A Topological Perspective:

Thermodynamic Irreversibility and the Arrow of Time

Applications of Continuous Topological Evolution

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Basic Ideas

- The presence of matter establishes a **topological structure** on a space-time manifold. (Different from a metric structure.)
- Topological evolution implies a change in topological properties.
- Irreversible thermodynamic evolution requires topological change.

Basic Ideas

- Properties that are independent from size and shape are topological properties. Such objects are deformation invariants.
- Topological evolution can take place continuously or discontinuously.
- Cutting is a discontinuous process. Pasting is a continuous process.

Simple Examples

- A condensed fluid is a thermodynamic phase with a connected components.
- A vapor is a thermodynamic phase with disconnected components.
- A change of phase implies a change of topology. Condensation is continuous. Vaporization is discontinuous.

Emphasis on

Continuous Topological Evolution

- **Continuous Evolution can change Topology.**
- **An arrow of time and thermodynamic irreversibility require Topological Change.**
- **Exterior Differential Forms, unlike tensors, are functionally well behaved - with respect to those C1 maps which are neither diffeomorpisms nor homeomorpisms.**
- **Hence Cartan's methods can be used to describe Continuous Topological Evolution.**

Objectives of CTE

(Continuous Topological Evolution)

- Establish the long sought for connection between Irreversible Thermodynamics and Dynamical Systems -- without Statistics!
- Demonstrate the connection between Thermodynamic Irreversibility and Topological (Pfaff) Dimension 4.

Axioms of CTE

(Continuous Topological Evolution)

- **Topological structure** of Physical Systems is encoded in an Action differential 1-form **A**.
- **Physical Processes** can be defined in terms of contravariant vector direction fields, **V.**
- **Continuous Topological Evolution** is encoded by Cartan's magic formula :

$$L_{(V)}A = i(V)dA + di(V)A$$

($L_{(V)}A$ is the Lie differential with respect to the direction field V acting on the 1-form A)

Theorems of CTE

- **Topological evolution** is a necessary condition for both time asymmetry and thermodynamic irreversibility
- **A unique extremal direction field** which represents a conservative reversible Hamiltonian process always exists on subspaces of topological dimension **2n+1**.
- **A unique torsional direction field** which represents a thermodynamically irreversible process always exists on subspaces of even topological dimension **2n+2**.

Cartan's Magic Formula

Define the exterior differential forms:

Work = W= i(V)dA, Energy = U= i(V)A, Heat = Q.

