A topological perspective of the Arrow of Time and Thermodynamic Irreversibility

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Abstract

Exterior differential forms, unlike tensor fields, are well behaved with respect to C1 differentiable, but not invertible, maps, such as those used to describe projections. Relative to these non-homeomorphic maps that induce topological change, a logical arrow of time is generated, for the functional coefficients of the exterior differential forms may be retrodicted from data on the final state, but not predicted from data on the initial state. It follows that Cartan's methods of exterior differential forms can be used to study continuous topological evolution, and thermodynamic irreversibility. The methods are applied to those physical systems that can be described by a 1-form of Action, A, and to processes that admit description in terms of a vector field, V. Cartan's evolutionary formula uses the Lie derivative with respect to the vector field, V, acting on the 1-form of Action, A, to create an evolutionary 1-form Q. These dynamical equations of topological evolution establish a formal connection to the first law of thermodynamics, and can be used to demonstrate that thermodynamic irreversibility is an artifact of Pfaff topological dimension of at least 4.

1. Introduction

In this article, a topological perspective will be used to establish the long sought for, non-statistical, connection between dynamic mechanical systems, thermodynamic irreversibility, and the arrow of time. Consider the following remarks concerning the "Boltzman paradox:

> "*The traditional reductionist view is that we should seek the explanation on the basis of the reversible mechanical equations of motion. But, as the physicist Ludwig Boltzmann discovered, it is not possible to base the arrow of time directly on equations that ignore it. His failed attempt to reconcile microscopic mechanics with the second law gave rise to the "irreversibility paradox" Peter Coveny and Roger Highfield, "*C*haos, entropy and the arrow of time", (WH Allen, London, 1990; Ballantine, New York, 1991).*
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> *"The arrow of time is one of the big unclaimed prizes of modern physics. The problem is to reconcile the temporal asymmetry of thermodynamics with the apparent temporal symmetry of fundamental physical theories" Hugh Price Nature 348 (22 November, 1990), p. 356*
>
> On the otherhand: *"...when they are correctly presented, the classical views of Boltzmann perfectly account for macroscopic irreversibility on the basis of deterministic, reversible, microscopic laws" J Bricmont 1996.*
>
> *Since in the differential equations of mechanics themselves there is absolutely nothing analogous to the Second Law of thermodynamics the latter can be mechanically represented only by means of assumptions regarding initial conditions. L. Boltzmann ([11], p.170)*
>
> *Indeed, the laws of physics are always of the form: given some initial conditions, here is the result after some time.*
>
> *And Fred Hoyle wrote: "The thermodynamic arrow of time does not come from the physical system itself. . . it comes from the connection of the system with the outside world"*

In this article, the Boltzman paradox will be resolved in terms of the point of view of Continuous Topological Evolution [1]. First consider the definitions:

1. Causal evolution is defined as a map of C1 functions from a domain of base variables to a unique range of base variables. The maps may be many to one and are not necessarily homeomorphisms.

2. Prediction implies that well behaved functional forms (not just numeric point data) on the range of base variables can be deduced from functional forms defined on the domain of base variables.

3. Retrodiction implies that functional forms on the domain can be deduced from functional forms on the range.

The fundamental axioms are:

Axiom 1. *The topological features of Physical Systems on a domain of independent base variables can be encoded in terms of exterior differential forms (symbolically represented by A).*

Axiom 2. *Physical Processes can be defined in terms of contravariant vector fields, which may or may not be generators of 1-parameter groups, and in particular need not be homeomorphisms (symbolically represented by V).*

Axiom 3. *Equations of Continuous Evolution describing both reversible and irreversible Processes acting on Physical Systems are encoded by Cartan's magic formula :*

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) \tag{1.1}$$

The result of employing these axioms will be to demonstrate that:

1. Topological evolution is a necessary condition for thermodynamic irreversibility.

2. Thermodynamic irreversibility is an artifact of topological dimension 4 or more, while topological dimension 3 is a necessary condition for chaos.

3. The assumption of Uniqueness of predicted solutions (which implies a Topological dimension 2 or less) and Homeomorphic evolution are different constraints on classical mechanics that eliminates any time asymmetry.

   The functional forms of tensor fields with arguments as the base variables of the final state are not well defined in terms of the functional forms of tensor fields with arguments as the base variables of the initial state, unless the map from initial to final state is a diffeomorphism, which preserves topology. On the other hand, the functional forms of those tensor fields which are coefficients of exterior differential forms, and with arguments as the base variables of the initial state, are well defined in terms of the functional forms of tensor fields with arguments as the base variables of the final state, even when the map from initial to final state represents topological evolution. Hence an Arrow of Time asymmetry is a logical result when topological evolution is admitted, but does appear if the evolution is restricted to be homeomorphic.

4. The insistence that a unique outcome can be predicted from given initial data implies that the minimum topological dimension for a given geometrical dimension is 2 or less. If the topological dimension is 3 or greater, and if solutions to a particular evolutionary problem exist, then the solution is not unique. Envelope solutions are classic examples of solution non-uniquess.

The combined thermodynamic-topological perspective presented herein uses the mathematical tools of exterior differential forms to describe the topological features of physical systems, and vector fields to describe the continuous evolutionary processes that may or may not change the topology of the physical system. Examples will demonstrate that topological change is a necessary condition for thermodynamic irreversibility, and continuous topological change establishes the arrow of time.

2. Topological Tools

2.1. Topological Structure

The idea that the presence of a physical system establishes a *topological structure* on a base space of independent variables is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. The topological features of the physical system are presumed to be encoded in terms of exterior differential forms, which - unlike tensors - are functionally well behaved with respect to differentiable maps that are not invertible. Note that a given base may support many different topological structures; hence a given base may support many different physical systems.

For maps, between base sets, that are C1 differentiable (hence smooth), but are not invertible, it is impossible to predict uniquely the functional forms of covariant or contravariant vector fields, constructed over a final base set, in terms of functional forms given on an initial base set [2]. Point-wise values of the tensor fields in certain cases may be predicted, but the functional forms are never predictable with respect to such non-invertible maps. Hence, classical theories based on tensor fields, which can describe geometrical evolution, will fail to describe topological evolution. It may be surprising to note that (with respect to such non-invertible, non-homeomorphic, maps) it is possible to retrodict the functional forms of covariant vectors and contravariant vector densities on the initial base set in terms of the given functional forms on the final base set. For differentiable evolutionary processes that are diffeomorphisms, topology does not change and both prediction and retrodiction of tensor fields is possible. For differentiable evolutionary processes which are not homemorphisms, topology changes, and deterministic prediction fails, but retrodiction remains possible. Hence the feature of topological evolution imposes a sense of asymmetry with respect to an evolutionary parameter.

2.2. Continuity

Although C1 non-invertible maps are not homeomorphisms, and therefore the topology of the initial state and the topology of the final state are not the same, such maps can be continuous. Continuous topological evolution is not an oxymoron, for topological continuity is defined such that the limit points of every subset in the domain (relative to the topology on the initial state) permute into the closure of the subsets in the range (relative to the topology on the final state). The initial and final state topologies need not be the same.

