$$

$$\left[\frac{(Lz\,Ay + Ax\,D2)\,(D2\,Ay - Ax\,Lz)}{\Phi^2\,(D2^2+Lz^2)^2}, \frac{Lz^2\,\Phi^2 + D2^2\,\Phi^2 + D2^2\,Ay^2 - 2\,Lz\,Ay\,Ax\,D2 + Lz^2\,Ax^2}{\Phi^2\,(D2^2+Lz^2)^2},\right.$$

$$-\frac{(\mathrm{D2}\,Ay-Ax\,Lz)\,Az}{\Phi^2\,(\mathrm{D2}^2+Lz^2)\,\mathrm{D1}},\ -\frac{\mathrm{D2}\,Ay-Ax\,Lz}{\Phi^2\,(\mathrm{D2}^2+Lz^2)}\Big]$$

$$\Big[-\frac{(Lz\,Ay+Ax\,\mathrm{D2})\,Az}{\Phi^2\,(\mathrm{D2}^2+Lz^2)\,\mathrm{D1}},\ -\frac{(\mathrm{D2}\,Ay-Ax\,Lz)\,Az}{\Phi^2\,(\mathrm{D2}^2+Lz^2)\,\mathrm{D1}},\ \frac{\Phi^2+Az^2}{\mathrm{D1}^2\,\Phi^2},\ \frac{Az}{\mathrm{D1}\,\Phi^2}\Big]$$

$$\Big[-\frac{Lz\,Ay+Ax\,\mathrm{D2}}{\Phi^2\,(\mathrm{D2}^2+Lz^2)},\ -\frac{\mathrm{D2}\,Ay-Ax\,Lz}{\Phi^2\,(\mathrm{D2}^2+Lz^2)},\ \frac{Az}{\mathrm{D1}\,\Phi^2},\ \frac{1}{\Phi^2}\Big]$$

From the Frame Field use the standard methods to compute the
Cartan Matrix of connection 1-forms.
See http://www.uh.edu/~rkiehn/pdf/projfram.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]):
> cartan:=(evalm(FFINV&*dFF)):
```

The Interior (space-space) Connection 1 forms

```
> Gamma11:=factor(wcollect(cartan[1,1]));Gamma21:=factor(wcollect(cartan[2,1]));Gamma31:=factor(wcollect(cartan[3,1]));
```

$$\Gamma 11 := -\frac{\mathrm{D2}\,\mathrm{d}(\mathrm{D2})+Lz\,\mathrm{d}(Lz)}{\mathrm{D2}^2+Lz^2}$$

$$\Gamma 21 := \frac{-Lz\,\mathrm{d}(\mathrm{D2})+\mathrm{D2}\,\mathrm{d}(Lz)}{\mathrm{D2}^2+Lz^2}$$

$$\Gamma 31 := 0$$

```
> Gamma12:=factor(wcollect(cartan[1,2]));Gamma22:=factor(wcollect(cartan[2,2]));Gamma32:=factor(wcollect(cartan[3,2]));
```

$$\Gamma 12 := -\frac{-Lz\,\mathrm{d}(\mathrm{D2})+\mathrm{D2}\,\mathrm{d}(Lz)}{\mathrm{D2}^2+Lz^2}$$

$$\Gamma 22 := -\frac{\mathrm{D2}\,\mathrm{d}(\mathrm{D2})+Lz\,\mathrm{d}(Lz)}{\mathrm{D2}^2+Lz^2}$$

$$\Gamma 32 := 0$$

```
> Gamma13:=factor(wcollect(cartan[1,3]));Gamma23:=factor(wcollect(cartan[2,3]));Gamma33:=factor(wcollect(cartan[3,3]));
```

$$\Gamma 13 := 0$$

$$\Gamma 23 := 0$$

$$\Gamma 33 := -\frac{\mathrm{d}(\mathrm{D1})}{\mathrm{D1}}$$

```
>
```

The "space-time" connection 1-forms are:

```
> hh1:=factor(wcollect(cartan[4,1]));hh2:=factor(wcollect(cartan[4,2]));hh3:=factor(wcollect(cartan[4,3]));
>
```

$$hh1 := \frac{\mathrm{d}(\Phi)\,Lz\,Ay+\mathrm{d}(\Phi)\,Ax\,\mathrm{D2}-Lz\,\mathrm{d}(Ay)\,\Phi-\mathrm{D2}\,\mathrm{d}(Ax)\,\Phi}{\Phi\,(\mathrm{D2}^2+Lz^2)}$$

$$hh2 := \frac{\mathrm{d}(\Phi)\,\mathrm{D2}\,Ay-\mathrm{d}(\Phi)\,Ax\,Lz-\mathrm{D2}\,\mathrm{d}(Ay)\,\Phi+Lz\,\mathrm{d}(Ax)\,\Phi}{\Phi\,(\mathrm{D2}^2+Lz^2)}$$

$$hh3 := -\frac{Az\,\mathrm{d}(\Phi)-\mathrm{d}(Az)\,\Phi}{\mathrm{D1}\,\Phi}$$

The "time-space connection" 1-forms are

