ADVENTURES IN APPLIED TOPOLOGY
Physical Applications of Cartan's Theory of Exterior Differential Systems.

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Contents

0.1 Preface

The purpose of this monograph is not to teach topology, but to sensitize the reader to topological aspects, rather than geometrical aspects, of physical systems and their evolution. The monograph will demonstrate how a topology perspective can be used in the practical world of the applied physical sciences, how a topological perspective can lead to a better understanding of phenomena (such as irreversibility) that appear be intractable in terms of geometrical concepts, and more over how a topological perspective can lead to new and useful devices and processes.

0.1.1 What is topology?

A somewhat imprecise, but useful, definition of topology is that it is the study of properties that do NOT depend upon size or shape. The most fundamentalof topological properties is the number – the number of parts, the number of intersections, the number of links, the number of holes, the number of dimensions..... Topological evolution is the study of when and how these numbers change.

0.1.2 What is Geometry?

Similarly, a somewhat imprecise, but useful, definition ofgeometry may be said to be the study of properties that depend upon size and shape. It is extraordinary, but most current scientific and engineering concepts are based upon the geometric tradition, where the concept of number is constant, and the notion of topological evolution is ignored. Non-uniqueness and discontinuities are abhored. Yet there is evidence that irreversibility and aging imply changing topology.

0.1.3 Various Parts of this mongraph

The monograph includes a number of separate parts each emphasizing certain important features of the Cartan method, but with some overlap of each part.

Part 1. The Topology of an Exterior Differential Systems:
(Pfaff dimension. Topological Torsion. Cartan's Topology. Cartan's Topological Structure.
Continuous Topological Evolution)

Part 2. Topological Defects.
(Period integrals. Links and Braids. Discontinuities and the Lorentz equivalence class.
Topological Refinements by Connections and Metric. Cartan's structural equations)

Part 3. Deformation Invariants. Anholonomic fluctuations. Thermodynamic Irreversibility

Part 4. The Many Faces of Torsion
(Frenet Torsion, Affine Torsion, Topological Torsion and Topological Spin. Classical Field Theory.
Calculus of Variations. LaGrange vs Hamiltonian methods.)

Part 5. Detailed Applications.
(Mechanics, Hydrodynamics, Electromagnetism, Thermodynamics)

Part 6. Differential topology and Differential geometry.
(Exterior differential forms and envelopes, hypersurfaces, tangential discontinuities. Minimal surfaces.
Wakes. Solitons.)

0.2 Geometry and Physics

0.2.1 What is Euclidean Geometry - Invariant Size and Shape - Rigid Body motion

Historically, fundamental physical theories have been based on geometrical models, geometrical relationships and geometrical properties of physical objects. In fact the very idea of a physical measurement is associated with the concepts of how big, how far, or how long. These ideas of measurement are intuitively geometrical ideas, and all involve a comparison of something of immediate interest to some legislated standard. However the idea of a comparison is not restricted to pure geometrical concepts of size and shape. An abstract comparison may be viewed as a mapping from a range to a domain, or a transformation from an initial state to a final state. The mapping or transformation may be an element of an equivalence class of mappings, and that class is determined by its invariant properties. Recall that, according to Felix Klein, a (euclidean) geometrical property is defined to be an invariant of a translation or rotation. A simple observation demonstrates that size and shape are such geometrical properties, and these properties are the fundamental invariants of that branch of physics that deals with rigid body motion.

0.2.2 Isometric Geometry - Invariant distance - Bending

With the advent of tensor calculus, mathematicians relaxed the constraints of pure (euclidean) geometry a bit in order to include the concept of bending without compression or shear. The fundamental physical invariant of interest became the distance between a pair of points: size is considered is an invariant in such geometries, but shape is not (necessarily) an invariant of the equivalence class of transformations that describe bending processes. The equivalence class of transformations based on the property of invariant distance are called isometries. The pure geometrical constraint of invariant shape is relaxed to include the possibility of bending. The idea of invariant distance has dominated physical theories since the turn of the century. In fact a new derivative concept, the covariant derivative, was defined in a such a manner that it preserves the concept of distance as an invariant. Displacements via a "covariant derivative" are constrained such the the distance between a pair of points is preserved as an invariant! However, be warned that the presciption of a covariant derivative has in its foundations the assumption that intrinsic compressions or shears are not considered!

0.2.3 Conformal Geometry - Invariant Phase - Twisting

rotational parallelism - rotational shear - a second torsion concept.

0.2.4 Affine Geometry - Invariant parallel planes - Shear

Linear parallelism affine torsion

0.2.5 Homothetic Geometry - Pressure

If the deformation is confined to compressions or expansions, then the distance between a pair of points is no longer invariant, and as both size and shape are not necessarily invariants, the covariant derivative concept becomes obsolete. The question arises as to what invariants might be used to classify such transformations. Such transformations are defined as conformal transformations that leave invariant the angle between a pair of lines.

The next class of transformations are those that are continuous reversible deformations that admit translations, rotations, bending, expansion, and finally shears. The invaraints of such transformations are defined as topological properties.

0.3 Topology and Physics

Indeed the geometric method has served physics well, but there are many things in nature that obey rules that are independent from size and shape. For example, the Planck blackbody radiation distribution in frequency is independent from the size, shape, and even chemical makeup of the hot body that is radiating. The simply connected space of a hollow wave guide of any finite size or shape will always have a low frequency cut-off, but the co-axial cable, which is topologically not simply connected, can support DC currents. A physical system is conservative in a thermodynamic sense if the cyclic work vanishes, independent from the length (size and shape) of the process path. The closed surface integral of Gauss' law does not depend upon the shape or size of the surface but only on the number of charges contained in its interior. A flowing fluid can be in a laminar streamline state which can evolve into a chaotic turbulent state. A measurement with a finite compact apparatus of a infinite or non-compact property will always have an uncertainty associated with the measurement. These qualities of nature that do not seem to depend upon size and shape, and seem to be independent of continuous reversible deformations, can be defined and studied as topological properties.

The idea is that basic physical properties exist that do not depend upon legislated standards, but are absolute in the sense that they are answers to questions such as:

How many?, or

Is it possible?, or more technically,

If a solution exists, is the solution unique?

If the solution is not unique, how many solutions are there?

A great deal of engineering and physical theories are built around the deterministic geometrical dogma which supposes that, given initial data, the name of the game is to be able to predict the outcome, and predict it uniquely. From a more topological perspective, it will become apparent that unique prediction may become impossible, but deterministic retrodiction can be achieved.

0.3.1 Observables as invariants of transformations

Concepts that do not depend upon size and shape can still be invariants of an equivalence class of transformations. Again, these invariants, which are not pure geometric invariants, may be used to define an equivalence class of transformations. The issue is how to define and observe these qualities of nature that donot depend upon size and shape. Consider a piece of notebook paper made out of flexible rubber material. The sheet has 3 holes along one side, and can be marked as 1,2,3,4 at its corners, in a prescribed sequence or orientation. Translate the sheet, and ask what are the invariants of the transformation. The answer is: the size, the shape, the number of holes and the orientation sequence, 1,2,3,4. are all invariant properties of the translation.

Now rotate the sheet; what stays the same? Again, size, shape, hole count, and orientation stay the same. However, in the case of rotation there exists one other invariant that is not in the class of translations. This additonal rotational invariant is the fixed point that defines where the axis of rotation intersects the sheet. Translations are said to be transitive because there is no fixed point, while rotations are intransitive because there must be one fixed point. Recall that by Klein's definition, the four properties of size,shape, hole count and orientation are geometric properties.

Now take the this rubber sheet and deform it by pulling and stretching the sheet. What stays the same? The answer is not the size and not the shape, but the hole count (distorted holes, of course, in the deformed case) and the orientation sequence 1,2,3,4 do stay the same under the deformation. Those properties that stay the same under continuous and reversible deformation are defined to be topological properties. Note that topological properties are included in the class of geometrical properties, but the class of geometrical transformations are included in the class of topological transformations. Pure geometric properties will be defined as those properties which are invariant under translations and rotations only.

A topological property is defined as an invariant of a homeomorphism, or in more simple terms, a topological property is an invariant of a continuous and reversible deformation, while pure goemetrical properties are not. Pure geometrical properties such as size and shape can evolve with respect to homeomorphisms. In this monograph, the process of studying invariants of transformations will be taken one step further, for of physical interest to dissipative systems are those processes that are continuous but not reversible. Pure topological properties are not invariants of continuous but irreversible transformations. As pure geometrical properties evolve with respect to continuous and reversible transformations, pure topological properties evolve with respect to continuous but irreversible processes. For example, if the rim of one of the holes in the rubber sheet was grasped and pulled out of the sheet into the shape of a long trumpet with the rim becoming smaller and smaller until it collapsed to a point that could be glued together, then the topological property of hole count in the rubber sheet would have been changed from three to two during the deformation and glueing process. Note that it was the absolute number of holes that changed during this process of topological evolution which effectively collapsed one of the holes. It is important to note the topological change is quantized, for you can never have half a hole. The question of how many holes is absolute, for it is in relation to the integers.

What are the invariants of the equivalence class of continous, but irreversible transformations? Examples of such invariant properties are connectivity, compactness, and most important to this monograph, the concept of closure. Rather than carrying the words "continuous but irreversible" throughout the monograph, a biological concept will be used to define such processes: A continuous but irreversible process will be defined as an element of an equivalence class of transformations, and will be defined as an aging process. Like all transformations, the equivalence class of aging processes will be defined in terms of its invariants. The ability to develop a physical understanding of the aging process must be built upon the observable invariants of such processes, and the dynamical theory of those topological invariants that can change during such processes. This dynamical theory will be called the theory of topological evolution.

0.4 Physical laws as topological statements

The ultimate goal of this monograph is to establish methods of distinguishing topological effects in physics from geometrical ones, to establish laws describing topological properties of matter, and in particular to establish the laws of physical topological evolution. Note that the first step is to go beyond the constraints of geometry and study strictly topological properties and the evolution of geometrical properties. The second step is to go beyond the constraints of topology to study the evolution of topological properties. The reader may not realize that he or she has often worked with topological concepts without knowing anything about topology, per se. For example, it will be demonstrated herein that the Maxwell theory of electromagnetism, without the geometrical constraint of a Lorentz symmetry group, is a statement about topological properties of space-time. It also will be demonstrated that the first law of thermodynamics is a topological statement of cohomology. The flow of a Navier-Stokes fluid can admit solutions which are examples of an irreversible but continuous topological evolution.

0.4.1 Point set topology and metric de-emphasis

In order to establish a foundation for topological evolution, an introduction to topological ideas and definitions is presented in terms of point set methods for which the topological concepts can be exhibited in terms of simple examples. This expose of topology given in this monograph will not be complete, and will not cover all of topological theory. Only those parts of topology that the author feels are necessary and useful for the development of physical and engineering applications will be presented. A conventional introduction to topology often starts with a metric topology, but herein the concept of a metric is purposely avoided, as the idea of a metric is the essence of those geometrical qualities of size and shape. The conventional procedure is to develop the topological ideas in terms of a space with a euclidean or some Riemannian metric. Then the topological concepts are shown to be independent of the choice of metric. However, the notion of a metric is not needed, and the point set approach takes that point of view that the metric is just extra baggage that can often confuse the issues.

0.4.2 Cartan's exterior calculus

After the basic concepts of topology are presented, the next step is to develop a thorough understanding of Cartan's theory of Exterior Calculus. Cartan developed his exterior calculus long before the word Topology became fashionable, but the key feature of Cartan's theory is that it transcends the geometrical constraints of tensor calculus and is truely a theory of topology and topological evolution. It was mentioned above that a topological property was an invariant of a homeomorphism. Technically, a homeomorphism is a map from an initial to a final state that has two qualities: 1) it must be continuous, and 2) it must be reversible in the sense that the inverse exists and is continuous. If topological evolution is to take place, then one or both of these qualities must not be true. Of particular interest to the developments in this monograph are those evolutionary processes which are continuous but not reversible. However continuity is not a geometrical idea; it is a concept that does not depend upon size and shape. A major goal will be the development of a useful topological structure, such that it can be decided whether or not a particular process is continuous, or not. Fortunately, the concept of a topological structure can be developed in terms of the Cartan calculus, such that a decision can be made if a process is continuous or not. If the process is determined to be continuous, and if it can be shown that the topological properties change during the process, then the process is an aging process. That is the process is continuous but irreversible.

As the ultimate interest is with evolutionary processes that do not have continuous inverses, the emphasis on group theoretic methods that have so dominated the development of current physical theory will be given low priority. As the group concept requires the property of an inverse, it seems apparent to this author that such concepts, although very useful to geometrical problems, can not be at the heart of a theory that does not support a continuous inverse.

Perhaps the most important property or idea to this monograph is the concept of closure. The idea of closure is an invariant of a continuous but irreversible process. From set theoretic ideas, the idea of closure means that any pair of elements of a subset can be combined by a rule such that the resultant is still an element of the subset. Closure is perhaps the most fundamental property of a group. Elements of a vector space can be added together such that each sum is an element of the set of all basis elements multiplied by real numbers. The process of addition is closed. However, if two polar elements of a vector space are multiplied together by the method of the Gibbs cross product of engineering science, the resultant axial vector is not an element of the original subset of polar vectors. The Gibbs product is not closed. No engineer would ever add a torque to a force, or a linear momentum vector to an angular momentum vector, because they are not vectors of the same species.

Key observables in the understanding of the aging process are the concepts of closure and connectivity. Experimental methods to observe "closure" concepts must be devised if the notion of topological evolution is to be made practical. These notions may sound abstract and not useful, but when it is realized that the production of defects in a physical system, and the change of phase from solid to liquid, are exhibitions of topological evolution, then the ideas become more concrete.

0.5 Philosophy of a Topological Approach to Physics.

The motivation is, and has been, to develop a better non-statistical, non-magical, understanding of the real world of irreversible processes, which if they take place continuously, must involve changing topology. Turbulent flow in fluids, dissipative mechanical systems, non-equilibrium thermodynamics, electromagnetic charge creation, chemical precipitation, and biological aging are all ready examples of topological evolution in physical systems. However, it is apparent that the concepts of topological evolution apply to synergetic systems of all types, including political and economic systems. Many topological properties have rational ratios, and therefor their evolution is in relation to the integers. It also would seem apparent that topology is at the basis of the quantum theory, and that "Bohr's miracle" of a quantum "jump" has its explanation in topological evolution.

Before it is possible to make practical use of the concepts of changing topology, it is necessary to understand what topology itself is all about. It is important to be able to identify topological properties, such that when these topological properties change, under the action of continuous but irreversible processes, their changes will be recognized. The topological methods emphasized in this monograph involve topological properties that are independent from the more geometric ideas of metric and/or connection.

Right up front, memorize the following definitions. The implications of these definitions will be clarified in that which follows (These concepts are best explained by examples to be found in Appendix I.).

1. A topological property is an invariant of a homeomorphism.

2. A homeomorphism is a map describing a process from an initial to a final state which is both continuous and reversible. Reversible means that the inverse map exists and is also continuous. A diffeomorphism is a special homeomorphism that is differentiable.

3. A topological structure is the specification of enough topological properties to permit a determination to be made if a map, or a process, is continuous, even though it may not be be a diffeomorphism and even though it may not be reversible!

4. A process or a map is continuous if the limit points of the intitial state, relative to the topology of the initial state, permute into the limit points of the final state, relative to the topology of the final state.

5. A limit point p of a subset, A, is a point such that that every open set of the given topology that contains p contains another point b, of the subset A, but not equal to p. Note that p is not necessarily a point in A.

6. Open sets, or their compliments which are closed sets, may be used to define a topology. A given collection of subsets defines an a open set topology if the intersection of any pair of elements in the subset collection is also member of the subset collection, and if every union, infinite or not, of elements in the subset collection is also an element of the given subset collection. In essence, topology is based upon the idea of closure under logical union and intersection.

7. The closure of a subset is equal to the union of the subset and its limit points, and is also equivalent to the union of the interior of a set and its boundary. However, a subset may have a boundary without limit points, and it may limit points without a boundary.

8. A given collection of sets can support many different topologies, just by choosing subset collections that obey the closure rules described above.

In the first chapter, a simple example is presented that will clarify many of those topological definitions that may not be familiar to the reader. The simple example, using a set of 4 elements, turns out to be homeomorphic to the Cartan Topology of an exterior differential system based on a single 1-form of Action. The Cartan Topology forms the foundations of the calculus of variations, and the LaGrange-Hamiltonian field theories.

There are several ways to approach the concepts of topology. The mathematicians of 1800 to 1900 discovered a number of "global" results and features of mathematical systems that were deformation invariants, but the bulk of the work in topology as we know it now began after 1900. It would appear that the name topology was accepted by mathematicians about 1925, although there were then two disjoint schools of topology (point set topology and combinatorial algebraic topology) that in the early days were assumed to be studies of different mathematical topics. It is now known that these different "ways" of getting at topological information are equivalent. Most of these classical approaches, however, are much too stilted to have widespread acceptance by applied scientists and engineers.

However, E. Cartan in the early 1900's developed an extraordinary set of ideas based on two simple extensions of the ordinary calculus. He exploited the notion of the Grassmann exterior product, and developed the formalism of a closed exterior algebra. To this (closed) algebraic system he added the concept of the exterior differential (at first called the exterior derivative). From the notion of the exterior product and the exterior differential acting on exterior differential forms almost all of the useful features of differential topology found in this monograph can be constructed: the science is called the Cartan theory of exterior differential forms.

Topology is intuitively a non-local, or "global", idea of how interior things, perhaps consisting of many parts, behave synergetically with their environment or exterior, and with their boundary. A topological property may be viewed intuitively as an invariant of a continuous deformation. Klein's view of geometrical properties (such as size and shape) was that geometrical properties were invariants of translations and rotations. Both viewpoints are perhaps a bit symplistic, but they do help in explaining the difference between a topological property and a geometrical property. The concept of physical coherence is intuitively a topological idea in that it corresponds to the notion of a synergetic interaction between parts not at the same point. The classic understanding of the coherence of a crystal is due to a very specialized and refined topology induced on space time by the presence of matter. The fact that a cylindrical wave guide has a low frequency cut-off, while the coaxial cable can transmit DC current is a topological idea demonstrating the interaction of a system and its boundary.

Note that during these discussions above no mention is made of how big or how small the system to be studied is; there has been no *geometric* definition of scales.

This notion that topological properties are independent from scales is the hardest concept to absorb for a scientist non-sensitized to topological thinking. After all, scientists are trained to "measure" something; they want to measure a size or shape. These geometrical hangups must be removed from the topological perspective, for if the system is too small, just stretch it out; if the system is to big, just compress it. Physical topology is the study of those properties that are independent from some legislated scale marks on a ruler, or on the number of time ticks of a clock. Physical topology deals with ideas that are in a sense the same on both the scale of the galaxy and on the scale of the atom. This is not to say that the world of microphysics is identical to the world of macrophysics, for when a physical system is constrained to yield geometric invariants, certain topological features will dominate over other topological features, depending on the scales chosen. The coaxial waveguide always has a low-frequency cut-off (a topological idea), but the actual wavelength depends upon the geometric idea of scales.

