HYDRODYNAMIC WAKES and MINIMAL SURFACES with FRACTAL BOUNDARIES

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Abstract: The observable features of hydrodynamic wakes can be put into correspondence with those characteristic surfaces of tangential discontinuities upon which the solutions to the evolutionary equations of hydrodynamics are not unique. Only the robust minimal surface subset, associated with a harmonic vector field, will be persistent and of minimal dissipation. Surprisingly, those minimal surfaces generated by iterates of complex holomorphic curves in four dimensions are related to fractal sets.

Introduction

A remarkable feature of hydrodynamic wakes and coherent structures in stratified flows is that their associated "instability" patterns seem to belong to two broad equivalence classes of spiral shapes. These two patterns are epitomized by the Kelvin-Helmholtz instability pattern (Figure 1a) and the Rayleigh-Taylor instability pattern (Figures 1b).

Figure 1a. The pattern of the Kelvin-Helmholtz instability.

Figure 1b. The pattern of the Rayleigh-Taylor Instability.

The two basic experimental patterns are often replicated and deformed by the fluid motion, but otherwise they have vividly sharp visible boundaries and remarkably long persistent lifetimes in what otherwise would be considered to be a diffusive and dissipative environment.

Note that the Kelvin-Helmholtz instability pattern is characterized by a replication of the primitive pattern of a Cornu spiral. The Rayleigh-Taylor instability pattern is characterized by a replication of the primitive pattern of a Mushroom spiral. Although an analytic description of the Cornu spiral has been known for more than 100 years, only recently has the present author become aware of a closed form analytic description for the mushroom spiral. This work was presented at the August 1992, IUTAM meeting at Poitier, and is summarized in section 4.

The essential questions are: Why do these spiral patterns appear almost universally in wakes? Why do they persist for such substantial periods of time? Why are they so sharply defined? What are the details of their creation? As H.K.Browand said [Browand, 1986] "There does not exist a satisfactory theoretical explanation for these wake patterns."

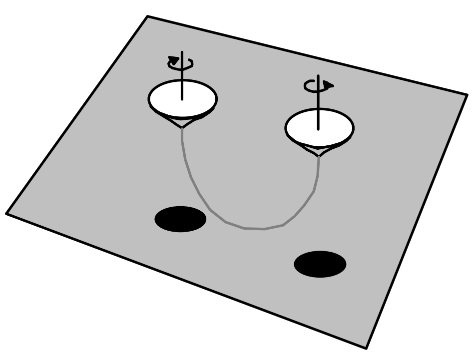
Although the theory and ideas presented in this article were initiated and motivated by topological arguments, the presentation will utilize only the most fundamental of topological notions to describe the basic physical phenomena.

2. Two Remarkable Observations.

Although the ubiquitous mushroom pattern and its topological features have fascinated this author for many years, it is important to note that there were two (relatively recent) experiments, or better, two observations of the mushroom spiral pattern in hydrodynamic wakes which motivated the current work and led to the concepts presented in this article.

The first observation was made by this author in 1986, while visiting an old friend in Rio de Janeiro, Brazil. To replicate the experiment, inject kinetic energy and angular momentum into a stratified fluid with a free surface by stroking a half submerged, flat, circular plate in a direction parallel to its oblate axis. Remove the plate at the end of the stroke to produce, initially, a pair of ordinary Rankine vortices in the surface of the density discontinuity. These Rankine vortices cause the initially flat surface of discontinuity to form a pair of parabolic concave indentations, indicative of the "rigid body" rotation of a pair of contra-rotating vortex cores of uniform vorticity. However, in a matter of a few seconds each concave shape will decay into the metastable soliton configuration of an inverted hyperbolic convex dimple of negative Gaussian curvature. The dimple depression is usually of the order of a few millimeters, but the circulation zone typically extends over a disc of some 10 to 15 centimeters or more, depending on the plate diameter. The configuration, or coherent structure, has been defined as the Falaco Soliton. For purposes of illustration , the vertical depression in Figure 2 has been greatly exaggerated. The Falaco Solitons will persist for many minutes in a still pool of water.

Dimpled indentations in free surface



Black Spots Refracted on Pool Floor

Figure 2. The Falaco Soliton

The effect is easily observed, for in strong sunlight the convex hyperbolic indentation will cause an intensely black circular disk (or absence of light) to be imaged on the bottom of the pool. A bright ring of focused light will surround the black disk, emphasizing the contrast. The optics of the problem are completely described by Snell refraction from a surface of revolution that has negative Gauss curvature. See Figure 3. This effect has been reported upon elsewhere, but the figures are replicated herein for clarity [Kiehn 1991, 1992a].

Optics of the FALACO SOLITON

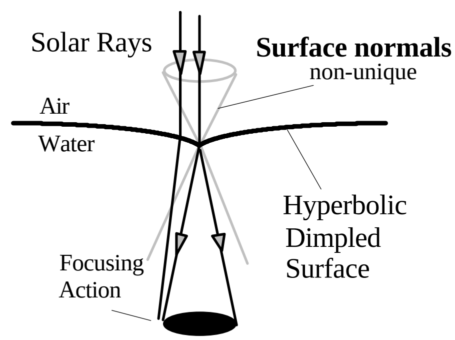


Figure 3. The Snell Black Disk

Dye injection near an axis of rotation during the formative stages indicates that there is a unseen thread, or 1-dimensional string singularity, in the form of a circular arc that connects the two 2-dimensional surface singularities or dimples (Figure 2). Transverse waves can be observed to propagate from one dimple vertex to the other dimple vertex, guided by the "string" singularity. If the string is "severed", the confined, two dimensional endcap singularities do not diffuse away, but instead disappear almost explosively. It is as if the Falaco soliton is the macroscopic topological equivalent of the illusive hadron in elementary particle theory, where two 2-dimensional surface defects (the quarks) are bound together by a string of confinement.