```
> gg1:=factor(wcollect(factor(wcollect(cartan[1,4]))));gg2:=factor(wcollect(cartan[2,4]));gg3:=factor(wcollect(cartan[3,4]));
```

$$gg1 := 0$$

$$gg2 := 0$$

$$gg3 := 0$$

The abnormality (time-time) connection 1-form

```
> Omega:=wcollect(subs(A=a,simplify(wcollect(cartan[4,4]))));
```

$$\Omega := -\frac{d(\Phi)}{\Phi}$$

```
> L:=factor(wcollect(hh1&^sigma1+hh2&^sigma2+hh3&^sigma3));d(omega);
```

$$L := (Az\,(d(\Phi)\ \&\wedge\ d(z)) - \Phi\,(d(Ay)\ \&\wedge\ d(y)) - \Phi\,(d(Ax)\ \&\wedge\ d(x)) - \Phi\,(d(Az)\ \&\wedge\ d(z)) + Ay\,(d(\Phi)\ \&\wedge\ d(y)) + Ax\,(d(\Phi)\ \&\wedge\ d(x)))\,/\,\Phi$$

$$(d(Ax)\ \&\wedge\ d(x)) + (d(Ay)\ \&\wedge\ d(y)) + (d(Az)\ \&\wedge\ d(z)) - (d(\Phi)\ \&\wedge\ d(t))$$

```
> S:=(wcollect(factor(hh1&^gg1+hh2&^gg2+hh3&^gg3)));
```

$$S := 0$$

There are in general two sets of torsion two forms.

1. Particle AFFINE (**PA**) torsion 2-forms which depend upon the product of little omega (the timelike part of the Vierbein) and the (time-space) connection components, little gamma.

2. WaveAFFINE (**WA**) torsion 2-forms which depend upon Big Omega (the time-time connection component or abnormality) and again the (time-space) connection components, little gamma.

If the time-space connection 1-forms vanish, neither form of torsion exists.

See http://www.uh.edu/~rkiehn/pdf/projfram.pdf

PA TORSION 2-forms

```
> Sigma1:=simplify(subs(wcollect(factor(omega&^gg1))));
  Sigma2:=factor(simplify(wcollect(factor(omega&^gg2))));Sigma3:=simplify(omega
  &^gg3);
```

$$\Sigma 1 := 0$$

$$\Sigma 2 := 0$$

$$\Sigma 3 := 0$$

WA TORSION 2-forms

```
> Phi1:=wcollect(subs(factor(Omega&^gg1)));Phi2:=wcollect(factor(Omega&^gg2));P
  hi3:=wcollect(factor(Omega&^gg3));
```

$$\Phi 1 := 0$$

$$\Phi 2 := 0$$

$$\Phi 3 := 0$$

Next compute the matrix of curvature 2-forms on the x,y,z subspace

Curvature 2-forms

```
> Theta:=array([[gg1&^hh1,gg1&^hh2,gg1&^hh3],[gg2&^hh1,gg2&^hh2,gg2&^hh3],[gg3&
  ^hh1,gg3&^hh2,(gg3&^hh3)]]);
>
```

$$\Theta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
>
```

DISCUSSION:

The projective frame field does not induce any curvature on the interior set {x,y,z},
if the exterior derivative of the abnormality is zero, the WA torsion is zero.
If the abnormality is not a constant, and if accelerations exist in either the expansion or in the Velocity, then the WA torsion need not be zero.

The PA torsion depends upon the non-uniformity of the Acceleration, d(Vz), and the non-uniformity of the expansion, d(G1). If d(Vz)=Az d(t), and if d(G1)=B1 d(t), then the PA Torsion is zero.
Note that the angular motion Lz does not enter into either of the torsion coeffients.

```
>
>
```

Example 5: A "simple" asymmetric Projective transformation

```
> FFINV:=array([[1,0,0,0],[0,1,0,0],[0,0,1,Vz],[0,0,Az,-Phi]]);
```

$$FFINV := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & Vz \\ 0 & 0 & Az & -\Phi \end{bmatrix}$$

```
> Z:=innerprod(FFINV,[d(x),d(y),d(z),d(t)]):sigma1:=Z[1];sigma2:=Z[2];sigma3:=Z[3];omega:=(Z[4]);d(omega);
```

The Vierbein 1-forms.

$$\sigma 1 := \mathrm{d}(x)$$
$$\sigma 2 := \mathrm{d}(y)$$
$$\sigma 3 := \mathrm{d}(z) + Vz\, \mathrm{d}(t)$$
$$\omega := Az\, \mathrm{d}(z) - \Phi\, \mathrm{d}(t)$$
$$(\mathrm{d}(Az) \mathbin{\&\wedge} \mathrm{d}(z)) - (\mathrm{d}(\Phi) \mathbin{\&\wedge} \mathrm{d}(t))$$

```
> Vol4:=wcollect(simplify(sigma1&^sigma2&^sigma3&^Z[4])):rho:=factor(getcoeff(Vol4));
```

The density (determinant)

$$\rho := -\Phi - Vz\, Az$$

The determinant cannot go to zero for the projective domain. The zero sets of the density function determine a hypersurface. IF the hypersurface is harmonic then it can be a boundary.