In these monographs the emphasis will be on the use of Cartan's theory of differential forms to describe topological features of physical systems that are independent of size and shape. (Appendix 2 of this monograph entitled "Cartan's methods of Exterior Calculus", may be used to bring the reader up to speed on the Cartan techniques utilized herein. A number of textbooks are available [Flanders-Bishop&Goldberg - Liebermann]) For those readers with some exposure to the exterior calculus, this means that the concepts of coordinate representations, or metric, or connection, or fiber bundles (with their more geometrical content) are going to be suppressed in favor of the Cartan theory, which does not require such constraints on the base variety of space and time.

A natural question arises as to how a "differential form" can be used to describe global information. The paradox is resolved when it is realized that the differential form can be considered as an entity "before" a limiting process has taken place. Recall that the limiting process of differential calculus requires some neighborhood constraint to be specified before the derivative is computed. In fact the limiting process depends upon the specification of a topology. The usual topology assumed is the homogeneous connected euclidean topology of R^N which consists of open sets defined in terms of open balls of domain less than r. Physical systems will require for their description topologies that are not equivalent to R^N, although in a small neighborhood in which certain "singularities" have been removed, it is generally assumed that the physical systems can be approximated by R^N for N large enough.

0.6 Why Exterior Differential Forms?

Cartan's exterior differential forms have amazing properties not contained in other mathematical entities:

1. Exterior differential forms on the final state are well behaved with respect to functional substitution of differentiable maps from the initial to the final state,

even though these maps are irreversible! [retrod] This fact is known as the "pull-back" and is the cornerstone of the investigations about irreversible processes. By functional substitution, not only is it possible to compute values of a differential form on the initial state from values given on the final state, but also it is possible to compute the *functional form* of the co-tensor fields on the initial state in terms of the functional forms of the co-tensor fields used to define the differential forms on the final state. Differential forms are defined on the co-tangent space of tensors over the base of space time, and pullback via the transpose of the Jacobian of the map from the initial to final state. Contra-variant tensor fields do not enjoy this unique functional behavior under the "pullback", unless the inverse map is well defined. However contravariant tensor *densities* do enjoy the pullback property even though the inverse map does not exist. They are retrodictive relative to the adjoint of the Jacobian of the map from initial to final state. Both the transpose and the adjoint of the Jacobian exist even though the inverse does not. The bottom line is that there are two species of exterior differential forms: those that behave as scalars under functional substitution and those that behave as scalar densities. These facts are often ignored in many physical theories, but they are easily observed experimentally in the realm of electromagnetism. The constraints of the Liouville theorem and the insistence on unimodular representations of processes strangle the differences between differential forms and differential form densities; hence these constraints are not used in a theory that involves topological change.

2. Exterior differential forms can carry information about singularities, and these singularities dictate much of the topological content of the differential form. The singularity to be studied is not of the type that blows up to infinity, necessarily, but instead is the more innocuous zero set. It is the zero points of functions and vector fields under the action of continuous maps that is of predominant interest herein. The zeros of a map become the infinities of the inverse map, but in these monographs the emphasis is on how much can be determined about processes that need not have an inverse. The singular infinities of the Dirac delta function are avoided in this book, if at all possible, but the "singularities" of the zeros are crucial.

3. Exterior differential forms may have preimages which are not unique. It is precisely this multi-valuedness of an integral preimage that allows differential forms to single out the physically interesting characteristics, or wave-fronts, or shock-fronts, or defects of dynamical systems. These topological singularities or defects are point sets (which may or may not be stationary) upon which a unique solution to some system of partial differential equations describing the physical system cannot be analytically continued from a nearby neighborhood. As will be determined later, dimension and the number of components are topological properties, and the generation of a defect will correspond to a topological change of dimension, or the creation of multiple components, over some domain.

A list is displayed below of a number of physical effects which - to first order - do not depend explicity on size or shape. Therefore, these phenomena must have a topological basis for their explanation. Key topological properties are the number of parts (connectivity), the dimension, and orientability. Concepts of size and shape, metric, or connection can not enter into the explanation of such phenomena, except in an auxiliary way. As the monograph progresses, the details of these topological features will be explained

Planck's Radiation Formula.

The distribution law of the radiation intensity as a function of frequency is independent from the size and shape of the hot body.

Coaxial Wave-Guide Propagation

A hollow wave guide is a high-pass filter, but a non-simply connected co-axial cable can pass DC.

Chaos does not occur in Dimension 2

A system of ODE's has a PDE equivalent. If the PDE satisfies the Frobenius integrability theorem, according to Darboux there is always a representation in terms of 2 functions.

Flux quanta and the Bohm-Aharanov effect (based on closed 1-forms)

Flux quanta in type II superconductors come in integer multiples of h/2e. The integral of $\int_{closed} \mathbf{A} \circ \mathbf{ds}$ over a closed path in closed domains, is a deformation invariant that can be used to explain the Kelvin circulation theorem (and why a wing flies) and the so-called Berry phase.

The **Charge quantum and Gauss' Law (based on 2-forms)**

The integral of $\iint_{closed} \mathbf{D} \circ \mathbf{dS}$ over a closed surface (not a boundary) depends only upon the number, n, of electrons of charge e= 1.6 x $10^{-19}C$ in the interior of the closed surface and is a deformation invariant.

The Poincare quanta of Topological Torsion and Topological Spin (based on 3-forms)

The Bohm-Aharonov effect is a 1-Dimensional period integral. Gauss's Law is a 2-Dimensional period integral. The Poincare quanta are two 3-Dimensional period integrals that have been little studied in science, but appear to have significance in hydrodynamics and plasmas as robust coherent vortex-like helical structures - insensitive to deformations in space time.

Quantum Transition Probability. (Cross ratios are independent from scale)

Fermi's Golden Rule demonstrates that the transition probability is a cross-ratio projective invariant, independent from scale.

Thermodynamic Irreversibility. **(The Heat 1-form, Q, does not admit an integrating factor.)**

Thermodynamic irreversibility is defined by the failure of the Frobenius Theorem for the 1 form of work, Q. It follows that the topological dimension of the associated Action 1-form, A, must be 4. In other words, thermodynamic irreversibility is an artifact of 4 dimensions.

Thermodynamic Phases

The difference between a vapor and a liquid is that the liquid is apparently connected, and the vapor consists of many disconnected parts. Then number of components is a topological property. Condensation implies a change in topology takes place. Condensation is a gluing or pasting process which often can be described continuously. Vaporization, on the other hand, is a discontinuous or cutting process, and does not permit a continuous description.

The Law of Corresponding States

In chemistry there is the law of corresponding states, which demonstrates the unversality of theremodynamics, independent form the size and configuration of the molecules under consideration.

Conductors vs Insulators.

Solid matter, when placed in an external electric field, and then separated into distinct topological components, exhibits two extreme features when the external electric field is removed.

1. Each of the separated topological components exhibit no net charge if the solid matter is an insulator.

2. Each of the separated topological components exhibits a finite plus or minus charge, with the total charge balancing to zero, if the solid matter is a conductor.

Part I

The Topology of Exterior Differential Systems.

Chapter 1

INTRODUCTION TO THE TOPOLOGY OF EXTERIOR DIFFERENTIAL SYSTEMS

In this presentation, a topological perspective will be used to extract those properties of physical systems and their evolution that are independent from the geometrical constraints of connections and/or metrics. It is subsumed that the presence of a physical system establishes a *topology* on an algebraic base space of independent variables. This concept is different from, but similar to, the geometric perspective of general relativity, whereby the presence of a physical system is presumed to establish a *metric* on a base space of independent variables. This concept is also different from, but similar to, a gauge theory whereby the presence of a physical system, or the lack thereof, establishes a *connection* on a base space of independent variables. Recall that a given base of independent variables may support many different topological structures; hence a given base may support many different physical systems.

The fundamental axioms utilized in this article are:

Axiom 1 *The presence of a Physical System establishes a topology on a domain of independent base variables which can be encoded in terms of exterior differential forms (symbolically represented by* Σ*).*

Axiom 2 *Physical Processes can be encoded in terms of contravariant vector direction fields, which may or may not be generators of 1-parameter groups, and in particular need not be homeomorphisms (symbolically represented by* V*).*

Axiom 3 *The closure of the exterior differential system (symbolically represented by* $\Sigma \cup d\Sigma$*) establishes a topological structure, such that equations of Continuous Evolution describing both reversible and irreversible Processes acting on Physical Systems are encoded by Cartan's magic formula :*

$$L_{(\mathbf{V})}\Sigma = i(\mathbf{V})d\Sigma + d(i(\mathbf{V})\Sigma) \tag{1.1}$$

The first axiom is similar, but not equivalent, to that Axiom of general relativity that implies that the presence of matter creates a geometric space with a Riemannian metric. The second axiom is similar, but not equivalent, to the axiom of tensor analysis that constrains the evolutionary processes to diffeomorphisms that describe geometrical, not topological change. The third axiom recognizes that the differential closure of the system of exterior differential forms, $\Sigma \cup d\Sigma$, creates a topological structure on the space. Recall that a topological structure is enough information on a space of sets such that it is possible to determine if a map (a process) from a space with topology T1 to a space with topology T2 is continuous. Such continuous maps for which topological change takes place cannot be homeomorphisms (for then the topology of the initial state T1 and the the topology of the final state T2 would be the same. A major success of the topological perspective is that continuous non-homeomorphic processes of topological evolution not only establish a logical basis for the arrow of time [?], but also can be exploited to explain the concept of thermodynamic irreversibility without the use of statistics.

In the period from 1899 to 1926, Eli Cartan developed his theory of exterior differential systems [1,2], which included the ideas of spinor systems [3] and the differential geometry of projective spaces and spaces with torsion [4]. The theory was appreciated by only a few contemporary researchers, and made little impact on the main stream of the physical sciences until about the 1960's. Even specialists in differential geometry (with a few notable exceptions [5]) made little use of Cartan's methods until the 1950's. Even today, many physical scientists and engineers have the impression that Cartan's theory of exterior differential forms is just another formalism of fancy.

However, Cartan's theory of exterior differential systems has several advantages over the methods of tensor analysis that were developed during the same period of time. The principle fact is that differential forms are well behaved with respect to functional substitution of C1 differentiable maps. Such maps need not be invertible even locally, yet differential forms are always deterministic in a retrodictive sense [6], by means of functional substitution. Such determinism is not to be associated with contravariant tensor fields, if the map is not a diffeomorphism. Cartan's theory of exterior differential systems contains topological information, and admits non-diffeomorphic maps which can describe topological evolution.

Although the word "topology" had not become popular when Cartan developed his ideas (topological ideas were described as part of the theory of analysis situs), there is no doubt that Cartan's intuition was directed towards a topological development. For example, Cartan did not define what were the open sets of his topology, nor did he use, in his early works, the words "limit points or accumulation points" explicitly, but he did describe the union of a differential form and its exterior differential as the "closure" of the form. With this concept, Cartan effectively used the idea that the closure of a subset is the union of the subset with its topological limit points. What was never stated (until 1990) is the idea that the exterior differential is indeed a limit point generator relative to a Cartan topology. The union of the identity operator and the exterior differential satisfy the axioms of a Kuratowski closure operator [7], which can be used to define a topology. The other operator of the Cartan calculus, the exterior product, also has topological connotations when it is interpreted as an intersection operator.

In a perhaps over simplistic comparison, it might be said that ubiquitous tensor methods are restricted to geometric applications, while Cartan's methods can be applied directly to topological concepts as well as geometrical concepts. Cartan's theory of exterior differential systems is a topological theory not necessarily limited by geometrical constraints and the class of diffeomorphic transformations that serve as the foundations of tensor calculus. A major objective of this article is to show how limit points, intersections, closed sets, continuity, connectedness and other elementary concepts of modern topology are inherent in Cartan's theory of exterior differential systems. These ideas do not depend upon the geometrical ideas of size and shape. Hence Cartan's theory, as are all topological theories, is renormalizeable (perhaps a better choice of words is that the topological components of the theory are independent from scale). In fact the most useful of Cartan's ideas do not depend explicitly upon the geometric ideas of a metric, nor upon the choice of a differential connection between basis frames, as in fiber bundle theories. The theme of this article is to explore the physical usefulness of those topological features of Cartan's methods which are independent from the constraints and refinements imposed by a connection and/or a metric.

In this article the Cartan topology will be constructed explicitly for an arbitrary exterior differential system, Σ. For a particular simple, but useful, system consisting of a single 1-form of Action, all elements of the Cartan topology will be evaluated, and the limit points, the boundary sets and the closure of every subset will be computed abstractly. Earlier intuitive results [7], which utilized the notion that Cartan's concept of the exterior product may be used as an intersection operator, and his concept of the exterior differential may used as a limit point operator acting on differential forms, will be given formal substance in this article. A major result of this article, with important physical consequences in describing topological evolutionary processes, is the demonstration that the Cartan topology is not necessarily a connected topology, unless the property of topological torsion vanishes, and that thermodynamic irreversibility is a consequence of 4 dimensions or more.

1.1 A Point Set Example of Topological Ideas

A more detailed discussion of topological ideas appears in Part 1, Appendix 1, but an important and useful example of pertinent topological ideas can be created from the set of 4 elements or points, $X = \{a, b, c, d\}$ and all possible subsets:

$$\emptyset, \tag{1.2}$$
$$\{a\}, \{b\}, \{c\}, \{d\}, \tag{1.3}$$
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \tag{1.4}$$
$$\{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, d\}, \tag{1.5}$$
$$\{a, b, c, d\} = X \tag{1.6}$$

Select the following subset elements as a topological basis,

$$\text{basis selection} \quad \{a\}, \{a, b\}, \{c\}, \{c, d\}, \tag{1.7}$$

and then compose a topology $T4$ of open sets from all possible unions of the selected basis elements:

$$T4\{open\} : \emptyset, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\} \tag{1.8}$$

The closed sets are the compliments of the open sets:

$$T4\{closed\} : \{a, b, c, d\}, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \emptyset \tag{1.9}$$

It is an easy exercise to demonstrate that the collections above indeed satisfy the axioms of a topology. (This is not the only topology that can be constructed over 4 elements. See Appendix 1).

This simple example of a point set topology permits explicit construction of all the topological concepts, which include limit sets, interiors, boundaries, and closures, for the all of subsets of X, relative to the topology, $T4$. The standard definitions are:

1. A limit point of a subset A is a point p such that all open sets that contain p also contain a point of A not equal to p.

2. The closure of a subset A is the union of the subset and its limit points, and is the smallest closed set that contains A.

3. The interior of a subset is the largest open set contained by the subset.

4. The exterior of a subset is the interior of its compliment.

5. A boundary of a subset is the set of points not contained in the interior or exterior.

6. The closure of a subset is also equal to the union of its interior and its boundary.

The results of applying these definitions to the $T4$ topology of 4 points are:

Table 1. **A T4 Topology of 4 points**

$X = \{a, b, c, d\}$

Basis subsets $\{a\}, \{a, b\}, \{c\}, \{c, d\}$

$T4\{open\} : \emptyset, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X$

$T4\{closed\} : X, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \emptyset$

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{c\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c, d\}$
$\{d\}$	$\emptyset$	$\emptyset$	$\{d\}$	$\{d\}$
$\{a, b\}$	$\{b\}$	$\{a, b\}$	$\emptyset$	$\{a, b\}$
$\{a, c\}$	$\{b\}, \{d\}$	$\{a, c\}$	$\{b, d\}$	X
$\{a, d\}$	$\{b\}$	$\{a\}$	$\{b, d\}$	$\{a, b, d\}$
$\{b, c\}$	$\{d\}$	$\{c\}$	$\{b, d\}$	$\{b, c, d\}$
$\{b, d\}$	$\emptyset$	$\emptyset$	$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{d\}$	$\{c, d\}$	$\emptyset$	$\{c, d\}$
$\{a, b, c\}$	$\{b\}, \{d\}$	$\{a, b, d\}$	$\{d\}$	X
$\{b, c, d\}$	$\{d\}$	$\{c, d\}$	$\{b\}$	$\{b, c, d\}$
$\{a, c, d\}$	$\{b\}, \{d\}$	$\{a, c, d\}$	$\{b\}$	X
$\{a, b, d\}$	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{b, c, d\}$
$\{a, b, c, d\}$	$\{b\}, \{d\}$	X	$\emptyset$	X

(1.10)

This $T4$ topology is quite interesting for many demonstrable reasons. First note that the all of the singletons of the topology are not closed. This implies that the topology is NOT a metric topology, NOT a Hausdorf topology, and even does NOT satisfy the separation axioms to be a T_1 topology. Note that all closed sets contain all of their limit points (which is a standard property of a closed topological set). Some open sets can contain limit points, but some open sets do not contain their limit points. Some subsets have boundaries that are composed of their limit points. Some subsets have limit points which are not boundary points. Certain subsets have a boundary, but do not have limit points, and in other cases there are subsets that have limit points, but do not have a boundary. There are certain subsets with a boundary, but without an interior. There are certain subsets with an interior, but without a boundary. Although the closure of a subset is the union of the subset and its limit points, or the union of the interior of a subset and its boundary, the intersection of a subset and its limit points may be empty or non-empty, while the intersection of the interior of a subset and the boundary of the subset is always empty. These situations, though topologically correct, are not always intuitive to those accustomed to metric based topological concepts, which impose a number of additional constraints on the sets of interest. Yet all of these topological ideas, including the non-intuitive ones, are easy to grasp from a study of the simple example of the $T4$ point set topology. Current physical theories often do not make use of the non-intuitive topological features - even though nature recognizes them.

One other very important observation is that there are subsets of the $T4$ topology, $\{a, b\}$ and $\{c, d\}$, (other than $\emptyset$ and X) which are both open and closed. The union of these two subsets $\{a, b\}$ and $\{c, d\}$ is X. Topological spaces with this property are said to be disconnected topological spaces. Note that subsets of a topological space can be disconnected, even though the topology on the space is a connected topology. What is important for describing evolutionary physical processes is that it is possible to construct a continuous map from a disconnected topology to a connected topology, but it is impossible to construct a continuous map from a connected topology to a disconnected topology. If the mapping process is interpreted as an evolutionary process, these facts establish a logical or topological basis for the arrow of time [?]. This idea will be exploited to explain the concept of thermodynamic irreversibility without the use of statistics.

What is even more remarkable is that the topology induced on a 4 dimensional space time variety by a differential 1-form of Action, A, emulates the properties of the $T4$ topology given above. The realization of a $T4$ topology in terms of exterior differential forms is herein defined as the "Cartan topology", and is detailed in the next section. The Cartan topology has far reaching consequences in applications to physical problems.

