Holonomic and Anholonomic Constraints and Coordinates, Frobenius Integrability and Torsion of Various Types

R. M. Kiehn

Emeritus, Phys Dept., Univ. Houston
11/29/99 Updated 08/25/2001
http://www.cartan.pair.com
rkiehn2352@aol.com

Abstract

Discussion and examples of holonomic and anholonomic constraints are presented in the languange of exteror differential forms.

1. Introduction

1.1. Preliminary: differential forms

The topics described below deal with things called Pfaffian equations and differential forms on a domain. Perhaps the simplest way to think of differential forms as to note that they are objects that form the integrands of multi-dimensional integrals. Differential forms, contrary to many exposes, are NOT covariant vectors. They do have coefficient functions that are presumed to transform as covariant tensors.

The most well known differential forms are 1-forms and 2-forms (related to 1 dimensional line integrals and 2 dimensional surface integrals) and N-1 form densities and N form densities (related the integrands of current densities and

density volume elements). An extraordinary feature of differential forms is that they are well behaved in a functional sense as invariants relative to admissable coordinate transformations called diffeomorophisms (which are maps that do not change topology). Perhaps even of greater importance is the fact that differential forms are well behaved in a functional sense with respect to C2 differentiable maps which are not homeomorphisms. Such maps can be used to describe *continuous topological evolution.*

When defined on a differential variety of coordinate functions, the differential forms can separate the domain into two sets. Regions of support (where the differential form is not zero) and the zero sets (regions where the differential form vanishes.) Pfaffian equations are of the latter category; they are essentially differential forms constrained to zero. Pfaff's problem is to find solutions to Pfaffian equations. Differential forms may be exact, closed, integrable or non-integrable. For exact forms, there exists a single global preimage whose exterior derivative reproduces the differential form. For 1-forms, an exact 1-form is represented by the exterior derivative of a unique global zero form, or function. The components (coefficients) of the exact 1-form behave as the components of a gradient covariant vector field. A Pfaffian equation, as an equation of constraint between coefficient functions and differentials of a 1-form, may or may not be exact. If the constraint is exact, then the constraint is defined as a holonomic constraint. All other constraints are called non-holonomic (or equivalently anholonomic) constraints. Pfaffian equations as anholonomic constraints appear throughout physical theories. Sometimes they appear in subtle and unappreciated ways. For example:

Kinematics as an anholonomic constraint A common set of anholonomic constraints (and a set that is usually unappreciated as an anholonomic constrait) is given by the zero set of the kinematic (particle) 1-forms (on the domain $\{q^k, t\}$). That is, consider a C2 map ϕ from $\{q^k, t\} \Rightarrow \{x^k\}$. It is useful to recognize that the map is from a domain (which can be defined as an intitial state) to a range (or a final state). The arguments of the map are the independent variables of the intitial state. Then consider the definitions.

$$\sigma^k \triangleq dx^k - V^k(q,t)dt = \{\partial\phi^k(q,t)/\partial q^m\}dq^m = \Delta^k \qquad (1.1)$$

$$With \quad : \qquad V^k(q,t) \triangleq \{\partial\phi^k(q,t)/\partial t\}. \qquad (1.2)$$

There are 5 cases that will be considered herein.: The 1-forms σ^k are zero, exact, closed, integrable, or non-integrable.

The kinematic hypothesis is that the 1-forms σ^k vanish on the domain of interest, which implies that the "anholonomic fluctuations", Δ^k, must vanish. The constraint implies that

$$\{\partial\phi^k(q,t)/\partial q^m\}dq^m = 0 \tag{1.3}$$

From which it follows that either all of the displacements vanish, $dq^m = 0$, such that the $q^m = $ constant "initial conditions", or the matrix $\{\partial\phi^k(q,t)/\partial q^m\}$ has a zero determinant, and the non-zero displacements define the Null eigenvector of the given matrix. In the latter case the initial conditions are not constants, but are constrained to a singular set. The kinematic constraint is satisfied if the anholonomic fluctuations vanish (the classic case).

A less restrictive constraint is that the 1-forms σ^k are closed, in the sense that $d\sigma^k = d(\Delta^k) = 0$, but $\Delta^k \neq 0$.

$$If\ d\sigma^k = 0,\ then\ dV^k\hat{\ }dt = (\partial\Delta^k/\partial t)\hat{\ }dt = \{\partial^2\phi^k(q,t)/\partial t\partial q^m\}dt\hat{\ }dq^m \Rightarrow 0 \tag{1.4}$$

Then either $t = $ constant, or the "Velocity" functions, V^k, can be functions of t alone. That is there exists a single parameter map from t into the functions, $V^k(t)$. The fluctuation 1-forms Δ^k are not zero, but are independent from time, t. Such is the foundation of the Langevin approach to fluctuations.

The 1-forms σ^k need not be exact perfect differentials, and contain information about certain (1-dimensional) topological properties (or defects) of the domain. Such domains of induced coordinate 1-forms do not support Affine torsion. A necessary requirement for Affine Torsion is that some or all of the induced coordinate 1-forms, σ^k, are not closed.

A third situation and even less restrictive constraint may be posed by the requirement that the Pfaffian equations are integrable. The implication is that there is a set of 3-forms that must vanish, $\sigma^k\hat{\ }d\sigma^k = 0$. Such a rank 3 Pfaffian anholonomic constraint is required if there exists and integrating factor $\lambda^k(x,t;\beta)$ for each σ^k, such that although $d\sigma^k \neq 0$, the product of the integrating factor and the 1-form is closed: $d\{\lambda^k(x,t;\beta)\ \sigma^k\} = 0$. If $\sigma^k\hat{\ }d\sigma^k \neq 0$, then the Pfaffian equations are not uniquely integrable integrable, and if the 3-form is closed $(d\sigma^k\hat{\ }d\sigma^k = 0)$, the 3-forms carry another set of topological properties (or defects) different from the defects generated by a closed, but not exact, system of 1-forms. When $\sigma^k\hat{\ }d\sigma^k \neq 0$, the domain is said to support "Topological Torsion".

