THERMODYNAMIC IRREVERSIBILITY AND THE ARROW OF TIME

A Topological Perspective

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Abstract: Exterior differential forms, unlike tensor fields, are well behaved with respect to C1 differentiable, but not invertible, maps, such as those used to describe projections. Relative to these non-homeomorphic maps that induce topological change, a logical arrow of time is generated, for the functional coefficients of the exterior differential forms may be retrodicted from data on the final state, but not predicted from data on the initial state. It follows that Cartan's methods of exterior differential forms can be used to study continuous topological evolution, and thermodynamic irreversibility. The methods are applied to those physical systems that can be described by a 1-form of Action, A, and to processes that admit description in terms of a vector field, V. Cartan's evolutionary formula uses the Lie derivative with respect to the vector field, V, acting on the 1-form of Action, A, to create an evolutionary 1-form Q. These dynamical equations of topological evolution establish a formal connection to the first law of thermodynamics, and can be used to demonstrate that thermodynamic irreversibility is an artefact of Pfaff topological dimension of at least 4.

1 APPLICATIONS OF CONTINUOUS TOPOLOGICAL EVOLUTION

In this article, a topological perspective will be used to establish the long sought for, non-statistical, connection between dynamic mechanical systems, thermodynamic irreversibility, and the arrow of time [1]. In effect, it will be demonstrated that the Boltzmann paradox can be resolved in terms of Continuous Topological Evolution [1], which differs from classical geometric theories of Continuous Evolution without topological change. Consider the definitions:

[1] *"The arrow of time is one of the big unclaimed prizes of modern physics. The problem is to reconcile the temporal asymmetry of thermodynamics with the apparent temporal symmetry of fundamental physical theories"* *Hugh Price*

1. Causal evolution is defined as a map of C1 functions from a domain of base variables to a unique range of base variables. The maps may be many to one and are not necessarily homeomorphisms.

2. Prediction implies that well behaved functional forms (not just numeric point data) on the range of base variables can be deduced from functional forms defined on the domain of base variables.

3. Retrodiction implies that functional forms on the domain can be deduced from functional forms on the range.

The fundamental axioms utilized in this article are:

Axiom 1 *The topological structure of Physical Systems on a domain of independent base variables can be encoded in terms of exterior differential forms (symbolically represented by A).*

Axiom 2 *Physical Processes can be defined in terms of contravariant vector direction fields, which may or may not be generators of 1-parameter groups, and in particular need not be homeomorphisms (symbolically represented by V).*

Axiom 3 *The topological structure of Physical Systems on a domain of independent base variables can be encoded in terms of exterior differential forms (symbolically represented by A).Equations of Continuous Evolution describing both reversible and irreversible Processes acting on Physical Systems are encoded by Cartan's magic formula :*

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) \tag{1}$$

These axioms and definitions will be used to demonstrate that :

Theorem 4 *Topological evolution is a necessary condition for both time asymmetry and thermodynamic irreversibility, but time asymmetry is not sufficient for thermodynamical irreversibility.*

Theorem 5 *There always exists on a topological space of odd dimension n = 2k+1 (k=1,2,3...) a unique extremal direction field which represents a Hamiltonian process that is conservative and reversible in a thermodynamic sense. Chaos requires $k >$ 0.*

Theorem 6 *There always exists on a topological space of even dimension n = 2k+2 (k=1,2,3...) a unique direction field which represents a process that is thermodynamically irreversible. Turbulence requires $k >$ 0.*

Due to space limitations, herein, the proofs of the theorems and detailed examples are to be found elsewhere [2].

Topological Structure: It is subsumed that the presence of a physical system establishes a *topological structure* on a (possibly geometric) base space of independent variables. This concept is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. Note that a given base of independent variables may support many different topological structures; hence a given base may support many different physical systems. For those C^1 differentiable, continuous maps which preserve topology (diffeomorphisms), both *Prediction,* and *Retrodiction,* of those functions that define tensor fields is possible. For those C1 differentiable, continuous maps which do not preserve topology and admit topological change, *Prediction* is impossible [3]. However, *Retrodiction* is possible for those anti-symmetric covariant tensor fields used to construct "pair" exterior differential forms, and those anti-symmetric contravariant tensor densities used to construct "impair" exterior differential forms. Hence the concept of continuous topological evolution establishes a time asymmetry and a logical arrow of time, while continuous evolution without topological change does not.