1.2 Algebraic and Differential Closure

The concept of (topological) closure is one of the most important ideas embedded in Cartan's theory. (For more detail on the Cartan methods, see Appendix 2.) His methods center on two procedures of closure, one algebraic, and one differential. Both processes are closed in the sense that when they operate on a subset of a (Cartan-Grassman) set of exterior differential forms, the objects created are also subsets of the set of exterior differential forms. There are no surprises. Cartan utilized the exterior algebra over a variety of dimension N to construct a vector space of exterior differential forms (and scalar functions) of dimension 2^N. The N+1 subspaces, Λ^p, of this (Grassmann) space are vector spaces of dimension equal to N things taken p at a time. Each subspace Λ^p has a basis set of $\binom{N}{p}$ independent p-form elements. The subspace Λ^0 is the space of functions with arguments in terms of the N independent (differentiable) variables of the variety. The Cartan-Grassmann set of exterior differential forms consists of objects constructed from these 2^N basis elements. The closure operations acting on elements of this vector space of dimension 2^N produce a vector within the same vector space of dimension 2^N. However, the closure operations, acting on elements of a given subspace, Λ^p, usually do not produce elements that reside within the same vector subspace, Λ^p. Geometric constraints of connection and metric are often used to produce operations that are closed within a subspace. The covariant derivative of tensor analysis is an example of such a constraint.

On a space of N=4 differentiable independent variables $\{x, y, z, s\}$, the primitive N *base* elements may be formulated as a differential position vector, and consists of the 1-forms,

$$\text{Differential position vector of a 4D variety:} \quad d\mathbf{R} = \left| \begin{matrix} dx \\ dy \\ dz \\ ds \end{matrix} \right\rangle. \tag{1.11}$$

The N dimensional set $\{dx^k\}$ forms a base for exterior differential 1-forms. All other 1-forms can be constructed from 0-forms, $A_k(x, y, z, s)$, which are scalar functions with basis element, $unity = 1$, multiplied, as in a vector space, times the differential primitive base elements. The vector space so created is defined as the space of 1-forms Λ^1. The classic 1-form is then the linear combination of base elements,

$$A = \Sigma_{k=1}^{4} A_k(x, y, z, s) dx^k = \mathbf{A} \circ d\mathbf{R} \ \subset \Lambda^1. \tag{1.12}$$

The intersections of these differential form base elements can be used to form the *basis* elements of other differential form vector subspaces, Λ^p. For example, the pairs (in a graded manner) $[dy\hat{}dz, dz\hat{}dx, dx\hat{}dy, dx\hat{}ds, dy\hat{}ds, dz\hat{}ds]$ will form the basis for the space of 2-forms, Λ^2. The triples, $[dx\hat{}dy\hat{}dz, dy\hat{}dz\hat{}ds, dz\hat{}ds\hat{}dx, ds\hat{}dx\hat{}dy]$ form a differential basis for 3-forms. The quadruple $[dx\hat{}dy\hat{}dz\hat{}ds]$ forms the basis for 4-forms. There are no p-forms with p > N. The collection of all of these basis sets forms the basis of the 2^N dimensional differential Grassmann algebra. All elements of the differential Grassman algebra can be composed from the functions Λ^0 and the primitive base elements, Λ^1.

As an example, the basis set for the Grassmann vector space over a differential variety of 4 independent variables $\{x, y, z, s\}$ is:

The Grassmann Vector Space
of exterior differential forms in 4D
5 vector subspaces of $\binom{N}{p}$ components

basis element	subspace Λ^p	Name
1	functions Λ^0	scalar
dx	1^{st} component Λ^1	vector
dy	2^{nd} component Λ^1	vector
dz	3^{rd} component Λ^1	vector
ds	4^{th} component Λ^1	vector
$dy\hat{}dz$	1^{st} "B-like" component Λ^2	tensor
$dz\hat{}dx$	2^{nd} "B-like" component Λ^2	tensor
$dx\hat{}dz$	3^{rd} "B-like" component Λ^2	tensor
$dx\hat{}ds$	1^{st} "E-like" component Λ^2	tensor
$dy\hat{}ds$	2^{nd} "E-like" component Λ^2	tensor
$dz\hat{}ds$	3^{rd} "E-like" component Λ^2	tensor
$dy\hat{}dz\hat{}ds$	1^{st} component Λ^3	psuedovector
$dz\hat{}ds\hat{}dx$	2^{nd} component Λ^3	psuedovector
$ds\hat{}dx\hat{}dy$	3^{rd} component Λ^3	psuedovector
$dx\hat{}dy\hat{}dz$	4^{th} component Λ^3	psuedovector
$dx\hat{}dy\hat{}dz\hat{}ds$	Λ^4	pseudoscalar

(1.13)

The closure operations are such that the operations on elements of this 2^N dimensional set produce elements in the 2^N dimensional set. As is true for most vector spaces, the 2^N dimensioanl differential basis set generated from the $\{dx^k\}$ and their exterior products is not unique. A sometimes useful construction is to form linear combinations of a new set of primitive basis elements by the rule,

$$\left|\sigma^k\right\rangle = \left[G^k_m\right] |dx^m\rangle\,, \tag{1.14}$$

where the matrix $\left[G^k_m(x^j)\right]$ is a matrix of functions. The exterior products of the σ^k can be used to form an equivalent Grassmann basis of dimension 2^N

The elements of the differential subspaces are called p-forms. Functions are defined as p=0 forms, and have the unit as a basis vector. In 4 dimensions, the 2^N subspace sets are Λ^0 = 1 dimensional, Λ^1 = N = 4 dimensional, Λ^2 = N(N+1)/2 = 6 dimensional, Λ^3 = N = 4 dimensional, and Λ^4 = N = 1 dimensional. The elements of the subspaces are often called scalars (0-forms, Λ^0), vectors (1-forms, Λ^1), tensors (2-forms, Λ^2), pseudovectors (3-forms, Λ^3), pseudo-scalars (4-forms, Λ^4) in relativistic physical theories. The Exterior differential (Grassmann) algebra has a *finite* 2^N dimensional basis (equal to 16 elements in a space of 4 independent variables). The concept of Cartan-Grassmann closure means that the operations on elements of the 2^N dimensional space yield results that are contained within the 2^N

dimensional space. When the operations are applied to elements of a subspace, the results usually are not contained in the same subspace, but they are contained within the 2^N dimensional vector space of p forms.

The algebraic operation of exterior multiplication is closed with respect the to the 2^N dimensional Grassman vector space of p forms. The exterior product (with symbol ˆ) takes elements of the 2^N base space and multiplies them together in a manner such that the result is contained as an element of the 2^N base space. This process of exterior multiplication is closed, for the action of the process on any subset of the 2^N base space produces another subset of the 2^N base space. However, the exterior product takes a p-form times a q-form into a p+q form. The elements of the product can be from different or from the same vector subspaces, but the resultant is always a linear combination of the subspaces of the Exterior algebra.

A compound element of the Cartan-Grassmann space will consist of sums of elements from different subspaces. The most general form will consist of elements from all subspaces. Denote an element of the Cartan-Grassman space as $Gr(1,2,3,4)$ if all subspaces are included in the compound element. For example, the Grassmann element, $Gr(1,0,3,0)$, consists of an element of the subspace, Λ^1, and the subspace, Λ^3. The Grassman Algebra and exterior product can evalute products that are combination of elements of different subspaces. For example, the exterior product creates the product of $Gr(1,2,0,0) = (A(dx) \subset \Lambda^1) + (B(dx\hat{}dy) \ \subset \Lambda^2)$ and $Gr(0,0,3,0) \subset C(dy\hat{}dz\hat{}ds)$ as an element

$$Gr(1,2,0,0)\hat{}Gr(0,0,3,0) = Gr(0,0,0,4) = AC(dx\hat{}dy\hat{}dz\hat{}ds) \subset \Lambda^4 \tag{1.15}$$

of the 2^N dimensional space of the Grassman Algebra.

Similarly the concept of exterior differentiation (with symbol d) is defined such that the operation produces a p+1 form from a p-form. This process of exterior differentiation is "closed", for the action of the process on any subset of the 2^N dimensional Cartan-Grassman space produces another subset of the 2^N dimensional Cartan-Grassman space. A differential ideal is defined as the union of a collection of given p-forms and their exterior derivatives.

An "interior" product with respect to a direction field $\mathbf{V}$ (with symbol $i(\mathbf{V})$ and of dimension N) can be defined on the Grassmann algebra of exterior differential forms. The interior product takes a p-form to a p-1 form, and in this sense is another operation which is closed within the Grassman algebra. The resultant product is still an element of the 2^N base space. Where the exterior differential raises the rank of a p-form to a p+1 form, the inner product lowers the rank of a p-form to a p-1form. (There are other useful operators that lower the rank of the exterior differential p-form, and involve integration.)

By composition of the exterior derivative and the inner product operators, the Lie differential operator (with symbol $L_{(\mathbf{V})} = i(\mathbf{V})d + di(\mathbf{V})$) can be constructed, such that when the Lie differential operates on an exterior p-form, the resultant object is another p-form. For a 1-form of Action, A, the process reads:

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = Q. \tag{1.16}$$

The resultant is not only closed relative to the Grassmann algebra, The resultant also remains within the same Grassman vector subspace. The Lie differential does not depend upon a metric nor upon a connection. When the Lie differential acting on a p-form vanishes, the p-form is said to be an invariant of the process, $\mathbf{V}$. When the Lie derivative of a p-form does not vanish, the topological features of the resultant p-form permit the processes, $\mathbf{V}$, that produce such a result, to be put into equivalence classes, depending on the Pfaff dimension of the resultant form, Q. For example, if the formula given above for a 1-form, A, yields a result Q such that $dQ = 0$, then the process $\mathbf{V}$ belongs to the class of process known as Hamiltonian processes in mechanics, and to the Helmholtz class of processes that conserve vorticity in Hydrodynamics. Of particular interest to this article are processes where Q is of Pfaff dimension* greater than 2. The Pfaff sequence constructed from Q contains three or more elements. Such processes are thermodynamically irreversible.

The Lie differential will be used extensively in physical applications of Cartan's theory, especially to the study of processes that involve topological evolution. The perhaps more familiar covariant derivative, highly constrained by connection or metric assumptions, is a special case of the Lie differential. The use of the covariant derivative leads to useful, but limited, physical theories for which the description of topological evolution is awkward, if not impossible.

The bottom line is to remember that the set over which the closure operations are valid is the finite 2^N dimensional Cartan-Grassman space constructed from the primitive N dimensional differential base.

1.2.1 The Exterior Product and Set Intersection

As mentioned above, Cartan's theory of exterior differential systems has its foundations in the Grassmann algebra, where two combinatorial processes are defined to produce algebraic and differential closure. The algebra is based upon the concepts of vector space addition, and an algebraic closure multiplication process now called the exterior product [8]. The Cartan calculus is defined in terms of the another closure operator now called the exterior differential†. In this monograph, the operators of the exterior product and exterior differential will be applied to objects defined as exterior differential p-forms, and which are elements of the Cartan-Grassmann space of exterior differential forms. The idea in this subsection is to demonstrate the close relationship of the exterior product acting on exterior differential forms and the intersection concept of topology. The idea in the next subsection is to demonstrate

*See section 2.1 for details of the concept of the Pfaff sequence, and 2.2 for the details of the concept of the Pfaff dimension.

†Cartan originally defined the calculus operation as the exterior derivative. Then in the later years defined calculus operation as the exterior differential.

the close relationship of the exterior differential, acting on exterior differential forms, and the limit set concept of topology.

An exterior differential p-form is a function of independent variables, x^v, and their differentials, dx^μ. Consider an exterior differential 1-form, A, given by the expression,

$$A = A_\mu(x^v)dx^\mu. \tag{1.17}$$

The Cartan operations of exterior multiplication (symbol ˆ) and exterior differential (symbol d), when operating on 1-forms, A and B, obey the rules

$$\begin{aligned} A\hat{}A &= 0, && (1.18)\\ A\hat{}B &= -B\hat{}A. && (1.19)\end{aligned}$$

and

$$\begin{aligned} dA &= d(A_\mu\, dy^\mu) = (dA_\mu)\hat{}dy^\mu + A_\mu\, d(dy^\mu) && (1.20)\\ &= (dA_\mu)\hat{}dy^\mu + 0 && (1.21)\\ d(A\hat{}B) &= dA\hat{}B - A\hat{}dB. && (1.22)\end{aligned}$$

The non-zero product, $A\hat{}B$, defines an exterior differential 2-form; the product of three 1-forms defines a 3-form; etc.. For more detail consult see Appendix 2, and the books by Flanders [?] or Liebermann [?].

Intersection of Implicit Surfaces

In simple cases, a 1-form can be constructed from the differential of an ordinary function. In such cases, the coefficients of the 1-form are proportional to the gradient of the function.

$$A = A_\mu dx^\mu = \nabla\phi \cdot d\mathbf{r} = (\partial\,\boldsymbol{\phi}/\partial x^\mu)dx^\mu \ = d\phi(x^\mu) \tag{1.23}$$

In surface theory, the gradient is classically interpreted as vector direction field orthogonal to the implicit surface, $\phi(x^\mu) = 0$. The total differential of the implicit surface function must also vanish.

$$\phi(x^\mu) = 0 \supset d\phi(x^\mu) = 0. \tag{1.24}$$

However, the condition, $d\phi(x^\mu) = 0$, does not mean that the components of the orthogonal (normal) gradient field are necessarily zero, identically. When all components of the gradient vanish simultaneously, the implicit surface is said to have a singularity. The Jacobian of the gradient field then has a zero determinant.

The condition, $d\phi(x^\mu) = 0$, also can be satisfied if the differential displacements, dx^μ, are constrained to reside on the surface, $\phi(x^\mu) = 0$. In such cases, there

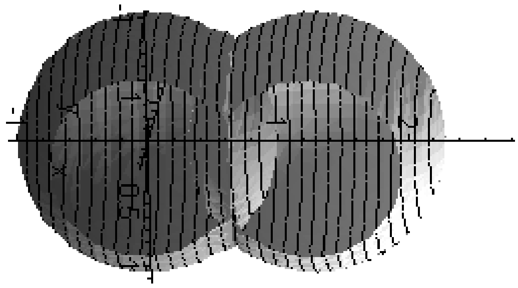

Figure 1 Intersecting spheres

are no displacements along the normal (gradient) direction; all displacements are constrainted to be orthogonal to the normal (gradient) direction field. Another important point is that there is a family of implicit surface functions, $\phi(x^\mu, \sigma) = 0$, that yield the same differential constraints, $d\phi(x^\mu) = 0$, subject to the constraint that the family parameter is a constant, $d\sigma = 0$. For more details see Appendix 3.

Now consider two families of implicit surfaces, $\phi(x^\mu, \sigma) = 0$ and $\psi(x^\mu, \rho) = 0$. Do the two surfaces intersect for appropriate values of the family parameters, σ and ρ? For example a spherical surface

$$\phi(x^\mu, \sigma) = (x^2 + y^2 + z^2 - \sigma^2) = 0 \tag{1.25}$$

of radius $\sigma = 5$ units, centered on the origin, does not intersect another spherical surface of radius

$$\psi(x^\mu, \rho) = ((x - 1.3)^2 + y^2 + z^2 - \rho^2) = 0 \tag{1.26}$$

$\rho = 1$, centered at $(x = 1.3, y = 0, z = 0)$. However, the spherical surfaces $\phi(x^\mu, 0.3 < \sigma < 2.3) = 0$ indeed have a space curve intersection with the sphere, $\psi(x^\mu, \rho = 1) = 0$.

A necessary condition for intersection of two implicit surfaces of constant parameters in 3D is that the exterior product of the differential 1-forms constructed from the exterior derivatives of the two implicit surface functions does not vanish:

$$d\phi(x^\mu)\hat{}d\psi(x^\mu) = \mathbf{J}_z dx\hat{}dy \ + \mathbf{J}_x dy\hat{}dz + \mathbf{J}_y dz\hat{}dx \neq 0. \tag{1.27}$$

Classic analysis in 3D says that the intersection of two implicit surfaces consists of a curve of points in common defined by solutions of ordinary differential equations (with s an arbitrary parameter of integration),

$$d\mathbf{R} = \mathbf{J}ds. \tag{1.28}$$

The vector field that generates the space curve of intersection is equivalent in 3D to Gibbs cross product of the two spatial gradient fields:

$$\text{Intersection of two implicit surfaces:} \quad \mathbf{J} = \nabla\phi \times \nabla\psi \neq 0. \tag{1.29}$$

For the two intersecting spheres, the cross product is a vector proportional to $\mathbf{J} = [0, -z, y]$. The problem of finding the intersection is not finished, for the non-zero 2-form (or the non-zero cross product) is a necessary but not sufficient condition. The initial conditions of the space curve representing the intersection must reside on both surfaces. These values are determined algbraically from the two implicit functions.

These concepts extend to 1-forms which are not representable by gradient fields, and to p-forms of higher rank. If the exterior product of two p-forms is not zero, then the p-forms have non-zero intersections. The coefficient functions are functionally independent, unless the exterior product vanishes.