There is an induced metric on R4

```
> FF:=inverse(FFINV):Gun:=subs(innerprod(transpose(FF),FF));
```

$$Gun := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{\Phi^2 + Az^2}{(\Phi + Vz\, Az)^2} & -\dfrac{-\Phi\, Vz + Az}{(\Phi + Vz\, Az)^2} \\ 0 & 0 & -\dfrac{-\Phi\, Vz + Az}{(\Phi + Vz\, Az)^2} & \dfrac{Vz^2 + 1}{(\Phi + Vz\, Az)^2} \end{bmatrix}$$

From the Frame Field use the standard methods to compute the
Cartan Matrix of connection 1-forms.
See http://www.uh.edu/~rkiehn/pdf/projfram.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]):
> cartan:=(evalm(FFINV&*dFF)):
```

The Interior (space-space) Connection 1 forms

```
> Gamma11:=factor(wcollect(cartan[1,1]));Gamma21:=factor(wcollect(cartan[2,1]));Gamma31:=factor(wcollect(cartan[3,1]));
```

$$\Gamma 11 := 0$$
$$\Gamma 21 := 0$$
$$\Gamma 31 := 0$$

```
> Gamma12:=factor(wcollect(cartan[1,2]));Gamma22:=factor(wcollect(cartan[2,2]))
  ;Gamma32:=factor(wcollect(cartan[3,2]));
```

$$\Gamma 12 := 0$$
$$\Gamma 22 := 0$$
$$\Gamma 32 := 0$$

```
> Gamma13:=factor(wcollect(cartan[1,3]));Gamma23:=factor(wcollect(cartan[2,3]))
  ;Gamma33:=factor(wcollect(cartan[3,3]));
```

$$\Gamma 13 := 0$$
$$\Gamma 23 := 0$$
$$\Gamma 33 := -\frac{Az\, \mathrm{d}(Vz)}{\Phi + Vz\, Az}$$

```
>
```

The "space-time" connection 1-forms are:

```
> hh1:=factor(wcollect(cartan[4,1]));hh2:=factor(wcollect(cartan[4,2]));hh3:=fa
  ctor(wcollect(cartan[4,3]));
>
```

$$hh1 := 0$$
$$hh2 := 0$$
$$hh3 := \frac{Az\, \mathrm{d}(\Phi) - \mathrm{d}(Az)\, \Phi}{\Phi + Vz\, Az}$$

The "time-space connection" 1-forms are

```
> gg1:=factor(wcollect(factor(wcollect(cartan[1,4]))));gg2:=factor(wcollect(car
  tan[2,4]));gg3:=factor(wcollect(cartan[3,4]));
```

$$gg1 := 0$$
$$gg2 := 0$$
$$gg3 := \frac{\mathrm{d}(Vz)}{\Phi + Vz\, Az}$$

The abnormality (time-time) connection 1-form

```
> Omega:=wcollect(subs(A=a,simplify(wcollect(cartan[4,4]))));
```

$$\Omega := -\frac{\mathrm{d}(\Phi)}{\Phi + Vz\, Az} - \frac{Vz\, \mathrm{d}(Az)}{\Phi + Vz\, Az}$$

```
> L:=factor(wcollect(hh1&^sigma1+hh2&^sigma2+hh3&^sigma3));d(omega);
```

$$L := \frac{Az\, (\mathrm{d}(\Phi)\ \&\hat{}\ \mathrm{d}(z)) + Az\, Vz\, (\mathrm{d}(\Phi)\ \&\hat{}\ \mathrm{d}(t)) - \Phi\, (\mathrm{d}(Az)\ \&\hat{}\ \mathrm{d}(z)) - \Phi\, Vz\, (\mathrm{d}(Az)\ \&\hat{}\ \mathrm{d}(t))}{\Phi + Vz\, Az}$$

$$(\mathrm{d}(Az)\ \&\hat{}\ \mathrm{d}(z)) - (\mathrm{d}(\Phi)\ \&\hat{}\ \mathrm{d}(t))$$

```
> S:=(wcollect(factor(hh1&^gg1+hh2&^gg2+hh3&^gg3)));
```

$$S := \frac{Az\, (\mathrm{d}(\Phi)\ \&\hat{}\ \mathrm{d}(Vz))}{(\Phi + Vz\, Az)^2} - \frac{\Phi\, (\mathrm{d}(Az)\ \&\hat{}\ \mathrm{d}(Vz))}{(\Phi + Vz\, Az)^2}$$

There are in general two sets of torsion two forms.

1. Particle AFFINE (**PA**) torsion 2-forms which depend upon the product of little omega (the timelike part of the Vierbein) and the (time-space) connection components, little gamma.

2. WaveAFFINE (**WA**) torsion 2-forms which depend upon Big Omega (the time-time connection component or abnormality) and again the (time-space) connection components, little gamma.

If the time-space connection 1-forms vanish, neither form of torsion exists.

See http://www.uh.edu/~rkiehn/pdf/projfram.pdf

PA TORSION 2-forms

