Notes on MINIMAL SURFACES AND SURFACES WITH PINCH POINTS
with
APPLICATIONS TO ELECTROMAGNETISM following Bateman

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Historical Preface

A remarkable result of differential topology that should have widespread engineering application (but an idea that has received almost no attention in practice) is that there exists a well-defined set of time dependent vector fields that generate minimal surfaces in 4 dimensions. These vector fields belong to the class of maps, Φ, given by the expression:

$$\Phi: \{x,y,z,t\} \rightarrow \{ \alpha(x,y,z,t), \beta(x,y,z,t), \varpi(\alpha(x,y,z,t) + i\beta(x,y,z,t)) \}, \tag{I}$$

where $\varpi = \phi(\alpha + i\beta) + i\,\psi(\alpha + i\beta)$ is an arbitrary complex analytic function of $\alpha + i\beta$. It is the thesis of this article that the understanding of wake production, induced drag and the onset of turbulence is to be associated with such vector fields. A slightly more general representation can be obtained by considering the class of complex 3-Dimensional vectors that satisfy the equation (1) given below.

This class of vector fields was first brought to this author's attention by the small 1914 monograph of H. Bateman entitled *Electrical and Optical Wave Motion*, Dover (1955). In this monograph Bateman published some extraordinary results, including examples of solutions to Maxwell's equations that emulate propagating singular strings (not plane waves). The key idea according to Bateman is the existence of complex 3-Dimensional vector solutions, **M**, to Maxwell's equations that satisfy the complex equation,

$$\mathbf{M} \bullet \mathbf{M} = 0. \tag{1}$$

Following Bateman, let $\mathbf{M} = \mathbf{B} \pm i\mathbf{E}/c$, with **E** and **B** possibly complex, such that (1) becomes

$$\mathbf{B} \bullet \mathbf{B} - \mathbf{E} \bullet \mathbf{E}/c^2 \pm i\, 2\, \mathbf{E} \bullet \mathbf{B}/c = 0\,. \tag{2}$$

The conjugate square, or norm, of **M** becomes over its domain of support,

$$\mathbf{M}^* \bullet \mathbf{M} \neq 0, \text{ or} \tag{3}$$

$$\mathbf{B}^* \bullet \mathbf{B} + \mathbf{E}^* \bullet \mathbf{E}/c^2 \neq 0 \tag{4}$$

From Osserman, a generalized minimal surface in E^N is a non-constant map from a 2-manifold, M, with a conformal structure (over regular regions) such that the coordinates of E^N are harmonic on M. Let the map be defined by

$$\{\alpha,\beta\} \rightarrow \mathbf{X}(F(\alpha+i\beta)) \quad k = 1,2,3,4... \ , \tag{5}$$

with F analytic, and $\mathbf{X}^k$ harmonic.

Then define **M** as

$$\mathbf{M} = \partial\, \mathbf{X}^k/\partial\alpha - i\, \partial\, \mathbf{X}^k/\partial\beta\,. \tag{6}$$

If (1) is true it follows that α and β form a set of isothermal coordinates on the minimal surface (and the induced metric, $g_{\alpha\beta} = \Sigma_k \{\partial\mathbf{X}^k/\partial\alpha \bullet \partial\mathbf{X}^k/\partial\beta\}$ generates a conformal structure), and if (3) is true, then that minimal surface is regular (without self intersections or pinch points). If **E** and **B** are real vectors, then the associated minimal surfaces are always regular, except at points where **E** and **B** are identically zero. If **E** and **B** are complex, then the associated minimal surfaces can have singularities. It is these singularities that are prime interest to this article.