It will be demonstrated below that both (1) holonomic coordinates (where each coordinate is a gradient) and (2) closed but not exact coordinates (typical

of multi-valued functions) are NOT to be associated with Frame matrices that produce Affine Torsion.

Thermodynamics as an anholonomic constraint   Cohomology is the study of forms whose difference is exact. The most famous such physical system is that given by the first law of thermodynamics: $Q - W = dU$. Neither the 1-form of heat, Q, nor the 1-form of work, W is (necessarily) closed, but their difference is exact! The first law is another example of a non-holonomic constraint:

$$Q - W - dU = 0. \tag{1.5}$$

Rolling without Friction as an anholonomic constraint   When a ball rolls on a surface without slipping of skidding, there is an anholonomic constraint imposed upon the dynamics. The constraint is often formulated by the Pfaffian equation $\lambda d\theta - dx = 0$.

1.1.1.   Exact 1-forms

A differential form, $\sigma(q) = A_b dq^b$, has a Pfaff dimension, or class, represented by an integer, m, that defines the minimum number of independent functions that are required (over a limited domain) to define the differential form. If the differential form has a global definition in terms of a single function $\phi(q)$ over the entire domain, then the form is said to be exact.

$$\text{An Exact 1-form:} \quad \sigma(q) = d\phi(q) = \{\partial\phi(q)/\partial q^b\}dq^b \tag{1.6}$$

The coefficients of the exact 1-form have the components of a gradient field. Exact forms are globally integrable.

1.1.2.   Closed 1-forms

A closed form is a form where the exterior derivative of the form vanishes. All exact 1-forms are closed (for C2 functions),

$$d\sigma(q) = d(d\phi(q)) = \{\partial^2\phi/\partial q^a \partial q^b - \partial^2\phi/\partial q^b \partial q^a\}dq^a {}^{\wedge}dq^b = 0. \tag{1.7}$$

However, there are 1-forms that are closed, but not exact. Closed, but not exact, 1-forms are integrable over a restricted domain, which may be finite but

not global. As an example, consider the 1-form constructed from two independent functions:

$$Example: \sigma(q) = \{\phi(q)d\chi(q) - \chi(q)d\phi(q)\}/\{a\phi^p + b\chi^p\}^{2/p} \qquad (1.8)$$

The example is closed (has zero "curl") but is not exact. $\qquad d\sigma(q) = 0 \qquad (1.9)$

The Pfaff dimension of a closed but not exact 1-form is 1, but the form cannot be globally represented by a gradient of a single function. The domain contains a topological defect(s), not typical of a Cartesian space. There are closed paths (cycles) that are not shrinkable and are not boundaries. A typical example is given by a planar disc with a central hole. Each hole is represented by a unique closed but not exact (deRham) that says something about the fact that the domain of definition is nto simply connected. Note that if the two functions in the example are taken to be $\phi = \Psi$ and $\chi = \Psi^\dagger$ then the closed (but not exact) 1-form becomes (for $a = 1, b = 1, p = 2$)

$$\sigma(q) = \{\Psi d\Psi^\dagger - \Psi d\Psi\}/(\Psi^\dagger\Psi) = \{\Psi\nabla_m\Psi^\dagger - \Psi\nabla_m\Psi\}dq^m/(\Psi^\dagger\Psi), \qquad (1.10)$$

which is recognized as being proportional to the "probability current" in quantum mechanics. The integrals around closed cycles of such closed but not exact 1-forms are the basis for the Bohm Aharanov effect, and the deRham formulas are the basis for what has been popularized as "flux quantization" or as the "Berry phase".

Closed forms which are not exact are often related to multivalued functions. The closed 1-form given above is not well defined when the denominator goes to zero. If the domain of definition excludes these regions, then the form is well defined, and there exist closed integration chains called cycles (which are not necessarily boundaries) for which the integral of the form along the closed chain is NOT zero. If the integration is performed about a boundary cycle(s) the value of the integral is zero. An exact form integrated about a closed cycle is zero, whether it is a boundary or not. These closed but not exact 1-forms carry topological information (in terms of the excluded points which are interpreted as defects or singularities). As mentioned above. these closed but not exact 1-forms have physical significance as the flux quantum. Similar closed, but not exact

2-forms, have physical significance as the charge quantum. Closed but not exact 3-forms have physical significance as the spin quantum.

See http://www22.pair.com/csdc/pdf/periods.pdf   or   R. M. Kiehn, J. Math Phys **18** 4 (1977)

1.1.3.   Not closed but integrable 1-forms

Even if a 1-form σ is not closed, such that $d\sigma \neq 0$, it still may be true that there exists an integrating factor λ that makes the form closed. That is

$$d(\lambda(q)\,\sigma) = d(\lambda(q)\,A_b dq^b) = d\lambda\,\hat{}\,\sigma + \lambda\,d\sigma \Rightarrow 0 \qquad (1.11)$$

It follows that a necessary condition for the existence of an integrating factor is that

$$\sigma\,\hat{}\,\{d\lambda\,\hat{}\,\sigma + \lambda\,d\sigma\} = \sigma\,\hat{}\,d\lambda\,\hat{}\,\sigma + \lambda\,\sigma\,\hat{}\,d\sigma = 0 + \lambda\,\sigma\,\hat{}\,d\sigma = 0 \qquad (1.12)$$

Hence, the 3-form $\sigma\,\hat{}\,d\sigma$ must vanish if an integrating factor exists. The condition is known as the Frobenius condition of integrability. The integrating factor does not necessarily convert the non-closed 1-form into an exact 1-form.

1.1.4.   Non-integrable 1-forms

If the associated 3-form $\sigma\,\hat{}\,d\sigma$ does not vanish, then an integrating factor does not exist, and the system is said to be non-integrable. The Frobenius condition of integrability fails.

2. FRAME MATRICES

The columns of a matrix of functions for which the determinant is non-zero on a finite domain $\{q^a\}$ can be used as a basis matrix for contravariant vectors. The columns of the frame matrix are treated as contravariant vectors with components designated by the upper (row) index. The rows of the Frame matrix are treated as the components of covariant vectors, with the component index being the lower (column) index. The designations follow in accord with the existence of integral maps from the domain to the target. However, the requirements for a Frame matrix do not require that the Frame be deducible from a map of functions. All that is required is that the Frame matrix form a linearly independent set of basis vectors at every point p. The requirement is that the determinant of the Frame

matrix is non-zero. Points on the domain where the determinant of the Frame matrix is not zero are called defects or singularities.