```
> Sigma1:=simplify(subs(wcollect(factor(omega&^gg1))));
  Sigma2:=factor(simplify(wcollect(factor(omega&^gg2))));Sigma3:=simplify(omega
  &^gg3);
```

$$\Sigma 1 := 0$$

$$\Sigma 2 := 0$$

$$\Sigma 3 := \frac{Az\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(Vz)) - \Phi\,(\mathrm{d}(t)\ \&\wedge\ \mathrm{d}(Vz))}{\Phi + Vz\,Az}$$

WA TORSION 2-forms

```
> Phi1:=wcollect(subs(factor(Omega&^gg1)));Phi2:=wcollect(factor(Omega&^gg2));P
  hi3:=wcollect(factor(Omega&^gg3));
```

$$\Phi 1 := 0$$

$$\Phi 2 := 0$$

$$\Phi 3 := -\frac{\mathrm{d}(\Phi)\ \&\wedge\ \mathrm{d}(Vz)}{(\Phi + Vz\,Az)^2} - \frac{Vz\,(\mathrm{d}(Az)\ \&\wedge\ \mathrm{d}(Vz))}{(\Phi + Vz\,Az)^2}$$

Next compute the matrix of curvature 2-forms on the x,y,z subspace

Curvature 2-forms

```
> Theta:=array([[wcollect(gg1&^hh1),wcollect(gg1&^hh2),wcollect(gg1&^hh3)],[gg2
  &^hh1,gg2&^hh2,gg2&^hh3],[gg3&^hh1,gg3&^hh2,(gg3&^hh3)]]);
>
```

$$\Theta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{Az\,(\mathrm{d}(Vz)\ \&\wedge\ \mathrm{d}(\Phi))}{(\Phi + Vz\,Az)^2} - \frac{\Phi\,(\mathrm{d}(Vz)\ \&\wedge\ \mathrm{d}(Az))}{(\Phi + Vz\,Az)^2} \end{bmatrix}$$

```
> AMAXWELL:=Az*d(z)-Phi*d(t);FMAXWELL:=d(AMAXWELL);HMAXWELL:=(AMAXWELL&^FMAXWEL
  L);Topological_Torsion:=(Phi*d(Az)-Az*d(Phi))*d(z)&^d(t);Curvature_2_forms:=(
  Phi*d(Az)-Az*d(Phi))*(d(Vz))/(Phi+Vz*Az)^2;PA_torison_2_forms:=AMAXWELL*d(Vz)
  /(Phi+Az*Vz);
```

Compute the Topological Torsion of the Frame relative to the 1-form AMAXWELL

$$AMAXWELL := Az\,\mathrm{d}(z) - \Phi\,\mathrm{d}(t)$$

$$FMAXWELL := (\mathrm{d}(Az)\ \&\wedge\ \mathrm{d}(z)) - (\mathrm{d}(\Phi)\ \&\wedge\ \mathrm{d}(t))$$

$$HMAXWELL := -Az\ \&\wedge(\mathrm{d}(z), \mathrm{d}(\Phi), \mathrm{d}(t)) - \Phi\ \&\wedge(\mathrm{d}(t), \mathrm{d}(Az), \mathrm{d}(z))$$

$$Topological\_Torsion := (\mathrm{d}(Az)\,\Phi - Az\,\mathrm{d}(\Phi))\,(\mathrm{d}(z)\ \&\wedge\ \mathrm{d}(t))$$

$$Curvature\_2\_forms := \frac{(\mathrm{d}(Az)\,\Phi - Az\,\mathrm{d}(\Phi))\,\mathrm{d}(Vz)}{(\Phi + Vz\,Az)^2}$$

$$PA\_torison\_2\_forms := \frac{(Az\,\mathrm{d}(z) - \Phi\,\mathrm{d}(t))\,\mathrm{d}(Vz)}{\Phi + Vz\,Az}$$

NOTE THAT THE CURVATURE 2-form COMPONENTS VANISH IF TOPOLOGICAL TORSION TENSOR IS ZERO.

THe two species of forms have a common factor. No topological torsion, no induced curvature. Note the interpaly between the 1-form of action and the torsion and curvature 2-forms. Note that both sets of 2-forms require the concept of non-uniform velocity , or acceleration.

DISCUSSION:

This projective frame field has curvature that depends upon the "interaction between A and V and the fact that the potential and Vz have orthogonal components , and/or Vz and Az have orthognal components.
if the exterior derivative of the abnormality is zero, the WA torsion is zero.
If the abnormality is not a constant, and if accelerations exist in either the expansion or in the Velocity, then the WA torsion need not be zero.
The PA torsion depends upon the non-uniformity of the Acceleration, d(Vz), and the non-uniformity of the expansion, d(G1). If d(Vz)=Az d(t), and if d(G1)=B1 d(t), then the PA Torsion is zero.
Note that the angular motion Lz does not enter into either of the torsion coeffients.

>

Example 6: An asymmetric Projective transformation

```
> FFINV:=array([[1,0,0,Vx],[0,1,0,Vy],[0,0,1,Vz],[Ax,Ay,Az,-Phi]]);
```

$$FFINV := \begin{bmatrix} 1 & 0 & 0 & Vx \\ 0 & 1 & 0 & Vy \\ 0 & 0 & 1 & Vz \\ Ax & Ay & Az & -\Phi \end{bmatrix}$$