Following Bateman, such vector fields generate a conserved 4-current, { **J** , ρ }, defined as

$$\mathbf{J} = \rho\,\mathbf{V} = \mathbf{E} \times \mathbf{B}, \qquad \rho\, c^2 = 1/2\,\{\, \mathbf{E} \bullet \mathbf{E} + c^2\, \mathbf{B} \bullet \mathbf{B} \,\} \tag{5}$$

If **E** and **B** are real, then almost every student of physics will recognize the real and imaginary parts of (2) to be the first and second Poincare invariants of the Lorentz transformations, and the scalar given by (4) as the energy density of the electromagnetic field. Any regular domain of a minimal surface associated with real **E** and **B** has null values for the first and second Poincare invariant, and a positive definite value for the field energy density. The **E** field is orthogonal to the **B** field, and the electric and magnetic energy densities are equal. The conclusion is reached that propagating electromagnetic waves must be associated with minimal surfaces. The associated minimal surface is always regular and without singularities for real **E** and **B**. When the **E** and **B** fields are complex, which in a physical sense implies the existence of elliptical polarization, another interpretation is possible.

This association of electromagnetic wave propagation with minimal surface theory was apparently unknown to Bateman, and not appreciated by the present author until only very recently, following a nth re-reading of Osserman's book on *A Survey of Minimal Surfaces*. According to Osserman, the complex 3-vector representations of minimal surfaces were known to Enneper and Weierstrass. A study of the minimal surfaces generated in E^4 by (I) is given by Kommerell.

The minimal surfaces so generated in E^4 by this class of vector fields will have 3-dimensional images that are not always regular. In general, two dimensional non-regular surfaces may have "singularities" consisting of "curves of double points" created by intersections of two local surface patches, or of "triple" points consisting of intersections of three local surface patches, or of curves of double points which terminate on "Pinch" points within the interior of the surface. These three types of self intersection singularities are the only three "stable" singularities in the sense of Whitney. Recall that Whitney proved that any N manifold can be embedded in 2N+1 euclidean space, and immersed in a 2N euclidean space. The induced surfaces may be orientable or non-orientable. The non-orientable examples are characterized by the Klein-Bottle, or the Projective Plane, and the orientable surfaces by the Sphere. Each surface may have tubular handles, holes and distortions. Of interest to this work are not just any surface, but those surfaces which in particular are minimal surfaces.

If the surface has no singularities, then the surface is said to be regular or *embedded*. The constraint of regularity implies that the surface normal vector never goes to zero over the surface, or the induced metric on the surface is always invertible. This implies that are always two linearly independent directions on a regular domain of the surface. If the lines of self intersection are divergence free on the domain (meaning that they stop or start only on boundary points, or are closed upon themselves, then the surface is said to be *immersed* in 3-Dimensions. The points where the divergence of the lines of intersection is not zero are defined as Pinch points. Such surfaces cannot be immersed in 3-D. The Pinch points are signatures of the fact the surface *resides* in 4-Dimensions (as an immersion), and cannot be immersed in 3-Dimensions.

A flow vector field may have domains where it is irrotational or solenoidal, and these domains may be separated by a surface. If the surface of separation is a minimal surface, then the flow on this surface is harmonic. The minimal surface need not be regular, and may have lines of self-intersection. These lines of surface self-intersections (lines of singular double points) are not necessarily solenoidal. In fact, the Pinch points are points where the lines of self-intersection terminate not on themselves and not on a boundary, but in the surface interior. The Pinch points may be viewed as the "sources" of the divergence of the lines of self-intersection. The classic example is given by Whitney's Umbrella, the last of the only three possible stable, but singular, mappings (See Figure 1).

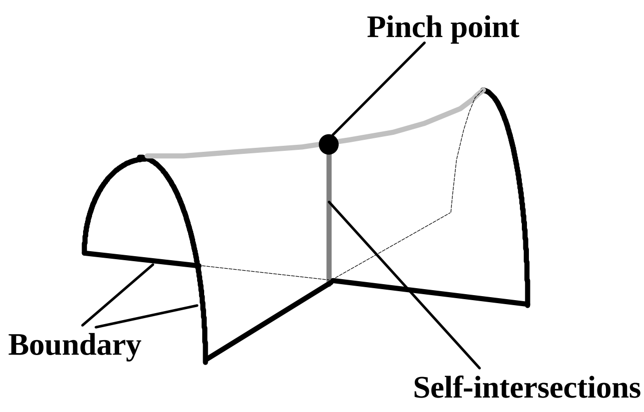

Figure 1 WHITNEY UMBRELLA

These ideas about singular surfaces are to be applied to those minimal surfaces that act as limit set attractors in dissipative systems; i.e., to those surfaces associated with the generation of observable wakes.