The spin pairing mechanism exhibited as the Falaco soliton is interpreted as a topological phenomenon independent from size and shape. Therefore the effect should occur at all scales, including microscopic as well as cosmic configurations. In fact, during the formative stages of the Falaco vertex pair, the decaying Rankine vortices exhibit more or less planar spiral arm features, easily visible as caustics emanating from each vortex core. This observation is so dramatic that it leads to the conjecture that there might be a singularity thread connecting the nuclei of similarly paired, almost flat, spiral arm galaxies, such as M31 and the Milky Way. The idea is given further credence by the recent analysis that seems to show that galactic formation is not dense, but instead appears to be confined to surface structures throughout the visible universe.

To this author the importance of the Falaco Solitons is that they offer clean experimental evidence that topological defects can be created in a fluid. Moreover, the experiments are easily replicated by anyone with access to a swimming pool. The Falaco solitons certainly are among the most easily reproduced solitons. Interactions with a pole placed vertically to the bottom of the pool experimentally emulate the coherent scattering of solitons observed in computer simulations. As the drifting soliton spin pair and its connecting thread interacts with the pole, the Snell black discs shimmer and disappear, only to coherently reappear after the soliton pair has passed beyond the interaction zone. For hydrodynamics, the observation firmly cements the idea that these objects are truly coherent structures. In fact, the effect leads to a precise definition of a coherent structure: A coherent structure is a deformable bounded domain with a uniform invariant topology.

The observation of the Falaco Solitons in 1986 greatly stimulated further theoretical research into the topological properties of hydrodynamic systems by the present author. Early on, it was recognized that the Snell refraction on the bottom of the pool produces a circular disk, more or less independent from the angle of solar incidence. This observation, as well as the negative Gauss curvature of the surface, lends further credence to the idea (an idea to be exploited in this article) that the topological surface distortion is a minimal surface. The argument is that only spheres and minimal surfaces have a conformal Gauss map (Snell projection) [Struik, 1961]. This spontaneous creation of a minimal surface by a wake was an idea that remained dormant until March of 1992, when a second observation stimulated the present author to generate the theory of spiral wake patterns first presented at the IUTAM conference in Poitier. Before discussing this second observation it should be noted that if chalk dust is sprinkled on the surface of the pool during the formative stages of the Falaco soliton, then the topological signature of the familiar Mushroom Spiral pattern is exposed.

The second observation that stimulated the development of this article was a picture that appeared on the cover of the March, 1992 issue of FLYING magazine. The cover photo showed a jet aircraft making a climb out through a stratified cloud bank off the coast of California. As the jet passed through the discontinuity layer, it produced once again the topological signature of the Mushroom Spiral pattern in its wake. A schematic of the wake pattern is presented in Figure 4.

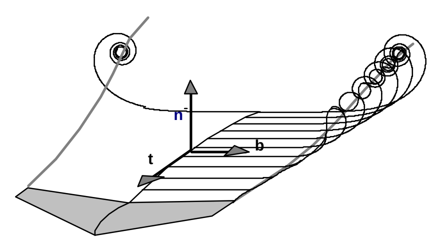


Figure 4. Tip Vortex Wake

The important point demonstrated by the photograph of the Mushroom Spiral was that it clearly showed several rolled up layers of sharply defined, non-diffusive interface between the fog and clear air. The sharpness of these boundaries, and their persistence, even under deformation, suggested that these features, like the Falaco Solitons, were due to topological effects. The effects of viscosity must be only of secondary importance to the topological issues. The photograph made the present author determined to find a simple explanation for the Mushroom Spiral pattern. The answer came in a surprisingly simple application of the more than 100 year old Frenet-Cartan theory of space curves. Although the methods used were simple, the details would have been impossible to obtain in a reasonable amount of time without the advantage of a modern

Personal Computer. An outline of the technique is described in section 4, below.

Following this recognition that the spiral patterns of observable wakes could be captured by such a simple processes of solving systems of ordinary differential equations, it then became apparent that the surfaces of hydrodynamic wakes must be associated with the characteristics of the partial differential equations of evolution. The persistent boundaries of the aircraft wake are to be interpreted as topological limit sets, representing characteristic surfaces of tangential discontinuities which can extend physical boundaries into the interior of a bulk fluid as topological boundaries. These surfaces behave as impermeable membranes across which mass can not flow, and pressure is constant. However, as demonstrated by Landau, all surfaces of tangential discontinuities are locally unstable, and should "lead to turbulence". Why then should these locally unstable surfaces of sharply defined tangential discontinuities persist for so long? The answer, given below in more detail, is that these surfaces must be minimal surfaces upon which the vector velocity field is harmonic. On such domains where the velocity vector field is harmonic,

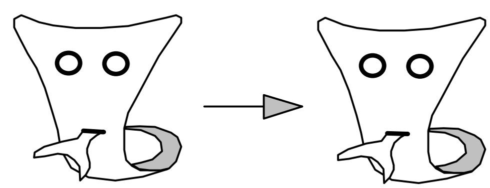
$$\text{grad div } \mathbf{V} - \text{curl curl } \mathbf{V} = 0.$$

It follows that the effects of viscosity and diffusion in the Navier-Stokes equations are annihilated by this zero factor. Not only are such tangential discontinuities minimal surfaces of least area, they are also surfaces of minimal dissipation. The study of a surface with minimum area that spans a given boundary was one of the first problems of the Calculus of Variations. The idea that observable wakes are also the result of such a minimization process is theoretically pleasing.