```
> Z:=innerprod(FFINV,[d(x),d(y),d(z),d(t)]):sigma1:=Z[1];sigma2:=Z[2];sigma3:=Z
  [3];omega:=(Z[4]);d(omega);
```

The Vierbein 1-forms.

$$\sigma1 := d(x) + Vx\, d(t)$$

$$\sigma2 := d(y) + Vy\, d(t)$$

$$\sigma3 := d(z) + Vz\, d(t)$$

$$\omega := Ax\, d(x) + Ay\, d(y) + Az\, d(z) - \Phi\, d(t)$$

$$(d(Ax)\ \&\hat{}\ d(x)) + (d(Ay)\ \&\hat{}\ d(y)) + (d(Az)\ \&\hat{}\ d(z)) - (d(\Phi)\ \&\hat{}\ d(t))$$

```
> Vol4:=wcollect(simplify(sigma1&^sigma2&^sigma3&^Z[4])):rho:=factor(getcoeff(V
  ol4));
```

The density (determinant)

$$\rho := -\Phi - Vz\, Az - Vy\, Ay - Ax\, Vx$$

The determinant cannot go to zero for the projective domain. The zero sets of the density function determine a hypersurface. IF the hypersurface is harmonic then it can be a boundary.

There is an induced metric on R4

```
> FF:=inverse(FFINV):Gun:=subs(innerprod(transpose(FF),FF));
```

$$Gun := \left[\frac{\Phi^2 + 2\,\Phi\, Vz\, Az + 2\,\Phi\, Vy\, Ay + Vz^2\, Az^2 + 2\, Vz\, Az\, Vy\, Ay + Vy^2\, Ay^2 + Ax^2\, Vy^2 + Ax^2\, Vz^2 + Ax^2}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2}, \right.$$

$$-\frac{Vx\, Ay\, \Phi + Vx\, Ay\, Vz\, Az + Vx\, Ay^2\, Vy + Ax\, Vy\, \Phi + Ax\, Vy\, Vz\, Az + Ax^2\, Vy\, Vx - Ax\, Vz^2\, Ay - Ay\, Ax}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2},$$

$$-\frac{Vx\, Az\, \Phi + Vx\, Az^2\, Vz + Vx\, Az\, Vy\, Ay - Ax\, Vy^2\, Az + Ax\, Vz\, \Phi + Ax\, Vz\, Vy\, Ay + Ax^2\, Vz\, Vx - Ax\, Az}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2},$$

$$\left. -\frac{-\Phi\, Vx - Vz\, Az\, Vx - Vy\, Ay\, Vx + Ax\, Vy^2 + Ax\, Vz^2 + Ax}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2} \right]$$

$$\left[-\frac{Vx\,Ay\,\Phi + Vx\,Ay\,Vz\,Az + Vx\,Ay^2\,Vy + Ax\,Vy\,\Phi + Ax\,Vy\,Vz\,Az + Ax^2\,Vy\,Vx - Ax\,Vz^2\,Ay - Ay\,Ax}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2},\right.$$
$$\frac{Vx^2\,Ay^2 + \Phi^2 + 2\,\Phi\,Vz\,Az + 2\,\Phi\,Ax\,Vx + Vz^2\,Az^2 + 2\,Vz\,Az\,Ax\,Vx + Ax^2\,Vx^2 + Vz^2\,Ay^2 + Ay^2}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2},$$
$$-\frac{-Vx^2\,Ay\,Az + Vy\,Az\,\Phi + Vy\,Az^2\,Vz + Vy\,Az\,Ax\,Vx + Vz\,Ay\,\Phi + Vz\,Ay^2\,Vy + Vz\,Ay\,Ax\,Vx - Ay\,Az}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2},$$
$$\left.\frac{-Vx^2\,Ay + Ax\,Vx\,Vy + Vz\,Az\,Vy + \Phi\,Vy - Vz^2\,Ay - Ay}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2}\right]$$
$$\left[-\frac{Vx\,Az\,\Phi + Vx\,Az^2\,Vz + Vx\,Az\,Vy\,Ay - Ax\,Vy^2\,Az + Ax\,Vz\,\Phi + Ax\,Vz\,Vy\,Ay + Ax^2\,Vz\,Vx - Ax\,Az}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2},\right.$$
$$-\frac{-Vx^2\,Ay\,Az + Vy\,Az\,\Phi + Vy\,Az^2\,Vz + Vy\,Az\,Ax\,Vx + Vz\,Ay\,\Phi + Vz\,Ay^2\,Vy + Vz\,Ay\,Ax\,Vx - Ay\,Az}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2},$$
$$\frac{Vx^2\,Az^2 + Vy^2\,Az^2 + \Phi^2 + 2\,\Phi\,Vy\,Ay + 2\,\Phi\,Ax\,Vx + Vy^2\,Ay^2 + 2\,Vy\,Ay\,Ax\,Vx + Ax^2\,Vx^2 + Az^2}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2},$$
$$\left.\frac{-Az\,Vx^2 - Az\,Vy^2 + \Phi\,Vz + Vz\,Vy\,Ay + Vz\,Ax\,Vx - Az}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2}\right]$$
$$\left[-\frac{-\Phi\,Vx - Vz\,Az\,Vx - Vy\,Ay\,Vx + Ax\,Vy^2 + Ax\,Vz^2 + Ax}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2}, \frac{-Vx^2\,Ay + Ax\,Vx\,Vy + Vz\,Az\,Vy + \Phi\,Vy - Vz^2\,Ay - Ay}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2}\right.$$
$$\left., \frac{-Az\,Vx^2 - Az\,Vy^2 + \Phi\,Vz + Vz\,Vy\,Ay + Vz\,Ax\,Vx - Az}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2}, \frac{Vx^2 + Vy^2 + Vz^2 + 1}{(\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx)^2}\right]$$

From the Frame Field use the standard methods to compute the
Cartan Matrix of connection 1-forms.
See http://www.uh.edu/~rkiehn/pdf/projfram.pdf
for details of the Cartan method for an arbitrary Repere Mobile.

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3]),d(FF[1,4])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3]),d(FF[2,4])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3]),d(FF[3,4])],[d(FF[4,1]),d(FF[4,2]),d(FF[4,3]),d(FF[4,4])]]):
> cartan:=(evalm(FFINV&*dFF)):
```