Then Cartan's Magic Formula of CTE,

$$L_{(V)}A = i(V)dA + di(V)A = Q,$$

becomes the First Law of Thermodynamics,

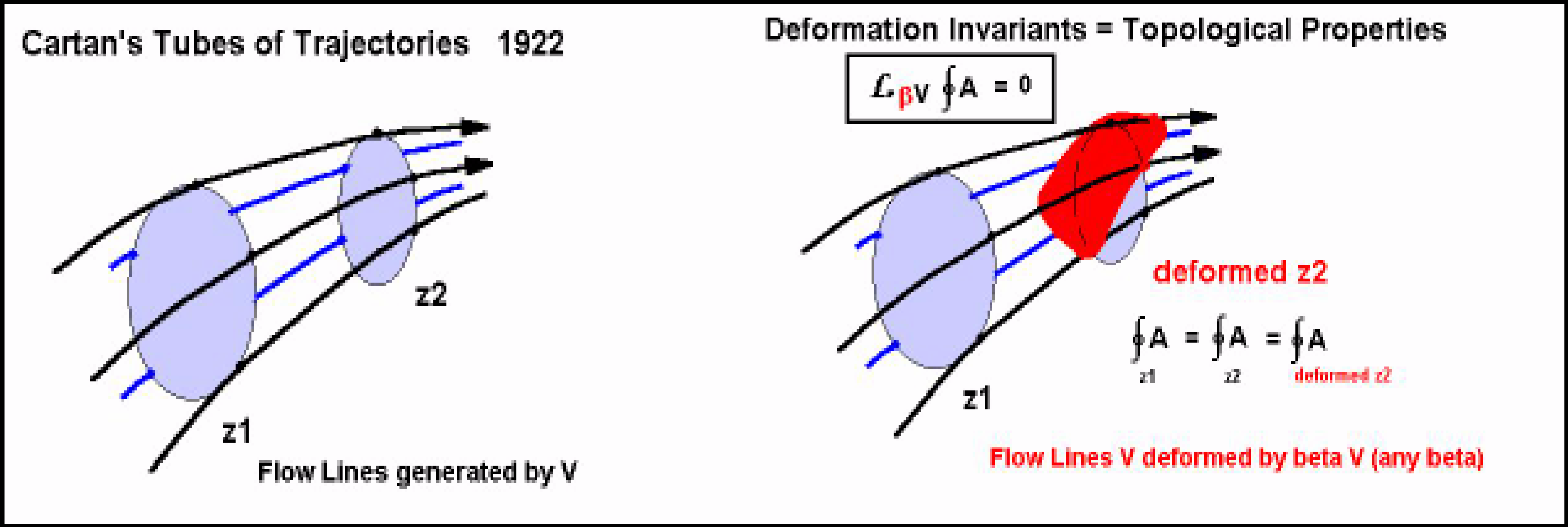
$$L_{(V)}A = W + dU = Q,$$

connecting Dynamical Systems and Thermodynamics.

($L_{(V)}A$ is the Lie differential with respect to the direction field V acting on the 1-form A)

Cartan's Magic Formula

Cartan used an integral version of his Lie derivative formula - in terms of hydrodynamic flow along a tube of trajectories - to prove that all Hamiltonian processes preserve the closed integrals of Action, and conversely.



Thermodynamic Processes page 1

- **Classical Thermodynamics**:

  A Process acting on a Physical System that creates a 1-form of Heat Q, is an irreversible process unless Q admits an integrating factor.

- **Frobenius Theorem**:

  An integrating factor exists iff $Q \wedge dQ = 0$.
  (Pfaff dimension of $Q < 3$.)

Thermodynamic Processes page 2

For a Process V acting on a Physical System represented by a 1-form of Action, A:

A Dynamical Test for a Reversible Process is

$$L_{(V)}A \wedge L_{(V)}\, dA = Q \wedge dQ = 0.$$

A Dynamical Test for an Irreversible Process is

$$L_{(V)}A \wedge L_{(V)}\, dA = Q \wedge dQ \neq 0.$$

($Q \wedge dQ = 0$ implies a Pfaff dimension of <3; $Q \wedge dQ \neq 0$ implies a Pfaff dimension of 3 or more.)

Thermodynamic Processes page 3

All classic Hamiltonian, Symplectic, Bernoulli and Stokes Processes, satisfy the Helmholtz-Poincare constraint ("conservation of vorticity").

$$L_{(V)}dA = dQ = 0.$$

and are therefore

Thermodynamically Reversible.

as

$$L_{(V)}A \wedge L_{(V)}\, dA = Q \wedge dQ = 0.$$

Pfaff Topological Dimension

Each physical system represented by a 1-form of Action, A, has a minimum number of functions required for its topological definition. This number, D_{pfaff} = Pfaff Topological Dimension, is equal or less than the geometrical dimension m of the domain of support.

The Top Pfaff dimension, n, is equal to the number of non-zero terms in the

Pfaff Sequence = {A,dA,A^dA,dA^dA…}.

Subspaces of lesser Pfaff dimension form coherent topological structures, or thermodynamic phases.

Contact Manifolds, $n = 2k+1$.

On subspaces of **Pfaff dimension $n = 2k + 1 \leq m$, called contact manifolds, the Principle of Least Action implies that the evolution obeys the Helmholtz-Poincare constraint ("conservation of vorticity")**

$$L_{(V)}dA = dQ = 0.$$

and such evolutionary processes are therefore

Thermodynamically Reversible.

A unique direction field, **V**, completely determined by the topological features of the Action, A, of odd Pfaff dimension, such that $W = i(V)dA=0$ is called the **Extremal Field,** and if $U = i(V)A = 1$, a **Reeb** field.

Symplectic Manifolds n = 2k+2.