Cartan's Topology

An earlier development in applied differential topology [3] used the notion of Pfaff dimension to classify hydrodynamic flows in terms of Cartan's theory of exterior differential forms . The flow vector field is mapped to a Cartan 1-form of action, A, and then domains of support for this 1-form are put into equivalence classes defined by the Pfaff dimension. Translated to engineering terms, the idea is that globally laminar or streamline, non-chaotic, hydrodynamic flows are of Pfaff dimension 2 or less, $A \hat{} dA = 0$. Such vector fields may be associated with the "normal" or gradient to an algebraic variety (a function set equal to zero) which defines a surface. Such surfaces never have lines of self intersection or pinch points, and are always orientable on the domain of support. If the domain of interest includes points where each coefficient of the 1-form vanishes identically, then these points are usually defined to be critical (or stagnation) points. It may also be true that such points correspond to points of self-intersection.

Consider a domain of Pfaff dimension 1. Then the 1-form of Action, A, may be generated from the gradient of a single scalar function, Θ: $A = d\Theta = \nabla\Theta \bullet d\mathbf{r}$. When $A \neq 0$, there are N-1 vectors, $\mathbf{X}_{(k)}$, such that $i(\mathbf{X}_{(k)})A = 0$. The vectors, $\mathbf{X}$, span an N-1 hypersurface. The associated 1-form, $A^{+} = i(\mathbf{X}_1)\, i(\mathbf{X}_2)...\, dx\hat{}dy\hat{}...$ is such that $A\hat{}A^{+} = 0$. In N dimensions, in order to obtain a two surface, N-2 1-forms must be specified to generate a 2-surface. The vectors that span the 2-surface are solutions of the simultaneous equations $i(\mathbf{X})A = 0$ and $i(\mathbf{X})B = 0$, ... When a 1-form is generated by a single scalar function as a gradient, then the field it represents is irrotational. The hypersurface it represents is transversal to the the field. The field has zero curl, but it has finite divergence, $dA^{+} \neq 0$, unless the single scalar function is harmonic.

Consider a 1-form constructed from N-1 gradient functions by the top down recipe

$$A = i(d\alpha)i(d\beta)...\Omega \quad \text{where } \Omega = dx\hat{}dy\hat{}....dx^{N} .$$

Then the field is solenoidal (has zero divergence) but can have non-zero curl. The field has zero curl if each of the gradients from which it is composed are harmonic.

Consider the three dimensional case. Then the gradient 1-form is of Pfaff dimension 1 and represents the normal field to a 2-surface except at critical points where the the 1-form vanishes identically.

Chaotic systems are represented by vector fields, or their equivalent action 1-forms, on domains of support which are of Pfaff dimension 3 (A^dA # 0, but dA^dA = 0). It has been suggested that chaotic flows differ from turbulent flows, in that turbulent flows are of Pfaff dimension 4 (dA^dA # 0) [6]. These equivalence classes of vector fields have been defined as domains that support topological torsion, and topological parity, respectively [7]. In order to have lines of surface self-intersection, the vector field associated with such surfaces must be of Pfaff dimension 3. In order to have Pinch points, the vector field must be of Pfaff dimension 4. In short, the concept of Pfaff dimension tells something about the regular, embedding, or immersive properties of a surface associated with a vector field.

The area of research proposed herein is based on the idea that the theory of wakes and turbulence (in terms of minimal surfaces generated by the class of maps of vector fields described above) must involve the generation of surfaces with self intersections and pinch points.

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