The Interior (space-space) Connection 1 forms

```
> Gamma11:=factor(wcollect(cartan[1,1]));Gamma21:=factor(wcollect(cartan[2,1]));Gamma31:=factor(wcollect(cartan[3,1]));
```

$$\Gamma 11 := -\frac{Ax\,\mathrm{d}(Vx)}{\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx}$$
$$\Gamma 21 := -\frac{Ax\,\mathrm{d}(Vy)}{\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx}$$
$$\Gamma 31 := -\frac{Ax\,\mathrm{d}(Vz)}{\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx}$$

```
> Gamma12:=factor(wcollect(cartan[1,2]));Gamma22:=factor(wcollect(cartan[2,2]));Gamma32:=factor(wcollect(cartan[3,2]));
```

$$\Gamma 12 := -\frac{Ay\,\mathrm{d}(Vx)}{\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx}$$
$$\Gamma 22 := -\frac{Ay\,\mathrm{d}(Vy)}{\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx}$$
$$\Gamma 32 := -\frac{Ay\,\mathrm{d}(Vz)}{\Phi + Vz\,Az + Vy\,Ay + Ax\,Vx}$$

```
> Gamma13:=factor(wcollect(cartan[1,3]));Gamma23:=factor(wcollect(cartan[2,3]));Gamma33:=factor(wcollect(cartan[3,3]));
```

$$\Gamma 13 := -\frac{Az\, d(Vx)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

$$\Gamma 23 := -\frac{Az\, d(Vy)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

$$\Gamma 33 := -\frac{Az\, d(Vz)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

\>

The "space-time" connection 1-forms are:

```
> hh1:=factor(wcollect(cartan[4,1]));hh2:=factor(wcollect(cartan[4,2]));hh3:=fa
  ctor(wcollect(cartan[4,3]));
>
```

$$hh1 := \frac{Ax\, d(\Phi) + Ax\, d(Ay)\, Vy + Ax\, Vz\, d(Az) - d(Ax)\, \Phi - d(Ax)\, Vz\, Az - d(Ax)\, Ay\, Vy}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

$$hh2 := \frac{d(\Phi)\, Ay - d(Ay)\, \Phi - d(Ay)\, Vz\, Az - Ax\, d(Ay)\, Vx + Vz\, d(Az)\, Ay + d(Ax)\, Vx\, Ay}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

$$hh3 := \frac{Az\, d(\Phi) + d(Ay)\, Az\, Vy - d(Az)\, \Phi - d(Az)\, Vy\, Ay - d(Az)\, Ax\, Vx + d(Ax)\, Az\, Vx}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

The "time-space connection" 1-forms are

```
> gg1:=factor(wcollect(factor(wcollect(cartan[1,4]))));gg2:=factor(wcollect(car
  tan[2,4]));gg3:=factor(wcollect(cartan[3,4]));
```

$$gg1 := \frac{d(Vx)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

$$gg2 := \frac{d(Vy)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

$$gg3 := \frac{d(Vz)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

The abnormality (time-time) connection 1-form

```
> Omega:=wcollect(subs(A=a,simplify(wcollect(cartan[4,4]))));
```

$$\Omega := -\frac{d(\Phi)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx} - \frac{Vy\, d(Ay)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx} - \frac{Vz\, d(Az)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx} - \frac{Vx\, d(Ax)}{\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx}$$

```
> L:=factor(wcollect(hh1&^sigma1+hh2&^sigma2+hh3&^sigma3));d(omega);
```