There are 4 major cases to consider:

1. The Frame matrix maps exact differentials into exact differentials (the holonomic case where the map from range to target is known.)

$$\left[F_a^k(q^b)\right] \circ |dq^a\rangle \Rightarrow |\sigma^k\rangle = |dx^k\rangle \tag{2.1}$$

2. The Frame matrix maps exact differentials into non-exact differentials (non-holonomic)

$$\left[F_a^k(q^b)\right] \circ |dq^a\rangle \Rightarrow |\sigma^k\rangle \tag{2.2}$$

There are three subcases to consider.

(a). The induced 1-forms $|\sigma^k\rangle$ are closed, integrable, or not integrable. Note that the term "non-holonomic" is used to imply that the induced 1-forms are not exact. As will be shown below, both holonomic (exact) and special non-holomomic (closed but not exact) Frames are Affine torsion free.

(b) Non-holonomic is a necessary, but not sufficient, condition to produce Affine torsion. The lack of closure (the 2-form vanishes: $d\sigma^k \neq 0$) is a sufficient requirement to produce Affine torsion. However, in certain circumstances the system may be integrable in the Frobenius sense, such that a new Frame can be constructed in terms of "integrating" factors, and that new frame will be of the types (a) or (b) above. A necessary condition for integrability is that the topological torsion 3-form must vanish: $(\sigma^k {}^{\wedge} d\sigma^k = 0)$.

(c) In the non-integrable cases, $(\sigma^k {}^{\wedge} d\sigma^k \neq 0)$.

3. The Frame matrix maps non-exact differentials into exact differentials

$$\left[F_a^k(q^b)\right] \circ |\omega^a\rangle \Rightarrow |dx^k\rangle \tag{2.3}$$

4. The Frame matrix maps non-exact differentials into non-exact differentials

$$\left[F_a^k(q^b)\right] \circ |\omega^a\rangle \Rightarrow |\sigma^k\rangle \tag{2.4}$$

Only the first case is holonomic, for in that case there exists a functional map from $\{q^a\}$ to $\{x^k\}$. The other cases are anholonomic. The first three cases are special cases of case 4.

7

Note that in *every* case there is a right Cartan connection such that

$$d\left[F_a^k(q)\right] = \left[F_c^k(q)\right] \circ \left[C_a^c(q)\right] = \left[F_c^k(q)\right] \circ \left[\sum_b C_{ab}^c(q)dq^b\right] \qquad (2.5)$$

This result is deduced from the condition that the Frame matrix $\left[F_c^k(q)\right]$ has an inverse $[G_m^c(q)]$. The result of a connection is not imposed as a definition out of the blue. The connection matrix of differential 1-forms is constructed as

$$\left[\sum_b C_{ab}^c(q)dq^b\right] = [C_a^c(q)] = -\left[dG_m^c(q)\right] \circ [F_a^m(q)] = [G_m^c(q)] \circ [dF_a^m(q)] \quad (2.6)$$

The construction is independent from a metric constraint imposed upon the system. It is important to note that the indices that denote the elements of the right Cartan matrix are defined on the initial state, and the arguments are in terms of initial state variables.

Consider the exterior derivative of the Frame matrix, without the imposition of the constraint that the determinant be non-zero. Then for case 2

$$d\left[F_a^k(q^b)\right] \circ |dq^a\rangle = \{\partial F_a^k/\partial q^b - \partial F_b^k/\partial q^a\}dq^b{}^\wedge dq^a \qquad (2.7)$$

When the inverse matrix exists, another expression for the vector of closure 2-forms is obtained as

$$d\left|\sigma^k\right\rangle = [G_k^c(q)] \circ \{\partial F_a^k/\partial q^b - \partial F_b^k/\partial q^a\}dq^b{}^\wedge dq^a = \qquad (2.8)$$

.

Consider the two definitions: (constructed from the connection generated by the Frame matrix)

The **Vector of Affine Torsion 2-forms** is defined as : $|AT^c\rangle = [C_a^c(q)]\ \hat{}\ |dq^a\rangle = \ \left|C_{[ab]}^c dq^{b\,\hat{}}\right.$

$$(2.9)$$

The **Matrix of Curvature 2-forms** is defined as $\quad [\Theta_a^c] = \{[dC_a^c(q)] + [C_b^c(q)]\ \hat{}\ [C_a^b(q)]\}$

$$(2.10)$$

8

2.1. Holonomic systems (Case 1)

The basis assumption is that Frame matrix maps exact differentials into exact differentials.

$$\left[F_a^k(q^b)\right] \circ \left|dq^a\right\rangle \Rightarrow \left|dx^k\right\rangle \tag{2.11}$$

Such a situation exists when there exists an explicit mapping from one set of (coordinate) variables $\{q^a\}$ to another set of variables $\{x^k\}$ is given in terms of C2 differentiable functions,

$$\phi : q^a \Rightarrow x^k = \phi^k(q^a). \tag{2.12}$$

The induced exact coordinate differentials $\left|dx^k\right\rangle$ are related linearly to the exact differentials $\left|dq^a\right\rangle$ by the Jacobian mapping:

$$d\phi : \left|dq^a\right\rangle \Rightarrow \left|dx^k\right\rangle = \left[\partial\phi^k(q^b)/\partial q^a\right] \left|dq^a\right\rangle \tag{2.13}$$

The exact coordinate differentials are said to be **holonomic**, as they have a unique functional pre-image. Note that a necessary condition (but not sufficient) for integrability to exactness is that the exterior derivatives of the coordinate differentials must vanish, $d(dx^k) = 0$. Often the target of the map is presumed to be a (pseudo) euclidean space with a diagonal metric of constants equal to ± 1. The reverse problem of mapping from a (pseudo) euclidean space can also be considered.