C*ontinuity:* Although C1 non-invertible maps are not homeomorphisms, and therefore the topology of the initial state and the topology of the final state are not the same, such maps can be continuous. Continuous topological evolution is not an oxymoron, for topological continuity is defined such that the limit points of every subset in the domain (relative to the topology on the initial state) permute into the closure of the subsets in the range (relative to the topology on the final state). The initial and final state topologies need not be the same. Pasting together is a continuous process for which the topology of the final system state is not necessarily the same as the topology of the initial system state. Separation or cutting into parts is a discontinuous process for which the system topology of the final state is not the same as the system topology of the initial state. Discontinuous topological evolution is not to be considered in this article.

Cartan's Magic Formula: Cartan's "magic formula" (a descriptive phrase introduced by Marsden [4]) representing the "evolution" of the 1-form of Action, A, with respect to the "flow" generated by the vector field, $\mathbf{V}$, is the cornerstone of the development. The Cartan formula does not depend upon the constraints of a geometric connection or metric, and has been called the "homotopy formula" by Arnold [5]. If the coefficient functions of A and V are C2 differentiable, then it is possible to prove that the evolutionary process described by Cartan's magic formula is continuous. The C2 constraints can be relaxed, but such special cases will not be studied at present.

Herein, the following definitions are made:

1. The term $W = i(\mathbf{V})dA$ is defined as the inexact 1-form of "virtual work".
2. The function $U = i(V)A$ is defined as the "internal energy".
3. The sum of the two terms, $W + dU$, define the 1-form of "heat", Q.

Q and W are not necessarily exact 1-forms.

From these definitions, it is apparent that Cartan's magic formula not only represents a topological evolutionary process, where the process V acts on the physical system A to produce the 1-form of heat, Q, but also is formally equivalent to the cohomological description of the First Law of Thermodynamics.

$$
\begin{aligned}
L_{(\mathbf{V})}A &= i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q \\
&= W + dU = Q
\end{aligned} \tag{2}
$$

In this article this formal correspondence is taken seriously. The magic in Cartan's formula is due to the fact that it can be used to describe both those evolutionary processes where topology is invariant and those evolutionary processes where the topology of the initial state is not the same as the topology of the final state. Moreover the formula can be used to determine equivalence classes of evolutionary processes, for a given physical system, which are thermodynamically reversible or irreversible.

Thermodynamic irreversibility: Following thermodynamic experience, a thermodynamic process is defined to be a reversible process if the heat 1-form, Q, admits an integrating factor. The integrating factor (in thermodynamics) defines the concept of temperature. Therefore, if the heat 1-form does not admit an integrating factor, the thermodynamic process is irreversible [6]. From a topological point of view, the heat 1-form admits an integrating factor if and only if Q satisfies the conditions of the Frobenius integrability theorem, $Q\hat{}dQ = 0$. This definition of thermodynamic irreversibility, when combined with Cartan's magic formula, permits the correspondence to be made between thermodynamics and mechanical systems. A simple test for thermodynamic irreversibility of a process acting on a system is given by the equations:

$$Q\hat{}dQ = (L_{(\mathbf{V})}A)\hat{}(L_{(\mathbf{V})}dA) = 0 \supset the\,process\;is\,reversible. \tag{3}$$

$$Q\hat{}dQ = (L_{(\mathbf{V})}A)\hat{}(L_{(\mathbf{V})}dA) \neq 0 \supset the\,process\;is\,irreversible. \tag{4}$$