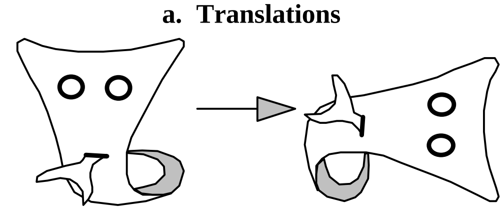
3. Topological Properties

For more than 25 years the present author has had an interest in delineating those properties of nature that are of topological origin from those properties of nature that have a more geometrical basis. The goal has been to derive the basic equations of topological evolution for dissipative irreversible systems [Kiehn, 1974, 1990, 1991]. The fact that the two fundamental scroll instability patterns of hydrodynamics are recognizable even though they may be strongly deformed by the evolutionary flow indicates that there is a topological explanation for their creation and persistence. Recall F. Klein's definition of a geometrical property: A geometrical property is a property that is invariant with respect to translations and rotations. Extending this idea, a practical definition of a topological property may be given: A topological property is a property that is an invariant of a continuous deformation (whose inverse is also continuous).

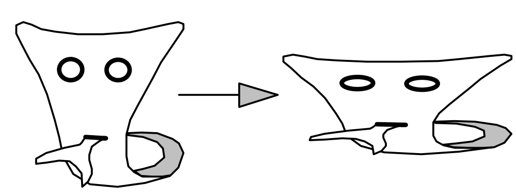
In Figure 5a, a surface with two holes and one curve of self-intersection is translated; in Figure 5b, the same object is rotated; in Figure 5c the original object is deformed. In the first two cases, the size, shape, number of holes, and the number of curves of self-intersection are invariants of translations and rotations. They are, by Klein's definition, geometrical properties.



a. Translations



b. Rotations



c. Deformations

Fig. 5 Geometrical Invariants

(Size and Shape)

vs. Topological Invariants

(Holes and Self-Intersections)

On the other hand, when the initial object is deformed, the size and shape are no longer invariant, but the number of holes and the number of curves of self-intersection are deformation invariants. The number of holes and the number of curves of self intersection are therefore topological properties.

The example demonstrates that many topological properties are related to the integers. After all, you cannot have half a hole. These integer properties are typical of closed topological sets and period integrals. As discussed in [Kiehn, 1977, 1991] there are three types of period integrals associated with one, two and three dimensional topological defects. It is not too difficult to demonstrate that the Kelvin "Conservation of Circulation" theorem of hydrodynamics is related to the conservation of holes, and the evolutionary invariance of topological period integrals of the velocity field around 1-dimensional closed loops. On the other hand the "Conservation of Helicity" theorem in hydrodynamics is related to the conservation of the number of curves of self intersection, and the evolutionary invariance of topological period integrals over closed 3-dimensional elements of space-time.

If changes in these topological properties take place, then it is to be expected that in many cases the observational data should be in relationship to the integers. Topological evolution implies that the evolutionary process is either discontinuous, irreversible (meaning that the inverse mapping does not exist, or if it exists, it is not continuous) or both. For hydrodynamics an argument can be made to show that the creation of the turbulent state from the laminar state requires discontinuous transitions, but the decay of turbulence can proceed by means of continuous (but irreversible) transformations. Condensation is such a continuous process of topological change, while gasification is a discontinuous process. Recent work communicated to me at the Barcelona conference by Voropayev indicates that the attenuation of light through an unstable Rayleigh-Taylor layer appears to change in discrete amounts as the initial turbulent, unstable, layer decays. This would correspond to a description of the initial state of the discontinuity surface as being in an excited state of high topological excitation, with a large number of mushroom defects. As time evolves, the surface state decays into a state of lesser topological excitation with the condensation of one mushroom defect into another. The topological change would be expected to occur in a discrete fashion. Voropayev indicates that as the process proceeds, the smaller mushrooms amalgamate into larger ones. This almagamation of vortices has been observed also in the numerical simulations of Montgomery [Montgomery, 1991]. This remarkable topological process, similar to Bose condensation, might serve as an alternate basis for an explanation of the inverse cascade observed in hydrodynamic systems.

4. Mushroom Spiral Patterns

At the IUTAM 1992 conference at Poitier, the author presented an analytic description of the mushroom spiral, a result which was obtained while studying certain topological qualities of systems of space curves. These space curves were assumed to be generated by single parameter solutions to a given vector field. The analysis was conducted using the classic equations of the Frenet-Cartan moving basis frame, or as Cartan describes it, the Repere Mobile.