For the integrable situations, the Jacobian matrix of functions may be used as a basis frame $\left[F_a^k(q^b)\right]$

$$\left[F_a^k(q^b)\right] = \left[\partial\phi^k(q^b)/\partial q^a\right] \tag{2.14}$$

on subspaces of $\{q^b\}$ where the determinant of the Jacobian matrix does not vanish. As the Frame matrix (in this case the Jacobian matrix) has an inverse on such domains, the exterior derivatives of the Jacobian matrix lead to the concept of a connection $[C]$ that linearly connects the differentials of the basis functions to linear combinations of the basis functions. This concept of a connection is consequence of the linear independence of the column vectors that define any basis Frame matrix, and is not a postulated definition:

$$d\left[F_a^k(q)\right] = \left[F_c^k(q)\right] \circ \left[\sum_{c=1..n} C_{ab}^c(q)dq^b\right]. \tag{2.15}$$

The matrix elements of the right Cartan connection matrix, $[C_a^c(q)]$, are differential 1-forms, $C_{ab}^c dq^b$.

If the functions of the holonomic mapping are C2 differentiable, then by direct computation,

$$
\begin{aligned}
d(dx^k) &= \left[F_c^k(q)\right] \circ [C_a^c(q)] \; \hat{} \; |dq^a\rangle \\
&= \left[F_c^k(q)\right] \circ \left[\sum_{c=1..n} C_{ab}^c(q)dq^b\right] \hat{} \; dq^a \\
&= \left[F_c^k(q)\right] \circ \left|\{C_{ab}^c - C_{ba}^c\}dq^{b}\hat{} dq^a\right\rangle = 0 \qquad (2.16)
\end{aligned}
$$

The vector of two forms based on the anti-symmetric components of the connection has been defined as the vector of Affine torsion 2-forms:

Vector of Affine Torsion 2-forms: $\left|C_{[ab]}^c dq^{b}\hat{} dq^a\right\rangle$ (2.17)

However, this vector of Affine Torsion 2 forms must vanish as d(dx) = 0.

BOTTOM LINE: I*f the exterior differential system has a preimage as a holonomic mapping, the Vector of Affine Torsion 2-forms must vanish.*

(Hence statements about "holonomic affine torsion" are misleading if not false (see Sarfatti Vigier 2000).)

Application of the exterior derivative one more time yields

$$
\begin{aligned}
d(d(dx^k)) &= d\{\left[F_c^k(q)\right] \circ [C_a^c(q)]\} \; \hat{} \; |dq^a\rangle \\
&= \left[F_e^k(q)\right] \circ \{[C_c^e(q)] \; \hat{} \; [C_a^c(q)] + [dC_a^c(q)]\}\hat{} \; |dq^a\rangle \\
&= \left[F_e^k(q)\right] \circ [\Theta^e] \; \hat{} \; |dq^a\rangle \Rightarrow 0. \qquad (2.18)
\end{aligned}
$$

The result is that the 3-form $[\Theta^e] \; \hat{} \; |dq^a\rangle$ must vanish for the holonomic case. This necessary condition does not imply that the matrix of curvature 2-forms must vanish, although that is a possibility.

In summary *The holonomic case is Affine torsion free, but may or may not admit non-zero curvature*

2.2. Anholonomic systems - Case 2

Even without knowledge of the unique integral solutions (which may not exist), it often possible to construct (or impose by other arguments) over a domain a Frame Field, $[F]$. The Frame field acts as a basis matrix of linearly independent functions with arguments on the variety, y^a. (Methods for constructing $[F]$ are described in "The Many Faces of Torsion"). The domain may be finite, but not global. Consider the anholonomic (non-holonomic) case where

$$\left[F_a^k(q^b)\right] \circ \left|dq^a\right\rangle \Rightarrow \left|\sigma^k\right\rangle. \tag{2.19}$$

Multiplying by the inverse frame yields

$$\left|dq^b\right\rangle = \left[G_k^b(q^b)\right] \circ \left|\sigma^k\right\rangle \tag{2.20}$$

Exterior differentiation of the first equation yields

$$\left[F_b^k(q^b)\right] \circ \left[C_a^b(q^b)\right] \hat{} \left|dq^a\right\rangle \Rightarrow d\left|\sigma^k\right\rangle \tag{2.21}$$

which implies that the vector of Affine Torsion 2-forms is given by the expression

$$\left[C_a^b\right] \hat{} \left|dq^a\right\rangle = \left[G_k^b(q^b)\right] \circ d\left|\sigma^k\right\rangle \tag{2.22}$$

The important result is that the vector of Affine Torsion 2-forms will vanish even if the induced 2-forms $\left|\sigma^k\right\rangle$ are closed, not merely exact. Affine Torsion requires that the induced 1-forms are **not closed**.

Exterior differentiation of the second equation yields

$$d\left|dq^b\right\rangle = 0 = \left[dG_k^b(q^b)\right] \circ \left|\sigma^k\right\rangle + \left[G_k^b(q^b)\right] \circ d\left|\sigma^k\right\rangle \tag{2.23}$$

So if the reciprocal frame field is used as a basis frame, there exists a reciprocal right Cartan connection, $\left[D_k^m(q^b)\right]$, such that

$$d\left|dq^b\right\rangle = 0 = \left[G_m^b(q^b)\right] \circ \left\{\left[D_k^m(q^b)\right] \circ \left|\sigma^k\right\rangle + d\left|\sigma^k\right\rangle\right\} \tag{2.24}$$

The term in the curly brackets defines the Vector of Cartan Torsion 2-forms based on $\left|\sigma^k\right\rangle$. It is apparent that this Vector of Cartan torsion 2-forms is zero and consists of two compensating parts, one of which is the Vector of Affine connection 2 forms.