1.2.2 The Exterior Differential and Limit Points

The second closure operator found in Cartan's theory of exterior differential systems is the exterior differential. The exterior differential, like the exterior product, also has topological connotations when applied to differential forms, but the results are sometimes surprising and unfamiliar. Where the exterior product is related to the topological concept of set intersection, the exterior differential is related to the topological idea of limit points. It will be demonstrated that:

Theorem 4 *With respect to the Cartan topology, the exterior differential is a limit point generator.*

The exterior differential is a differential operator which takes the p-forms into p+1 forms. Hence, like the exterior product, the exterior differential generates a vector in a different vector subspace of the exterior algebra.

$$d(\omega^p) \Rightarrow \omega^{p+1}. \tag{1.30}$$

The exterior differential of a function (0-form) is equivalent to the total differential of a scalar function, and yields a 1-form with coefficients proportional to the gradient field. The exterior differential of a 1-form is defined as

$$\begin{aligned} d\omega^1 &= d(A_b\, dy^b) = (dA_b)\hat{}\,dy^b + A_b\, d(dy^b) \\ &= (\partial A_b/\partial y^e\, dy^e)\hat{}\,dy^b + 0 \\ &= (\partial A_b/\partial y^e - \partial A_e/\partial y^b)\, dy^e\,\hat{}\,dy^b \\ &= F_{[eb\ldots]}\, dy^{[eb\ldots]} = F_{[H]} dy^{[H]}. \end{aligned} \tag{1.31}$$

It has been assumed that $dd(\omega^p) = 0$. The collective index notation $[H] = [eb...]$ permits the formula defining exterior differentiation to be generalized:

$$\begin{aligned} d\omega^p &= d(A_H\, dy^H) = (dA_H)\hat{}dy^H && (1.32)\\ &= (\{\partial A_H/\partial y^e\}\, dy^e)\hat{}dy^H && (1.33)\\ \text{where } dy^{[H]} &= dy^{[mnjk..]} = dy^m\hat{}dy^n\hat{}dy^j\hat{}dy^k... && (1.34) \end{aligned}$$

Other properties of the exterior differential will be exemplified by the rules for distributing the operator over a product of 1-forms, A and B,

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB. \quad (1.35)$$

It can be shown that the operator $KCl = I \cup d$, where I is the identity and d is the exterior differential, acting on a system of differential forms satisfies the "Kuratowski closure" axioms [?], and therefor can be used to define a topology. Starting from a single 1-form, A, on a 4 dimensional space, it is possible to generate the Pfaff Sequence

$$\begin{aligned} Pfaff\ Sequence &: \{A, dA, A\hat{}dA, dA\hat{}dA\} && (1.36)\\ &= \{A, F, H, K\}. && (1.37) \end{aligned}$$

The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology will be constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures:

$$\textit{the Cartan topological base}: \{A, KCl(A), A\hat{}dA, KCl(A\hat{}dA)\}. \quad (1.38)$$

When applied to the Pfaff sequence generated by a single 1-form of Action, A, on a space of 4 dimensions, the base elements correspond to the set

$$\begin{aligned} \textit{the Cartan topological base} &: \{A, A \cup F, H, H \cup K\} && (1.39)\\ &\approx \{a, b, c, d\} \text{ of the point set example above} && (1.40) \end{aligned}$$

When it is realized that the exterior differential acts a limit point generator, it becomes apparent why Cartan referred to the union of Σ and $d\Sigma$ as the closure of Σ,

$$\text{Closure} = (KCl) \circ \Sigma = (I \cup d) \circ \Sigma = \Sigma + d\Sigma = \text{subset} + \text{limit points}. \quad (1.41)$$

In the next section, the topological features of the Cartan topology, based on the Cartan topological base, will be worked out in detail. It will turn out that the Cartan topology can be put into correspondence with the T4 topology of 4 points displayed in a previous section. It will be evident, indeed, that the exterior differential is a limit point generator for any subset relative to the Cartan topology. This is a remarkable result, for as will be demonstrated below, all C2 vector fields acting through the concept of the Lie differential on a set of differential forms, with C2 coefficients, generate continuous transformations with respect to the Cartan topology. Moreover, the Cartan topology is disconnected if $A\hat{}dA \neq 0$. As the conditions for unique integrability of the 1-form A are given by the Frobenius theorem, which requires $A\hat{}dA = 0$, it should be expected that one of the features of the disconnected Cartan topology is that if solutions exist, they are not unique.

1.3 Maxwell's Equations of Electrodyanamics

1.3.1 The Maxwell Exterior Differential System

Maxwell's Electrodynamics, unfettered by geometric Lorentz constitutive relations, may be used as an excellent example to grasp the concepts of Cartan's methods. Maxwell's PDE's are topological statements deduced from an exterior differential system. The two postulates of el;ectrodynamics are:

$$\text{The Postulate of Potentials:} \quad F - dA = 0, \tag{1.42}$$

$$\text{The Postulate of conserved Charge Current densities:} \quad \overline{J} - d\overline{G} = 0. \tag{1.43}$$

No constraints of geometrical connection or metric are required. Such geometric constraints refine the Maxwell topology, and are useful for understanding constitutive equations that distinguish, for example, birefringent media from optically active media. The Maxwell-Faraday PDE's are not restricted to spaces of topological dimension N = 4. For an exterior differential system $F - dA = 0$ on a space of any dimension N>3, the closure conditions, $ddA = dF = 0$, always yield the same identical Maxwell-Faraday PDE's for the first 4 variables, without mofications or added terms do, even if N >4. Additional PDE's are also generated for N>4, but the system of PDE's created forms a nested set, with the 4 Maxwell-Faraday equations as topological kernel, of invariant format for any dimension N.

The PDE's resulting from the exterior differential system do not depend upon the symbols used in the formlation, and are always of the same format. The remarkable result is that Faraday induction is a topological idea, and does not depend upon metric or connection. The concept of Faraday induction applies to any system that satisfies the Postulate of Potentials.

As demonstrated below, the Postulate of Potentials establishes the field intensities, **E** and **B**, (think forces),and the Postulate of Conserved Charge current densities establishes the field excitations, $\mathbf{D}$ and $\mathbf{H}$, (think sources). The topological perspective subsumes that the two species are independent ideas. The experimental justification of such ideas can be demonstrated with a simple parallel plate capacitor experiment. First connect the plates to a battery of constant potential and let it remain connected. Insert a slab of plastic dieletric halfway between the plates. Release the plastic slab. Does the slab remain motionless, or is the motion such that the slab is expelled or attracted? For a second experiment, attach the plates of the capacitor to a battery and then disconnect the battery after charging the capacitor. Now insert the plastic slab halfway, and release it. Does the slab remain motionless, or is the motion such that the slab is expelled or attracted? In the first case, the $\mathbf{E}$ field remains constant (the potential does not change), and motion of the dielectric slab causes the $\mathbf{D}$ field to change (the battery adjusts the charge distribution). In the second experiment, the charge distribution is constant, so that the $\mathbf{D}$ field remains constant, but the $\mathbf{E}$ field changes. Consider the simple constitutive constraint, $\mathbf{D} = \varepsilon\mathbf{E}$. In the first experiment, insertion would cause the average ε to increase, hence even though $\mathbf{E}$ remains constant, the $\mathbf{D}$ field would increase. However, the total energy density $\mathbf{D} \circ \mathbf{E}$ would decrease if the slab was expelled, and that is what happens. In the second experiment, motion of the slab would cause the $\mathbf{E}$ field to change, as the $\mathbf{D}$ field remains constant, and the minimum energy density occurs when the slab is fully inserted.

Current electromagnetic dogma presents the idea that from a given charge current density distribution, $[\mathbf{J}, \rho]$,it is possible to deduce the $\mathbf{E}$ and $\mathbf{B}$ fields. However, the Postulate of conserved Charge-Current densities indicates that it is $\mathbf{D}$ and $\mathbf{H}$ that are the related quantities, not $\mathbf{E}$ and $\mathbf{B}$. The Postulate of Potentials indicates that the field intensities $\mathbf{E}$ and $\mathbf{B}$ are deduced from the potentials $[\mathbf{A}, \phi]$. It takes some constitutive constraint to convert $\mathbf{D}$ and $\mathbf{H}$ into $\mathbf{E}$ and $\mathbf{B}$, or $[\mathbf{J}, \rho]$ into $[\mathbf{A}, \phi]$. Both types of constraints appear in the literature in great detail and variety. Such assumptions obscure the topological basis and differences between exterior differential forms and exterior differential form densities.

The postulate of potentials indicates that the domain of support for the 2-form F is not compact without boundary[‡]. The postulate also demonstrates that magnetic monopoles are not compatible with the assumption of C2 differentiability. Such a statement does not apply to the density N-2 form $\overline{G}$, which can have closed and non-closed components. The closed but not exact components of $\overline{G}$ lead to the quantization of charge as a topological result. As $\overline{G}$ is a density, it also follows that quantized charge is a pseudo-scalar [?Post]. The historical assumptions of charge as a scalar are not compatible with the topological format. Experiments with piezo electric crystals indicate that volume deformations can cause electrical phenomena. If $\overline{G}$ was not a density, there would be no Piezo electricity.

[‡]There are two exceptions: the Klein bottle and the torus.

1.3.2 The D H field excitations: differential N-2 form densities.

For example consider the exterior differential of the N-1 form density[§], $\overline{D}$, in three dimensions, given by the expression,

$$\begin{aligned} d\overline{D} &= d(\overline{D}^x dy\hat{}dz - \overline{D}^y dz\hat{}dx + \overline{D}^z dx\hat{}dy) \\ &= div_3(\overline{D})dx\hat{}dy\hat{}dz \Rightarrow \rho(x,y,z)dx\hat{}dy\hat{}dz \end{aligned} \tag{1.44}$$

where ρ has been defined as the resultant of the action of the exterior differential, $div_3(\overline{\mathbf{D}})$. The usual interpretation of Gauss' law is that the field lines of the vector (density) $\overline{\mathbf{D}}$ terminate (or have a limit or accumulation point) on the charges, Q. Gauss' law generates both the intuitive idea that sources are related to limit points, and demonstrates the novel concept that the exterior differential is a limit point operator. The exterior differential creates limits points when the operation is applied to a differential form. However, as demonstrated above, the concept that the exterior differential is a limit point operator relative to the Cartan topology is a general idea, and is not restricted to Gauss' law.

Maxwell's PDE's are topological statements deduced from an exterior differential system. The two postulates are

$$\text{The Postulate of Potentials:} \quad F - dA = 0. \tag{1.45}$$

$$\text{The Postulate of conserved Charge current densities:} \quad \overline{J} - d\overline{G} = 0. \tag{1.46}$$

Extending this idea to four dimensions for the N-2 form density, $\overline{G}$, of Maxwell excitations $(\mathbf{D}, \mathbf{H})$,

$$\overline{G} = -D^x dy\hat{}dz + D^y dz\hat{}dx - D^z dx\hat{}dy + H^x dx\hat{}dt + H^y dy\hat{}dt + H^z dz\hat{}dt, \tag{1.47}$$

the exterior differential $d\overline{G}$ of $\overline{G}$ yields a three form, J, defined as the electromagnetic current 3-form,

$$\overline{G} = J^x dy\hat{}dz\hat{}dt - J^y dx\hat{}dz\hat{}dt + J^z dx\hat{}dy\hat{}dt - \rho dx\hat{}dy\hat{}dt \tag{1.48}$$

[§] There are two species of differential forms considered in this article. The first specie transforms as a scalar with respect to diffeomorphisms. The second specie transforms as a scalar density, and is proportional to the determinant of the diffeomorphism. The coefficients pull back with respect to the transpose of a differential Jacobian mapping, whether it is a diffeomorphism or not. The second species, the differential form densities, pull back with respect to the adjoint of a differential Jacobian mapping.

where in 3-vector language,

$$curl\ \mathbf{H} - \partial\mathbf{D}/\partial t = 0 \qquad div\,\mathbf{D} = \rho. \tag{1.49}$$

The charge current density act as the "limit points" of the Maxwell field excitations. Note that $d\overline{J} = 0$ for C2 functions by Poincare's lemma.

However, consider the N-1 current, C (not necessarily equal to $\overline{J}$as defined above) in four dimensions

$$C = \rho\{V^x dy\hat{}dz\hat{}dt - V^y dx\hat{}dz\hat{}dt + V^z dx\hat{}dy\hat{}dt - 1dx\hat{}dy\hat{}dt\} \tag{1.50}$$

and its exterior differential as given by the expression,

$$dC = \{div_3(\rho\mathbf{V}) + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt. = Rdx\hat{}dy\hat{}dz\hat{}dt = R\,\Omega_{4\_vol} \tag{1.51}$$

When the 4-form R vanishes, the resultant expression is physically interpreted as the "equation of continuity" or as a "conservation law". Over a closed boundary, that which goes in is equal to that which goes out (when $dC = 0$). Note that the concept of the conservation law is a topological constraint: the "limit points" of the "current 3-form" in four dimensions must vanish if the conservation law is to be true. If the RHS of the above expression is not zero, then the current 3-form is said to have an "anomaly", or a source (or sink) . The anomaly acts as the source of the otherwise conserved quantity. The limit points, R, of the 3-form, C, are generated by its exterior differential, $dC = \{div_3(\rho\mathbf{V}) + \partial\rho/\partial t\}\Omega_4$. When the RHS is zero, the current "lines" do not stop or start within the domain. (It is possible for them to be closed on themselves in certain topologies).

1.3.3 The E B Field Intensities: differential 2-forms

On a four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (constrained by the postulate of potentials, $F - dA = 0$) can be written in the form

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A}\circ d\mathbf{r} - \phi dt. \tag{1.52}$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes:

$$\begin{aligned} F = dA &= \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k \qquad (1.53)\\ &= \mathbf{B}_z dx\hat{}dy + \mathbf{B}_x dy\hat{}dz + \mathbf{B}_y dz\hat{}dx + \mathbf{E}_x dx\hat{}dt + \mathbf{E}_y dy\hat{}dt + \mathbf{E}_z dz\hat{}dt. \end{aligned}$$

where in usual engineering notation,

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{1.54}$$

The closure of the exterior differential system, $dF = 0$, vanishes for C2 differentiable p-forms, to yield

$$dF = ddA = \{curl\ \mathbf{E} + \partial \mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B} dx\hat{}dy\hat{}dz\} \Rightarrow 0. \tag{1.55}$$

Equating to zero all four coefficients leads to the Maxwell-Faraday partial derivative equations,

$$\{curl\ \mathbf{E} + \partial \mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}. \tag{1.56}$$

This topological development of the Maxwell-Faraday equations has made no use of a connection nor of a metric.

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds.

This now almost classic generation of the Maxwell field equations [9] has another less familiar interpretation: The **E** and **B** field intensities are the topological limit "points" of the 1-form of potentials, $\{\mathbf{A}, \phi\}$, relative to the Cartan topology! The limit points of the 2-form of field intensities, F, are the null set. For C2 vector fields, the Cartan topology admits flux quanta, charge quanta, and spin quanta, but excludes magnetic monopoles [10]. When the differential system of interest is built upon the forms A, F and G, it is possible to show that superconductivity is to be associated with the constraints on the limit point sets of A, $A\hat{}F$, *and* $A\hat{}G$ [11]. That is, superconductivity has its origins in topological, not geometrical, concepts. This remarkable idea that the exterior differential is a limit point operator is based upon Kuratowski's closure operator is equivalent to the union of the identity and the exterior differential.

Chapter 2

TOPOLOGY GENERATED BY A 1-FORM OF ACTION, A

2.1 The Pfaff sequence

A specific 1-form of Action can be constructed from an ordered set of M functions, $A_m(x^k)$ and differentials, dx^m on a base space, or variety, of M arbitrarily chosen independent differentiable variables, $\{x^1, ..., x^M\}$. A typical format in terms of four coordinates $\{x, y, z, s\}$ is given by the expression,

$$A = A_x(x^k)dx + A_y(x^k)dy + A_z(x^k)dz + A_s(x^k)ds. \tag{2.1}$$

To construct the Pfaff sequence for the 1-form, A, requires one (exterior) differential process, dA, and a number of (exterior) algebraic processes. The process is rather simple and leads to the finite set defined as the

$$\text{Pfaff Sequence:} \quad \{A, dA, A\hat{}dA, dA\hat{}dA, ...\}. \tag{2.2}$$

For example, consider the 1-form, A, on the 4D variety, $\{x, y, z, s\}$ with coefficient functions $A_x(x^k)$ given as .

$$A_x = (z^2 + s^2)2x, \tag{2.3}$$
$$A_y = (z^2 + s^2)2y, \tag{2.4}$$
$$A_z = -(x^2 + y^2)2z, \tag{2.5}$$
$$A_s = -(x^2 + y^2)2s, \tag{2.6}$$

Construct the elements of the Pfaff sequence:

$$A = (z^2 + s^2)2xdx + (z^2 + s^2)2ydy - (x^2 + y^2)2zdz - (x^2 + y^2)2sds, \tag{2.7}$$
$$dA = 4(zdz + sds)\hat{}(xdx + ydy) - 4(xdx + ydy)(zdz + sds) \tag{2.8}$$
$$= 8zxdz\hat{}dx + 8zydz\hat{}dy + 8sxds\hat{}dx + 8syds\hat{}dy \tag{2.9}$$
$$A\hat{}dA = 0 \tag{2.10}$$
$$dA\hat{}dA = 0. \tag{2.11}$$

The Pfaff sequence for the prescribed 1-form has only two elements, even though the number of base variables is four. This result means that the 1-form can be represented in terms of not less than two independent functions of the base variables. Indeed, the 1-form composed of two independent functions,

$$A = \phi d\chi - \chi d\phi, \tag{2.12}$$

is equivalent to the initial formulation, if

$$\phi = x^2 + y^2 \tag{2.13}$$
$$\chi = z^2 + s^2. \tag{2.14}$$

2.2 The Pfaff topological dimension

In general, the Pfaff sequence for a 1-form of Action is readily computed, and will contain $N \leq M$ elements, where N is defined as the Pfaff topological dimension (or class) of the given 1-form, A. (Dimension is a topological property.) One of the fundamental Axioms of this article is the idea that the topological properties of a physical system can be encoded in terms of an exterior differential system. It is remarkable that many physical systems appear to be represented by a single 1-form of Action, and almost immediately a topological property can be produced: the Pfaff topological dimension. The importance and application of this topological property will be developed in detail in that which follows.

A specific 1-form of Action can be constructed from an ordered set of M functions, $A_m(x^k)$ and differentials, dx^m on a base space of M arbitrarily chosen independent variables, $\{x^1, ..., x^M\}$. A typical format in terms of coordinates then is given by the expression,

$$A = \sum_1^m A_m(x^k)dx^m. \tag{2.15}$$

The general question arises as to the possible redundancy in this preliminary description of the physical system. In particular, it might be possible to find a lesser number of independent variables and covariant functions, $A_m(x^k)$, and still adequately describe the topological features of the 1-form, A. The implication is that there exists a $C1$ map, ϕ, from the original space of M independent variables to a space of N independent variables, with $N < M + 1$. On the N dimensional space, the 1-form of Action has a "canonical" format which is different depending on when $N = 2n + 1$, or $N = 2n + 2$. This number N is defined as the Pfaff Dimension.

As an example consider the Action 1-form for LaGrangian dynamics, with Lagrange multipliers p_k, of the form

$$A = L(x^k, V^k, t)dt + p_k(dx^k - V^k dt) \tag{2.16}$$

for $0 < k < n+1$. At first glance, it would appear that there are 3n+1 independent functions, $\{x^k, V^k, p_k, t\}$ in the description, as the LaGrange function depends on the 2n+1 variables, $\{x^k, V^k, t\}$. However, when the Pfaff sequence is computed, the Topological Pfaff dimension is N=2n+2, not 3n+1. (To prove this result is a good exercise.)

What is even more remarkable is that properties of the $T4$ topology, discussed in earlier sections, can be replicated in terms of the Pfaff sequence of exterior differential sets,

$$\text{Pfaff Sequence} : \{A, dA, A\hat{}dA, dA\hat{}dA...\}, \tag{2.17}$$

In fact, the Action 1-form on a domain of independent variables $\{x^1, ..., x^M\}$ may be used to define a (Cartan) topology on the domain. The first step is to compute the exterior derivative of the Action, dA. The second step is to construct, algebraically, the elements of the Pfaff sequence $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$. Then elements of this sequence will terminate (become zero) at either $(dA)^n$ or $A\hat{}(dA)^n$. The number of terms in the sequence defines the topological Pfaff dimension. In short, the Pfaff Dimension determines the irreducible minimum number of functions that are required to describe the Action 1-form. If the Pfaff dimension is odd, N=2n+1, then the 1-form of Action is said to generate a contact manifold. A canonical (Darboux) representation is of the form,

$$\text{Contact Manifold (Pfaff dimension = 2n+1):} \quad A = \sum_{k=1}^{n} p_k dq^k + d\tau \tag{2.18}$$

where the 2n+1 functions, $\{p_k,\, q^k, \tau\}$ are well defined functions of $\{x^1, ..., x^M\}$. If the Pfaff dimension is even, N=2n+2, then the 1-form of Action is said to generate an exact symplectic manifold. This symplectic manifold of dimension N=2n+2, by Stokes theorem, can not be compact without boundary. It is either a manifold with boundary, or it is an open manifold, a perfect candidate for studying thermodynamic non-equilibrium systems. (There are two exceptions: the Klein bottle and the Torus).