Topological Dimension, n, versus Geometrical Dimension, m: Of key importance for any particular physical system is the choice of the "correct" 1-form of Action, A, which encodes the topological features of a specific physical system. Experience (guesswork) and the degree of agreement with measurement of both invariance or evolution of topological (quantum) numbers will satisfy the working scientist. For any given 1-form, A, functionally defined on a (perhaps geometrical) base space, or variety, of dimension m, it is possible to compute the "Pfaff sequence", {A, dA, A^dA, dA^dA...}. It is remarkable that this sequence terminates at a minimum number, $n \leq m$, representing the irreducible minimum number of independent functions required to define the topological encoding. This number n is called the Pfaff topological dimension, and the last non-zero element of the Pfaff sequence is defined as the top Pfaffian. The topological dimension, n, is less than or equal to the geometric dimension, m. Note that the requirement for thermodynamic irreversibility implies that the Pfaff dimension of the heat 1-form Q must be 3 or greater, which means the the Frobenius theorem of *unique* integrability fails. The idea is that the Pfaff topological dimension implies the existence of a continuous map from the variety of dimension m > n to a variety of dimension n.

An explicit physical 1-form, A, will generate a "Cartan topology" on the domain. It is easy to demonstrate that the Cartan topology is a connected topology if the Pfaff dimension is 2 or less, and a disconnected topology if the Pfaff dimension is 3 or more [7]. Hamiltonian mechanics and Eulerian streamline flows in hydrodynamics (on base spaces of geometric dimension 4) are associated with Action 1-forms of Pfaff topological dimension 2 or less, and are thermodynamically reversible. Turbulence, being the antithesis of streamline flow, must be represented by a topology of Pfaff dimension 4 or more, and is thermodynamically irreversible. Again, the constraint of continuous topological evolution induces a logical "Arrow of Time". Note that the decay of turbulence can be studied by continuous methods, but the creation of turbulence cannot. In both cases, topological evolution takes place as the Pfaff topological dimension changes, but the creation of turbulence cannot be continuous, where the decay of turbulence can be continuous.

Contact manifolds: When the 1-form A is of odd topological dimension (n= 3 or greater), then the 2-form dA can be put into correspondence with an odd-dimensional n=2k +1 antisymmetric matrix of functions of maximal rank. This matrix has one unique eigenvector with a null eigen value. Hence the topological encoding of a physical system determines a unique direction field defined as the "extremal" direction field (on the 2k + 1 dimensional variety). Evolution in the direction of this unique "extremal" vector field, $\mathbf{V}_E$, implies that the virtual work, W, vanishes, and that the exterior derivative of heat 1-form dQ vanishes, as Q is exact. Such extremal vector fields always have a Hamiltonian generator, and are not thermodynamically irreversible, as Q^dQ=0. The extremal Hamiltonian evolution preserves the even dimensional topological features of the physical system (the Poincare invariants). If the extremal process is also adiabatic, such that both Q = 0 and dQ = 0, then the process preserves both odd and even topological features, and is a homeomorphism. The equivalence class of processes that satisfy the closure requirement, (dQ=0) includes not only extremal fields, $i(\mathbf{V}_E)dA = 0$, but also those that can have a Casimir generator (Bernoulli flows) or those that can generate limit cycles.

Symplectic manifolds: When the 1-form A is of even topological dimension (n = 4 or greater), then the 2-form dA can be put into correspondence with an even-dimensional antisymmetric matrix of functions of maximal rank. Extremal fields (with null eigenvalue) do not exist, but there is a unique evolutionary direction field, $\mathbf{V}_T$, on the n = 2k + 2 dimensional variety that is completely determined from the topology of the physical system, induced by the 1-form, A. On a n = 4 dimensional base manifold, this unique direction field is defined by the equations,

$$A\hat{}dA = i(\mathbf{V}_T)dx\hat{}dy\hat{}dz\hat{}dt. \tag{5}$$

This vector field $\mathbf{V}_T$ is defined as the Topological Torsion vector. As $A\hat{}A\hat{}dA = 0$ the Topological Torsion vector is transverse with respect to the 1-form of