There are four cases to consider depending on the properties of the induced 1-forms:

a. If the non-exact 1-forms are closed, such that $d\left|\sigma^k\right\rangle = 0$, then the vector of affine 2-forms, $[C_a^c(q)] \mathbin{\char`\^} |dq^a\rangle$, must vanish. **This anholonomic domain does not support Affine Torsion**. Moreover the 3-form, $[\Theta_a^c] \mathbin{\char`\^} |dq^a\rangle$ must vanish. A sufficient but not necessary condition is that the matrix of curvature 2-forms is zero. In other words such anholonomic domains could have curvature without torsion.

b. If the non-exact 1-forms are not closed then $d\left|\sigma^k\right\rangle \neq 0$. The constraint implies that this type of anholonomic domain necessarily supports Affine Torsion. A second application of the exterior derivative implies that, again, the 3-form $[\Theta_a^c] \mathbin{\char`\^} |dq^a\rangle$ must vanish. Hence, such anholonomic cases necessarily support Affine Torsion, but may or may not have zero curvature.

c. If the $\left|\sigma^k\right\rangle$ are not closed but integrable, then the vector of 3-forms, $\left|\sigma^k \mathbin{\char`\^} d\sigma^k\right\rangle$ must vanish.

d. If induced 1-forms are not integrable. Such domains are said to support topological torsion. The 3-form $\left|\sigma^k \mathbin{\char`\^} d\sigma^k\right\rangle \neq 0$. In simple terms, Affine Torsion is related to the "curl" of the induced coordinate 1-forms, σ^k. If the 1-forms have zero curl, the domain does not support Affine Torsion. If the 1-forms are integrable, then the topological torsion is zero. Non-zero topological torsion implies that the $\left|\sigma^k\right\rangle$ are not integrable.

2.3. Anholonomic systems - Case 3

Consider the anholonomic (non-holonomic) case where

$$\left[F_a^k(q^b)\right] \circ |\omega^a\rangle \Rightarrow \left|dx^k\right\rangle. \tag{2.25}$$

Exterior differentiation yields

$$d\{\left[F_a^k(q^b)\right] \circ |\omega^a\rangle\} = \left[F_a^k(q^b)\right] \circ \{[C_a^c(q)] \mathbin{\char`\^} |\omega^a\rangle + |d\omega^a\rangle\} = d\left|dx^k\right\rangle \Rightarrow 0 \tag{2.26}$$

It must be true that the vector of Cartan torsion 2-forms based ω^a on must vanish

$$\text{Vector of Cartan Torsion 2-forms:} \quad [C_a^c(q)] \mathbin{\char`\^} |\omega^a\rangle + |d\omega^a\rangle = C_{ab}^c dq^b \mathbin{\char`\^} \omega^a \tag{2.27}$$

A second exterior differentiation leads to the condition:

$$\left[F_a^k(q^b)\right] \circ \{[C_a^c(q)] \,\hat{}\, [C_a^c(q)] \,\hat{}\, |\omega^a\rangle + 2\,[C_a^c(q)] \,\hat{}\, |d\omega^a\rangle + [dC_a^c(q)] \,\hat{}\, |\omega^a\rangle\} = 0 \quad (2.28)$$

or

$$\left[F_a^k(q^b)\right] \circ \{[\Theta_a^c(q)] \,\hat{}\, |\omega^a\rangle + 2\,[C_a^c(q)] \,\hat{}\, |d\omega^a\rangle\} = 0 \quad (2.29)$$

The bracket term 3-form has to vanish under the starting hypothesis, which implies that the curvature component $[\Theta_a^c(q)] \,\hat{}\, |\omega^a\rangle$ can be compensated by the Coriolis term $2\,[C_a^c(q)] \,\hat{}\, |d\omega^a\rangle$, if each component is not identically zero. No conditions are place upon the vector of Affine Torsion 2-forms which is not at all equivalent to the vector of Cartan Torsion 2-forms.

Now the 1-forms $|\omega^a\rangle$ may be composed of linear combinations $|\omega^a\rangle = A_b^a dq^b$ such that

$[C_a^c(q)] \,\hat{}\, |\omega^a\rangle = C_{ae}^c dq^e \,\hat{}\, A_b^a dq^b$

As will be seen (below) when the system is not holonomic, and no unique integral equivalent exists, then the spaces involved can have connections which are not free of affine torsion. The bottom line is that there is a correspondence between integrability and no- torsion. This correspondence will be demonstrated below.

2.4. Suggested Reading

There are three books on Pfaffian systems that cover the topics in some detail. The first is by

Schouten, J. A. and Van der Kulk, W., "Pfaff's Problem and its Generalizations", (Oxford Clarendon Press, 1949)

Another is by

M. Zhitomirski, "Typical Singularities of differential 1-forms and Pfaffian Equations" Mathematical Monographs 113, AMS 1992.

Neither of these texts is easy reading.

A third book of interest in "Exterior Differential Systems" by Griffiths, Bryant, Chern et al. (I will update this reference later.)

A paper by P. Fiziev and H. Kleinert gr-qc/9605046 May 1996 does a detailed job explaining "Anholonomic Transformations of Mechanical Action Principle(s)"

A paper by Manual del Leon and David de Diego "On the geometry of non-holonomic Lagrangian systems" J Math Phys. **37** (7) 1996, p.3389 gives some examples, and a set of references, but is mostly restricted to what are called

semi-holonomic constraints below. Also see "Solving non-Holonomic Lagrangian Dynamics...." Extract Mathematica **11**, 2 1996 p.325, by the same authors

Also see David C. Robinson and W. F. Shadwick "The Griffiths-Bryant algorithm and the Dirac theory of Constraints" Fields Institute Communications **7** 1996, p189

**

3. APPENDIX

3.1. Basic Ideas

First a few definitions.

> Definition: An exterior differential form, ϖ, is a function of independent variables (functions) and the differentials of the independent variables. (There are also a few restrictions of anti-symmetry and homogeneity in the differentials that lead to algebraic closure. These conditions need not be discussed now. See Flanders "Exterior Differential Forms")

$$A\ differential\ form: \quad \varpi = f(x^k, dx^k) \tag{3.1}$$

Example: The classic Cartan Hilbert Action is a (linear) exterior differential form

$$\varpi = f(x^k, dx^k) \Rightarrow \omega = p_k dq^k - H(p_k, q^k, t)dt.$$

> Definition: A constraint is an exterior differential form set equal to zero.

$$\varpi = f(x^k, dx^k) \Rightarrow 0. \tag{3.2}$$

> Definition: The zero set of a differential form (which is homogeneous in the differentials) defines a constraint called a Pfaffian equation. (The differential form, especially when linear in the differentials, often is referred to as the "Pfaffian").

$$\varpi \Rightarrow \omega = f(x^k, dx^k) = \sum_k A_k(x^k)dx^k \Rightarrow 0. \quad Pfaffian\ Equation \tag{3.3}$$

14

Classic examples of Pfaffian equations of constraint are given by the kinematic and dynamic expressions subsumed as axioms of freshman mechanics:

$$dx^k - V^k dt = 0, \quad dV^k - A^k dt = 0, \quad dp_k - F_k dt = 0. \tag{3.4}$$

Definition: A constraint that does not depend upon differentials is said to be a holonomic constraint. A constraint that involves the differentials is said to be an anholonomic or a non-holonomic constraint. If only one differential is involved, then the constraint is called a semi-holonomic constraint.