The classic Frenet idea is to assume that a space curve can be generated by a position vector, R(t), in Euclidean space, whose components are functions of a single parameter, say t. As t evolves, the position vector sweeps out a curve in space. At each point, P, given by **R**(t), differential and algebraic processes may be used to construct a basis set for a vector space with origin at P. In 3-dimensions, this basis set is constructed from the unit tangent, **t**, to the curve, and its derivatives. The constraint, d**R** - **t**ds = 0, is a kinematic (and topological) constraint that is used to define the differential of arc length, ds, in terms of the unit tangent, **t**. By differentiating **t** with respect to the *arc length*, s, a new vector, κ **n**, is produced, with a line of action orthogonal to **t** in the Euclidean space. The unit vector, **n**, defined as the normal vector satisfies the equations

$$d\mathbf{t}/ds = \kappa\,\mathbf{n}\,, \qquad (1)$$
$$\mathbf{t} \bullet \mathbf{t} = 1,\ \ \mathbf{n} \bullet \mathbf{n} = 1,\ \ \mathbf{n} \bullet \mathbf{t} = 0\,.$$

The coefficient, κ, is defined as the curvature. By using the Gram-Schmidt process, a third unit vector, **b**, defined as the binormal in the classic literature, may be constructed to the complete the basis set: { **t,n,b** }. It is of interest to compute the derivatives of each of the basis vectors with respect to the arc length, s, demonstrating how the basis set pivots and rotates about the moving point, P. The results are

$$\begin{aligned} d\mathbf{t}/ds &= \quad\quad + \kappa\,\mathbf{n} \\ d\mathbf{n}/ds &= -\kappa\,\mathbf{t} \quad + \tau\,\mathbf{b} \\ d\mathbf{b}/ds &= \quad\quad -\tau\,\mathbf{n}\ . \end{aligned} \quad (2)$$

the vectors {**t, n, b**} form an orthogonal moving frame at the point p along the space curve. The variables s, κ, τ, are the arc length, curvature and torsion, respectively, and may be used as a set of intrinsic coordinates for a space curve. The intrinsic variables are the same for any given space curve independent from a particular choice of coordinate system. If two space curves have the same intrinsic expressions, they are congruent. They intrinsic variables are geometric properties of the space curve, but can be used to extract topological information.

There are two extreme situations. In the first extremal case the torsion is negligible ($\tau = 0$) and the space curve is confined to the plane described by the vectors **t** and **n**. In the second case the curvature is neglected ($\kappa = 0$). Both extremes lead to motion confined to a plane and are governed by ordinary differential equations of the form:

$$\begin{aligned} d\mathbf{t}/ds &= +\kappa\,\mathbf{n} \\ d\mathbf{n}/ds &= -\kappa\,\mathbf{t}\ . \end{aligned}$$

For $\mathbf{t} = \{u,v\} = \{dx/ds,\ dy/ds\}$, a particularly useful representation for the unit tangent vector is given by the expressions for the unit tangent in terms of the phase equations:

$$\mathbf{t}_x = dx/ds = u = \sin(Q(s)) \quad (3)$$

$$\mathbf{t}_y = dy/ds = v = \cos(Q(s)). \quad (4)$$

The first of the Frenet equations leads to an expression for the curvature, κ, as

$$\kappa = dQ(s)/ds \quad (5)$$

For $Q(s) = s$, $\kappa = 1$, and the resulting space curve, found by integrating the phase equations, is a circle. For $Q(s) = \ln s$, $\kappa = 1/s$ and the resulting space curve is a logarithmic spiral. For $Q(s) = s2/2$, $\kappa = s$ and the resulting space curve is the Cornu spiral. For the case $\kappa = 1$ and the case $\kappa = s$, the infinite interval from $s = -\infty$ to $+\infty$ is mapped into a bounded region of the plane. These results have been known for more than 100 years. However, a simple sequence is to be recognized:

... $k = s\text{-}1$, $k = s0$, $k = s1$...

The question arises as to what are the shapes of the space curves for arbitrary integer exponents, $k = s^n$.

Through the power of the PC these questions may be answered quickly by integrating the phase equations. The results of the numerical integrations are presented in Figure 6a for n = 1 and in Figure 6b for n = 2.

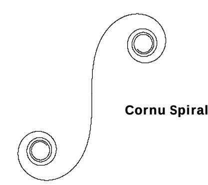


Figure 6a. The Cornu spiral is a harmonic function for which the phase function, $Q = s^2$.

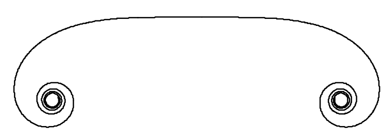


Figure 6b. The Mushroom spiral is a harmonic function for which the phase function $Q(s) = s^3$.

Perhaps more important is the fact that the Cornu spiral of Figure 6a is the deformable equivalent for all odd-integer $n > 0$, and the Mushroom spiral of Figure 6b is the deformable equivalent for all even-integer $n > 0$. Not only has the missing analytic description of the mushroom spiral been found, but also a raison d'être has been established for the universality of the two spiral patterns. They belong to the even and odd conjugate classes of arc length exponents describing, as plane curves, a map of the infinite interval into a bounded region of the plane. A similar sequence can be generated for the half-integer exponents with n = +3/2, 7/2, 11/2... giving the Mushroom spirals and 5/2, 9/2, 13/2... giving the Cornu spirals.