Pasting together is a continuous process for which the topology of the final system state is not necessarily the same as the topology of the initial system state. Separation or cutting into parts is a discontinuous process for which the system topology of the final state is not the same as the system topology of the initial state. The obvious topological property that changes is the number of parts. Projections from higher dimensions to lower dimensions are classic examples of many to one differentiable maps that are not invertible. The obvious topological property that changes is the property of dimension. Consider a flat putty disc in the shape of an annulus. Deform the putty continuously such that the points that make up the central hole are pasted together. On the other hand make an interior cut in a disk of putty and discontinously separate the points to make a hole. The obvious topological property that changes is the number of holes. (Discontinuous processes are ignored in this presentation.)

2.3. Cartan's Magic Formula

Cartan's "magic formula" (a descriptive phrase introduced by Marsden [1]) representing the "evolution" of the 1-form of Action, A, with respect to the "flow" generated by the vector field, V, is the cornerstone of the development. The Cartan formula does not depend upon connection or metric, and has been called the "homotopy formula" by Arnold [2]. If the coefficient functions of A and V are C2 differentiable thbe it ispossible to prove that Cartan's formula describes continuous evolution. The C2 constrtaints can be relaxed, but will be studied elsewhere.

Herein, the following definitions are made:

1. The term $W = i(\mathbf{V})dA$ is defined as the inexact 1-form of "virtual work".
2. The function $U = i(V)A$ is defined as the "internal energy".
3. The sum of the two terms, $W + dU$, define the inexact 1-form of "heat", Q.

From these definitions, it is apparent that Cartan's magic formula not only represents an evolutionary process, where the process V acts on the physical system A to produce the 1-form of heat, Q, but also is formally equivalent to the cohomological description of the First Law of Thermodynamics.

$$\begin{aligned} L_{(\mathbf{V})}A &= i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q \\ &= W + dU = Q \end{aligned} \tag{2.1}$$

In this article this formal correspondence is taken seriously. Moreover the formula can be used to determine equivalence classes of evolutionary processes, for a given physical system, which are thermodynamically reversible or irreversible.

The fundamental theme is to study processes that describe continuous topological evolution. Such evolutionary processes are not necessarily invertible and do not admit unique deterministic prediction of tensor fields from initial data. However, they do permit the deterministic retrodiction of tensor fields by means of functional substitution and pullbacks.[3] The magic in Cartan's formula is that it can be used to describe such evolutionary processes where the topology of the initial state is not the same as the topology of the final state, as well as for adiabatic processes for which the topology does not change.

Both the heat and the work 1-forms as defined above are not necessarily exact, and therefore can lead to non-zero cyclic integrals. The symbol $L_{(\mathbf{V})}$ stands for the "Lie derivative" with respect to $\mathbf{V}$,a term evidently coined by Slebodzinsky.[4] The symbol dA stands for the "exterior derivative" of A, and the symbol $i(\mathbf{V})A$ is used to designate the "interior product" of the contravariant $\mathbf{V}$ with the covariant A in a tensorial sense, producing a diffeomorphic invariant. However, no constraints of metric or connection are applied a priori to the domain of definition. For more detail see [5]

For physical systems of measurement it is presumed that the ultimate or fundamental domain (or base) of independent variables will be designated by the ordered quadruplet $\{x, y, z, t\}$. Most useful applications will be constructed from both covariant and contravariant vector fields and functions ultimately defined over this base. However, an initial domain of definition may be conveniently of higher dimension; that is, the initial variety may consist of 2n+1 or 2n+2 independent variables. Note that the initial variety may consist of both "coordinates" and "parameters", and the notation is suitable for application of Fiber bundle theory.

2.4. Thermodynamic Irreversibility $Q\hat{}dQ \neq 0$

Following the lead of thermodynamic experience, a thermodynamic process which is reversible it to be associated with a heat 1-form, Q, which admits an integrating factor. The integrating factor (in thermodynamics) defines the concept of temperature. Therefore, if the heat 1-form does not admit an integrating factor, the thermodynamic process is irreversible.[7] From a topological point of view, the heat 1-form admits an integrating factor if and only if Q satisfies the conditions of the Frobenius integrability theorem, $Q\hat{}dQ = 0.$ This definition of thermodynamic irreversibility, when combined with Cartan's magic formula, permits the link to be made between thermodynamics and mechanical systems.

To repeat: It is subsumed that the physical system can be represented by a 1-form of Action, A, and a physical process can be represented by the vector field $\mathbf{V}$. As the system (A) is propagated via the action of the Lie derivative with respect to the process, $\mathbf{V}$, the outcome is to produce the heat 1-form, Q. Hence a simple test for thermodynamic irreversibilty of a process acting on a system is given by the equations:

$$Q\hat{}dQ = (L_{(\mathbf{V})}A)\hat{}(L_{(\mathbf{V})}dA) = 0 \supset the\,process\ is\,reversible. \tag{2.2}$$

$$Q\hat{}dQ = (L_{(\mathbf{V})}A)\hat{}(L_{(\mathbf{V})}dA) \neq 0 \supset the\,process\ is\,irreversible. \tag{2.3}$$

The technique is as follows: First start with a reasonable description of a physical system in terms of a 1-form of Action, A, and then for a given vector field, $\mathbf{V}$, representing a process, construct Q from Cartan's formula. Finally, use the Frobenius test to see if the given process is reversible or not.

2.5. The Pfaff Topological Dimension, n, versus Geometrical Dimension, m:

Of key importance for any particular physical system is the choice of the "correct" 1-form of Action, A, which encodes the topological features of a specific physical system. Experience (guesswork) and the degree of agreement with measurement will satisfy the working scientist. By measurement, it is meant that certain geometrical and topological features will be "observable" evolutionary invariants of a process, or of an equivalence class of processes. In physics, the equivalence class of processes is often specified as solutions to a system of partial differential equations; herein, the alternative view is taken that the equivalence class is generated by an exterior differential system of constraints acting on exterior differential forms. Exterior differential systems are, in effect, specifications of topological constraints on the physical system. For example, the constraint $F - dA = 0$ is topologically a constraint that says the 2-form object is globally closed in an exterior differential sense. The 2-form F is constrained to be equal to the limit sets of A, relative to the Cartan topology generated by the topological structure of the given 1-form of Action, A.

Perhaps one of the most important, and yet easily computed, ideas is the concept of Pfaff Topological dimension, or class, of an exterior differential 1-form, A. Recall that dimension is a topological property, hence if the Pfaff Topological dimension changes during an evolutionary process, the process is describing a process of continuous topological change - a necessary requirement for thermodynamic irreversibility. Examples of such processes will be given below. It is also important to remember that the same set of elements can support more than one topology. In the case at hand, the Pfaff Topological dimension of the three 1-forms, A, W, and Q will be of importance to the analysis of non-equilibrium thermodynamics, and the arrow of time. A given differential form, A, geometrically defined in terms of $2n$ differentials and functions on a variety of geometric base dimension n, may require $m \preceq 2n$ independent functions and differentials for its topological description. The Pfaff topological dimension can be less than the Geometric dimension of the space over which the 1-form of Action has been constructed. The concept implies that their exists a differentiable projective map from a space of dimension $2n$ to the space of dimension m. An exterior differential form defined on the (final) target space induces a functionally well defined exterior differential form on the (initial) domain space, by means of functional substitution and the "pullback" of the projection. The topological features of investigations on the lower dimensional space can be retrodicted back to the initial higher dimensional manifold, even though the projective mapping is not a homeomorphism and therefor implies topological change. Remarkably, the Pfaff topological dimension of the 1-form on the target space is the same as the Pfaff topological dimension of the pulled back 1-form on the initial space.