Example 1 A Holonomic constraint: $f(x^k) - c \Rightarrow x^2 + y^2 + z^2 - 1 = 0$. This constraint, as a function of coordinate (independent) variables, if applied to a mechanical system, implies the motion must be confined to a sphere of unit radius. Holonomic constraints define hypersurfaces which are oriented, global, and often of more than one component. A Mobius band cannot be represented by a holonomic constraint (submersion) on a three dimensional space (It can be represented by a two dimensional immersion into a three dimensional space.)

Example 2 An Anholonomic constraint: $g(x, y, z, dx, dy, dz) = 0$ is a non-holonomic or anholonomic constraint. Such an equation, if linear in the differentials, is defined as a Pfaffian equation. This definition is a bit more general than that definition used in many control theory texts. (see the definition for semi-holonomic constraints given below where the differentials are replaced by velocity functions). The constraint of rolling without slipping is a anholonomic constraint.

$$\lambda d\theta - dx = 0 \tag{3.5}$$

Often it is not appreciated that the kinematic definitions given above are indeed anholonomic constraints on the domain.

Example 3 A Semi Holonomic constraint: $g(x, y, z; V^x, V^y, V^z) = 0$. It is usually assumed that V^x, V^y, V^z are defined as differentials of x,y,z with respect to time. Note that this assumption implies that the system admits the additional anholonomic kinematic constraints mentioned above. Under the assumption that $d\theta - \Lambda dt = 0$ and $dx - V dt = 0$, the anholonomic constraint of rolling without slipping becomes a semi-holonomic constraint.

$$(\lambda\Lambda - V)dt = 0 \supset (\lambda\Lambda - V) = 0. \tag{3.6}$$

15

Particle motion subsumes the kinematic constraints. Wave (or fluid motion) does **not**. Particles and waves are equivalent on domains where the kinematic equations are valid for the velocity field. These domains form a submanifold of the space of coordinates and velocity functions. The space of "waves" is much larger than the space of "particles". (It turns out that when the system is integrable, the two developments are alias-alibi related, and then can be said to be equivalent. A Hamiltonian formulation and a Lagrange formulation are strictly equivalent only when they are integrable. A Lagrange formulation with anholonomic constraints is equivalent to a Hamiltonian formulation if they are of the same class or Pfaff dimension. (see below) As will be shown below, integrability implies that either the differential 1-form can be represented in terms of at most two independent functions of the original independent variables: $\omega = \varphi d\psi$, or $\omega = d\psi$. The associated Pfaffian equations of constraint are the same for either representation. However, the exterior derivative of the first possibility is not zero, where the exterior derivative of the second possibility is zero.

Every differentiable holonomic constraint, $\phi(x^j) - c = 0$, leads (by differentiation) to a non-holonomic Pfaffian equation:

$$d\phi(x^j) = \sum_k (\partial\phi/\partial x^k) dx^k \Rightarrow \sum_k A_k(x^j) dx^k = 0 \qquad (3.7)$$

but not conversely. Differential forms, ω, not equal to zero define a module of vector fields such that $i(\mathbf{V})\omega = 1$. Pfaffian equations (differential forms constrained to zero) define a module of vector fields such that $i(\mathbf{V})\omega = 0$

> Definition: A differential 1-form, $\omega = (\sum_k A_k(x^j) dx^k)$ is said to be integrable on domains where either $\sum_k A_k(x^j) dx^k = d\phi(x^j)$ or $\Theta(x^j)(\sum_k A_k(x^j) dx^k) = d\phi(x^j)$. The function $\Theta(x^j)$ is defined as an integrating factor.

A Pfaffian equation of constraint constructed from an integrable 1-form has a unique direction field, in the sense that the coefficient functions, $A_k(x^j)$, are proportional to the gradient of a single function, $\phi(x^j)$. The hypersurface, $\phi(x^j) = 0$, defines the subspace (N-1 dimensional domain) of points which are isolated from the larger (N dimensional) domain.

Theorem: (Frobenius) An integrable differential 1- form $\omega = A_k(x^j)dx^k$ satisfies the exterior differential equation (an equation of constraint!)

$$\omega \hat{} d\omega = 0. \qquad (3.8)$$

Definition: The 3 -form $\omega \hat{} d\omega$ is defined as the "topological torsion". Integrable 1-forms have zero topological torsion.

The kinematic constraints mentioned above satisfy the Frobenius theorem. As the differentials dx are presumed to be exact, their exterior derivatives vanish. Hence the exterior derivative of the differential form $\omega = dx - Vdt$, whose zero set represents the Pfaffian constraint, has only one component equal to the 2-form, $d\omega = dV^x \hat{} dt$. If the Pfaffian is integrable, it must be true that $d\omega = 0$. Hence this single 2-form must vanish, which implies that $V^x = V^x(t)$ a function of t alone. That is, the kinematic assumption implies that the velocities admit a representation in terms of a single parameter (usually called time) Consider a more general case, where there exists a differentiable map of more than 1-parameter (string or m-branes) from $\{t, y^m\} \Rightarrow x^k = f^k(t, y^m)$. Then

$$\omega = dx^k - \{\partial f^k(t, y^m)/\partial t\}\, dt = \sum \{\partial f^k(t, y^m)/\partial y^j\}\, dy^j \qquad (3.9)$$

This equation has the appearance of the kinematic formula, if the function $\{\partial f^k(t, y^m)/\partial t\}$ is identified with $V^k(t, y^m)$, and if the RHS is zero: $\sum \{\partial f^k(t, y^m)/\partial y^j\}\, dy^j = 0$. If the matrix $[F(t, y^m)] = \{\partial f^k(t, y^m)/\partial y^j\}$ is of maximal rank (det $[F] \neq 0$) then the only possibility is that the y^m are fixed constants in the sense that $dy^m = 0$. If the y^m are not constants (constant initial conditions) then the kinematic equations will have fluctuations where the RHS is not zero.