Periodic patterns can be obtained by examining phase functions of various functional forms. For example, the Kelvin-Helmholtz instability of Figure 1a is homeomorphic to the extraordinarily simple choice

$$Q(s) = 1/\cos^2(s) , \qquad (6)$$

and, similarly, the Rayleigh-Taylor instability of Figure 1b is homeomorphic to the case

$$Q(s) = \tan(s)/\cos(s). \qquad (7)$$

In fact the curves of Figure 1a and Figure 1b were generated by numeric solutions to the phase equations for phase functions given by (6) and (7). Other choices for the phase function are described in [Kiehn, 1992b].

In summary, the simple phase analysis captures many of the features of the observed spiral wake patterns. The association of wakes with the Cornu and the Mushroom patterns also leads to the idea that this phenomena can be associated with first and second order diffraction theory of wave scattering. In fact, as described below, it would appear that the synergetic interaction by waves is the physical feature that globally stabilizes these surfaces of discontinuities, giving them their robust persistence.

5. Wakes are Characteristic Surfaces or Domains of Non-unique Solubility.

The theory is still in the development stages, but a number of basic results have surfaced. As agreed upon by many hydrodynamicists, wakes should be considered as domains with topological boundaries of tangential discontinuities. However, the key theoretical feature to remember is that tangential discontinuities are a special subset of those characteristic domains of space time upon which the partial differential equations of evolution do not admit unique solubility. The two species of characteristic sets are shocks and tangential discontinuities. Only tangential discontinuities are of interest to this article.

Partial differential systems may have characteristics over any domain which is hyperbolic. For example, consider the two dimensional Landau equation for a compressible isentropic fluid,

$$\{(c^2-\phi_x^2)\phi_{xx}+(c^2-\phi_y^2)\phi_{yy}-2\phi_x\phi_y\phi_{xy}\}=0 \qquad (8)$$

where c is interpreted as the speed of sound, and the velocity field is represented by the gradient of the potential function, ϕ. This partial differential equation is a quasi-linear equation of the form,

$$A\phi_{\eta\eta}+2B\phi_{\eta\xi}+C\phi_{\xi\xi}=D, \qquad (9)$$

and following Landau [Landau, 1959, p.380], the characteristic surfaces contain embedded curves which are given by solutions to ordinary differential equations,

$$d\eta/d\xi = [B \pm (B2 - AC)^{1/2}]/C . \qquad (10)$$

These ordinary differential equations have real solutions when the argument of the square root is positive. A short analysis will show these (hyperbolic) domains are where the local velocity exceeds the local characteristic speed, c. It may be argued that the formation of the characteristics is due to the finite speed of sound in every real fluid. The assumption of incompressibility (div **V** = 0) is too strong of a constraint to be used in the analysis of wakes. Around sharp physical boundaries, such as the trailing edge of a wing, or any surface defect, accelerations of the fluid will always cause local domains of hyberbolicity to occur. It is to be expected that the characteristic surfaces will emanate from such sharp edges. These arguments furnish a method for the creation of spiral wakes in the interior of a bulk fluid. Surprisingly, they do not invoke the concept of viscosity explicitly.

Families of these characteristic curves form the characteristic surfaces, and represent point sets upon which the solutions to the PDE's are not unique; i.e., the characteristic surfaces include surfaces of tangential discontinuities (on which the normal components of the vector field are continuous, but the tangential components are not continuous), and shocks (on which the normal components are discontinuous, but the tangential components are continuous). Landau states that tangential discontinuities are to be ignored, for they will lead to turbulence. However, as described in section 7, certain subsets of characteristic surfaces (which always are locally unstable) may be globally stabilized, or better said, are robust. It is these special characteristic surfaces, those associated with minimal surfaces, that are of interest to this article.

The key feature for engineers and scientists to understand is that the problem of wakes will not yield to the dogma that given initial data, only a unique outcome is of interest scientifically. In fact, in order to understand wakes it is necessary to study the opposite problem. Ask the question: Where are the solutions NOT UNIQUE?

6. Wakes are Locally Unstable, but Non-permeable, Tangential Discontinuity Subsets of Characteristic Domains

The two species of characteristic surfaces are shocks and tangential discontinuities. The different physical properties of each species have been succinctly described by [Landau, 1959]. Characteristic surfaces of tangential discontinuities are special in that they act as mass membranes. No mass flow takes place transversely through a tangential discontinuity (which is the opposite to a shock wave). The reason that wakes have their sharp visual appearance is due to the fact that they are surfaces of tangential discontinuities; mass diffusion is not permitted through such surfaces. Shocks do permit the flow of mass across the discontinuity and do permit pressure discontinuities; tangential discontinuities do not. Shocks can be dissipative and involve entropy change; tangential discontinuities are not dissipative. Shocks are generally stable, tangential discontinuities are generally unstable. Tangential discontinuities are essentially adiabatic (entropy change is of third order according to Landau).

7. Wakes are Robust Minimal Surface Subsets of locally unstable Tangential Discontinuities.

All such surfaces of tangential discontinuities are locally unstable [Landau, 1959 p.114], for they are associated with a hyperbolic domains. In fact Landau claims that these surfaces are the precursors of turbulence. However, it is argued in this article that it is the special subset of locally unstable surfaces of tangential discontinuities associated with minimal surfaces which can, like soap films, exhibit domains of coherence and global stability [Barbossa, 1976]. The word robust is used to describe this concept of global stability, where synergetic interaction of its parts causes the system to have a finite lifetime.