For any given 1-form, A, functionally defined on a (perhaps geometrical) base space, or variety, of dimension n, it is possible to compute the "Pfaff sequence", $\{A, dA, A\hat{}dA, dA\hat{}dA...\}$. It is remarkable that this sequence terminates at a minimum number, $m \leq n$, representing the irreducible minimum number of independent functions required to define the topological encoding. This number m is called the Pfaff topological dimension, and the last non-zero element of the Pfaff sequence is defined as the top Pfaffian. The topological dimension, m, is less than or equal to the geometric dimension, n. Note that the requirement for thermodynamic irreversibility implies that the Pfaff dimension of the heat 1-form Q must be 3 or greater, which means the Frobenius theorem of *unique* integrability for the Pfaffian expression, $A = 0$, fails. The idea is that the Pfaff topological dimension implies the existence of a continuous differentiable map from the variety of dimension $n > m$ to a variety of dimension m.

An explicit physical 1-form, A, will generate a "Cartan topology" on the domain. It is easy to demonstrate that the Cartan topology is a connected topology if the Pfaff dimension is 2 or less, and a disconnected topology if the Pfaff dimension is 3 or more [7]. Hamiltonian mechanics and Eulerian streamline flows in hydrodynamics (on base spaces of geometric dimension 4) are associated with Action 1-forms of Pfaff topological dimension 2 or less, and are thermodynamically reversible. Turbulence, being the antithesis of streamline flow, must be represented by a topology of Pfaff dimension 4 or more, and is thermodynamically irreversible.

Again, the constraint of continuous topological evolution induces a logical "Arrow of Time" related to a change in Topology. Note that the decay of turbulence can be studied by continuous methods, but the creation of turbulence cannot. It is possible to demonstrate that a map from a continuous topology to a discontinuous topology is not continuous, but a map from a continuous topology to a discontinuous topology is continuous. In both evolutionary maps, topological evolution takes place as the Pfaff topological dimension changes, but note that the creation of turbulence cannot be continuous, where the decay of turbulence can be continuous.

2.6. Contact manifolds:

When the 1-form A is of odd topological dimension (n= 3 or greater), then the 2-form dA can be put into correspondence with an odd-dimensional n=2k +1 antisymmetric matrix of functions of maximal rank. This matrix has one unique eigenvector with a null eigen value. Hence the topological encoding of a physical system determines a unique direction field defined as the "extremal" direction field (on the 2k + 1 dimensional variety). Evolution in the direction of this unique "extremal" vector field, $\mathbf{V}_E$, implies that the virtual work, W, vanishes, and that the exterior derivative of heat 1-form dQ vanishes, as Q is exact. Such extremal vector fields always have a Hamiltonian generator, and are not thermodynamically irreversible, as Q^dQ=0. The extremal Hamiltonian evolution preserves the even dimensional topological features of the physical system (the Poincare invariants). If the extremal process is also adiabatic, such that both Q = 0 and dQ = 0, then the process preserves both odd and even topological features, and is a homeomorphism. The equivalence class of processes that satisfy the closure requirement, (dQ=0) includes not only extremal fields, $i(\mathbf{V}_E)dA = 0$, but also those that can have a Casimir generator (Bernoulli flows) or those that can generate limit cycles.

2.7. Symplectic manifolds:

When the 1-form A is of even topological dimension (n = 4 or greater), then the 2-form dA can be put into correspondence with an even-dimensional antisymmetric matrix of functions of maximal rank. Extremal fields (with null eigenvalue) do not exist, but there is a unique evolutionary direction field, $\mathbf{V}_T$, on the n = 2k + 2 dimensional variety that is completely determined from the topology of the physical system, induced by the 1-form, A. On a n = 4 dimensional base manifold, this unique direction field is defined by the equations,

$$A\hat{}dA = i(\mathbf{V}_T)dx\hat{}dy\hat{}dz\hat{}dt. \tag{2.4}$$

This vector field $\mathbf{V}_T$ is defined as the Topological Torsion vector. As $A\hat{}A\hat{}dA = 0$ the Topological Torsion vector is transverse with respect to the 1-form of Action: $i(\mathbf{V}_T)A = 0$. By direct calculation it is possible to show that $W = i(\mathbf{V}_T)dA = \Gamma\, A$. In otherwords the 1-form of virtual work is proportional to the 1-form of Action. Cartan's magic formula becomes

$$L_{(\mathbf{V}_T)}A = \Gamma\, A \tag{2.5}$$

where Γ equals 1/2 the coefficient of the non-zero 4-form

$$dA\hat{}dA = 2 * \Gamma(x, y, z, t)dx\hat{}dy\hat{}dz\hat{}dt = \{div_4(\mathbf{V}_T)\}dx\hat{}dy\hat{}dz\hat{}dt. \tag{2.6}$$

As the 2-form is of maximal rank, $\Gamma(x, y, z, t) \neq 0$. It follows that evolution in the direction of the Torsion Vector is thermodynamically irreversible, as

$$Q\hat{}dQ = L_{(\mathbf{V}_T)}A\hat{}L_{(\mathbf{V}_T)}dA = \Gamma^2\, A\hat{}dA \neq 0, \tag{2.7}$$

The factor Γ^2 plays a role related to the entropy production rate.

2.8. Cartan's development of Hamiltonian systems.

Recall that Cartan proved that if the 1-form of Action is taken to be of the classic format, $A = p_k dq^k + H(p_k, q^k, t)dt$, on a 2n+1 dimensional domain of variables $\{p_k, q^k, t\}$, then a subset of all vector fields, V, that satisfy his magical equation would generate "Hamiltonian flows" of classical mechanics.[6] The necessary and sufficient constraint for the vector field to be of the Hamiltonian format was that the closed integrals of the Action $\int_{z1} A$ must be evolutionary invariants of the process, V.

$$\text{Cartan's Constraint:} \qquad L_{(\mathbf{V})}\int_{z1} A \Rightarrow 0. \tag{2.8}$$

The symbol, $\int_{z1}$, is used to designate that the integration chain is a closed cycle, z1; $\int_{z2}$, would be used to designate a two dimensional closed cycle; etc.. The cycle may or may not be a boundary.

The Cartan criteria does not constrain the Hamiltonian function $H(p_k, q^k, t)$ to be independent from time, but as will be described below, it does insure that the Cartan topology of the initial state is the same as the Cartan topology of the final state. The same criteria to generate "Hamiltonian flows" can be used on 2n+2 dimensional domains, (p_k, q^k, t, s). The key difference is that on the odd dimensional domain (a contact manifold) the Hamiltonian flow is a unique "extremal" field. The generator of the flow is the Hamiltonian function, $H(p_k, q^k, t)$. On the 2n+2 dimensional domain (a symplectic manifold), a unique extremal field does not exist. There do exist (many) "Hamiltonian flows", but they are generated, not from $H(p_k, q^k, t, s)$, but from other functions, known as Bernouilli-Casimir functions, Θ. There does, however, exist a unique vector direction field of evolution on the symplectic 2n+2 domain, but it is not a Hamiltonian flow. In fact, it will be demonstrated below that this unique vector field (defined as the Topological Torsion current) represents thermodynamically irreversible processes.