Definition: A Pfaff sequence of a differential 1-form A consists of a finite number n of ordered non-zero elements, constructed as $\{A, dA, A\hat{}dA, ...\}$. The class, or Pfaff dimension, of the 1-form A at a point $\{x^k\}$ is equal to n, the number of non-zero elements of the sequence.

Example: The differential form $A = y^1 dy^2 + dy^3$ defined on the 4 dimensional domain, $\{y^1, y^2, y^3, y^4\}$ generates the Pfaff sequence,

$$\{A = y^1 dy^2 + dy^3, dA = dy^1 \hat{} dy^2, A\hat{}dA = dy^1 \hat{} dy^2 \hat{} dy^3, dA\hat{}dA = 0\}. \qquad (3.10)$$

The Pfaff dimension of the example is 3. The first zero element in the sequence is the 4 form $dA\hat{}dA$. The class or Pfaff dimension of an integrable form is 2 or less. The proof follows from the Frobenius theorem. The Pfaff dimension in effect determines the minimum number of independent functions which are required to describe the 1-form. Integrable 1-forms can be written in the format $A = \varphi d\psi$ where $\varphi(x^k)$ and $\psi(x^k)$ are independent functions of the independent variables.

Definition: An Anholonomic Kinematic Fluctuation is defined as

$$dx^k - V^k\, dt = \Delta x^k \neq 0. \quad dV^k - A^k\, dt = \Delta V^k \neq 0. \qquad (3.11)$$

The Cartan-Hilbert invariant integral has an Action integrand, A, which is based upon Anholonomic Kinematic Fluctuations. The "momenta" p_k, play the role of Lagrange multipliers.

$$
\begin{aligned}
Action &= L(x,V,t)dt + p_k\Delta x^k = L(x,V,t)dt + p_k(dx^k - V^k\, dt) \quad (3.12)\\
&= p_k dx^k - (p_k V^k - L)dt = p_k dx^k - H(p_k, V^k, x^k, t)dt \qquad (3.13)
\end{aligned}
$$

Note that the "Hamiltonian" function $H(p_k, V^k, x^k, t)$ depends, in general, on the velocity functions, V^k, as well as the Lagrange multipliers, p_k, and the coordinates and time, x^k, t. The Maximum Pfaff dimension of the Action is 2n+2 although though there are what appear to be 3n+1 independent functions used in its construction.

The exterior derivative of the Cartan Hilbert Action is given by the expression

$$dA = (p_k - \partial L/\partial V^k)(\Delta V^k)\hat{}dt + (dp_k - \partial L/\partial x^k dt)\hat{}(\Delta x^k) \qquad (3.14)$$

which demonstrates the influence of the fluctuations (in both velocity and position) on the 2-form, dA. It is apparent that the anholonomic fluctuations in velocity, (ΔV^k), are unimportant if the momenta are presumed to be canonical, $(p_k - \partial L/\partial V^k) \Rightarrow 0$. Intuitively, fluctuations in Velocities are attributed to temperature, where fluctuations in position are associated with pressure.

Definition: Topological torsion of a 1-form of Action, A,(on a 4 dimensional domain) is defined as the 3 form $A\hat{}dA$

A differential 1-form with non-zero topological torsion is of Pfaff dimension 3 or more. The 1-form can be expressed in terms of not less than 3 independent functions. Therefore, according to the Frobenius theorem, the 1-form A which supports a non-zero Topological Torsion 3 form is not uniquely integrable.

3.2. Subtleties

3.2.1. Exactness and Closure

A constrained differential 1-form, A, usually does not have a unique primitive function, ϕ, whose total differential generates a Pfaffian equation to within a factor, Θ. A necessary but not sufficient condition for the existence of the unique global holonomic function is that the exterior derivative of A, or at least $\{\Theta A\}$, must vanish. For $\Theta = 1$, the covariant vector field of components, A_k, must have zero "curl". If A represents a vector potential, then such potentials produce no field intensities, B. In a fluid, one would say that there is no vorticity. (The integrating factor sometimes can be considered as a conformal factor)

When the exterior differential of a form vanishes, $d\omega = 0$, the form is said to be closed, and the union of the form and its closure forms a differential ideal. If the differential form is exact then there is a unique pre-image (in the example, a function ϕ) such that $\omega = d\phi$. What is the difference between and exact form and a closed form? The answer is that on a simply connected domain, there is no difference. However, if the domain is not simply connected (think of holes in a piece of paper) then there is a difference. For each hole there is a closed *but not exact component* to the differential form. As an example consider the 1-form which as it stands is neither exact nor closed.

$$\sigma = (ydx - xdy) \ \ with \ \ d\sigma = 2dy\char`^dx \tag{3.15}$$

Multiply the form σ by the closure factor $1/(\pm x^2 \pm y^2)$.

$$\gamma = \sigma/(\pm x^2 \pm y^2) = (ydx - xdy)/(\pm x^2 \pm y^2) \ \ with \ \ d\gamma = 0 \tag{3.16}$$

The resulting 1-form is now closed but not exact. The domain of support must exclude a small set where the denominator goes to zero. If the signs are the same, the excluded set is a point at the origin. The original euclidean plane now has a hole, and is no longer simply connected. The form γ is called a harmonic form (in the sense of deRham).

What is remarkable is that the integral of a closed and exact form on a cycle, $z1$, is zero, but the integral of a closed but not exact form on a cycle is an integer multiple of some constant.

$$\int_{z1} d\phi = 0 \quad but \quad \int_{z1} \gamma = n \, 2\pi \tag{3.17}$$

19

Gauge conditions as exact differential additions to a 1-form, are trivial. Gauge conditions of the closed but not exact type are NOT trivial. They contain topological information (such as the hole count in a non-simply connected domain. (Bohm-Aharanov, Joukowski airfoil, Meissner expulsion, Sommerfeld quantum conditions, etc.)