To be more specific, consider a position vector {u,v,w} to a point on a surface in terms of the characteristic coordinates, {ξ,η}, and with parametrization, s, as defined by the equations

$$u = d\eta/ds = A(s)\sin(Q(s)),$$
$$v = d\xi/ds = A(s)\cos(Q(s)), \qquad (11)$$
$$\text{and} \qquad w = F(u,v) = f(s).$$

Note the similarity in form between these equations, the phase equations (3) and (4), and the characteristic equations given by (10).

If F(u,v) satisfies the equation

$$(1+F_v^2)F_{uu} + (1+F_u^2)F_{vv} - 2F_uF_vF_{uv} = 0, \qquad (12)$$

then F(u,v) defines a minimal surface [Osserman, 1986]. There is a solution of (12) which is unique in that it is the only harmonic minimal surface; it is the solution F(u,v) = tan-1(u/v) = Q(s), or the right helicoid. It is this special subset of characteristic surfaces that is to be associated with the spiral solutions generated by Equations (3-4). Working backwards, assert that the surface of characteristics is a minimal surface; then to generate the ordinary differential equations (3-4), the minimal surface must be the right helicoid.

More precisely, the fundamental result is that of the infinite number of surfaces of tangential discontinuities, there is a special subset whose sections yield spiral space curves that are solutions of Equations (3-4). It turns out that this special subset can be related to the unique harmonically generated minimal surface, the right helicoid. In the domain of global stability, the locally unstable surfaces create the persistent spiral wake patterns observed at moderate Reynolds numbers. When the Reynolds number exceeds a certain value such that the global stability of the minimal surface is lost, then these special surfaces of tangential discontinuity lose their synergetic robustness and the flow becomes turbulent.

It is the thesis of this article that it is these globally stabilized minimal surfaces of tangential discontinuities which are those surfaces that generate the distorted but persistent mushroom patterns in the Von Karman wake, or the Lanchester tip vortices in the wake of an aircraft in flight, or the Kelvin-Helmholtz instability pattern of a shear layer. These minimal surface subsets, like soap films, are ROBUST. Like all minimal surfaces, they are to be associated with harmonic vector fields, for the coordinate functions on a minimal surface are harmonic [Osserman, 1986]. Harmonic vector field solutions to the Navier-Stokes equations are extraordinary for they are solutions that do not depend upon the magnitude of the viscosity coefficient. The diffusive and dissipative terms cancel out for harmonic vector fields, a fact which explains the long life and persistence of observable wakes.

8. Minimal Surfaces and Fractals

Encouraged by the arguments presented above the present author gathered motivation to learn more about the theory of minimal surfaces. An extraordinary theorem, utilized by S. Lie, is the fact that every holomorphic function of a complex variable will generate a minimal surface in four dimensions [Nitsche, 1989 p.139]. This means that there exists an interesting class of vector fields in space and time which are solutions to the Navier-Stokes equations, and at the same time generate minimal surfaces. For example, consider the analytic function U(z) of $z = x + iy$. The two components of U and the two components of z form a four dimensional neighborhood, and according to the Lie theorem, there is an associated minimal surface! To be specific, consider the function, $U(z) = z^2 + (a+ib)$, where a and b are constants. The analytic function, according to the Lie theorem, generates a minimal surface. Now construct the next functional iterate of U; i.e., construct $U(U(z)) = (z^2 + a+ib)^2 + a+ib$. It is an analytic function, and so it generates another minimal surface. Let the process continue, constructing repeated iterates and successive minimal surfaces.

Now recall that this iteration procedure on a complex analytic function, U(z), is precisely the method used (for a given a+ib) to construct the Julia set of U(z). That is, values of z separate into those which are repelled by the Julia set either to a finite attracting basin or to infinity. The Julia set forms the repelling boundary in the z plane, and can be fractal. In fact, the sample function is the example used by Mandelbrot to generate the Mandelbrot set. These results are not fully understood as yet, but indicate an extraordinary connection between the theory of self-similar fractals, the geometry of minimal surfaces, and the calculus of variations.

The important fractal property of self-similarity is replicated by minimal surfaces. To understand this idea, consider the implicit function, ϕ, whose zero set, $\phi(x,y,z) = 0$, defines a surface in a Euclidean space of three dimensions. Construct the unit normal to this surface by dividing the gradient of this function by the square root of the inner product of the gradient with itself; i.e., define

$$\mathbf{n} = \mathrm{grad}\,\phi \,/ < \nabla\phi \bullet \nabla\phi >^{1/2} \qquad (13)$$