2.9. Thermodynamic Reversibility $Q\hat{}dQ = 0$

The Cartan constraint $(L_{(\mathbf{V})}\int_{z1} A = 0)$ thereby partitions all possible vector fields of evolution into two equivalence classes, those representing processes that were "Hamiltonian", and those that were not Hamiltonian. Herein the idea is to exploit the Cartan's magic formula to obtain a better understanding of the non-Hamiltonian processes $(L_{(\mathbf{V})}\int_{z1} A \neq 0)$, and how they may represent dissipative and irreversible physical phenomena. Hamiltonian processes may be time-dependent, hence decaying energy alone is not a sufficient criteria to insure thermodynamic irreversibility.

Following the lead of thermodynamic experience, a thermodynamic process which is reversible it to be associated with a heat 1-form, Q, which admits an integrating factor. The integrating factor (in thermodynamics) defines the concept of temperature. Therefore, if the heat 1-form does not admit an integrating factor, the thermodynamic process is irreversible.[7] From a topological point of view, the heat 1-form admits an integrating factor if and only if Q satisfies the conditions of the Frobenius integrability theorem, $Q\hat{}dQ = 0$. This definition of thermodynamic irreversibility, when combined with Cartan's magic formula, permits the link to be made between thermodynamics and mechanical systems.

Rather than applying the method to many examples, it is possible to consider equivalence classes determined by the Pfaff dimension, or class, $Pfaff(W)$, of the 1-form of virtual work, W. The Pfaff dimension of the virtual work 1-form, $W = i(\mathbf{V})dA$, depends on both the process $(\mathbf{V})$ and the system (A). Conservative Hamiltonian processes belong to the $Pfaff(W) = 0$ or $Pfaff(W) = 1$. Processes that belong to the $Pfaff(W) = 4$ are always irreversible.

3. The Pfaff Dimension of the 1-form of Virtual Work

Given any 1-form, $W = i(\mathbf{V})dA$, defined over a set of independent variables, it is always possible to construct its Pfaff sequence from the form, W, its exterior differential, dW, and algebraic exterior products of these objects;

$$\text{Pfaff sequence of W} \quad \{W, dW, W\hat{}dW, dW\hat{}dW, ..\}. \tag{3.1}$$

At some integer M+1 the remaining elements of the Pfaff sequence are zero. The Pfaff dimension, or class, $Pfaff(W)$ of the form, W, is defined as the integer M equal to the number of non-zero terms in the Pfaff sequence. This integer is always less than or equal to the number of independent variables. The Pfaff dimension specifies the irreducible number of functions required to specify the 1-form of interest. As the 1-form of Work is constructed from the 1-form of Action, the number of contravariant components of a vector field, V, required to define the 1-form of virtual work, W, need not exceed the Pfaff dimension of the Action 1-form. However, the components of an arbitrary contravariant vector field on the original domain of definition may not be fully expressible in terms of projected functions of the 1-form of Action. In modern language, the Pfaff dimension of the 1-form of Action determines the base, but the contravariant vector field has additional components along the fibers of the vector bundle.

Cartan's magic formula takes note of this difference, for the 1-form of virtual work, W, is transversal to the process, while the 1-form of heat is not.

$$i(\mathbf{V})W = i(\mathbf{V})i(\mathbf{V})dA = 0 \quad \textit{but} \quad i(\mathbf{V})Q = i(\mathbf{V})d(i(\mathbf{V})A) \neq 0 \tag{3.2}$$

In the language of fiber bundles, this result gives a precise definition to the differences between the concepts of work and heat. Heat can have components along the fibers, work does not.

In that which follows, the features of the various equivalence classes defined by the Pfaff dimension of the 1-form of virtual work are explored. In all classes considered, the trivial case $dA = 0$, is ignored, for then every vector field representing a process on such physical systems is such that the virtual work vanishes. All such cyclic processes are adiabatic, and if the process is such the the internal energy is constant, $dU = d(i(\mathbf{V})A) = 0$, then such processes are locally and globally adiabatic. If the process is an associated vector, (such the $U = i(V)A \Rightarrow 0$) then the process resides on the "equi-potential" surface defined by the Pfaffian equation, $dA = 0$.

3.1. Reversible Case 1: Pfaff(W) = 0. Cyclically adiabatic extremal processes.

When $dA \neq 0$, the constraint, $Pfaff(W) = 0$, implies that the virtual work 1-form vanishes, $W = i(\mathbf{V})dA = 0$, and the 2-form $dW \Rightarrow 0$. Recall that the 2-form of "vorticity", or field intensities, dA, consists of an anti-symmetric matrix of coefficients. Hence, only when the Pfaff dimension of the Action is an odd-integer, 2n+1, is it possible for work 1-form to vanish. In such cases the processes, $\mathbf{V}$, are defined as extremals (a word borrowed from the calculus of variations) and are *uniquely* determined (to within a projective factor) as the null eigen vector of the anti-symmetric matrix of functions that are used to represent the coefficients of the 2-form dA. As this extremal constraint determines the "equations of motion", it should be noted that there is a large equivalence class of physical systems that will have the "same" orbital motion. In the extremal case the 1-form of Action is not unique, for any closed 1-form, γ, with $d\gamma = 0$, may be added to the initial 1-form, A, without changing the structure of the 2-form, dA. It is the form dA that determines the virtual work, W.

$$dA = d(A_0 + \gamma) = dA_0 + d\gamma = dA_0. \tag{3.3}$$

The "equations of motion" are said to be "gauge" invariant in the sense that the virtual work 1-form is the same for all physical systems which are elements of the large equivalence class of Actions which differ from one another by a closed 1-form (the "gauge"). Note that the gauge differences between the elements of different actions are not necessarily exact differentials; the class of actions that produce gauge invariant fields, or equations of motion, can belong to different cohomology classes. In short, the same W has many precursors A

However, from a thermodynamic point of view, the heat 1-form, Q, and how the system interacts with its surroundings, is sensitive to the closed 1-form additions to the Action 1-form. The heat 1-form, Q, and the internal energy, U, are **not** gauge invariant.

$$\begin{aligned} L_{(\mathbf{V})}A &= i(\mathbf{V})dA + d(i(\mathbf{V})A) \qquad (3.4)\\ &= 0 + dU = d\{i(\mathbf{V})A_0 + (i(\mathbf{V})\gamma\} = Q \end{aligned}$$

However, what is remarkable, is that any closed integral of the Action, $\int_{z1} A$, is a (relative) integral invariant of the extremal evolutionary processes generated by $\mathbf{V}$ of this equivalence class.

$$L_{(\mathbf{V})}\int_{z1} A = \int_{z1} Q = 0 \quad (W = 0). \tag{3.5}$$

Hence, any cyclic integral of the heat 1-form is "gauge" invariant. During portions of the cycle, Q may be positive and negative, such that over the cycle, the net Q is zero. (Such systems are sometimes called "breathers" and can be related to limit cycles that occur in dissipative systems.)

If Q vanishes identically, the process is said to be locally adiabatic. For a given system, the constraint that the process be locally adiabatic, can be satisfied by an extremal vector field, which is also "associated". The two constraints,

$$(i(\mathbf{V})A) = 0 \ \ (associated) \tag{3.6}$$

$$i(\mathbf{V})dA = 0 \ \ (extremal) \tag{3.7}$$

form a subclass of processes defined as "characteristic" processes, It follows that such characteristic processes are locally adiabatic.