On subspaces of Pfaff dimension $n = 2k + 2 \leq m$, called symplectic manifolds, extremal fields do not exist. However, a unique direction field T can be defined in terms of the topological features of the physical system, A.

$$L_{(T)} A = \Gamma A \neq 0, \qquad L_{(T)} dA = d\Gamma\hat{}A + \Gamma dA \neq 0.$$

Processes in the direction of the Torsion Vector, T, are Thermodynamically Irreversible, as

$$L_{(T)} A\hat{}\, L_{(T)} dA = Q\hat{}dQ = \Gamma^2 A\hat{}dA \neq 0.$$

Entropy production rate

Construct the 3-form A^F = A^dA

Topological Torsion vector T

A ^ dA = i[T, h](dx^dy^dz^dt)

with the 4 component Torsion Direction Field

Γ^2 ≈ Entropy Production Rate

≈(1/2 divergence of T) ^{2}

Connectivity and the Arrow of time

Regions where $D_{Pfaff} \leq 2$ generate a connected topology;

$D_{Pfaff} \geq 3$ generate a disconnected topology.

Continuous Processes can represent the evolution from

disconnected topology (≥ 3) to a connected topology (≤ 2) .

Continuous Processes can NOT represent the evolution from a

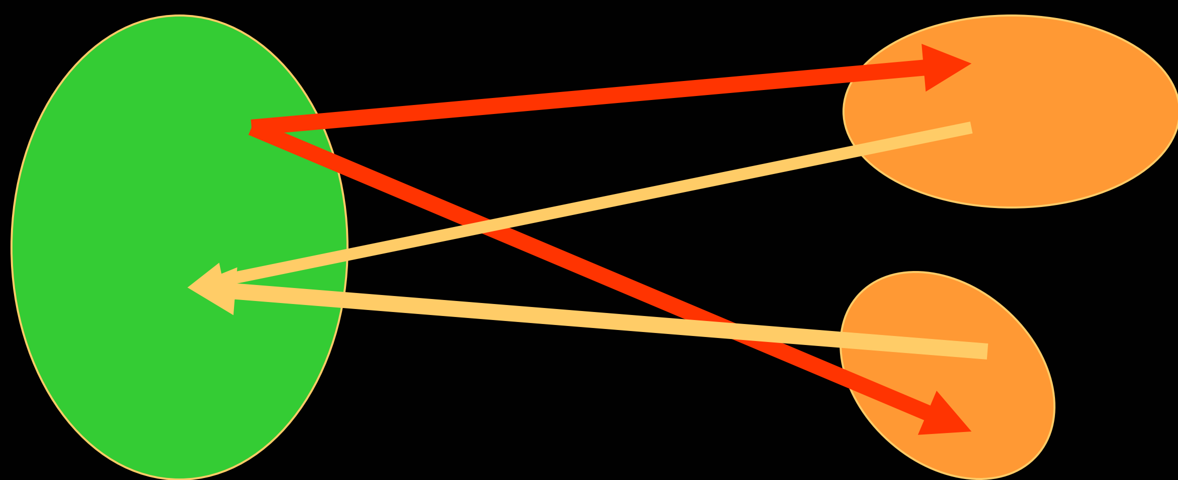
connected topology (≤ 2) to a disconnected topology (≥ 3) .

(An Arrow of Time.)

You can describe the decay of turbulence continuously, but NOT the creation of turbulence.