A canonical representation for the 1-form of Action is of the form on the symplectic manifold is,

$$\text{Symplectic Manifold (Pfaff dimension = 2n+2):} \; A = \sum_{k=1}^{n} p_k dq^k + H d\tau, \tag{2.19}$$

but the function H is independent from the functions, $\{p_k,\, q^k, \tau\}$. The key issue is that evolutionary behavior of the physical system, as represented by the 1-form of

Action, on the contact manifold is different from the evolutionary behavior of the physical system on the symplectic manifold. On a contact manifold, there exists a unique extremal direction field, determined by the single null eigen vector of the matrix of coefficients of the canonical 2-form dA. Such extremal fields do not exist on a symplectic manifold, for the anti-symmetric matrix of coefficients of the canonical 2-form dA does not have null eigenvalues.

In most historical physical theories, the constraint of unique integrability and predictability implies that the associated Pfaff sequence has only two terms:

$$\text{Integrable Pfaff Sequence} : \{A, dA, 0, 0, ...\}, \tag{2.20}$$

The Pfaff topological dimension of such canonical systems is 2. However, in this monograph the systems of most interest are those for which the Pfaff sequence will have 3 or 4 non-zero terms. For such cases, the concept of unique integrability fails.

2.3 The Cartan "Point Set" Topology.

Cartan built his theory around an exterior differential system, Σ, which consists of a collection of 0- forms, 1-forms, 2-forms, etc. [12]. He defined the closure of this collection as the union of the original collection with those forms which are obtained by forming the exterior differentials of every p-form in the initial collection. In general, the collection of exterior differentials will be denoted by $d\Sigma$, and the closure of Σ by the symbol, $KCl(\Sigma)$, where

$$\text{Kuratowski Closure operator:} \quad KCl(\Sigma) = \Sigma \cup d\Sigma \tag{2.21}$$

For notational simplicity in this article the systems of p-forms will be assumed to consist of the single 1-form, A. The exterior differential of A is defined as the 2-form $F = dA$, and the closure of A is the union of A and $F : KCl(A) = A \cup F$. The other logical operation is the concept of intersection, so that from the exterior differential it is possible to construct the set $A\hat{}F$ defined collectively as $H : H = A\hat{}F$. The exterior differential of H produces the set defined as $K = dH$, and the closure of H is the union of H and $K : KCl(H) = H \cup K$.

For any given 1-form, A, an entire Pfaff sequence of exterior differential sets,

$$\text{Pfaff Sequence} : \{A, dA, A\hat{}dA, dA\hat{}dA...\}, \tag{2.22}$$

can be generated on a N dimensional variety. The construction of the elements of the Pfaff sequence will continue until a zero element is produced. The Pfaff sequence will contain $M \leq N$ non-zero elements, where M is defined as the Pfaff topological dimension (or class of the form, [13]), and this dimension is a topological invariant of the given 1-form, A. The largest p-form so constructed is defined as the top Pfaffian.

The idea is that the 1-form A of maximal Pfaff dimension generates on space-time four equivalence classes of points that act as domains of support for the elements of the Pfaff sequence, A, F, H, K. The union of all such points will be denoted by $X = A \cup F \cup H \cup K$. The fundamental equivalence classes are given specific names:

$$\text{Topological ACTION} \quad : \quad A = A_\mu dx^\mu \tag{2.23}$$
$$\text{Topological VORTICITY} \quad : \quad F = dA = F_{\mu\nu} dx^\mu \hat{}\, dx^\nu \tag{2.24}$$
$$\text{Topological TORSION} \quad : \quad H = A\hat{}\, dA = H_{\mu\nu\sigma} dx^\mu \hat{}\, dx^\nu \hat{}\, dx^\sigma \tag{2.25}$$
$$\text{Topological PARITY} \quad : \quad K = dA\hat{}\, dA = K_{\mu\nu\sigma\tau} dx^\mu \hat{}\, dx^\nu \hat{}\, dx^\sigma \hat{}\, dx^\tau . \tag{2.26}$$

The Cartan topology* is constructed from a basis of open sets, which are defined as follows: first consider the domain of support of A. Define this "point" by the symbol A. A is the first open set of the Cartan topology. Next construct the exterior differential, $F = dA$, and determine its domain of support. Next, form the closure of A by constructing the union of these two domains of support, $KCl(A) = A \cup F$. $A \cup F$ forms the second open set of the Cartan topology.

Next construct the intersection $H = A\hat{}\, F$, and determine its domain of support. Define this "point" by the symbol H, which forms the third open set of the Cartan topology. Now follow the procedure established in the preceding paragraph. Construct the closure of H as the union of the domains of support of H and $K = dH$. The construction forms the fourth open set of the Cartan topology. In four dimensions, the process stops, but for N > 4, the process may be continued.

Now consider the topological basis collection of open sets that consists of the subsets,

$$B = \{A,\ KCl(A),\ H,\ KCl(H)\} = \{A,\ A \cup F,\ H,\ H \cup K\} \tag{2.27}$$

The collection of all possible unions of these base elements, and the null set, $\emptyset$, generate the Cartan topology of open sets:

$$T(open) = \{X,\ \emptyset,\ A,\ H, A \cup F,\ H \cup K,\ A \cup H,\ A \cup H \cup K,\ A \cup F \cup H\}. \tag{2.28}$$

These nine subsets form the open sets of the Cartan topology constructed from the domains of support of the Pfaff sequence constructed from a single 1-form, A, in 4 dimensions. The compliments of the open sets are the closed sets of the Cartan topology.

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*These results with application were presented as a talk given in August, 1991, at the Pedagogical Workshop on Topological Fluid Mechanics held at the Institute for Theoretical Physics, Santa Barbara UCSB.

$$T(closed) = \{\emptyset,\ X,\ F \cup H \cup K,\ A \cup F \cup K, A \cup F,\ H \cup K, F \cup K, F,\ K\}. \quad (2.29)$$

From the set of 4 "points" $\{A, F, H, K\}$ that make up the Pfaff sequence it is possible to construct 16 subset collections by the process of union. It is possible to compute the limit points for every subset relative to the Cartan topology. The classical definition of a limit point is that a point p is a limit point of the subset Y relative to the topology T if and only if for every open set which contains p there exists another point of Y other than p [14]. The results of this and other standard definitions are presented in Table 2, and are to be compared to the simple topology of Table 1.

Table 2. The Cartan T4 Topology

A 1-form in 4D: $A = A_k(x)dx^k$

$X = \{A, \quad F = dA, \quad H = A\hat{}F, \quad K = F\hat{}F\}$

Basis subsets {A, $KCl(A)$, H, $KCl(H)$} $= \{A, A \cup F, H, H \cup K\}$

$T(open) = \{X,\ \emptyset,\ A,\ H, A \cup F,\ H \cup K,\ A \cup H,\ A \cup H \cup K,\ A \cup F \cup H\}$

$T(closed) = \{\ \emptyset, X, F \cup H \cup K, A \cup F \cup K,\ H \cup K,\ A \cup F,\ F \cup K,\ F, K\}$

(2.30)

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
σ	$d\sigma$	.	$\partial\sigma$	$\sigma \cup d\sigma$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
A	F	A	F	$A \cup F$
F	$\emptyset$	$\emptyset$	F	F
H	K	H	K	$H \cup K$
K	$\emptyset$	$\emptyset$	K	K
$A \cup F$	F	$A \cup F$	$\emptyset$	$A \cup F$
$A \cup H$	F, K	$A \cup H$	$F \cup K$	X
$A \cup K$	F	A	$F \cup K$	$A \cup F \cup K$
$F \cup H$	K	H	$F \cup K$	$F \cup H \cup K$
$F \cup K$	$\emptyset$	$\emptyset$	$F \cup K$	$F \cup K$
$H \cup K$	K	$H \cup K$	$\emptyset$	$H \cup K$
$A \cup F \cup H$	F, K	$A \cup F \cup K$	K	X
$F \cup H \cup K$	K	$H \cup K$	F	$F \cup H \cup K$
$A \cup H \cup K$	F, K	$A \cup H \cup K$	F	X
$A \cup F \cup K$	F	$A \cup F$	K	$A \cup F \cup K$
X	F, K	X	$\emptyset$	X

By examining the set of limit points so constructed for every subset of the Cartan system, and presuming that the functions that make up the forms are C2 differentiable (such that the Poincare lemma is true, $dd\omega = 0, any\ \omega$), it is easy to show that for all subsets of the Cartan topology every limit set is composed of the exterior differential of the subset, thereby proving the claim that the exterior differential is a limit point operator relative to the Cartan topology. For example, the open subset, $A \cup H$, has the limit points that consist of F and K. The limit set consists of $F \cup K$ which can be derived directly by taking the exterior differentials of the elements that make up $A \cup H$; that is, $(F \cup A = d(A \cup H) = (dA \cup dH)$. Note that this open set, $A \cup H$, does not contain its limit points. Similarly for the closed set, $A \cup F$, the limit points are given by F which may be deduced by direct application of the exterior differential to $(A \cup F) : (F) = d(A \cup F) = (dA \cup dF) = (F \cup \emptyset) = (F)$.

2.3.1 Topological Torsion and Connected vs. Non-connected Cartan topologies.

Topological torsion of a 1-form is defined as the exterior product of the 1-form and its exterior derivative. Topological torsion is different from, but can be related to, the Frenet torsion of a space curve and the affine torsion of a connection. If non-zero, Topological torsion has important topological properties. The Cartan topology as given in Table 2 is composed of the union of two sub-sets which are both open and closed,

$$(X = KCl(A) \cup KCl(H) = \{A \cup F\} \cup \{H \cup K\}), \tag{2.31}$$

a result that implies that the Cartan topology is not necessarily a connected topology. An exception exists if the topological torsion, H, and hence its closure, vanishes, for then the Cartan topology is connected. This extraordinary result has broad physical consequences. The connected Cartan topology based on a vanishing topological torsion is at the basis of most physical theories of equilibrium. In mathematics, the connected Cartan topology corresponds to the Frobenius integrability condition for Pfaffian forms. In thermodynamics, the connected Cartan topology is associated with the Caratheodory concept of inaccessible thermodynamic states [15], and the existence of an equilibrium thermodynamic surface. If the non-exact 1-form, Q, of heat generates a Cartan topology of null topological torsion, $H = Q\hat{}dQ = \emptyset$, then the Cartan topology built on Q is connected. Such systems are "isolated" in a topological sense, and the heat 1-form has a representation in terms of two and only two functions, conventionally written as: $Q = TdS$. Note again that a fundamental physical concept, in this case the idea of equilibrium, is a topological concept independent from geometrical properties of size and shape. Processes that generate the 1-form Q such that $Q\hat{}dQ = \emptyset$ are thermodynamically reversible. If $Q\hat{}dQ \neq \emptyset$, the process that generates Q is thermodynamically irreversible.

When the Cartan topology is connected, it might be said that all forces are extendible over the whole of the set, and that these forces are of "long range". Conversely when the Cartan topology is disconnected, the "forces" cannot be extended indefinitely over the whole domain of independent variables, but perhaps only over a single component. The components are not arc connected. In this sense, such forces are said to be of short range, as they are confined to a specific component. Note that this notion of short or long range forces does not depend upon geometrical size or scale. The physical idea of short or long range forces is a topological idea of connectivity, and not a geometrical concept of how far.

In an earlier article, these ideas were formulated intuitively in order to give an explanation of the "four forces" of physics. The earlier work was based upon experience with differential geometry [16]. The features of the Pfaff sequence were used to establish equivalence classes for 1-forms constructed from known example metric field solutions, $g_{\mu v}$, to the Einstein field equations. The original ideas, based upon experience with systems in differential geometry, can now be given credence based upon differential topology. The construction of a 1-form, $A = g_{\mu 4} dx^{\mu}$, whose coefficients are the space time components of a metric tensor, will divide the topology into equivalence classes depending upon the number of non-zero elements of its Pfaff sequence. This number has been defined above as the Pfaff topological dimension. Long range parity preserving forces due to gravity (Pfaff dimension 1) and electromagnetism (Pfaff dimension 2) are to be associated with a Cartan Topology that is connected ($H = A\hat{}F = A\hat{}dA = 0$). Both the strong force (Pfaff dimension 3) and the weak force (Pfaff dimension 4) are "short" range ($H \neq 0$) and are to be associated with a disconnected Cartan topology. The strong force is parity preserving ($K = 0$) and the weak force is not ($K \neq 0$). The fact that the Cartan topology is not necessarily connected is the topological (not metrical) basis that may be used to distinguish between short and long range forces.

In much of our physical experience with nature, it appears that the disconnected domains of Pfaff dimension 3 or more are often isolated as nuclei, while the surrounding connected domains of Pfaff dimension 2 or less appears as fields of charged or non-charged molecules and atoms. However, part of the thrust of this article is to demonstrate that such disconnected topological phenomena are not confined to microscopic systems, but also appear in a such mundane phenomena as the flow of a turbulent fluid. Physical examples of the existence of topological torsion (and hence a non-connected Cartan topology) are given by the experimental appearance of what appear to be coherent structures in a turbulent fluid flow.

To prove that a turbulent flow must be a consequence of a Cartan topology that is not connected, consider the following argument: First consider a fluid at rest and from a global set of unique, synchronous, initial conditions generate a vector field of flow. Such flows must satisfy the Frobenius complete integrability theorem, which requires that $A\hat{}dA = 0$. The Cartan topology for such systems is connected, and the Pfaff dimension of the domain is 2 or less. Such domains do not support topological torsion (the Helicity vanishes). Such globally laminar flows are to be distinguished from flows that reside on surfaces, but do not admit a unique set of connected sychronizeable initial conditions. Next consider turbulent flows which, as the anti-thesis of laminar flows, can not be integrable in the sense of Frobenius; such turbulent domains support topological torsion ($A\hat{}dA \neq 0$), and therefore a disconnected Cartan topology. The connected components of the disconnected Cartan topology can be defined as the (topologically) coherent structures of the turbulent flow.

Note that a domain can support a homogeneous topology of one component and then undergo continuous topological evolution to a domain with some interior holes. The domain is simply connected in the initial state, and multiply connected in the final state, but still connected. However, consider the dual point of view where the originally connected domain consists of a homogeneous space that becomes separated into multiple components. The evolution to a topological space of multiple components is not continuous. It follows that the case of a transition from an initial laminar state ($H = 0$) to the turbulent state ($H \neq 0$) is a transition from a connected topology to a disconnected topology. Therefore the transition to turbulence is NOT continuous. However, note that the decay of turbulence can be described by a continuous transformation from a disconnected topology to a connected topology. Condensation is continuous, gasification is not. It is demonstrated below that relative to the Cartan topology all C2 differentiable, $\mathbf{V}$, acting on C2 p-forms by means of the Lie differential are continuous. The conclusion is reached that the transition to turbulence must involve transformations that are not C2, hence can occur only in the presence of shocks or tangential discontinuities.

Chapter 3
CONTINUOUS TOPOLOGICAL EVOLUTION

3.1 Introduction

An objective of this article is to develop a theory of topological evolution that may be used to describe the irreversible evolution of dissipative non-conservative physical systems. The ideas will utilize topological concepts for it is postulated that a necessary condition for irreversible evolution involves topological change. The basis for such a postulate follows from the fact that if an evolutionary process is described by a map, Φ, between initial and final states, and if the map is not continuously reversible, then the observable topology of the final state is different from the observable topology of the initial state. Cartan's methods can be used to extend these concepts to the dynamics of physical systems that admit description in terms of exterior differential forms. It is remarkable that the mathematical development leads to recognizable thermodynamic features which permit the determination of classes of processes which are reversible or irreversible. For example, all Hamiltonian processes are thermodynamically reversible. An essential feature of irreversible processes is that they involve the evolution of what has been defined as Topological Torsion.

The observation of topological change, with the production and destruction of defects and holes, lines of self-intersection and other obstructions, will be the signature of topological irreversible evolution. Topological change can occur discontinuously as in a cutting process, or continuously, as in a pasting process. Such continuous but irreversible processes can be used to study the decay of turbulence, but not its creation. The production of disconnected components will be the signature of those discontinuous processes which are necessary to describe the creation and evolution of chaotic but perhaps reversible evolution, or turbulent, irreversible evolution. In this article, emphasis will be place upon those processes which are continuous, but not reversible.

Processes or maps that preserve topology are technically described as homeomorphisms [1]. Homeomorphisms are both continuous and reversible. Homeomorphic reversibility means that the inverse function, Φ^{-1}, must exist and must be continuous. Topological properties, such as orientability, compactness, connectivity, hole count, lines of self-intersection, pinch points, and Pfaff dimension are invariants of homeomorphisms, but geometrical properties such as size and shape are not necessarily invariants of homeomorphic deformations. In fact an elementary method of recognizing topological properties is to observe those properties that stay the same under continuous deformations that do not preserve size and shape.

The theory of Continuous Topological Evolution is developed herein in terms of physical systems that undergo certain thermodynamic processes. The physical system is assumed to be modeled in terms of the topological features inherent in Cartan's theory of exterior differential systems. The thermodynamic process will be defined in terms of a vector field, V, and its effect on the differential forms that make up the exterior differential system. The action of the process will be defined in terms of the Lie differential with respect to V acting on the differential forms that make up the exterior differential system, and which in turn approximate the physical system. The methods lead to concepts that are coordinate free and are well behaved in any reference system. A precise non-statistical definition of thermodynamic irreversibility will be stated, and a cohomological equivalent of the first law of thermodynamics will be derived and studied relative to the single constraint of continuous but irreversible topological evolution. Remarkably, many intuitive thermodynamic concepts can be stated precisely, without the use of statistics, in terms of the theory of continuous topological evolution based on the Cartan topology.

Given a topology on the final state and a map from an initial state to the final state it is always possible to define a topology on the initial state such that the given transformation, or even a given set of transformations, is continuous. However, the topologies of the initial and final states need not be the same; hence the map need not be reversible. Recall that with respect to a discrete topology all maps from the initial to final state are continuous, while relative to the concrete topology, only the constant functions are continuous [2]. A first problem of a theory of topological evolution is to devise a rule for constructing a topology that is physically useful and yet is neither too coarse nor too fine. Such a rule is necessary for the concept of continuity of an evolutionary transformation is defined relative to the topologies of the initial and final states. In this article the topological rules will be made by the specification of an exterior differential system that will model the physical system of interest. Many physical systems appear to be adequately modeled by 1-form of Action.