As mentioned above, on an even dimensional manifold of maximal rank, extremal fields do not exist. However, as will be discussed below, on the 2n+2 symplectic manifold for which there is no unique extremal field, there does exist a unique direction field defined as the (topological) torsion current. Evolution in the direction of the torsion current can decay to a domain where (through topological evolution) the 2n+2 domain becomes a 2n+1 domain, and from then on the evolution can take place along an extremal direction. The initial decay is essentially a transient process that dies out (irreversibly) to a steady state conservative process.

The extremal evolutionary processes form the basis for classical mechanics on state-space. It is apparent that the net heat around any closed path is cyclically zero. If in addition the internal energy is a constant, $dU = 0$,then such processes are locally adiabatic, $Q = W + dU = 0 + 0 = 0$. As the extremal vector field is determined only up to a factor, ρ, it is possible to choose this function such that the internal energy is a constant,

$$U = \rho(i(\mathbf{V})A) = Const. \tag{3.8}$$

For such choices of ρ the extremal process is locally adiabatic.

Suppose the initial domain of independent variables $\{E, t, p_k, q^k\}$ is of dimension 2n+2, with a Darboux representation for the 1-form of Action given by the expression

$$A = p_k dq^k - Edt. \tag{3.9}$$

The top Pfaffian, $dA\hat{\ }dA...$ is a 2n+2 form

$$dA\hat{\ }dA... = dE\hat{\ }dt\hat{\ }dp_1...\hat{\ }dp_n\hat{\ }dq^1...\hat{\ }dq^n. \tag{3.10}$$

If the Pfaff dimension of the Action 1-form is to be 2n+1, then this 2n+2 form must vanish. Hence the variable, E, cannot be functionally independent from the remaining (presumed to be independent) variables; it follows that $E = H\{p, q, t\}$

on the 2n+1 dimensional domain. The Action 1-form is then written in the Cartan-Hamiltonian format

$$A = p_k dq^k - H\{p, q, t\}dt \tag{3.11}$$

Relative to the 2n+1 "coordinates" $\{p_k, q^k, t\}$, consider the vector field $\mathbf{V} = \{f_k, V^k, 1\}$ and find the solution to the equation, $W = i(\mathbf{V})dA = 0$. The result is

$$\mathbf{V} = \{f_k = -\partial H/\partial q, V^k = \partial H/\partial p, 1\} \tag{3.12}$$

and the extremal field is said to be Hamiltonian.

By constucting the exterior derivative of Cartan's magic formula,

$$\begin{aligned} L_{(\mathbf{V})}dA &= di(\mathbf{V})dA + dd(i(\mathbf{V})A) \\ &= dW + 0 = dQ \end{aligned} \tag{3.13}$$

As $dW = 0$ for $Pfaff(W) = 0$, it follows that $dQ = 0$. Hence, all even dimensional elements of the Pfaff sequence generated by the Action, $\{dA, dA\hat{}dA, ...\}$, and their integrals, are absolute invariants of the equivalence class of extremal fields, a result known to Poincare.

Note that all processes for which the work 1-form is of Pfaff class 0 are reversible, for $Q\hat{}dQ = 0$.

3.2. Reversible Case 2: Pfaff(W) = 1. Symplectic processes.

When the virtual work 1-form is closed but not zero, $W \neq 0, dW = 0$, then the Pfaff dimension is equal to 1. The closure constraint forces the virtual work 1-form to be composed of a perfect differential and/or a harmonic part. When the virtual work 1-form is exact such that

$$W = i(\mathbf{V})dA = d\Theta(x, y, z, t), \tag{3.14}$$

then the function Θ is defined as a Bernouilli-Casimir function, and is an invariant (first integral) of those evolutionary process, $\mathbf{V}$, that belong to the $Pfaff(W) = 1$.

$$L_{(\mathbf{V})}\Theta = i(\mathbf{V})d\Theta = i(\mathbf{V})(i(\mathbf{V})dA) = 0. \tag{3.15}$$

In hydrodynamics, the Berenoulli function is a constant along any streamline, but neighboring streamlines will have different values for the Bernoulli function, hence $\Theta = \Theta(x, y, z, t)$ but $i(V)d\Theta = 0$.

When the virtual work 1-form is exact, the processes are not only reversible ($dQ = 0$), but they are also cyclically adiabatic. On the otherhand, if the virtual work 1-form is closed, but not exact, then the processes, although reversible, are not cyclically adiabatic.

If the Pfaff class of the Action is even, then there exists a unique vector field, $\mathbf{V}$, that defines a locally reversible adiabatic process. For if $Q = 0$,

$$W = i(\mathbf{V})dA = -dU = -d(i(\mathbf{V})A) \tag{3.16}$$

then

$$L_{(\mathbf{V})}A = W + dU = -dU + dU = Q = 0. \tag{3.17}$$

As an example consider the domain $\{x, y, z, t\}$ and the Action $A = \mathbf{A}\bullet\, d\mathbf{r} - \phi dt$. The adiabatic condition becomes the partial differential system

$$-\partial\mathbf{A}/\partial t - grad\phi + \mathbf{V} \times curl\mathbf{A} = -grad(\mathbf{V} \cdot \mathbf{A} - \phi) \tag{3.18}$$

$$\mathbf{V} \cdot (-\partial\mathbf{A}/\partial t - grad\phi) = \partial(\mathbf{V} \cdot \mathbf{A} - \phi)/\partial t \tag{3.19}$$

However, when the work 1-form in not exact, then the process is not cyclically adiabatic, and there will exist non-zero cyclic contributions to the work and heat. The ratio of these integrals is rational [deRham].

The evolutionary vector field is again said to be "Hamiltonian", for $dp - (-\partial\Theta/\partial q)dt = 0$ and $dq - (\partial\Theta/\partial p)dt = 0$. If the Action is written in the Cartan format,

$$A = pdq - H(p, q, t, \sigma)dt),$$

then the Hamiltonian energy, $H(p, q, t, \sigma)$, is not neccessarily an invariant of the flow generated by the Bernouilli-Casimir function, Θ. The Bernoulli-Casimir is, however, an evolutionary invariant, and its gradient is transversal to the evolutionary process.

However, when the work 1-form in not exact, but may have harmonic components, γ, representing topological obstructions. In these cases, the process is not adiabatic in a cyclic sense, for

$$\int_{z1} Q = \int_{z1} W + d(U) = \int_{z1} \{d(\Theta + U) + \gamma\} = 0 + \int_{z1} \gamma \neq 0$$

There will exist non-zero cyclic contributions to the work and heat. The ratio of these cyclic integrals is rational [deRham].

Note that all processes for which the work 1-form is of class 1 are reversible, for $Q\hat{}dQ = 0$.

3.3. Reversible Case 3: Pfaff(W) = 2 or 3

When $Q\hat{}dQ = 0$, but $dQ = dW \neq 0$, the first law implies that

$$W\hat{}dW + dU\hat{}dW = 0. \tag{3.20}$$

Then either $W\hat{}dW = 0$ (and the Pfaff dimension of W is 2) or $W\hat{}dW \neq 0$ (and the Pfaff dimension of W is 3 or more). In the first case there is a functional relationship between thae variables $U = U(P, V)$ or $U = U(T, s)$. In both cases $dW\hat{}dW = 0,$ hence the Pfaff dimension of W is 3 or less.