Action: $i(\mathbf{V}_T)A = 0$. By direct calculation it is possible to show that $W = i(\mathbf{V}_T)dA = \Gamma\, A$. In otherwords the 1-form of virtual work is proportional to the 1-form of Action. Cartan's magic formula becomes

$$L_{(\mathbf{V}_T)}A = \Gamma\, A \tag{6}$$

where Γ equals 1/2 the coefficient of the non-zero 4-form

$$dA\hat{}dA = 2 * \Gamma(x, y, z, t)dx\hat{}dy\hat{}dz\hat{}dt = \{div_4(\mathbf{V}_T)\}dx\hat{}dy\hat{}dz\hat{}dt. \tag{7}$$

As the 2-form is of maximal rank, $\Gamma(x, y, z, t) \neq 0$. It follows that evolution in the direction of the Torsion Vector is thermodynamically irreversible, as

$$Q\hat{}dQ = L_{(\mathbf{V}_T)}A\hat{}L_{(\mathbf{V}_T)}dA = \Gamma^2\, A\hat{}dA \neq 0, \tag{8}$$

The factor Γ^2 plays a role related to the entropy production rate.

An Electromagnetic Example : On the four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action can be written in the form $A{=}\Sigma_{k=1}^3 A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt$ which generates the 2-form $dA = \mathbf{B}_z dx\hat{}dy ... + \mathbf{E}_x dx\hat{}dt .....$ The 3-form of Topological Torsion becomes $A\hat{}dA = i(\mathbf{V}_T)dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{S}^x dy\hat{}dz\hat{}dt..... - hdx\hat{}dy\hat{}dz$,such that in engineering language,

$$\mathbf{V}_T = -\{(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi); \mathbf{A} \bullet \mathbf{B}\} \equiv \{\mathbf{S}, h\} \text{ and } \Gamma = (\mathbf{E} \bullet \mathbf{B}). \tag{9}$$

The 4-form of Topological Parity becomes $dA\hat{}dA = 2(\mathbf{E} \bullet \mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt = (div\, \mathbf{S} + \partial h/\partial t)dx\hat{}dy\hat{}dz\hat{}dt$. From (7) and (8), evolution in the direction of the Topological Torsion vector is thermodynamically irreversible.

Dissipative Evolution to States Far from Equilibrium: Starting with a physical system of even topological dimension, dissipative irreversible evolution in the direction of the Topological Torsion vector, $\mathbf{V}_T$, can ultimately achieve a state where the dissipative coefficient Γ vanishes over certain subsets of the base. The physical system undergoes a "topological phase change" from a symplectic manifold to an attractor contact manifold. The odd dimensional contact submanifold admits a unique extremal direction field, $\mathbf{V}_E$, representing a reversible, non dissipative, evolutionary process. These contact manifold regions act as topological defects in the symplectic manifold, and can represent long lived conservative topological states that are far from equilibrium. As an example, consider the billiard ball given initial angular momentum and translational kinetic energy and then placed on a surface that is not free of "friction". At first the ball skids or slips as the system evolves irreversibly on a symplectic manifold. Then, when the constraint of rolling without slipping is established, the subsequent motion is no longer dissipative, and the system evolves on a contact manifold [8]. The ultimate equilibrium state would be associated when the linear and angular momentum are zero - the state of rest.

Lagrangian methods, Anholonomic Fluctuations, Non-Canonical Momentum. Consider a physical system that can be defined in terms of the Cartan-Hilbert choice for the Action 1-form

$$A = L(t;q,v,p)dt + p_j(dq^j - v^j dt), \tag{10}$$

defined on the m = 3k +1 variety $\{t;q,v,p\}$. Note that the Lagrange function is presumed to be dependent upon m = 3k + 1 variables, not just the 2k+1 variables of state space. Do not assume that p is constrained to be a jet; e.g., $p_j \neq \partial L/\partial v^j$. Instead, consider p_k to be a Lagrange multiplier to be determined later. In addition, do not assume that the topological limit $(dq - vdt) = 0$ is necessarily valid over the whole domain. This lack of kinematic perfection is defined as an anholonomic fluctuation, Δq.