Physical exhibitions of continuous and discontinuous transformations can be achieved through the deformations of a soap film attached to a wire frame. For example, a soap film attached to a single closed, but double, loop of wire can be deformed from a non-orientable surface into an orientable surface continuously (the topological property of orientability is changed). That is, the soap film can be transformed continously from a Moebius band into a cylindrical strip. As another example, consider an initial state where a soap film is attached to two slightly separated but concentric circular wire loops. The resulting surface is a minimal surface of a single component. As the separation distance of the concentric rings forming the boundary of the soap film is slowly increased, the minimal surface is stretched until a critical separation is reached. Then, without further displacement, the surface spontaneously continues

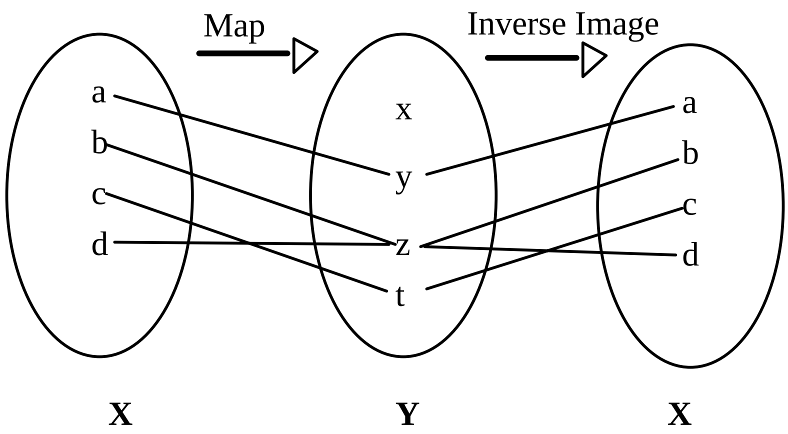

Figure 1

to deform to form a "two sheeted" cone connected at a singular vertex point. The surface separates at the conical singularity, and the two separate sheets of the cone continue to collapse to form a minimal surface of two components. The final state consists of two flat films attached, one each, to each ring. The originally connected minimal surface undergoes a topological (phase) change to where it becomes two disconnected (still minimal) surfaces. An example of this topological transition in the surface of null helicity density has been described in conjunction with the parametric saddle node Hopf bifurcation of a Navier-Stokes flow [3].

In this article the fundamental set, X, will be the points $\{x, y, z, t, ...\}$ that make up an N-dimensional space. Upon this fundamental set will be constructed arbitrary subsets, such as functions, tensor fields and differential forms. Many different topologies may be constructed on the fundamental set in terms of special classes of subsets that obey certain rules of logical closure. In fact the very existence of subsets can be used to define a course topology on X in terms of a topological base. The topological base consists of those subsets whose unions form a special collection of all possible subsets that is closed under logical union and intersection. This special collection of subsets will be defined as the open sets of a topology. The topological base can be used to define a topological structure. A space is said to have a topological structure if it is possible to determine if a transformation on the space is continuous [4].

3.2 Continuity

The classic definition [5] of a continuous transformation between a set X with topology $T1$ to a set Y with a topology $T2$ states that the transformation is continuous if and only if the inverse image of open sets of $T2$ are open sets of $T1$. This definition can be made transparent by use of a simple point set example.

Consider two sets of 4 points, an initial state, $\{a, b, c, d\}$ and a final state

$\{x, y, z, t\}$. Define an open set topology on the initial state $T1 = [X, \emptyset, a, ab, abc]$ and a open set topology on the final state $T2 = [X, \emptyset, x, y, xy, yzt]$. The transformation considered is exemplified by the Figure 1.

The open set (y) has a preimage (a) which is open. The open set (yzt) has a preimage (abc) which is open. Hence the Map is continuous. (The open sets that involve x are not included as the map does involve x.) However, the Inverse Image mapping is not continuous for the open set (ab) has a preimage as (yz) but (yz) is not an open set of Y. The point set example demonstrates the idea of a continuous but not homeomorphic mapping. The objective herein is to examine such maps in terms of exterior differential systems.

There exists another more useful method of defining continuity which does not depend explicitly on being able to define open sets and their inverse images. This second method of defining continuity is based on the concept of closure. The closure of a set can be defined in (at least) two ways:

1. The closure of a set is the union of the interior and the boundary of a subset.
2. The closure of a set is the union of the set and its limit points.

The first definition of closure is perhaps the most common, and is often exploited in geometric situations, where a metric has been defined and a boundary can be computed easily. The second definition of closure is independent from metric and is the method of choice in this article, both for defining continuity and establishing a topological structure. In terms of the concept of closure, a transformation is continuous if and only if for every subset, the image of the closure of the initial subset is included in the closure of the image of that subset [5]. Another way of stating this idea is

3. A map is continuous iff the limit points of every subset in the domain permute into the closure of the subsets in the range.

If a method for constructing a closure operator (a Kuratowski closure operator K of a subset relative to a topology) can be defined, then a strong version of continuity would imply that the Kuratowski closure operator commutes with those transformations which are continuous. The test for continuity would be to construct the closure of an arbitrary subset on the initial state, and then to propagate the elements of the closure to the final state by means of a transformation. If this result is the same as the result obtained by first propagating the subset to the final state by means of the transformation, and then constructing its closure on the final state, then the map is continuous. Note that such a procedure has defined a topological structure which will be exploited in this article, for the subsets of interest will be defined as a Cartan system of exterior differential forms, Σ, on X. The topological base defined by this class of sets is too course to be of interest. Hence the Cartan exterior derivative will be used to generate additional sets of forms, $d\Sigma$, which when adjoined to the initial system of forms defines the Kuratowski closure of the Cartan system as the system of forms, $K(\Sigma) = \{\Sigma \cup d\Sigma\}$.

The Cartan exterior product may be used as a convenient intersection operator between sets of differential forms. Starting from the system, $\{\Sigma\}$, the Cartan topology is then determined by the construction of the Cartan-Pfaff sequence, which consists of all possible intersections that may be constructed from the subsets of the closure of the differential system:

$$Pfaff\ \ Sequence: \ \{\Sigma, d\Sigma, \Sigma\hat{}d\Sigma, d\Sigma\hat{}d\Sigma, ...\}. \tag{3.1}$$

The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology will be constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures:

$$the\ \ Cartan\ \ topological\ base: \{\Sigma, K(\Sigma), \Sigma\hat{}d\Sigma, K(\Sigma\hat{}d\Sigma), ...\}. \tag{3.2}$$

With respect to a topological base constructed from a single 1-form of Action it has been shown [34] that the Cartan exterior derivative may be viewed as a closure or limit point operator. Given any subset of the Cartan topological space, the exterior derivative of that subset generates its limit points, if any. This is a remarkable result, for as will be demonstrated below, all C2 vector fields acting through the concept of the Lie differential on a set of differential forms, with C2 coefficients, generate continuous transformations with respect to the Cartan topology. Moreover, the Cartan topology is disconnected if $\Sigma\hat{}d\Sigma \neq 0$ is not zero.

3.3 Cartan's Topological Structure

An exterior differential 1-form A is deterministic, as a predictive (or retrodictive) invariant, with respect to all tensor diffeomorphisms. The coefficient functions, $A_\mu(x^\upsilon)$, are presumed to behave as a co-variant vector, and the differentials, dx^μ, behave as a contravariant vector, with respect to tensor diffeomorphisms. (Exterior differential form densities will be discussed later.) However, the exterior differential 1- form, and hence all p-forms, are also well behaved with respect to a larger class of transformations, which contain the tensor diffeomorphisms as special cases. The exterior differential 1- form is deterministic in a retrodictive sense (but not in a predictive sense) with respect to C1 mappings that do not have a local or a global inverse. These C1 mappings do not preserve all topological features, where diffeomorphisms of tensor theory, are special cases of homeomorphisms, which do preserve all topological properties. These extraordinary features of exterior differntial forms demonstrate that Cartan's theory is not just another formalism of fancy, and goes well beyond the theory of tensor analysis. In fact, these features of exterior differential forms can be exploited to develop something that has slipped through the net of tensor analysis: a non-statistical theory of thermodynamic irreversibility.

A topological structure is defined to be enough information to decide whether a transformation is continuous or not [18]. The classical definition of continuity depends upon the idea that every open set in the range must have an inverse image in the domain. This means that topologies must be defined on both the initial and final state, and that somehow an inverse image must be defined. Note that the open sets of the final state may be different from the open sets of the initial state, because the topologies of the two states can be different.

There is another definition of continuity that is more useful for it depends only on the transformation, and not its inverse, explicitly. A transformation is continuous if and only if the image of the closure of every subset is included in the closure of the image. This means that the concept of closure and the concept of transformation must commute for continuous processes. Suppose the forward image of a 1-form A is Q, and the forward image of the set $F = dA$ is Z. Then if the closure, $KCl(A) = A \cup F$ is included in the closure of $KCl(Q) = Q \cup dQ$, for all sub-sets, the transformation is defined to be continuous. The idea of continuity becomes equivalent to the concept that the forward image Z of the limit points, dA, is an element of the closure of Q [18]:

> A function that produces an image $f[A] = Q$ is continuous iff for every subset A of the Cartan topology, $Z = f[dA] \subset KCl(Q) = (Q \cup dQ)$.

The Cartan theory of exterior differential systems can now be interpreted as a topological structure, for every subset of the topology can be tested to see if the process of closure commutes with the process of transformation. For the Cartan topology, this emphasis on limit points rather than on open sets is a more convenient method for determining continuity. A simple evolutionary process, $X \Rightarrow Y$, is defined by a map Φ. The map, Φ, may be viewed as a propagator that takes the initial state, X, into the final state, Y. For more general physical situations the evolutionary processes are generated by vector fields of flow, $\mathbf{V}$. The trajectories defined by the vector fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this article is the Lie differential with respect to a vector field, $\mathbf{V}$, acting on differential forms, Σ [19].

The Lie differential has a number of interesting and useful properties.

1. The Lie differential does not depend upon a metric or a connection.

2. The Lie differential has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part:

$$L_{(\mathbf{V})}\omega = i(V)d\omega + di(V)\omega. \tag{3.3}$$

This formula is known as "Cartan's magic formula". For those vector fields V which are "associated" with the form ω, such that $i(V)\omega = 0$, the Lie differential becomes equivalent to the covariant differential of tensor analysis. Otherwise the two differential concepts are distinct.

3. The Lie differential may be used to describe deformations and topological evolution. Note that the action of the Lie differential on a 0-form (scalar function) is the same as the directional or convective differential of ordinary calculus,

$$L_{(\mathbf{V})}\Phi = i(V)d\Phi + di(V)\Phi = i(V)d\Phi + 0 = \mathbf{V} \cdot grad\Phi. \tag{3.4}$$

It may be demonstrated that the action of the Lie differential on a 1-form will generate equations of motion of the hydrodynamic type. In fact Arnold calls the Lie differential the "convective" or "Fisherman's" differential.

4. With respect to vector fields and forms constructed over C2 functions, the Lie differential commutes with the closure operator. Hence, the Lie differential (restricted to C2 functions) generates transformations on differential forms which are continuous with respect to the Cartan topology. To prove this claim:

First construct the closure, $\{\Sigma \cup d\Sigma\}$. Next propagate Σ and $d\Sigma$ by means of the Lie differential to produce the decremental forms, say Q and Z,

$$L_{(\mathbf{V})}\Sigma = Q \quad and \quad L_{(\mathbf{V})}d\Sigma = Z. \tag{3.5}$$

Now compute the contributions to the closure of the final state as given by $\{Q \cup dQ\}$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the final state, and the evolutionary process defined by $\mathbf{V}$ is continuous. However,

$$dQ = dL_{(\mathbf{V})}\Sigma = di(V)d\Sigma + dd(i(V)\Sigma) = di(V)d\Sigma \tag{3.6}$$

as $dd(i(V)\Sigma) = 0$ for C2 functions. But,

$$Z = L_{(\mathbf{V})}d\Sigma = d(i(V)d\Sigma) + i(V)dd\Sigma = di(V)d\Sigma \tag{3.7}$$

as $i(V)dd\Sigma = 0$ for C2 p-forms. It follows that $Z = dQ$, and therefore $\mathbf{V}$ generates a continuous evolutionary process relative to the Cartan topology. QED It is to be noticed that this concept of a topological structure is developed in terms of the action of the Lie differential acting on a 1-form. The method does not depend upon metric or connection.

Certain special cases arise for those subsets of vector fields that satisfy the equations, $d(i(\mathbf{V})\Sigma) = 0$. In these cases, only the functions that make up the p-form, Σ, need be C2 differentiable, and the vector field need only be C1. Such processes will be of interest to symplectic processes, with Bernoulli-Casimir invariants.

By suitable choice of the 1-form of action it is possible to show that the action of the Lie differential on the 1-form of action can generate the Navier Stokes partial differential equations [20]. The analysis above indicates that C2 differentiable solutions to the Navier-Stokes equations can not be used to describe the transition to turbulence. The C2 solutions can, however, describe the irreversible decay of turbulence to the globally laminar state.

3.4 The evolutionary process

An arbitrary evolutionary process, $X \Rightarrow Y$, is defined by a map Φ. The map, Φ, may be viewed as a propagator that takes the initial state, X, into the final state, Y. In this article the evolutionary processes to be studied are asserted to be generated by vector fields, $\mathbf{V}$. However, evolutionary vector fields need not be topologically constrained such that they are generators of a single paramenter group. In other words, kinematics without fluctuations is not imposed a priori. The local trajectories defined by the vector fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this article is the Lie differential with respect to a vector field ,$\mathbf{V}$, acting on differential forms, Σ [6]. The Lie differential has a number of interesting and useful properties.

1. The Lie differential does not depend upon a metric or a connection.
2. The Lie differential has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part:

$$L_{(\mathbf{V})}\Sigma = i(\mathbf{V})d\Sigma + di(\mathbf{V})\Sigma. \tag{3.8}$$

   Marsden [43] calls this Cartan's Magic Formula (see below).

3. The Lie differential may be used to describe deformations and topological evolution.
4. If the Lie differential of Σ is zero, then Σ is a (Bernouilli type) invariant along the flow trajectories generated by $\mathbf{V}$.
5. With respect to vector fields and forms constructed over C2 functions, the Lie differential commutes with the Kuratowski closure operator. Hence, the Lie differential generates transformations on differential forms which are continuous with respect to the Cartan topology.

For example, the action of the Lie differential on a 0-form (scalar function) is the same as the directional derivative of ordinary calculus,

$$L_{(\mathbf{V})}\varphi = i(\mathbf{V})d\varphi + 0 \Rightarrow \mathbf{V} \cdot grad\varphi. \tag{3.9}$$

3.4.1 The Covariant derivative vs. the Lie differential.

The covariant derivative of tensor analysis, and as used in General Relativity, is often defined in terms of isometric diffeomorphic processes (that preserve the differential line element) and can be used to describe rigid body motions and isometric bendings, but not deformations and shear processes associated with convective fluid flow. Another definition of the covariant derivative is based on the concept of a connection, such that the differential process acting on a tensor produces a tensor. The definition of the covariant derivative usually depends upon the additional structure (or constraint) of a metric or a connection placed on a given variety, while the Lie differential does not. As the Lie differential is not so constrained, it may be used to describe non-diffeomorphic processes for which the topology changes continuously. The covariant derivative is more or less avoided in this monograph as it hides certain topological features of evolutionary processes.

In the examples given below, it will be demonstrated that the action of the Lie differential on a 1-form of Action typically will generate hydrodynamic equations of motion. As mentioned above, the Lie differential is not the same as the classic metric dependent covariant derivative (based upon Christoffel symbols), or generalizations of the metric connection used in certain gauge or fiber bundle theories. The abstract reason is that the Lie differential satisfies the equations

$$L_{(f\mathbf{V})}\Sigma = f \cdot L_{(\mathbf{V})}\Sigma + df \hat{} \; i(\mathbf{V})\Sigma, \tag{3.10}$$

while the covariant derivative, $\mathcal{D}$, and its generalizations are constrained [12] such that the second term on the right vanishes:

$$\mathcal{D}_{(f\mathbf{V})}\Sigma = f \cdot \mathcal{D}_{(\mathbf{V})}\Sigma. \tag{3.11}$$

This latter equation is often interpreted by saying that f represents the action of some "group", and the covariant derivative is defined such that it commutes with the action of the group. The Lie differential is not limited to the constraint of a specified group. However, there may exist a special sub-class of vector fields relative to a specific differential form, Σ, that permit the Lie differential to be identified with a covariant derivative. This special class of vector fields are called associated vectors (relative to the exterior differential form Σ), and are defined by the equation,

$$\text{Class of associated vectors}: \quad i(\mathbf{V})\Sigma = 0. \tag{3.12}$$

Those vector fields that satisfy $i(\mathbf{V})d\Sigma = 0$ are defined as *extremal* vector fields relative to Σ, a term that comes from the calculus of variations and its close correspondence to evolution defined by the Lie differential.

$$\text{Class of extremal vectors}: \quad i(\mathbf{V})d\Sigma = 0. \tag{3.13}$$

Vector fields that are both extremal and associated are defined as characteristic vector fields.

$$\text{Class of characterisic vectors}: \quad i(\mathbf{V})\Sigma = 0 \quad \text{and} \quad i(\mathbf{V})d\Sigma = 0 \tag{3.14}$$

Characteristic vector fields admit propagating discontinuities, which form the precise definition of a signal in electromagnetism [Fock Luneberg].

3.4.2 The Lie differential and continuity

The first four properties of the Lie differential appear in the literature, but the extraordinary property that all C2 vector fields that propagate C2 differential forms in the manner of a convective flow (Lie differential) are continuous relative to the Cartan topology requires proof: Given Σ, first construct the closure, $\Sigma \cup d\Sigma$. Next propagate Σ and $d\Sigma$ by means of the Lie differential to produce the decremental or residue forms, say Q and Z,

$$L_{(\mathbf{V})}\Sigma = Q \quad and \quad L_{(\mathbf{V})}d\Sigma = Z. \tag{3.15}$$

Now compute the contributions to the closure of the final state as given by $Q \cup dQ$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the final state, and the evolutionary process defined by $\mathbf{V}$ is continuous. However,

$$dQ = dL_{(\mathbf{V})}\Sigma = di(\mathbf{V})d\Sigma + dd(i(\mathbf{V})\Sigma) \tag{3.16}$$

and

$$Z = L_{(\mathbf{V})}d\Sigma = (i(\mathbf{V})dd\Sigma) + di(\mathbf{V})d\Sigma. \tag{3.17}$$

The difference becomes

$$Z - dQ = (i(\mathbf{V})dd\Sigma) - dd(i(\mathbf{V})\Sigma). \tag{3.18}$$

The concept of continuity requires that $Z - dQ \Rightarrow 0,$ forming an exterior differential system. For vector fields and differential forms with coefficient functions that are twice differentiable, the continuity condition is always satisfied relative to the Cartan topology (the Poincare lemma states that $dd\omega = 0$ where ω is any differential p-form with C2 coefficients). Therefore, subject to the constraint of C2 differentiability, every vector field, $\mathbf{V}$, generates a continuous evolutionary process relative to the Cartan topology. The set $\{\Sigma, d\Sigma\}$ forms a differential ideal (closure) which is permuted into the differential ideal $\{Q, dQ\}$ by the action of the Lie differential with respect to $\mathbf{V}$. *QED.*

The Lie differential also can be used to make some sense out of certain classes of discontinuous evolutionary processes (which are not C2). For example, consider a vector field $\mathbf{V} = \rho\mathbf{v}$ where the support function, ρ, is not C2. Then, the action of the Lie differential produces the discontinuity or excess function,

$$Z - dQ = -dd(i(\rho\mathbf{v})\Sigma) = d(d\rho\hat{}(i(\mathbf{v})\Sigma) + d\rho\hat{}d(i(\mathbf{v})\Sigma). \qquad (3.19)$$

This equation is of use in the study of shock waves and other discontinuous processes.