Arrow of Time and Turbulence

Creation of Turbulence is a Discontinuous Process



Decay of Turbulence is a Continuous Process

Streamline Flow

Connected Topology $D_{pfaff} \leq \mathbf{2}$

Turbulent Flow

Disconnected Topology $D_{pfaff} > \mathbf{3}$

Decay to Equilibrium in 4D

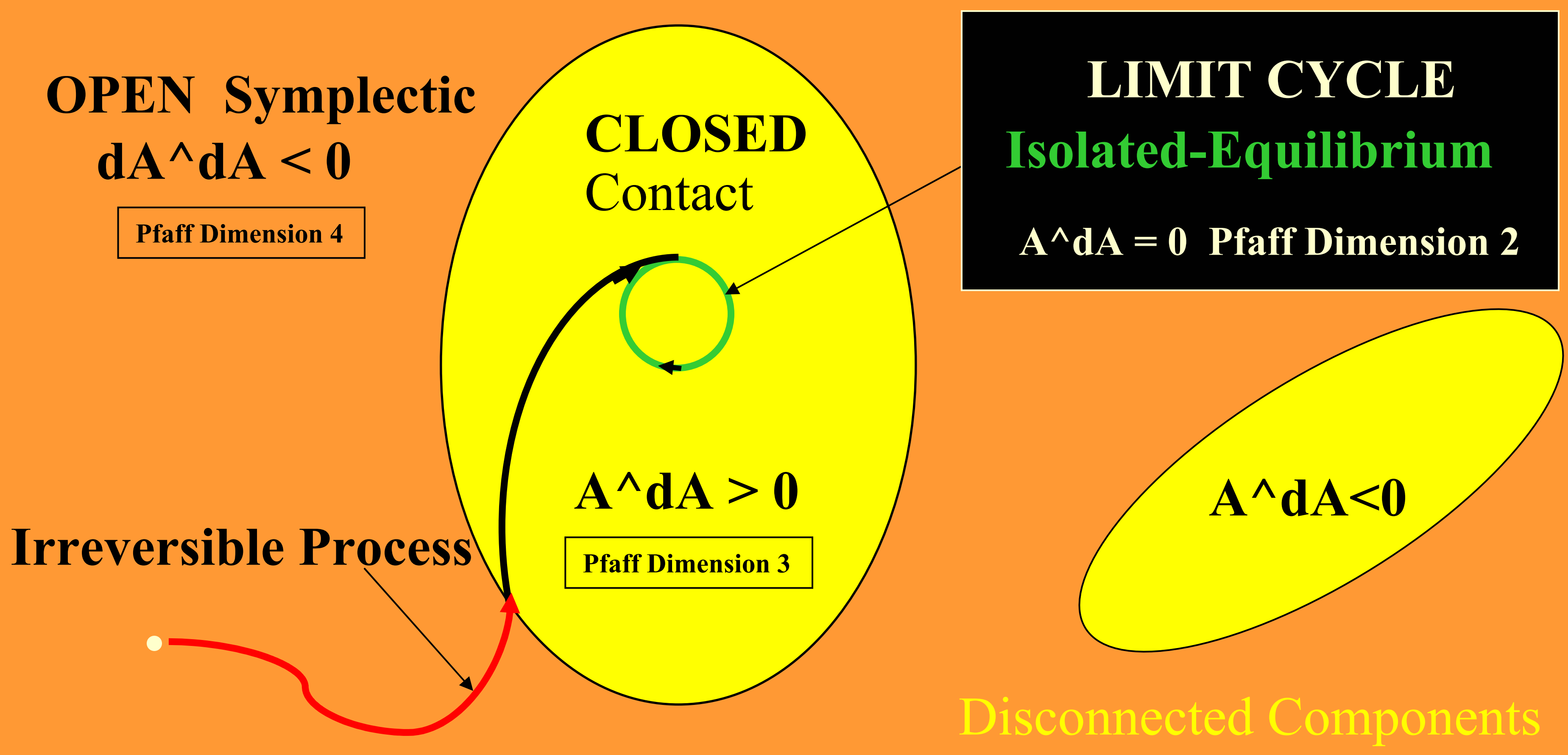
Example: **Continuous evolution** can describe the irreversible evolution on an

"Open" symplectic domain of Pfaff dimension 4, with evolutionary orbits being attracted to a contact

"Closed" domain of Pfaff dimension 3, with an ultimate decay to the

"Isolated-Equilibrium" domain of Pfaff dimension 2 or less (integrable Caratheodory surface).

Irreversible Decay on a Symplectic Manifold to a Contact Manifold of disconnected components, then to an Isolated-Equilibrium State.



Electromagnetic Example page 1

Use the 4D electromagnetic 1-form of Action:

A =A (x,y,z,t) • dr -ϕ(x,y,z,t)dt.

Define: E = -curl A - grad ϕ(x,y,z,t), B = curl A

Construct the 2-form F = dA,

$F=dA = B_z\, dx \wedge dy \ldots - E_z\, dz \wedge dt \ldots$

Then the 3-form dF = ddA = 0, and generates the Maxwell-Faraday PDE's.