$$\begin{aligned}L := -(&-Az\,(d(\Phi)\ \&\!\wedge\ d(z)) - Vy\, Az\,(d(Ay)\ \&\!\wedge\ d(z)) - Ax\, Vx\,(d(\Phi)\ \&\!\wedge\ d(t)) - Az\, Vz\,(d(\Phi)\ \&\!\wedge\ d(t))\\ &- Ay\, Vy\,(d(\Phi)\ \&\!\wedge\ d(t)) + \Phi\,(d(Ay)\ \&\!\wedge\ d(y)) + Vz\, Az\,(d(Ay)\ \&\!\wedge\ d(y)) + Ax\, Vx\,(d(Ay)\ \&\!\wedge\ d(y))\\ &+ \Phi\,(d(Ax)\ \&\!\wedge\ d(x)) + Vz\, Az\,(d(Ax)\ \&\!\wedge\ d(x)) + Vy\, Ay\,(d(Ax)\ \&\!\wedge\ d(x)) + \Phi\,(d(Az)\ \&\!\wedge\ d(z))\\ &+ Vy\, Ay\,(d(Az)\ \&\!\wedge\ d(z)) + Ax\, Vx\,(d(Az)\ \&\!\wedge\ d(z)) - Ay\,(d(\Phi)\ \&\!\wedge\ d(y)) - Ax\,(d(\Phi)\ \&\!\wedge\ d(x))\\ &- Ax\, Vy\,(d(Ay)\ \&\!\wedge\ d(x)) - Vx\, Az\,(d(Ax)\ \&\!\wedge\ d(z)) - Vx\, Ay\,(d(Ax)\ \&\!\wedge\ d(y)) + \Phi\, Vx\,(d(Ax)\ \&\!\wedge\ d(t))\\ &- Ax\, Vz\,(d(Az)\ \&\!\wedge\ d(x)) + \Phi\, Vy\,(d(Ay)\ \&\!\wedge\ d(t)) + \Phi\, Vz\,(d(Az)\ \&\!\wedge\ d(t)) - Vz\, Ay\,(d(Az)\ \&\!\wedge\ d(y)))\,/\,(\\ &\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)\end{aligned}$$

$$(d(Ax)\ \&\!\wedge\ d(x)) + (d(Ay)\ \&\!\wedge\ d(y)) + (d(Az)\ \&\!\wedge\ d(z)) - (d(\Phi)\ \&\!\wedge\ d(t))$$

```
> S:=(wcollect(factor(hh1&^gg1+hh2&^gg2+hh3&^gg3)));
```

$$\begin{aligned}S := &\frac{Ax\, Vy\,(d(Ay)\ \&\!\wedge\ d(Vx))}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2} + \frac{(-\Phi - Vz\, Az - Vy\, Ay)\,(d(Ax)\ \&\!\wedge\ d(Vx))}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2}\\ &+ \frac{Ax\,(d(\Phi)\ \&\!\wedge\ d(Vx))}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2} + \frac{Ax\, Vz\,(d(Az)\ \&\!\wedge\ d(Vx))}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2} + \frac{(-\Phi - Vz\, Az - Ax\, Vx)\,(d(Ay)\ \&\!\wedge\ d(Vy))}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2}\\ &+ \frac{Vz\, Ay\,(d(Az)\ \&\!\wedge\ d(Vy))}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2} + \frac{Ay\,(d(\Phi)\ \&\!\wedge\ d(Vy))}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2} + \frac{Vx\, Ay\,(d(Ax)\ \&\!\wedge\ d(Vy))}{(\Phi + Vz\, Az + Vy\, Ay + Ax\, Vx)^2}\end{aligned}$$

$$+\frac{Az\,(d(\Phi)\ \&\wedge\ d(Vz))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{(-\Phi-Vy\,Ay-Ax\,Vx)\,(d(Az)\ \&\wedge\ d(Vz))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vy\,Az\,(d(Ay)\ \&\wedge\ d(Vz))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vx\,Az\,(d(Ax)\ \&\wedge\ d(Vz))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}$$

There are in general two sets of torsion two forms.

1. Particle AFFINE (**PA**) torsion 2-forms which depend upon the product of little omega (the timelike part of the Vierbein) and the (time-space) connection components, little gamma.

2. WaveAFFINE (**WA**) torsion 2-forms which depend upon Big Omega (the time-time connection component or abnormality) and again the (time-space) connection components, little gamma.

If the time-space connection 1-forms vanish, neither form of torsion exists.

See http://www.uh.edu/~rkiehn/pdf/projfram.pdf

PA TORSION 2-forms

```
> Sigma1:=simplify(subs(wcollect(factor(omega&^gg1))));
Sigma2:=factor(simplify(wcollect(factor(omega&^gg2))));Sigma3:=simplify(omega
&^gg3);
```

$$\Sigma 1 := \frac{Ax\,(d(x)\ \&\wedge\ d(Vx))+Ay\,(d(y)\ \&\wedge\ d(Vx))+Az\,(d(z)\ \&\wedge\ d(Vx))-\Phi\,(d(t)\ \&\wedge\ d(Vx))}{\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx}$$

$$\Sigma 2 := \frac{Ax\,(d(x)\ \&\wedge\ d(Vy))+Ay\,(d(y)\ \&\wedge\ d(Vy))+Az\,(d(z)\ \&\wedge\ d(Vy))-\Phi\,(d(t)\ \&\wedge\ d(Vy))}{\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx}$$

$$\Sigma 3 := \frac{Ax\,(d(x)\ \&\wedge\ d(Vz))+Ay\,(d(y)\ \&\wedge\ d(Vz))+Az\,(d(z)\ \&\wedge\ d(Vz))-\Phi\,(d(t)\ \&\wedge\ d(Vz))}{\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx}$$