Note that special situation arise when $(i(\mathbf{v})\Sigma) = 0$. Such special vector fields were defined above to be *associated* vector fields, and have the properties that the Lie differential has the same abstract form as the covariant derivative. It can be shown that for even dimensional symplectic manifolds, there is a unique vector direction field that satisfies $i(\mathbf{T})\Sigma = 0$ and $L_{(\mathbf{V})}\Sigma = \Gamma\ \Sigma$. This direction field will generate thermodynamically irreversible evolution, and is continuous if C0.

3.5 Topological Evolution

3.5.1 Evolutionary Invariants.

If the flow field generated by $\mathbf{V}$ acting on a Cartan system of forms satisfies the equations

$$L_{(\mathbf{V})}\Sigma = 0 \quad and \qquad (3.20)$$
$$L_{(\mathbf{V})}d\Sigma = 0. \qquad (3.21)$$

then, with respect to such evolutionary processes, the forms of the closure are said to be absolute invariants. It follows that each element that makes up the Cartan topological base is also invariant, such that the whole Cartan topology is invariant. As $\mathbf{V}$ is continuous, and the topology is preserved, those vector fields, $\mathbf{V}$, that satisfy the equations above must be homeomorphisms, and are reversible. In other words, $Q = 0$ and $dQ = 0$ are sufficient conditions that $\mathbf{V}$ be reversible.

However, for continuous transformations on the elements of the C2 Cartan topology the general equations of topological evolution become,

$$L_{(\mathbf{V})}\Sigma = Q \qquad (3.22)$$

and

$$L_{(\mathbf{V})}d\Sigma = dQ, \qquad (3.23)$$

from which it follows that

$$L_{(\mathbf{V})}\Sigma\hat{}d\Sigma = Q\hat{}d\Sigma + \Sigma\hat{}dQ \qquad (3.24)$$

and

$$L_{(\mathbf{V})}d\Sigma\hat{}d\Sigma = 2dQ\hat{}d\Sigma. \tag{3.25}$$

As these equations of continuous topological evolution imply that the elements of the topological base may not be constant, then specific tests must be made to determine what features of the topology are changing, if any. For if it can be determined that the topology is indeed modified by the evolutionary process, then the process generated by this class of vector fields, $\mathbf{V}$, is continuous, but need not be reversible.

When $dQ \neq 0$, the limit points are not invariants, and it would be natural to expect that the topology is not constant. However, even if Q is closed, such that $dQ = 0$, it may be true that Q contains harmonic components, such that DeRham cohomological classes of Σ are not evolutionary invariants. Even though the topology of the initial state is not the same as the topology of the final state (for the "hole" count of the initial state is not the same as the hole count of the final state) it is not necessarily true that such continuous processes are thermodynamically irreversible.

3.5.2 Deformation Invariants.

Consider the flow lines tangent to a given vector direction field, $\mathbf{V}(x, y, z, t...)$ that generates a dynamical system, $d\mathbf{r} - \mathbf{V}d\tau = 0$. By reparameterization, $\mathbf{V} \Rightarrow \beta(x, y, z, t...)\mathbf{V}$, the "speed" at which points move down the lines of flow can be changed, but the points that start on a particular flow line, remain upon the same flow line. Next consider a closed curve, Z1, intersecting the flow lines transversely for say $\tau = 0$. The flow lines that intersect Z1 form a "tube of trajectories" As τ increases to some value, say $\tau = 1$, the points of the closed curve appear to flow down the "tube of trajectories". The result of this convective evolution is to produce a new closed curve, Z2 Now choose another parameterizarion function β‘, which is equal to the original β at $\tau = 0$. The points that make up the closed curve Z1 now flow down the same tube of trajectories, but at $\tau = 1$ form a new closed curve $deformed\ Z2$ that may be considered as a deformation of the closed curve Z2.

Next consider the propagation by means of the Lie differential relative to the direction field, $\beta\mathbf{V}$, of the closed integral of a 1-form, $\int_{z1} A$. The integration chain $z1$ is defined as a 1 dimensional cycle, or a closed curve of points. The action of the Lie differential on the closed integral of action can be written, for arbitrary parameterization, as:

$$L_{(\beta\mathbf{V})}\int_{z1} A = \int_{z1} i(\beta\mathbf{V})dA + \int_{z1} di(\beta\mathbf{V})A = \int_{z1} i(\beta\mathbf{V})dA + 0 = \int_{z1} \beta i(\mathbf{V})dA, \tag{3.26}$$

It follows that if the term, $\beta i(\mathbf{V})dA$, is closed, such that $d(\beta i(\mathbf{V})dA) = 0$, then the Lie differential of the closed integral vanishes, and does not depend upon the choice of β. In such special cases, the closed integral may be viewed as a deformation invariant, and becomes a topological invariant property of the evolution.

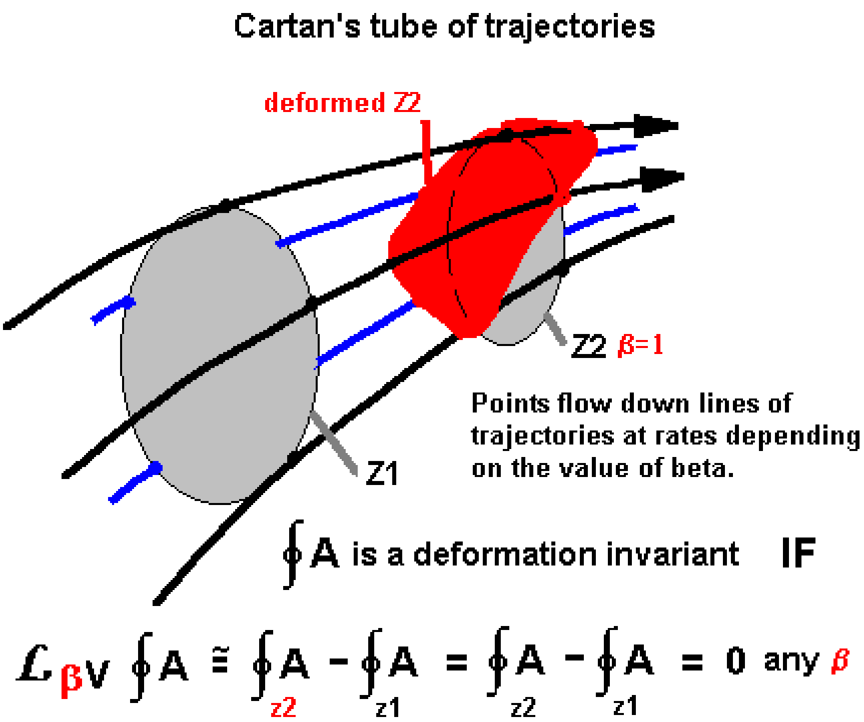
Cartan's tube of trajectories
deformed Z2
Z2 β=1
Z1
Points flow down lines of trajectories at rates depending on the value of beta.
∮A is a deformation invariant IF
$\mathcal{L}_{\beta V} \oint A \cong \oint_{z2} A - \oint_{z1} A = \oint_{z2} A - \oint_{z1} A = 0$ any β

Figure 2

The same arguments may be used to deduce topological properties of arbitrary p-forms. For example consider the 2-form $F = dA$. Then the Lie derivative of the closed integral of F becomes

$$L_{(\beta\mathbf{V})}\int_{z2} F = \int_{z2} i(\beta\mathbf{V})dF + \int_{z1} di(\beta\mathbf{V})F = \int_{z1} i(\beta\mathbf{V})ddA + 0 = 0. \tag{3.27}$$

The result (for C2 functions) is zero for any evolutionary vector field acting on the closed integral of a closed p-form (in this case, the 2-form F is exact.). Hence the closed integrals of closed p-forms are deformation invariants, or topological properties, of the evolutionary process generated by $\beta\mathbf{V}$. The values of the closed integrals (deRham period integrals) depend upon the integration chains, and have ratios which are rational. Cartan developed these methods to prove that the necessary and sufficient condition that a vector field have a Hamiltonian generator, was that the closed integral of the Action 1-form was a deformation invariant. [14]

3.6 Simple Systems

3.6.1 The Action 1-form and its Pfaff Sequence

Consider an arbitrary 1-form, A, on an n dimensional variety of independent functions. The exterior derivative of A produces a 2-form of closure points, $F = dA$, whose components are given by the expression, $F_{\mu\nu}dx^{\mu}\hat{}dx^{\nu}$. The combined set $\{A, F\}$ forms the closure of the set $\{A\}$. All possible intersections of the closure, $\{A, F, A\hat{}F, F\hat{}F...\}$, form what is defined herein as the Pfaff sequence for the domain $\{x, y, z, t\}$. In this article (for a 4 dimensional variety) these elements are defined as

$$\text{Topological ACTION} : A = A_{\mu}dx^{\mu} \tag{3.28}$$

$$\text{Topological VORTICITY} : F = dA = F_{\mu\nu}dx^{\mu}\hat{}dx^{\nu} \tag{3.29}$$

$$\text{Topological TORSION} : H = A\hat{}dA = H_{\mu\nu\sigma}dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\sigma} \tag{3.30}$$

$$\text{Topological PARITY} : K = dA\hat{}dA = K_{\mu\nu\sigma\tau}dx^{\mu}\hat{}dx^{\nu}\hat{}dx^{\sigma}\hat{}dx^{\tau}. \tag{3.31}$$

The union of all elements of the Pfaff sequence and their closures forms the elements of the Cartan topological base:

$$\{A, A \cup F, H, H \cup K...\}. \tag{3.32}$$

In order to take into account projective (and certain discontinuous) features, the vector fields of interest often will be scaled by a support function, ρ, such that $\mathbf{J} = \rho\mathbf{V}$. The fundamental equations of continuous evolution become

$$L_{(\rho\mathbf{V})}A = Q \tag{3.33}$$

$$L_{(\rho\mathbf{V})}F = dQ \tag{3.34}$$

$$L_{(\rho\mathbf{V})}H = Q\hat{}F + A\hat{}Q$$

$$L_{(\rho\mathbf{V})}K = 2(dQ\hat{}F) = 2d(Q\hat{}F) \tag{3.35}$$

Note that for the even dimensional elements of the Pfaff sequence, (F and K), the action of the Lie differential is to produce an exact form: dQ, for the Lie differential of F, and $2d(Q\hat{}F)$ for the Lie differential of K. As integrals of exact forms over closed cycles or boundaries of support vanish, then it is possible to formulate the first theorem.

Theorem 5 *All even dimensional Pfaff classes of p-forms, $dA = F, dA\hat{}dA = K$ ... are relative integral deformation invariants of continuous evolutionary processes relative to the Cartan topology.*

The closed integrals of F, K, ... are invariants of a continuous process as each integrand is exact, and the integral of an exact form over a closed domain vanishes. Hence if the functions are twice differentiable,

$$L_{(\rho\mathbf{V})}\int_{z2} F = \int_{z2}\{i(\rho\mathbf{V})dF + di(\rho\mathbf{V})F\} = \int_{z2} dQ \Rightarrow 0. \tag{3.36}$$

The closed integrals of $F, K, ...$ are invariants of any process generated by $\rho\mathbf{V}$ for integration domains, $z2$, that are boundaries or cycles.

This theorem is an extension of Poincare's theorem for even dimensional p-forms which are absolute integral invariants (the integration domain is not necessarily closed) with respect to the restricted set of Hamiltonian processes. It is important to realize that the theorem expresses the existence of (relative) integral deformation invariants (topologocal properties) with respect to processess that may be thermodynamically reversible or irreversible. It should be noted that the domains of support of the even dimensional Pfaff classes can not be compact without boundary.

3.6.2 The Action 1-form and fluctuations

For purposes of expose, the Cartan system, Σ, will be limited to a single 1-form of action, A, and perhaps a single pseudoscalar field, or N form density, ρ. The 1-form of Action, A, can be written in several equivalent formats known as the Cartan-Hilbert action:

$$A = A_\mu dx^\mu = \mathbf{p} \cdot d\mathbf{x} - \mathcal{H}(\mathbf{x}, \mathbf{v}, \mathbf{p}, t)dt = \mathcal{L}(\mathbf{x}, \mathbf{v}, t)dt + \mathbf{p} \cdot (d\mathbf{x} - \mathbf{v}dt) \tag{3.37}$$

The last representation indicates that the Action may be viewed abstractly in terms a Lagrangian function, $\mathcal{L}(\mathbf{x}, \mathbf{v}, t)$, and the kinematic fluctuations in position,

$$\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt). \tag{3.38}$$

It is to be noted that the usual assumption for physical systems is to assume that there are zero kinematic fluctuations. In this sense, kinematic perfection prevails:

$$\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0. \tag{3.39}$$

It is rarely appreciated that kinematic perfection is equivalent to an exterior differential system which imposes topological restrictions on the variety. For this example, the fluctuations, $\Delta\mathbf{x}$,are not presumed to be zero.

A simple count of the independent variables that are used to define the Cartan-Hilbert 1-form of action indicates that the "fluctuation" space is a variety of 3n+1⇒10 dimensions $(t, \mathbf{x}, \mathbf{v}, \mathbf{p})$. (For simplicity, the "particle" index n has been chosen to be unity). The coefficients, $\mathbf{p}$, act as Lagrange multipliers for the fluctuations, $\Delta\mathbf{x}$. However, it can be determined that the maximum Pfaff dimension of the sequence $\{A, dA, A\hat{}dA, dA\hat{}dA...)$ is of dimension 2n+2⇒8 and not dimension 10. Hence the 10 dimensional space is redundant, and an 8 dimensional space is adequate to describe the physical system in terms of a 1-form of Action. The given 1-form of Action therefore generates a non-compact symplectic manifold of dimension 8.

If the Lagrange multipliers $\mathbf{p}$ of the kinematic fluctuations $(d\mathbf{x} - \mathbf{v}dt)$ are restricted to be the canonical momenta, as defined by the ubiquitous formula, $\mathbf{p} = \partial\mathcal{L}/\partial\mathbf{v}$, the maximum Pfaff dimension is 7, forming a contact manifold historically defined as state space. If the Lagrange function $\mathcal{L}(\mathbf{x}, \mathbf{v}, t)$ is homogeneous of degree 1 in $\mathbf{v}$, then the maximal Pfaff dimension is 6, forming a symplectic Finsler manifold of dimension 6, the phase space of classical mechanics. This manifold cannot be compact without boundary.

If the contact manifold of dimension 7 is constrained by the equations of kinematic closure,

$$d(\Delta\mathbf{x}) = d(d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0, \tag{3.40}$$

then the space of interest becomes the configuration space of 4 dimensions, a submanifold of the original symplectic structure of 8 dimensions. The constraints of kinematic closure imply that the velocity field is expressible as functions of a single variable, t; $\mathbf{v} \Rightarrow \mathbf{v}(t)$. Note that the more severe constraint of kinematic perfection, $\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0$, implies that the maximal Pfaff dimension is 2, as in this case $A\hat{}dA = \mathcal{L}(\mathbf{x}, \mathbf{v}, t)dt\hat{}d\mathcal{L}(\mathbf{x}, \mathbf{v}, t)\hat{}dt = 0$. The Action defines a completely integrable 2 dimensional submanifold that, in this circumstance, is not compact without boundary. These concepts will be exploited in other examples given below.

3.7 Cohomology and the Evolution of Energy

3.7.1 Cartan's Magic Formula and the first law.

The evolutionary processes considered in this section are limited to processes defined by vector fields, $\rho\mathbf{V}$, and physical systems that are adequately modeld in terms of a 1-form of Action, A. The evolutionary equation(s) is defined in terms of Cartan's magic formula, which employs the Lie differential relative to $\rho\mathbf{V}$ acting on the 1-form A to produce a 1-form Q :

$$L_{(\rho\mathbf{V})}A = i(\rho\mathbf{V})dA + di(\rho\mathbf{V})A \Rightarrow Q \tag{3.41}$$

Define $i(\rho\mathbf{V})A$ as the function, U, and W as the inexact 1-form $i(\rho\mathbf{V})dA$:

$$\text{"internal energy"}: U = i(\rho\mathbf{V})A \tag{3.42}$$

and

$$\text{(Virtual) Work:} \quad W = i(\rho\mathbf{V})dA. \tag{3.43}$$

Then, formally, the Cartan magic formula becomes equivalent to the statement of cohomology: the difference between the inexact 1-form Q and the inexact 1-form W is a perfect differential, dU.

$$L_{(\rho\mathbf{V})}A = W + dU \Rightarrow Q. \tag{3.44}$$

Cartan's Magic formula, expressing the propagation of the 1-form of Action down the tube of trajectories generated by the vector field $\rho\mathbf{V}$, becomes the dynamical equivalent of the first law of thermodynamics, when the inexact 1-forms Q and W are interpreted as heat $= Q$ and work $= W$, respectively. These definitions are neither accidental nor whimisical, for it will be demonstrated below that they have utilization in many of the familiar formulas of classical physics.

Fundamentally, the Cartan magic formula is a topological law describing the evolution of energy. It is remarkable that the first law follows, without axiomatization, from the single and simple constraint that the 1-form of action, A, undergoes continuous topological evolution in terms of a dynamical system. It is also intuitively pleasing to see that the inexact 1-forms, Q and W, are defined in terms of a process. Elementary discussions of heat and work often emphasize the energy content of the first law, rather than the engineering idea that heat and work are related to processes.

Other authors have emphasized the topological foundations of thermodynamics [7], and from the time of Caratheodory have noted the connection to Pfaff systems [8]. However, these authors did not have access to, or did not utilize, the Cartan topology and DeRham cohomology. A remark by Tisza, "... the main content of thermostatic phase theory is to derive the topological properties of the sets of singular points in Gibbs phase space" [9], greatly stimulated the early developments of the theory presented in this article.