Electromagnetic Example page 2

Construct the 3-form $A\wedge F = A\wedge dA$

Topological Torsion: $A \wedge dA = i[T, h](dx\wedge dy\wedge dz\wedge dt)$

with the 4 component Torsion Direction Field

$$T = [T, h] = [E \times A + \phi B, A \bullet B].$$

Then,

$$L_{(T)} A = (E \bullet B) A$$

(The 4th component $A \bullet B$ is often defined as the Helicity density)

Electromagnetic Example page 3

Construct the 4-form F^F = dA^dA :

Topological Parity: **dA ^ dA = -2(E • B) dx^dy^dz^dt).**

On regions where Γ = (E • B) ≠ 0,

- **the Pfaff dimension is 4,**
- **evolution in the direction of the Torsion vector is thermodynamically irreversible.**

$$L_{(T)} A\wedge L_{(T)} dA = Q\wedge dQ = (-E \bullet B)^2 A\wedge dA \neq 0.$$

(The divergence of the Torsion vector is equal to -2(**E • B.**)

Electromagnetic Example page 4

On regions where (E • B) = 0,

- the Pfaff dimension is 3,
- and the evolution can proceed in the direction of the extremal field, which is reversible. (The Torsion vector on the 4D base space has zero divergence).

The closed integrals of the 3-form A^dA are deformation (topological) invariants for all processes V on domains of Pfaff dimension 3

(The values of the closed integrals are "topologically quantized")!

Darboux Format 4D

Consider a map Φ from **{x,y,z,t}** ⇒ **{P,H,Q,T}** and a 1-form of Action on the target space in Darboux Format,

$$A = PdQ + HdT.$$

By Functional Substitution, pull back the target to the domain

$$A = \mathbf{A}(x,y,z,t) \bullet d\mathbf{r} - \phi(x,y,z,t)dt \Leftarrow A = PdQ + HdT.$$

Then the vector and scalar potentials of the previous electromagnetic example are well defined functions of **{P(x,y,z,t),H(x,y,z,t), Q(x,y,z,t), T(x,y,z,t)}** and their differentials.

Result: All of the topological features of the Darboux representation now have an electromagnetic interpretation.

Darboux Format 4D

To reduce the algebraic complexity, constrain the map such that "time" on the final state is equal to "time" on the intial state: **T= t.** Then the important topological properties of the Darboux format in electromagnetic interpretation are:

$$\mathbf{E} = (\partial P/\partial t\ \nabla Q - \partial Q/\partial t\ \nabla P) - \nabla H, \quad \mathbf{B} = \nabla P \times \nabla Q$$

Frenet Torsion $\Rightarrow$ 0: $\mathbf{A} \bullet \mathbf{B} = 0$

Topological Torsion: $\mathbf{T} = -P\,(\partial Q/\partial t\ \nabla P - \nabla H) \times \nabla Q$

Topological Parity : $\mathbf{E} \bullet \mathbf{B} = -\nabla H \bullet \nabla P \times \nabla Q$

Topological Fluctuations 4D

Recall: The kinematic assumption is a Topological Constraint:

$$\Delta \mathbf{x} = d\mathbf{x} - \mathbf{v}dt \Rightarrow \mathbf{0}.$$

Define a transverse **topological fluctuation in position** as

$$\Delta \mathbf{x} = d\mathbf{x} - \mathbf{v}dt \neq \mathbf{0}. \qquad \textbf{(\sim Pressure)}$$

Define a transverse **topological fluctuation in velocity** as

$$\Delta \mathbf{v} = d\mathbf{v} - \mathbf{A}dt \neq \mathbf{0}. \qquad \textbf{(\sim Temperature)}$$

On a variety {P,v,x,t} define the 1-form of Action as:

$$A = L(v,x,t)\,dt + P\,(dx - vdt) = P\,dx + H\,dt \Rightarrow$$

$$A = L(v,x,t)\,dt + P\,\Delta x$$

Topological Fluctuations 4D

Define the topological fluctuation in momenta as:

$$\Delta p = dP - (\partial L/\partial x)\, dt - f\, \Delta x\ .$$

Then compute the elements of the Pfaff sequence:

Action: $A = L(v,x,t)\, dt + P\, \Delta x$

Vorticity: $dA = (\partial L/\partial v - P)\, \Delta v \wedge dt + \Delta p \wedge \Delta x$

Torsion: $A \wedge dA = L\, \Delta p \wedge \Delta x \wedge dt - P(\partial L/\partial v - P)\, \Delta v \wedge \Delta x \wedge dt$

Parity: $dA \wedge dA = -2\, (\partial L/\partial v - P)\, \Delta p \wedge \Delta v \wedge \Delta x \wedge dt$

Topological Fluctuations 4D

The Topological constraint of Canonical Momenta, $P = \partial L/\partial v$ reduces the Pfaff dimension from 2n+2 to 2n+1.