This vector, **n**, generates the unit normal field, everywhere transversal to the surface, $\phi(x,y,z) = 0$. If div **n** = 0, the surface is a minimal surface. That is, solutions, ϕ, to the equation

$$\mathrm{div}\,\mathbf{n} = \{ \nabla^2 \phi <\nabla \phi \bullet \nabla\phi > - <\nabla \phi \bullet \mathbb{J} \bullet \nabla\phi >\} / <\nabla \phi \bullet \nabla\phi >^{3/2} = 0$$

are functions whose zero sets define a minimal surface. $\mathbb{J}$ is defined to be the Jacobian matrix of the function, ϕ: $\mathbb{J} = \phi_{\mu\nu} = \partial^2\phi/\partial x^\mu \partial x^\nu$. This definition works in any number of dimensions, and may be used to define a minimal hyper surface in a space of dimension, N. Note that if the components of the vector, grad $\phi(x,y,z)$, are rescaled by an arbitrary function, $\lambda(x,y,z,t)$, then this scaling factor cancels out in the definition of **n**. The divergence condition is obviously related to the calculus of variations and is always associated with some form of a conservation law. (A somewhat more general condition would be to consider the divergence of the renormalized velocity vector, $\mathbf{n} = \mathbf{V}/<\mathbf{V} \bullet \mathbf{V}>^{1/2}$.)

For hydrodynamics, there are several important applications of this renormalization procedure. First, if the velocity field is multiplied by the density to produce the momentum field, $\rho\,\mathbf{V}$, then for any density function, $\rho(x,y,z,t)$, the formula (13) gives the same unit normal field, n. The zero divergence condition on n implies that the conservation of momentum is more important than the conservation, or invariance, of energy (physically, div **n** = 0 is more important than div **V** = 0).

Secondly, many useful solutions of the hydrodynamic equations are time harmonic. That is, the velocity vector field may be expressed as a vector field over the coordinates, with its time dependence determined by a common factor, $\lambda = \lambda(x,y,z,t)$. In this situation, the vector field, $\lambda\mathbf{V}$, is said to form a 1-parameter group, and generates trajectories (streamlines) that are independent from the common factor of renormalization, λ. There exist solutions to the equations of hydrodynamics which do not form a 1-parameter group (streamlines), and the simple kinematic time-harmonic interpretation is not precise. Such solutions may form a two parameter group, and define propagating surfaces with spatial sections called "strings".

Cartan has shown that a necessary and sufficient condition for any dynamical system, $\lambda\mathbf{V}$, to be a Hamiltonian system is that the closed loop integrals of the Action, (translate to the "number of topological holes", or to "the Kelvin circulation integral") must be an invariant of the motion for any renormalization (or re-parametrization) factor, λ [Kiehn, 1974] .

A third interesting case is when the factor, $\lambda(x,y,z,t) = 0$ is used to define a moving surface. Then the vector field $\lambda\mathbf{V}$ satisfies the "no-slip" condition on the material boundaries.

This general idea of scale independence for a vector field is typical of a projective geometry, and corresponds to the idea that the Lagrange function of the variational calculus is homogeneous of degree zero. It is this thermodynamic-like homogeneity feature that yields the properties of self-similarity for minimal surfaces.

The idea of a minimal surface extends to hyper surfaces, where the "surfaces" are of dimension greater than two. A particularly interesting case occurs in four dimensions where a minimal 2-surface may combine with a "plane wave", or another minimal surface. For example, the partial differential equations that must be satisfied by the divergence condition in four dimensions for the surface $\Psi(x,y,z,t) = 0$ are:

$$\begin{aligned}
&\{(\psi_t^2 + \psi_z^2 + \psi_y^2)\psi_{xx} + (\psi_t^2 + \psi_z^2 + \psi_x^2)\psi_{yy} \\
&+(\psi_t^2 + \psi_x^2 + \psi_y^2)\psi_{zz} + (\psi_z^2 + \psi_x^2 + \psi_y^2)\psi_{tt} \\
&-2\psi_x\psi_y\psi_{xy} - 2\psi_y\psi_z\psi_{yz} - 2\psi_z\psi_x\psi_{zx} \\
&-2\psi_x\psi_t\psi_{xt} - 2\psi_y\psi_t\psi_{yt} - 2\psi_z\psi_t\psi_{zt}\} \;=\; 0.
\end{aligned}$$

When the variables are constrained to the harmonic sets, $\Psi_x^2 + \Psi_y^2 = 1$ and $\Psi_z^2 + \Psi_t^2 = 1$, the equation for the three dimensional minimal hyper surface in the space of four dimensions reduces to a *pair* of coupled two dimensional minimal surface equations. Note that the quadratic harmonic condition is satisfied by the phase functions of (3) and (4).

For the special case where $\Psi = t - \phi(x,y,z) = 0$, the expression for the three dimensional hyper surface in a space of four dimensions reduces to the equation:

$$\begin{aligned}
&\{(1+\phi_z^2 + \phi_y^2)\phi_{xx} + (1+\phi_z^2 + \phi_x^2)\phi_{yy} + (1+\phi_x^2 + \phi_y^2)\phi_{zz} \\
&-2\phi_x\phi_y\phi_{xy} - 2\phi_y\phi_z\phi_{yz} - 2\phi_z\phi_x\phi_{zx}\} = 0.
\end{aligned}$$

This last expression for a minimal hyper surface in four dimensions can be deduced from Landau's formula for a three dimensional compressible gas

$$\begin{aligned}
&\{(c^2 - \phi_x^2)\phi_{xx} + (c^2 - \phi_y^2)\phi_{yy} + (c^2 - \phi_z^2)\phi_{zz} \\
&- 2\phi_x\phi_y\phi_{xy} - 2\phi_y\phi_z\phi_{yz} - 2\phi_z\phi_x\phi_{zx}\} = 0
\end{aligned}$$

by substitution of the expression $c^2 = c_0^2 + \mathbf{v}\bullet\mathbf{v}$, with $v = c_0$ grad ϕ. The conclusion is that there are many ways to generate minimal surfaces in hydrodynamic flows.