By direct computation it follows that the Pfaff sequence of the Cartan-Hilbert Action is of dimension n = 2k + 2 and not 3k + 1. The actual formula for the top Pfaffian in the sequence is,

$$\begin{aligned}
(dA)^{k+1} &= (k+1)!\{\Sigma_{j=1}^{k}(\partial L/\partial v^j - p_j) \bullet dv^j\}\hat{}\,\Omega_p\hat{}\,\Omega_q\hat{}\,dt, && (11)\\
where\, \Omega_p &= dp_1\hat{}\ldots\hat{}\,dp_n, && (12)\\
and\, \Omega_q &= dq^1\hat{}\ldots\hat{}\,dq^n. && (13)
\end{aligned}$$

The Cartan-Hilbert Action 1-form is associated with a symplectic manifold of dimension 2k + 2. The symplectic manifold supports a unique direction field, T, such that evolution in the direction of T is thermodynamically irreversible.

$$\begin{aligned}
A\hat{}(dA)^k &= k!\{L(t,q,v,p) - p_j\partial L(t,q,v,p)/\partial p_j\}\hat{}\,\Omega_p\hat{}\,\Omega_q\hat{}\,dt\\
&\quad -k!\{\Sigma_{j=1}^{k}(\partial L/\partial v^j - p_j) \bullet dv^j\}\hat{}(i(p_\sigma)\Omega_p)\hat{}\,\Omega_q\hat{}\,dt && (14)\\
where\ (i(p_\sigma)\Omega_p) &= p_1 dp_2\hat{}\,dp_3\ldots\hat{}\,dp_n - p_2 dp_1\hat{}\,dp_3\ldots\hat{}\,dp_n + \ldots && (15)
\end{aligned}$$

If the Cartan-Hilbert 1-form of Action is constrained such that

$$\{\Sigma_{j=1}^{k}(\partial L/\partial v^j - p_j) \bullet dv^j\} = 0, \text{ mod } \Omega_p\hat{}\,\Omega_q\hat{}\,dt \tag{16}$$

then the Pfaff sequence generates a contact structure on 2k+1 state space with the formula for the top Pfaffian:

$$A\hat{}(dA)^k = k!\{L(t,q,v,p) - p_j\partial L(t,q,v,p)/\partial p_j\}\Omega_p\hat{}\,\Omega_q\hat{}\,dt. \tag{17}$$

One ubiquitous method for such a reduction from n = 2k + 2 dimensions to n = 2k + 1 dimensions is to assume that the momenta be defined canonically; i.e., $(\partial L/\partial v^j - p_j) = 0$. The canonical space of 2k + 1 dimensions then can be reduced to a 2k phase space (and ultimately to thermodynamic equilibrium) only if the Lagrangian is homogeneous of degree 1 in the p_j. Otherwise the Action 1-form defines a contact structure of dimension n = 2k+1. Note that the canonical constraint does not constrain the anholonomic fluctuations.

Consider evolutionary processes defined on the n = 2k+2 dimensional symplectic space in terms of a direction vector field $\gamma\mathbf{V} = \gamma\ [1, v, a,\ f]$, relative to $[t; q, v, p]$. Construct the 1-form W of virtual work to yield in every case:

$$W = i(\mathbf{V})dA = \tag{18}$$
$$= \{p - \partial L/\partial v\}\Delta v + \{f - \partial L/\partial x\}\Delta q \tag{19}$$
$$\text{with fluctuations } \Delta v = dv - adt \neq 0, \quad \text{and } \Delta q = dq - vdt \neq 0. \tag{20}$$

When the 2-form dA is symplectic, the work 1-form (which can not vanish) has two terms; the first involves Δv and the second involves Δq. The symplectic structure requires that the the first term cannot vanish: the momenta cannot be canonical, there must be fluctuations in velocity, and the non-canonical momenta must not be orthogonal to the fluctuations in velocity. The unique direction field defined by the Topological Torsion vector, $\mathbf{V}_T$, on the symplectic space produces an evolutionary process which is thermodynamically irreversible. If fluctuations in position are identified with pressure, and fluctuations in velocity are identified with temperature, then it appears that non-zero velocity (temperature) fluctuations are necessary for thermodynamic irreversibility, but non-zero position (pressure) fluctuations, are not. This interesting result will be investigated further.

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