- Theorem 1. There exists a unique extremal direction field on the 2n+1 Contact manifold, a Hamiltonian conservative representation, such that the Virtual Work 1-form is zero.
- Theorem 2 (analogue to Heisenberg), The existence of Topological Vorticity and Topological Torsion require that the product of fluctuations in momenta and fluctuations in position is NOT zero.

$$\Delta p \wedge \Delta x \neq 0$$

Topological Fluctuations 4D

Consider the Work 1-form on a 2n+2 manifold, $(\partial L/\partial v-P) \neq \mathbf{0}$.

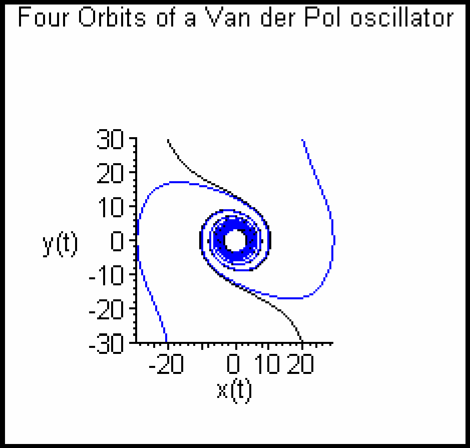
$$\mathbf{W = i(V)dA = (\partial L/\partial v\text{-}P)\, \Delta v + \Delta p \wedge \Delta x}$$

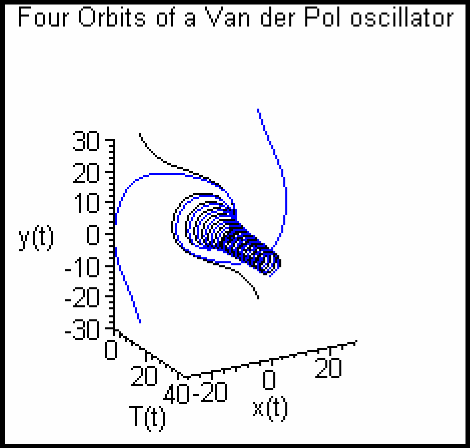
Processes on a 2n+2 symplectic manifold require $W \neq \mathbf{0}$.

- To be a symplectic manifold requires that the first term in the expression for work, W, is not zero. The momenta cannot be canonical, and the Velocity fluctuations must be non-zero. This implies the existence of a non-zero temperature, and leads to the analogue of the Planck concept of a zero point energy on the symplectic 2n+2 topological manifold.

Van der Pol Example

Perhaps the most famous limit cycle is that of the Van der Pol Oscillator. The evolutionary decay from arbitrary initial conditions {x,y,t} to a unique attracting limit cycle {f(x,y),t} is an obvious example of continuous topological change.





Lagrangian Example page 1

A Cartan-Hilbert 1-form of Action, A, for a physical system can be written as

$$A = L(t,x,v,p)dt + p \bullet (dx - v\,dt)$$

The k+1 base variables are {t, x}.

The 2k "fiber" variables are {v,p}.

The Lagrange function L(t,x,v,p) is a function of the 3k+1 variables, (t,x,v,p) .

Lagrangian Example page 2

However, the maximum Pfaff topological dimension is 2k+2 and the top Pfaffian on the symplectic manifold is

$$(dA)^{k+1} = (k+1)!\{\partial L/\partial v - p)\bullet dv\} \wedge \Omega_p \wedge \Omega_q \wedge dt$$

$$\Omega_p = dp_1 \wedge .. \wedge dp_n \qquad \Omega_q = dq^1 \wedge .. \wedge dq^n$$

Note that the "symplectic momenta" are not canonically defined: $p - \partial L/\partial v \neq 0.$

Lagrangian Example page 3

Evolution starts on the 2k+2 symplectic manifold with orbits being attracted to 2k+1 domains where the momenta become canonical: $p - \partial L/\partial v \Rightarrow 0$.

- Topological evolution can either continue to reduce the Pfaff topological dimension, or
- the process on the Contact 2k+1 manifold can become “extremal”, and the topological change stops.