3.4. (The Ideal Gas)

First consider the reversible Pfaff dimension 2 case, where $Q\hat{}dQ = 0$ and $W\hat{}dW = 0$. Then locally, $Q \equiv TdS$. It follows that

$$(P/T)dV + (1/T)dU = dS \tag{3.21}$$

By taking the exterior derivative of both sides,

$$d(P/T)\hat{}dV + d(1/T)\hat{}dU = 0 \tag{3.22}$$

A simple class of solutions would impose the "quadrature" conditions that each of the two forms vanish separately. It follows that V must be a function of (T/P) ,alone, and U must be a function of T, alone. Such conditions establish the equivalence class of the "ideal gas" with a linear representation in the format

$$V = nRT/P \quad and \quad U = nC_vT. \tag{3.23}$$

These representations are not the only possibilities for the reversible processes where the Pfaff dimension of the work 1-form is 2. Another realization implies that

$dU\hat{}dT\hat{}dS = 0$, such that the internal energy, U, is not a function of temperature alone, but unlike the ideal gas, the internal energy, in this reversible situation, also depends upon the entropy function, S.

3.5. Case 4 : The Pfaff dimension of the Work 1-form is 4.

In this case $dW\hat{}dW = dQ\hat{}dQ \neq 0$ and the process is never reversible. Examine the case where $dA\hat{}dA \neq 0$, on a domain of 4 dimensions. Then there exists a unique direction field T such that

$$A\hat{}dA = i(T)dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.24}$$

This vector field T is defined as the Topological Torsion vector. As $A\hat{}A\hat{}dA = 0$ the Topological Torsion vector is associated with the 1-form of Action:

$$i(T)A = 0. \tag{3.25}$$

By direct calculation it is possible to show that

$$W = i(T)dA = \Gamma\, A. \tag{3.26}$$

In otherwords the 1-form of virtual work is proportional to the 1-form of Action. Cartan's magic formula becomes

$$L_{(\mathbf{V})}A = \Gamma\, A \tag{3.27}$$

where Γ equals the coefficient of the non-zero 4-form

$$dA\hat{}dA = \Gamma(x, y, z, t)dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.28}$$

As the 2-form is of maximal rank, $\Gamma(x, y, z, t) \neq 0$.

It follows that evolution in the direction of the Torsion Vector yields

$$Q\hat{}dQ = L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA = \Gamma^2\, A\hat{}dA \neq 0, \tag{3.29}$$

which implies that the process is thermodynamically irreversible.

4. Anholonomic Fluctuations.

Consider a physical system that can be defined in terms of the Cartan-Hilbert 1-form

$$A = L(t; q, v)dt + p_k(dq^k - v^k dt), \tag{4.1}$$

defined on the 3n+1 variety $\{t; q^k, v^k, p_k\}$. Do not assume that p_k is constrained to be a jet; e.g., $p_k \neq \partial L/\partial v^k$. Instead, consider p_k to be a Lagrange multiplier to be determined later. It follows that the exact two form dA satisfies the equations

$$(dA)^{n+1} \neq 0, \; but \quad A\hat{}(dA)^{n+1} = 0. \tag{4.2}$$

The actual formula for the top Pfaffian (which is of dimension 2n+2 and not 3n+1) is:

$$\begin{aligned} (dA)^{n+1} &= (n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k) \bullet dv^k\}\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt & (4.3)\\ where\, \Omega_p &= dp_1\hat{}...\hat{}dp_n, & (4.4)\\ and\, \Omega_q &= dq^1\hat{}...\hat{}dq^n. & (4.5) \end{aligned}$$

It is to be noted that the unconstrained top Pfaffian of the Cartan-Hilbert Action is always associated with a symplectic (even dimensional) manifold, but not of the maximum dimension of the space of the 3n+1 variables. For n = 3 degrees of freedom, the top Pfaffian indicatea that the topological of Pfaff dimension of the 2-form, dA is $2n + 2 = 8$.

If the domain of definition is constrained such that the momenta are defined canonically, $\partial L/\partial v^k - p_k = 0$, then the 2-form dA is not symplectic on its maximal dimension 2n+2, but becomes a contact structure on 2n+1 with the formula

$$A\hat{}(dA)^n = n!\{p_k v^k - L(t, q, v\}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{4.6}$$

The coefficient in brackets is the Legendre transform of the Lagrangian producing the format of the classic Hamiltonian energy. The resulting 2n+1 (state) space always has a contact structure if the "total energy" is never zero, and the momenta are canonically defined. The space is reducible to a 2n phase space only if the Lagrangian is homogeneous of degree 1 in the v^k, otherwise it is a contact structure of dimension 2n+1.

Consider evolutionary processes defined interms of a vector field $\gamma\mathbf{W} = \gamma\{1, v, a, \; f\}$, relative to $\{t; q, v, p\}$. Construct the 1-form W of virtual work by contracting the exact two form dA with the vector field. For every case, the 1-form of virtual work has the format

$$W = i(\mathbf{W})dA = \{p - \partial L/\partial v\}\Delta v + \{f - \partial L/\partial x\}\Delta q. \tag{4.7}$$

where

$$\Delta v = dv - adt \neq 0 \tag{4.8}$$

and

$$\Delta q = dq - vdt \neq 0. \tag{4.9}$$

When the 2-form dA is symplectic, the work 1-form (which can not vanish) has two terms for any n; the first involves Δv and the second involves Δq. The work 1-form cannot vanish if dA is symplectic for there are no null eigenvectors of an anti-symmetric matrix of maximal rank. This fact implies that the following 4 situations are NOT allowed when dA is symplectic.

1. $\{p - \partial L/\partial v\} = 0$ and $\{f - \partial L/\partial x\} = 0$

   (Canonical momentum and gradient forces.)

2. $\{p - \partial L/\partial v\} = 0$ and $\Delta q = 0$

   (Canonical momentum and zero kinematic fluctuations in position.)

3. $\Delta v = 0$ and $\{f - \partial L/\partial x\} = 0$

   (Zero kinematic fluctations in velocity and gradient forces.)

4. $\Delta v = 0$ and $\Delta q = 0$

   (Zero kinematic fluctuations in velocity and Zero kinematic fluctuations in position.)

Conversely, when dA generates a contact manifold, one of the four cases above must be true. An elementary case is based upon the assumption that 4 is valid. That is, there exists a kinematic description of the process at both the first and the second order. Another case that is common is based on the assumption that the momentum is canonically defined. Then, for the Contact extremal case to exist, and as $p - \partial L/\partial v = 0$, it is necessary that the work 1-form reduces to vanishing expression

$$W = \{f - \partial L/\partial x\}\Delta q \Rightarrow 0 \text{ in the extremal case.} \qquad (4.10)$$

The extremal constraint is satisfied when the bracket factor vanishes, which is then the equivalent of the Lagrange-Euler equations of classical mechanics. However, the Contact constraints are also satisfied when the force is a gradient field, or there exist zero fluctuations in position, or the non-zero components of the force (the otherwise dissipative components) are orthogonal to the kinematic fluctuations in position.