If in the wave equation, the wave function is harmonic (and satisfies the complex minimal surface condition) in the first two coordinates, then the wave equation reduces to a "plane" wave in the remaining two coordinates. The phase of the entire complex wave leads to phase singularities and defects discussed by Nye and Berry.

9. What about viscosity?

Throughout the discussion above almost no mention was made of the conventional hypothesis that wakes are somehow "due to viscosity". The viscous approach to wake theory [Batchelor, 1967] is dominated by the assumption that the flow vector field is solenoidal: div **V** = 0, curl **V** # 0. In this article, the opposite assumption is studied; viscous effects are ignored, and the creation of wakes is associated with hyperbolic domains that depend upon the fact that the vector field is irrotational: div **V** # 0, and curl **V** = 0. Somehow the two methods have to be brought together. The meeting place is where the vector fields are both solenoidal and irrotational, and this domain forms a boundary between the two sets. However, note that this boundary is the point set upon which the vector field is harmonic. Further recall that the vector field that describes a minimal surface must be harmonic.

To be more specific, consider a solenoidal stream function solution to the Euler equations of hydrodynamics. Take the divergence of the Euler equations to obtain the equation

$$\nabla^2 P / \rho = -\{\Psi_{xx}\Psi_{yy} - \Psi_{xy}\Psi_{yx}\}$$

This equation has been utilized recently by M. Larchveque in an attempt to define a coherent structure [Larchveque, 1990] and by Coudy, et.al. [Douady, 1992] in a study of "string" structures in a turbulent flow. When the pressure is harmonic, the RHS of this equation implies that the Hessian determinant of the stream function vanishes, and according to Weingarten [Nitsche, 1989, p.19], the stream function then can be used to define a minimal surface.

These results imply that a wake may be viewed as a double layered minimal surface separating solenoidal domains from irrotational domains. A section of the Von Karman wake behind a cylinder is magnified in Figure 7 in order to demonstrate these ideas.

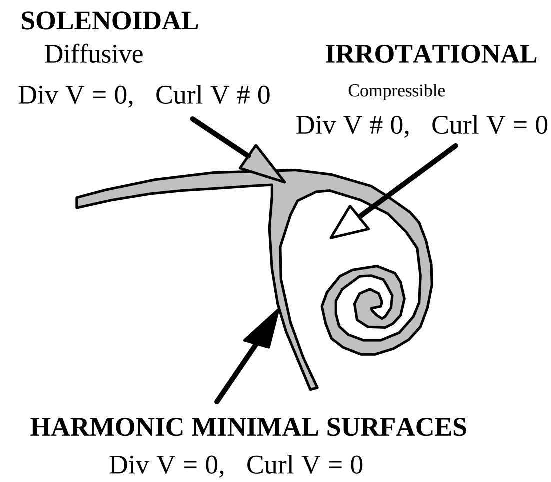


Fig 7. Section of a Mushroom Wake

As time evolves the effect of the viscosity is to thicken the wake by diffusion. For large Reynolds numbers, the double layer wake is very thin, and appears to persist because the diffusion time constant is very long compared to the diffraction time constant of waves associated with the minimal surface.

Summary

In summary, and to first order:

1. Wakes are sharply defined because they are surfaces of tangential discontinuities, which act as impermeable membranes that do not permit a transversal flow of mass.

2. Wakes are robust because they are related to minimal surfaces which like soap films enjoy a global synergetic stability.

3. Wakes are not diffusive and are persistent because their minimal surfaces are associated with harmonic vector fields, which act as zero factors or any viscosity coefficient in the Navier-Stokes equations.

4. Wakes are expected to produce spiral sections of the Cornu and Mushroom classes, as these sections are characteristic limit sets related to the minimal surface of the right helicoid.

5. Transitions between iterates of minimal surfaces can behave in relation to the integers as the number of holes or self-intersections changes.

6. Complex minimal surfaces can approximate fractal sets. It is to be expected that the tangential discontinuities that define

boundaries of such natural objects as clouds would form fractal sets.

7. Topological limit sets that form minimal surface topological boundaries can occur in the interior of a bulk fluid, extending material boundaries, and forming coherent structures and domains of re-entrant flow.

The sequence of ideas is summarized in Figure 8.

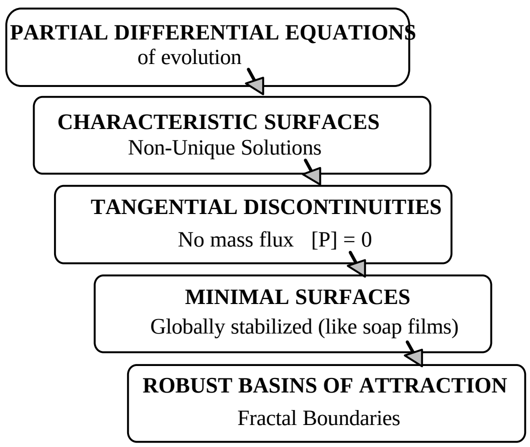


Fig 8 HYDRODYNAMIC WAKES

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