These same concepts work for differential forms that are not linear in the differentials. Hence the postulate of electromagnetism that $F - dA = 0$ is a strong topological anholonomic constraint, that says over the domain of support, the 2-form of F (E and B) is exact; that is, the 2-form does not have harmonic parts (although the 1-form, A, can have harmonic parts which are the "flux quanta"). The second postulate of Maxwell electrodynamics is the statement $J - dG = 0$. The idea again is that the 3-form J is exact without harmonic parts. The 2-form G can have harmonic parts, which serve as the charge quanta.

Now consider the topological torsion for the 1-form A which is defined as the 3-form $H = A \hat{\ } F = A \hat{\ } dA$. If $dH = 0$, then the question arises: Does H have harmonic parts? If the answer is yes then the harmonic parts serve as "topological" torsion quanta. A necessary condition for existence of such quanta is that the second Poincare invariant must be zero.

Similarly for the 3-form of topological spin $S = A \hat{\ } G$. The necessary condition for existence of EM spin quanta is that the $dS = 0$, or in other words that the First Poincare invariant must vanish. These points are exemplified at

http://www22.pair.com/csdc/car/carhomep.htm

For an interesting solution to the Maxwell postulates, further constrained by the Lorentz vacuum conditions. See

http://www22.pair.com/csdc/maple/reed21.html

Here, the Torsion field is not closed, but the Spin field is closed. In fact the Spin field in the example is the torsion field multiplied by an integrating factor. One would be led to say torsion is source of spin. However, the solution is a special case and the conclusion is not general, for the next example demonstrates that you can have finite topological torsion with zero topological spin. See

http://www22.pair.com/csdc/maple/reed31.html

3.3. Equations of Motion

Given an arbitrary 1-form of Action, which is not closed, Cartan has shown that the equations of motion generating a vector field V are of a Hamiltonian form if the Lie derivative of the Action, A, with respect to the vector field V is exact.

Theorem: Solutions V to the equation $L_{(V)}A = d(\Theta)$ are Hamiltonian vector fields.

This theorem is equivalent to the statement that the closed integrals of the harmonic components of A are constants of the motion. The number of holes does not change. The 1-dimensional topological property expressed as the harmonic 1-form is an invariant of processes that are generated from a Hamiltonian function.

Writing out the theorem shows that it is a statement in the form of an anholonomic differential constraint

$$i(V)dA - d(\Theta - i(V)A) = 0 \tag{3.18}$$

Use the symbols $W = i(V)dA$ defined as the work 1-form, and $U = i(V)A$ defined as the "internal energy". The work 1-form W is closed and exact. Hamiltonian systems are systems where the Pfaff dimension of the Work 1-form is 1.

3.4. Equations of motion for Non-Hamiltonian dynamics

It is apparent that to find equations of motion for non-Hamiltonian systems, the fundamental anholonomic constraint

$$i(V)dA = d(\Theta - i(V)A) \tag{3.19}$$

must be modified to include harmonic parts, and non-closed parts.

$$W = i(V)dA = d(\Theta - i(V)A) + \gamma + Z \tag{3.20}$$

$$dW = dZ \neq 0 \tag{3.21}$$

The last equation destroys the Helmholtz theorem, and the Poincare even dimensional integrals are no longer evolutionary invariants. An example of such a non-Hamiltonian mechanics was suggested in 1974. See

http://www22.pair.com/csdc/pd2/pd2fre5.htm

The formula is the anholonomic constraint

$$W - \Gamma A = i(V)dA - \Gamma A = 0. \tag{3.22}$$

It is know that this equation requires that the Pfaff dimension of the Action 1-form be even (2n+2). Hence the Pfaff space supports the topological torsion 3-form. Moreover, a unique solution vector V does exist for this problem. In

a space of 4 variables this vector is equivalent to the Torsion current (with components proportional to those of the 3-form of non-zero topological torsion, $A\hat{}dA$). Evolution in the direction of the Torsion current is thermodynamically irreversible, as the heat 1-form, Q, does not satisfy the Frobenius integrability theorem, and therefor does not admit an integrating factor.

3.5. Extremals and the Calculus of variations.

For integrals of the Action around closed loops, the values at the "endpoints" cancel out. Similar constraints are often placed on open integrals, forcing the cancellation of contributions at boundary points. The solutions of the problem are then given by vector fields that generate paths such that

$$L_{(V)} \int_{z1} A = \int_{z1} i(V)dA + \int_{z1} di(V)A \Rightarrow \int_{z1} i(V)dA = 0 \qquad (3.23)$$

It is apparent that the if equation is satisfied (giving the equation for an extremal as the "Lie derivative" of the integral must vanish) then the work 1-form must vanish. The bottom line is that Extremals are associated with anholonomic constraints, $W = f_k dx^k - Pdt = 0$. Extremal solutions say that there are vector fields such that $f_k V^k - P = 0$. This equation is the freshman definition of power as the product of force times velocity.

What is even more remarkable is that this equation, $W = 0$, has solutions only in spaces (as defined by the Pfaff sequence for the Action 1-form) of odd Pfaff dimension, 2n+1. (e.g. State Space).

> **Theorem:** Unique extremals, defined as solutions to the equation i(V)dA=0 for a given A, do not exist on domains of Pfaff dimension 2n+2

The theorem is easy to prove, for if the Pfaff space is a symplectic manifold of even dimension then the 2-form dA has an anti-symmetric matrix representation with no zero eigenvalues. On the other hand if the Pfaff space is an odd dimensional contact manifold, then the anti-symmetric matrix representation of dA has a unique eigen vector with eigen value zero. Hence on 2n+1 Pfaff space, the extremal exists and is unique.

3.6. Homework Problems

1. Discuss the torsion induced by A as compared to the torsion induced by W.

2. Discuss the affine torsion associated with an integrable but not exact coordinate system and how the idea intertwines with conformal maps and dilatations and chirality.

3. Compare Cartan Torsion 2-forms, affine torsion, topological torsion and anholonomic constraints.