Of current interest are those situations when the work one form is closed, but not zero. Such constraints define symplectic (not extremal) evolutionary processes which occur on even dimensional symplectic manifolds. Locally, as $W = i(\mathbf{W})dA = d\Theta$, it can be shown that such evolutionary fields belong to Lie groups, and that the non-constant functions, Θ, are Casimirs. A hydrodynamicist would use a different set of words. He would describe the Casimir as a Bernoulli function, a function which is constant along a particular flow line, but which will vary from flow line to flow line. Symplectic processes create conservation theorems of the Helmholtz type (conservation of vorticity, conservation of angular momentum..). In such systems, the Hamiltonian energy need not be an evolutionary invariant, and the system can decay to singular points where the symplectic structure condition fails. Such points will be defined as "equilibrium" points of a symplectic process. An example is given in reference [9] to show how the Navier-Stokes equations generate evolutionary vector fields of the symplectic type, but the Euler equations (without pressure) generate extremal vector fields. Numerical studies indicating such phenomena appear in [10]

If Δv is interpreted as "anholonomic differential fluctuations" in velocity, and Δq is interpreted as "anholonomic differential fluctuations" in position, then it is intuitive to state that fluctuations in velocity relate to temperature and fluctuations in position relate to temperature. Following this train of thought implies that the first term in the expression for W must be related to Enthalpy (functions of the type $-TS$ that involve temperature) and the second term to Helmholtz free energy (functions of the type $+PV$ that involve pressure). The combination defines the Gibbs free energy (functions of the type $-TS + PV$) of closed thermodynamic systems, and reversible processes. These thermodynamic ideas, more than 100 years old, are essentially the Casimirs of the symplectic vector fields of irreducible dimension 2n+2, and are not evident in extremal systems. When the evolutionary vector fields are symplectic, such that $dW = dQ = 0$, they define thermodynamic reversible processes. The Cartan evolutionary equation of a symplectic process becomes

$$\begin{aligned} L_{(\mathbf{W})}A &= W + dU = d\Theta + dU = \{p - \partial L/\partial v\}\Delta v + \{f - \partial L/\partial x\}\Delta q + dU \\ &\Rightarrow d(-TS + PV + U) = d(G) = Q, \end{aligned} \tag{4.11}$$

which defines the heat 1-form Q as the "gradient" of the Gibbs free energy, $G = TS - PV + U$. The Gibbs function is an evolutionary invariant by construction, for all Bernoulli-Casimir functions have transversal gradients.

$$L_{(\mathbf{W})}(G - U) = i(\mathbf{W})d(G - U) = i(\mathbf{W})i(\mathbf{W})dA = 0. \tag{4.12}$$

Under the classic assumption that $dU - TdS + PdV = Q$, it follows that the symplectic evolution generates a Pffafian form of the type $-SdT + VdP = 0$, which if integrated yields Gibbs version of an equation of state.

When the work 1-form is not closed, then the process can become thermodynamically irreversible. In this case, the evolution is on a symplectic manifold, but the process is not sumplectic (as $dW \neq 0$). To test for irreversibility, the usual engineering requirement is that the heat 1-form Q does not admit an integrating factor. Hence, as described above, a given process, $\mathbf{W}$, acting on a physical system, A, is irreversible when

$$Q\hat{}dQ = L_{(\mathbf{W})}A\hat{}L_{(\mathbf{W})}dA \neq 0. \tag{4.13}$$

It is remarkable that the symplectic systems of irreducible dimension 2n+2 seem to solve the Boltzmann - Loschmidt-Zermelo paradox of why canonical Hamiltonian mechanics does not seem to be able to describe the decay to an equilibrium state, and why the usual (extremal) methods of Hamiltonian mechanics do not give any insight into the concept of Pressure, Temperature, or the Gibbs free energy. It is extraordinary that answers to these 150 year old paradoxes of physics seem to follow without recourse to statistics if one utilizes Gromov's work on symplectic systems.

The interpretation of the fact that the top Pfaffian is of dimension 2n+2 and not 3n+1 is an open problem. The implication is that there must exist 3n+1-2n+2 = n-1 topological invariants in these systems.

5. Dissipative Evolution to States Far from Equilibrium

5.1. Irreversible Process in the direction of the Topological Torsion vector

On the four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action can be written in the form $A{=}\Sigma_{k=1}^{3}A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt$ which generates the 2-form $dA = \mathbf{B}_z dx\hat{}dy... + \mathbf{E}_x dx\hat{}dt.....$ The 3-form of Topological Torsion becomes $A\hat{}dA = i(\mathbf{V}_T)dx\hat{}dy\hat{}dz\hat{}dt = \mathbf{S}^x dy\hat{}dz\hat{}dt..... - hdx\hat{}dy\hat{}dz$,such that in engineering language,

$$\mathbf{V}_T = -\{(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi); \mathbf{A}\bullet\mathbf{B}\} \equiv \{\mathbf{S}, h\} \text{ and } \Gamma = (\mathbf{E}\bullet\mathbf{B}). \tag{5.1}$$

The 4-form of Topological Parity becomes $dA\hat{}dA = 2(\mathbf{E}\bullet\mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt = (div\,\mathbf{S} + \partial h/\partial t)dx\hat{}dy\hat{}dz\hat{}dt$. From (7) and (8), evolution in the direction of the Topological Torsion vector is thermodynamically irreversible.

5.2. The Skidding-Slipping Bowling Ball example.

As an application of the topological theory of irreversibility and anholonomic fluctuations, consider the experiment of a bowling ball given an initial amount of translational energy and given amount of rotational energy. Upon contact with the bowling alley, the ball slips or skids irreversibly dissipating both its translational and rotational, momentum and energy (the dissipative force is obscurely defined as friction). During the irreversible dissipation, the Pfaff dimension is 6. The dissipation process continues until a condition is reached where by the translational velocity of the center of mass is equal to the instantaneous tangential velocity of the contact point relative to the center of mass (equal to the product of the angular velocity and the radius of the ball). This condition is defined in engineering textbooks as rolling without slipping. At this point the Pfaff dimension of the Action 1-form becomes 5. Still far from equilibrium (Pfaff dimension 2), the contact manifold admits an extremal Hamiltonian evolutionary process, which without fluctuation proceeds for ever. In other words, once the condition of rolling without slipping is reached, the motion proceeds (essentially) without further dissipation (neglecting air resistance, etc.).

The objective of this section is demonstrate how these observations can be put into correspondence with the idea that the irreversible portion of the evolution begins on a symplectic manifold of dimension 2n+2, follows an irreversible trajectory in the direction of the Topological Torsion vector, and topologically decays into (or is attracted to) a contact manifold of dimension 2n+1, where the subsequent evolution can be described by a conservative Hamiltonian process.

So, consider a mechanical system with initial rotational energy $\beta m\lambda^2\omega^2/2$ and a translational energy $mv^2/2$. Define a Lagrange function,

$$L(x,\theta,t,v,\omega) = \{\beta m\lambda^2\omega^2/2 - mv^2/2\}, \tag{5.2}$$

where m is a constant mass, β is a numeric factor representing the geometrical features of the moment of inertia, and λ is characteristic constant length of the extended rotating and translating object (e.g., the radius of a ball). Initial conditions $\{v_0, \omega_{0\}}$ determine the initial relative amounts of rotational and translational energy. Now place the spinning ball on a surface that exerts a pressure on the boundary (on a surface orthogonal to a uniform gravitational field). Empirically it is observed that the initial amounts of kinetic energy and momentum decay irreversibly until the "no-slip" condition $\{dx - \lambda d\theta = 0\}$ is achieved. (The no-slip condition is an anholonomic constraint.) From this point on the ball rolls "without slipping" and without further reduction of translational and rotational energy. It is also noted that depending upon the initial conditions there can be a reversal of the direction of the translational motion, or there can be a reversal of the sense of rotation. The problem at hand is to define a dynamical system that replicates these observations.