WA TORSION 2-forms

```
> Phi1:=wcollect(subs(factor(Omega&^gg1)));Phi2:=wcollect(factor(Omega&^gg2));P
hi3:=wcollect(factor(Omega&^gg3));
```

$$\Phi 1 := -\frac{d(\Phi)\ \&\wedge\ d(Vx)}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vy\,(d(Ay)\ \&\wedge\ d(Vx))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vz\,(d(Az)\ \&\wedge\ d(Vx))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vx\,(d(Ax)\ \&\wedge\ d(Vx))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}$$

$$\Phi 2 := -\frac{d(\Phi)\ \&\wedge\ d(Vy)}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vy\,(d(Ay)\ \&\wedge\ d(Vy))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vz\,(d(Az)\ \&\wedge\ d(Vy))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vx\,(d(Ax)\ \&\wedge\ d(Vy))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}$$

$$\Phi 3 := -\frac{d(\Phi)\ \&\wedge\ d(Vz)}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vy\,(d(Ay)\ \&\wedge\ d(Vz))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vz\,(d(Az)\ \&\wedge\ d(Vz))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}-\frac{Vx\,(d(Ax)\ \&\wedge\ d(Vz))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}$$

Next compute the matrix of curvature 2-forms on the x,y,z subspace

Curvature 2-forms

```
> Theta:=array([[wcollect(gg1&^hh1),wcollect(gg1&^hh2),wcollect(gg1&^hh3)],[gg2
&^hh1,gg2&^hh2,gg2&^hh3],[gg3&^hh1,gg3&^hh2,(gg3&^hh3)]]);
>
```

$\Theta :=$

$$\left[\frac{Ax\,Vz\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{(-\Phi-Vz\,Az-Vy\,Ay)\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ax\,Vy\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ax\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2},\ \frac{Vz\,Ay\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vx\,Ay\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{(-\Phi-Vz\,Az-Ax\,Vx)\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ay\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2},\ \frac{(-\Phi-Vy\,Ay-Ax\,Vx)\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vx\,Az\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vy\,Az\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Az\,(\mathrm{d}(Vx)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}\right]$$

$$\left[\frac{(-\Phi-Vz\,Az-Vy\,Ay)\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ax\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ax\,Vz\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ax\,Vy\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2},\ \frac{Vx\,Ay\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ay\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vz\,Ay\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{(-\Phi-Vz\,Az-Ax\,Vx)\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2},\ \frac{Vx\,Az\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Az\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{(-\Phi-Vy\,Ay-Ax\,Vx)\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vy\,Az\,(\mathrm{d}(Vy)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}\right]$$

$$\left[\frac{Ax\,Vy\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ax\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ax\,Vz\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{(-\Phi-Vz\,Az-Vy\,Ay)\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2},\ \frac{(-\Phi-Vz\,Az-Ax\,Vx)\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Ay\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vz\,Ay\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vx\,Ay\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2},\ \frac{Vy\,Az\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Ay))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Az\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(\Phi))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{(-\Phi-Vy\,Ay-Ax\,Vx)\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Az))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}+\frac{Vx\,Az\,(\mathrm{d}(Vz)\,\&\wedge\,\mathrm{d}(Ax))}{(\Phi+Vz\,Az+Vy\,Ay+Ax\,Vx)^2}\right]$$

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DISCUSSION:

This projective frame field has curvature that depends upon the "interaction between A and V and the fact that the potential and Vz have orthogonal components , and/or Vz and Az have orthognal components.

if the exterior derivative of the abnormality is zero, the WA torsion is zero.

If the abnormality is not a constant, and if accelerations exist in either the expansion or in the Velocity, then the WA torsion need not be zero.

The PA torsion depends upon the non-uniformity of the Acceleration, d(Vz), and the non-uniformity of the expansion, d(G1). If d(Vz)=Az d(t), and if d(G1)=B1 d(t), then the PA Torsion is zero.

Note that the angular motion Lz does not enter into either of the torsion coeffients.

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Bottom Line: Curvature is not induced on 3 space by either form of the Affine subsets of a Projective transformationan, alone.

1. A first necessary condition for Curvature is the existence of "accelerations in the sense that d(V) is not zero.

2. A second necessary condition is the existence of a non zero Topological Torsion vector which is not collinear with the acceleration. The Topological Torsion vector is computed from the components of the 1-form A.dr - Phi dt

The Projective transformation with both A and V are required to induce curvature on the 3-space.

However, PA torsion can be achieved from Frame fields that are Affine subsets.

Zero curvature and non-zero Torsion are classic examples of Eisenhart-Cartan theory of paths.

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Other Examples : Lorentz transformations

Other Frame fields built upon the generators of Lorentz transformations are to found at http://www22.pair.com/csdc/maple/lorentz.htm

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