The resulting contact manifold becomes a “stationary” non-dissipating Hamiltonian state,

“Far from Equilibrium”.

The Sliding - Rolling Ball page 1

Consider a bowling ball with initial translational and rotational energy, thrown to the floor of the bowling alley.

Initially the ball skids or slips on a 2k + 2 symplectic manifold irreversibly reducing its energy and angular momentum via “friction” forces.

From arbitrary initial conditions, the evolution is attracted to a 2k + 1 contact manifold, where the ball rolls without slipping, and the anholonomic constraint vanishes.

$$dx - \lambda\, d\Theta = 0$$

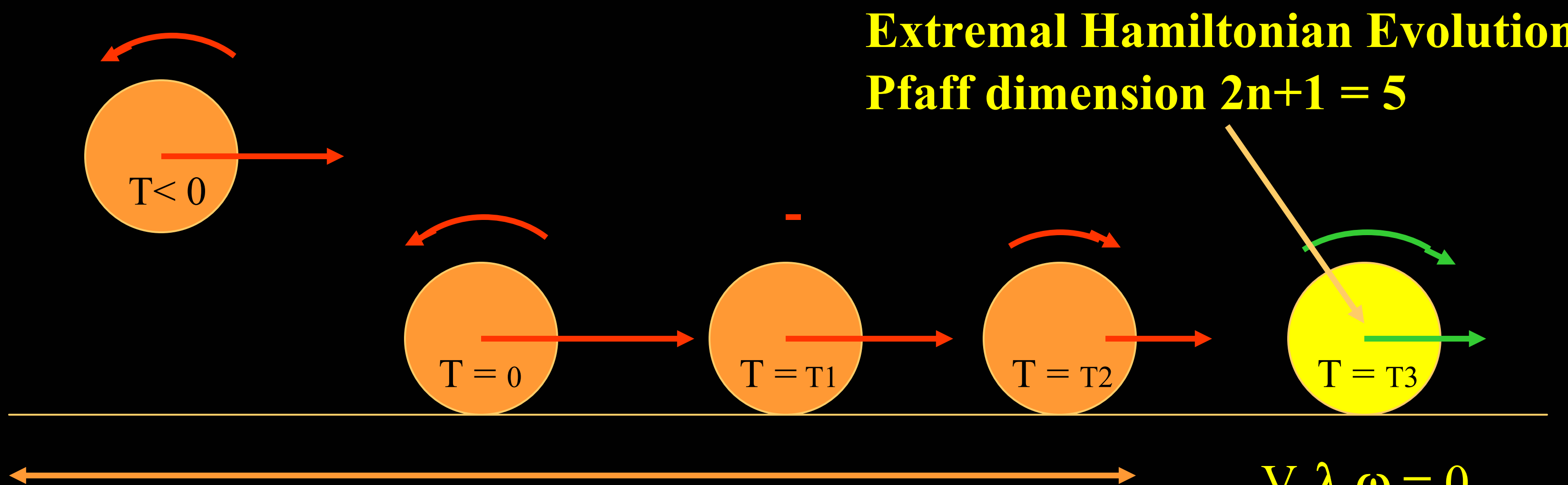
The Sliding - Rolling Ball page 2

The subsequent motion, neglecting air resistance, continues in a Hamiltonian manner without change of Kinetic Energy or Angular Momentum.

The 1-form of Action can be written as:

$$A = L(t,\mathbf{x}, \Theta,v,\omega)dt + \ldots + \mathbf{s} \bullet (d\mathbf{x} - \lambda\, d\Theta)$$

The Sliding - Rolling Ball page 3



$V\text{-}\lambda\ \omega = 0$

Irreversible Evolution - Pfaff dimension 2n+2 = 6

ω	ω	ω
$V\text{-}\lambda\ \omega > 0$	$V\text{-}\lambda\ \omega > 0$	$V\text{-}\lambda\ \omega > 0$
$\lambda\ \omega < 0$	$\lambda\ \omega = 0$	$\lambda\ \omega > 0$

Note how friction changes Angular Momentum

Summary

- **Without Topological Evolution, there is no Arrow of Time and no Thermodynamic Irreversibility.**
- **Physical Systems of Pfaff dimension 4 generate a unique continuous evolutionary process which is thermodynamically irreversible.**
- **Cartan's Magic formula combines continuous topological evolution and thermodynamics**