Sophomoric analysis explains the decay of rotational and translational energy as being due to "frictional forces" whose magnitude and direction "adjust" such as to achieve the desired result. The Hamiltonian extremal technique does not seem to apply to the irreversible state, but does seem to apply to the steady state of no-slip.

The 1-form of Action is presumed to be of the form

$$A = (L(x,\theta,t,v,\omega)dt + p_v(dx - vdt) + p_\omega(d\theta - \omega dt) + s(\lambda d\theta - dx). \tag{5.3}$$

The coefficients $\{p_v, p_\omega, s\}$ are to be considered as prolongation variables, or Lagrange multipliers. Re-arranging the 3n+1 =9 variables $\{t;\ x, \theta, v, \omega, p_v, p_\omega, s, L\}$, and substituting the assumption for the functional form of the Lagrangian given above, yields the Cartan format in terms of 8 independent variables, $\{x, \theta, t, s, v, \omega, p_v, p_\omega\}$,

$$A = (p_v - s)dx + (p_\omega + \lambda s)d\theta - \{p_v v + p_\omega \omega - (\beta m\lambda^2\omega^2/2 - mv^2/2)\}dt. \tag{5.4}$$

By direct computation of the Pfaff sequence on the set of 8 independent variables, $\{x, \theta, t, s, v, \omega, p_v, p_\omega\}$ the form

$$A\hat{}dA\hat{}dA\hat{}dA \Rightarrow 0, \tag{5.5}$$

which implies that the Pfaff dimension of the 1-form, A, is six. By comparison to the Darboux theorem, it is also apparent that this 1-form is of $Pfaff((A) = 6$.

Now redefine the Action 1-form in terms of new momenta

$$\pi_x = (p_v - s), \tag{5.6}$$
$$\pi_\theta = (p_\omega + \lambda s), \tag{5.7}$$
$$e_t = \{p_v v + p_\omega \omega - (\beta m \lambda^2 \omega^2/2 - m v^2/2)\}, \tag{5.8}$$

The 1-form of Action becomes

$$A = \pi_x dx + \pi_\theta d\theta - e_t dt \tag{5.9}$$

which is of the standard Cartan form and is of Pfaff dimension $2n + 2 = 6$, with a volume element

$$\Omega = d\pi_x\hat{}d\pi_\theta\hat{}de_t\hat{}dx\hat{}d\theta\hat{}dt. \tag{5.10}$$

On this 2n+2 space the unique Topological Torsion vector direction field can be computed from the definition,

$$i(\mathbf{T})\Omega = A\hat{}dA\hat{}dA, \tag{5.11}$$

and will have non-zero components only in the momentum (or vertical) subspace $\{d\pi_x\hat{}d\pi_\theta\hat{}de_t\}$. The Topological Torsion vector is orthogonal to the coordinate (or horizontal) subspace, $\{dx\hat{}d\theta\hat{}dt\}$.

For the problem at hand the algebra becomes simplified if the two Lagrange multipliers (momentum components), p_v and p_ω are assumed to be canonical, but the Lagrange multiplier, s, is presumed to be non-canonical. In this case, for $m =$ constant, $\beta =$ constant, $\lambda =$ constant,

$$p_v = -mv, \tag{5.12}$$
$$p_\omega = \beta m \lambda^2 \omega. \tag{5.13}$$

The Action 1-form becomes

$$A = \beta m\lambda^2\omega d\theta - mvdx - (\beta m\lambda^2\omega^2/2 - mv^2/2)dt + s(\lambda d\theta - dx) \tag{5.14}$$

with a 6D volume element (Top Pfaffian)

$$\Omega = dA\hat{}dA\hat{}dA = -6m^2\beta\lambda^2(v - \lambda\omega)dx\hat{}d\theta\hat{}dt\hat{}dv\hat{}d\omega\hat{}ds$$

The unique Torsion vector has 6 components

$$\mathbf{T} = [0, 0, 0, T_v, T_\omega, T_s,] \tag{5.15}$$

$$\text{with respect to } \mathbf{R} = [x, \theta, t, v, \omega, s]. \tag{5.16}$$

The functional components are proportional to:

$$T_v = m^2\lambda(-2\beta\lambda^2\omega v + \lambda\beta v^2 + \lambda^3\beta^2\omega^2) \tag{5.17}$$

$$T_\omega = -m^2\lambda(-2\beta\lambda\omega v + v^2 + \lambda^2\beta\omega^2) \tag{5.18}$$

$$T_s = -m^2\lambda(-2\beta\lambda^2\omega v + \lambda m\beta v^2 - \lambda^3 m\beta^2\omega^2 + 2\lambda\beta vs - 2\lambda^2\beta s\omega) \tag{5.19}$$

Relative to motion along the direction field of the unique Topological Torsion vector, the components of the work 1-form become

$$i(\mathbf{T})dA = W = \Gamma A = m^2\beta\lambda^2(v - \omega\lambda)A, \tag{5.20}$$

$$\text{with} \quad i(\mathbf{T})A = U = 0, \tag{5.21}$$

$$L_{(\mathbf{T})}dA = W = \Gamma A = m^2\beta\lambda^2(v - \omega\lambda)A, \tag{5.22}$$

It is apparent that the virtual work 1-form, W, is not zero except at the point when the system satisfies the "no-slip" condition:

$$\text{No slip condition } : (v - \omega\lambda) = 0. \tag{5.23}$$

For motion in this unique direction not only is the work 1-form not zero, it also is non-integrable.

Hence, before the system decays to the "no-slip" condition, the process is thermodynamically irreversible, as

$$Q\hat{}dQ = L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = W\hat{}dW = (m^2\beta\lambda^2(v-\omega\lambda))^2A\hat{}dA \neq 0.$$

The heat 1-form does not admit an integrating factor. However, after the system decays to the "no-slip" condition, the evolutionary process becomes adiabatic, for then $L_{(\mathbf{T})}A = Q = 0$.

It is apparent that the Torsion vector is completely determined by the system. That is the 1-form of Action is either of even or odd Pfaff dimension. If odd, there exists a unique extremal field, and a Hamiltonian representation. If the Pfaff dimension is even, there is a unique Torsion field, which is either expanding or contracting. (The sign of Γ determines the dilatation). The evolutionary process proceeds irreversibly until (possibly) the divergence of the Torsion vector vanishes ($\Gamma \Rightarrow 0$). From then on the system evolution proceeds in an adiabatic fashion. (This remark must be modified if the system momenta are not canonical, for then there will be temperature effects, not just pressure effects).

The Torsion vector generates a dynamical system that describes the irreversible evolutionary process.

$$\frac{dv}{Tv} = \frac{d\omega}{T\omega} = \frac{ds}{Ts} = d\tau \equiv \frac{dt}{\gamma}$$

$$\frac{dx}{v} = \frac{d\theta}{\omega} = \frac{dt}{1}$$

The torsion coefficients are the dissipative forces and torques. The singular set $6m^2\beta\lambda^2(v-\lambda\omega) \Rightarrow 0$ reduces the top Pfaffian to a contact manifold, which has a unique extremal.

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