3.7.2 Thermodynamic processes

In thermodynamics, a reversible process is defined as a process for which the 1-form of heat, Q, admits and integrating factor, and an irreversible process is a process for which the 1-form of heat does not admit an integrating factor (of reciprocal temperature). [7]. This definition may be made precise in terms of Cartan's magic formula and the Frobenius theorem, for if the 1-form of heat, Q, does not admit an integrating factor then the three form, $Q\hat{}dQ$, does not vanish. However, for a given physical system defined in terms of a 1-form of Action, A, and its Pfaff sequence, those processes, $\mathbf{V}$, that satisfy the equation $L_{(\mathbf{V})}A\ \hat{}\ L_{(\mathbf{V})}dA = 0$ are reversible.

$$\text{Definition of an reversible process, } \mathbf{V}: \qquad L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA = Q\hat{}dQ = 0 \tag{3.45}$$

This precise definition of thermodynamic reversibility will be subsumed, and the cohomological equivalent of the first law of thermodynamics will be studied relative to the constraint of continuous reversible or continuous irreversible topological evolution. Many intuitive thermodynamic concepts can be stated precisely in terms of the theory of continuous topological evolution based on the Cartan topology. For example, those processes for which $L_{(\mathbf{V})}A = Q = 0$ are adiabatic.

$$\text{Definition of a local adiabatic process, } \mathbf{V}: \qquad L_{(\mathbf{V})}A = Q = 0. \tag{3.46}$$

As must be the case in thermodynamics, there is a fundamental difference between the 1-form W and the 1-form Q. From the definition $W = i(\rho\mathbf{V})dA$, it follows that

$$i(\rho\mathbf{V})W = i(\rho\mathbf{V})i(\rho\mathbf{V})dA \Rightarrow 0 \ (transversality) \tag{3.47}$$

This fact implies that the 1-form W must be constructed from first integrals, ϕ, of the flow V, or from transversal fluctuations in the kinematics:

$$W = d\phi + \mathbf{f} \circ (d\mathbf{x} - \mathbf{v}dt). \tag{3.48}$$

Although W can be included in the concept of Q, there are parts of Q that are not transformable into W. A precise difference between the 1-form of (virtual) work and the 1-form of heat can be established: the 1-form of work is necessarly transversal to the process, while the 1 form of heat is not. This issue is at the heart of the second law of thermodynamics. The argument is pleasing for it gives formal substance to the intuitive differences between the thermo-dynamic concepts of heat and work.

$$i(\mathbf{V})W = i(\mathbf{V})i(\mathbf{V})dA = 0 \qquad but \qquad i(\mathbf{V})Q = -dU \neq 0. \tag{3.49}$$

Continuous processes on isolated systems satisfy the (extremal) equations

$$W = i(\mathbf{V})dA = 0\,. \tag{3.50}$$

Continuous processes on closed but not isolated systems satisfy the (Helmholtz or symplectic) equations

$$dW = di(\mathbf{V})dA = 0\,. \tag{3.51}$$

Continuous processes on open systems satisfy the equations

$$W\hat{\ }dW \neq 0. \tag{3.52}$$

It should be noted that if a process is such that the closed integral of the Action 1-form is a deformation invariant (a topological property that is preserved) then $\beta W = \beta i(\mathbf{V})dA$ must be closed. Hence the process acting on the physical system must be such that the work 1-form satifies the integrability conditon, $W\hat{\ }dW = 0$. In general, a hierarchy of processes will be defined by the sequence Pfaff equivalence classes constructed from the 1-form opf Work, W:

$$\{W = i(\mathbf{V})dA = 0, \quad dW = di(\mathbf{V})dA = 0, \quad i(\mathbf{V})dW = 0, \quad di(\mathbf{V})dW = 0, ...\}. \tag{3.53}$$

All continuous processes may be put into equivalence classes as determined by the vector fields, V, that generate the flow. For example, for the 1-form, A, those vector fields that satisfy the transversal equation,

$$Associated : i(\rho\mathbf{V})A = 0 \tag{3.54}$$

are said to be elements of the associated class of vector fields relative to the form A. For such processes, the internal energy is zero.

Those vectors that satisfy the equations,

$$Extremal : i(\rho\mathbf{V})dA = 0 \tag{3.55}$$

are said to be elements of the extremal class of vector fields. For such processes, the virtual work vanishes, $W = 0$. It should be noted that the 2-form dA admits a unique extremal vector only on spaces of odd dimensions, a 2n+1 dimensional state space which is defined as a contact manifold. If the Pfaff dimension of the 1-form A is 4, then a unique extremal vector does not exist. The domain is a symplectic manifold of even dimension. However, on the symplectic manifold it then follows that there does exists a unique vector field, the Torsion vector described above, but no extremal vector.

Vectors which are both extremal and associated are said to be elements of the characteristic class of vector fields [10].

$$Characteristic : i(\rho\mathbf{V})A = 0 \quad and \quad i(\rho\mathbf{V})dA = 0 \tag{3.56}$$

Note that characteristic flow lines generated by V of the Characteristic class preserve the Cartan topology, for each form of the Cartan topological base is invariant with respect to the action of the Lie derivative relative to characteristic flows. Characteristics are often associated with wave phenomena.

3.7.3 Thermodynamic Irreversibility and the Torsion vector.

It is important to realize that Q represents the inexact 1-form of heat, and its integral is the measurable quantity. . When $Q\hat{}dQ \neq 0$, then the heat 1-form is said to be non-integrable. The implication is that there does not exist an integrating factor for Q. Recall that classical thermodynamics states that a process that creates a heat 1-form which does not admit an integrating factor is thermodynamically irreversible. Hence, given a physical system described in terms of a 1-form of Action, A, it is possible, for a given process, to compute Q and dQ. If $Q\hat{}dQ \neq 0$, then that process $\mathbf{V}$ is thermodynamically irreversible. It is also possible to solve for those processes V that are thermodynamically irreversible when applied to a specified physical system.

$$\text{Definition of an irreversible process, } \mathbf{V}: \qquad L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA = Q\hat{}dQ \neq 0 \tag{3.57}$$

Given an Action 1-form in 4D, construct the 3 form of Topological Torsion, $H = A\hat{}dA$. Then there exists a vector field $\mathbf{T}$ such that $i(\mathbf{T})dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA$.

This vector field $\mathbf{T}$ is defined as the Topological Torsion vector. The properties of the Topological Torsion vector are such that

$$i(\mathbf{T})dA = \Gamma A, \quad and \quad i(\mathbf{T})A = 0. \tag{3.58}$$

Such vector fields are said to be homogeneous and associated relative to the 1-form A. The evolution of the physical system defined by A, in the direction of $\mathbf{T}$, yields the formula

$$L_{(\mathbf{T})}A = \Gamma A = Q, \quad and \quad L_{(\mathbf{T})}dA = dQ = d\Gamma\hat{}A + \Gamma dA \tag{3.59}$$

The heat 3-form becomes

$$Q\hat{}dQ = L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = \Gamma^2 A\hat{}dA \tag{3.60}$$

If the physical system admits Topological torsion, then $A\hat{}dA$ is not zero. Hence if the coefficient Γ^2 is not zero then the Heat 3-form $Q\hat{}dQ$ is not integrable and the process is thermodynamically irreversible. Examples will show that Γ is equal to the coefficient of the Topological Parity 4 form, $dA\hat{}dA$. When $dA\hat{}dA$ is not zero, such that Γ is not zero, the Torsion vector is uniquely defined, for then the coefficients of the 2-form $F = dA$ form an anti-symmetric matrix with an inverse. The physical system is said to define a symplectic 4D manifold. The conclusion is that thermodynamic irreversibility is an artifact of 4 dimensions [42].

It is important to realize that Q represents the inexact 1-form of heat, and its integral is the measurable quantity. When $L(V)A = Q = 0$ then the topological evolution process is defined to be an adiabatic process. When $L(V)Q = R \neq 0$, the process is defined to be radiative. When $Q\hat{}dQ \neq 0$, then the heat 1-form is said to be non-integrable. The implication is that there does not exist an integrating factor for Q. However, classical thermodynamics states that a process that creates a heat 1-form which does not admit an integrating factor is thermodynamically irreversible. Hence, given a physical system described in terms of a 1-form of Action, A, it is possible for a given process to compute Q and dQ. If $Q\hat{}dQ = 0$ then the process acting on the specified physical system is thermodynamically reversible. If $Q\hat{}dQ \neq 0$, then that process is thermodynamically irreversible. Given a specific physical system, it is also possible to solve for those processes V that are thermodynamically irreversible. Note that the same process acting on a different physical system need not be irreversible. Examples of this idea and its expression in terms of the Topological Torsion 3-form will be given below.

3.8 Continuous Processes

3.8.1 Closed Continuous Processes.

The continuous processes are naturally divided into two main categories: those for which $dQ = 0$ (closed processes) and those for which $dQ \neq 0$ (open processes). Closed flows also will be defined as uniformly continuous flows, to distinguish them from open flows, which are also continuous relative to the C2 constraint: Closed processes also will be defined as uniformly continuous processes, to distinguish them from open flows, relative to the C2 constraint. Therefore, relative to the Cartan Topology,

$$Closed\,process : L_{(\mathbf{V})}dA = dQ = 0 \tag{3.61}$$

defines a uniformly continuous closed process, while

$$Open\,process : L_{(\mathbf{V})}dA = dQ \neq 0 \tag{3.62}$$

defines an open process. Flow in the direction of the Torsion vector is an open flow.

Uniform continuity implies that the limit sets are invariant. Continuity only requires that the limit points permute amongst themselves. For example a fold into pleats which are then pasted together is a processes that rearranges the limit points and is not therefor uniformly continuous. Hence uniform continuity is a more constrained situation. When $dQ = 0$, it is possible to formulate immediately the following theorem (Poincare) for closed flows:

Theorem 6 *All even dimensional Pfaff classes of p-forms, $dA = F, dA\hat{}dA = K, ...$ are invariants of evolutionary processes that satisfy $L_{(\mathbf{V})}(dA) = dQ = 0$ relative to the Cartan topology. The forms $F, K, ...$ form a set of absolute integral invariants with respect to uniformly continuous processes.*

The difference between Theorem 1 and Theorem 2 is that in Theorem 2, the integration chains need not be closed. The proof of the theorem follows immediately by application of the Leibniz rule, using the constraint, $dQ = 0$:

$$L_{(\mathbf{V})}(dA\hat{}dA\hat{}dA..\hat{}dA) = \text{integer} \times \{L_{(\mathbf{V})}(dA)\}\hat{}dA\hat{}dA..\hat{}dA) = 0. \tag{3.63}$$

The integrands of the selected integrals are local invariants and so are their convected integrals.

The first application of theorem II gives,

$$L_{(\mathbf{V})}(dA) = L_{(\mathbf{V})}F = 0 \tag{3.64}$$

which is the equivalent of Helmholtz' theorem [14]. The theorem often is interpreted as the local conservation of angular momentum per unit moment of inertia, or the conservation of Topological Vorticity.

The second application of theorem II gives:

$$L_{(\mathbf{V})}(dA\hat{}dA) = L_{(\mathbf{V})}F\hat{}F = L_{(\mathbf{V})}K = 0 \tag{3.65}$$

which leads to the local conservation of Topological Parity, with respect to uniformally continuous flows.

In general,

$$L_{(\mathbf{V})}(dA\hat{}dA\hat{}...dA) = 0 \tag{3.66}$$

which expresses the invariance of a 2N dimensional area with respect to uniformally continuous flows.

3.8.2 Continuous Hydrodynamic Processes

Consider the domain of four independent variables of space time, $\{x, y, z, t\}$, and the three form of topological torsion

$$H = A\hat{}dA = A\hat{}F = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.67}$$

The continuous evolution of this 3-form is determined relative to an arbitrary process, $\mathbf{V}_4 = [\mathbf{V}, 1]$, by the equation:

$$L_{(\beta\mathbf{V}_4)}H = L_{(\beta\mathbf{V}_4)}(A\hat{}dA) = i(\beta\mathbf{V}_4)dH + di(\beta\mathbf{V}_4)H = Q\hat{}F + A\hat{}dQ \tag{3.68}$$

For local invariance of the 3-form with respect to arbitrary parameterizations, the evolutionary vector $\beta\mathbf{V}_4$ must be collinear with the topological torsion vector $(\mathbf{T}_4)$ such that the term $i(\beta\mathbf{V}_4)H \Rightarrow 0$. This constraint implies that the three form H then must be of the format:

$$H = A\hat{}F \approx \rho(x, y, z, t)(dx - V^x dt)\hat{}(dy - V^y dt)\hat{}(dz - V^z dt) = \rho \;\; i(\mathbf{V}_4)dx\hat{}dy\hat{}dz\hat{}dt \tag{3.69}$$

The invariance of the 3-form H then requires that a function $\rho(x, y, z, t)$ exist such that $dH \Rightarrow 0$. But this constraint becomes the equivalent of the famous hydrodynamic equation of continuity:

$$dH = \{div_3\, \rho\mathbf{V} + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt \Rightarrow 0 \tag{3.70}$$

which is interpreted physically as the conservation of mass. The implication is that those vector fields, $\beta\mathbf{V}_4$, that define a continuous hydrodynamic current, need not satisfy necessarily the formulas of topological kinematic constraint, $d\mathbf{x} - \mathbf{V}dt = 0$, but instead must be collinear with the topological torsion vector, $\mathbf{J}_4 = \lambda(x, y, z, t)\mathbf{T}_4$, if it exists. The important idea is that local deformable conservation of mass is to be associated with the conservation of the 3-form of Topological torsion as an absolute evolutionary invariant.

These results are to be compared with the even dimensional Poincare absolute integral invariants [12] for the more restrictive case of Hamiltonian (extremal) evolution of a Hamiltonian action,

$$A = A_\mu dx^\mu = \mathbf{p} \cdot d\mathbf{x} - H(\mathbf{x}, \mathbf{p}, t)dt \tag{3.71}$$

on a 2N+1 dimensional state space. It is the result (8.4) which is interpreted in statistical mechanics as the invariant area of phase space with respect to extremal, or Hamiltonian, evolution. The fact of the matter is that uniform continuity alone produces a set of absolute integral invariants for any action, in Hamiltonian format or not. Hamiltonian extremal flows satisfy the equation $dQ = 0$, and are therefore uniformly continuous, but they are not the only flows that satisfy this constraint. The invariance of "phase space area" is a consequence of uniform continuity alone, and does not require the additional constraints of constant homogeneity that limit the set of continuous flows to that subset of continuous vector fields which are extremal, and Hamiltonian.

3.8.3 DeRham categories of Closed Vector Fields

DeRham's cohomology theory [13] may be used to classify p-forms, and such ideas may be applied to the 1-form W defined by $W = i(\rho\mathbf{V})F$. Correspondingly, the vector fields that are used to construct the 1-forms W of virtual work perit processes to be put into the following categories, depending on whether the virtual work, W, is null, exact, closed, or not closed with respect to exterior differentiation. These categories are defined as:

Closed Flows $Categories\ for\, Q - W = dU$	$W = i(\rho\mathbf{V})F$	Q	dW	dQ
$Hamiltonian - extremal$	0	dU	0	0
$Bernoulli - Eulerian$	$d\Theta$	$d(U + \Theta)$	0	0
$Helmholtz - Symplectic$	$d\Theta + \gamma$	$d(U + \Theta) + \gamma$	0	0
Open Flows $Navier - Stokes - Torsion$	$arbitrary$	$arbitrary$	$dW \neq 0$	$dQ \neq 0$

(3.72)

The Bernoulli-Casimir functions, Θ, must be first integrals as in general,

$$i(\mathbf{V})W = i(\mathbf{V})d\Theta = 0. \tag{3.73}$$

For closed flows the first law insures that the 1-form W is closed, $dW = dQ = 0$, but W need not be exact and may contain harmonic components. That is, the 1-form W is not necessarily representable over the variety x, y, z, t in terms of the gradient of a single scalar function. The classic example of a non-exact 1-form is given by the expression,

$$\Gamma = \sigma_z(ydx - xdy)/(x^2 + y^2) \tag{3.74}$$

for which $d\Gamma = 0, but \int_{z1} \Gamma = 2\pi\sigma_z$. The coefficient σ_z is assumed to be a constant. Such forms, Γ, generate period integrals and the DeRham cohomology classes. The number of independent forms of the type given by equation (20) determine the Betti numbers of a variety for which the singular point (at the origin in the example) has been excised. The Betti numbers can be interpreted as a method for counting the number of holes or handles in the variety. It is these contributions to the general differential form that carry topological information about the domain of support. The duals to these forms are also closed, leading to the definition, harmonic forms.

From the first law the harmonic contributions to W are equal to the harmonic contributions to Q. If the harmonic contributions to Q are not zero, then the number of "holes and handles" in the Cartan topology of the final state is different from the number of holes and handles in the Cartan topology of the initial state, and the evolutionary process is continuous but not reversible.

In order to make (20) transversal, use the Cartan trick of substituting $dx^i - V^i dt$ for each dx^i. The transversal harmonic form becomes

$$\Gamma = \sigma_z\{ydx - xdy + (\mathbf{r} \times \mathbf{V})_z dt\}/(x^2 + y^2) \tag{3.75}$$

which demonstrates the close relationship to transversal harmonic forms and angular momentum. The format may be extended to a spin vector of components

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3] = [\sigma_x/(y^2 + z^2), \sigma_y/(z^2 + x^2), \sigma_z/(x^2 + y^2)] \tag{3.76}$$

such that the harmonic form becomes

$$\Gamma = \boldsymbol{\sigma}_1(zdy - ydz) + \boldsymbol{\sigma}_2(xdz - zdx) + \boldsymbol{\sigma}_3(ydx - xdy) + (\boldsymbol{\sigma} \circ \mathbf{r} \times \mathbf{V})dt. \tag{3.77}$$

The last term is recognized as a "spin orbit" coupling term. The idea of harmonic contributions to a 1-form is closely related to the concept of a complex number or ordered pair representation; i.e., the form cannot be represented by a map to a space of 1 dimension. Other formats for harmonic 1-forms are given by the expressions:

$$\Gamma = \{\phi d\chi - \chi d\phi\}/(a\phi^p + b\chi^p)^{2/p}, \tag{3.78}$$

where ϕ and χ are arbitrary functions on the base space, and for the complex function, ψ,

$$\Gamma = \{\psi d\psi^* - \psi^* d\psi\}/(\psi^*\psi). \tag{3.79}$$

The last representation of a harmonic form is in the format of the "probability current" of quantum mechanics, and gives a clue as how to adapt the formalism of this article to quantum systems. Such a development is deferred to a later article.

For closed flows on space time, the fundamental equations of evolution are given by the expressions for the odd 1-form and the odd 3-form. The even forms are invariant. The two fundamental equations of uniformly continuous evolution are:

$$L_{(\rho\mathbf{V})}A = Q \ and \tag{3.80}$$
$$L_{(\rho\mathbf{V})}H = Q\hat{}F \tag{3.81}$$

It should be remarked that if the 1-form of Action, A, is completely integrable in the sense of Frobenius, then the 3-form H is evanescent, and the evolutionary equation for H has no applicability. Such evolutionary processes ($H = 0$) are the equivalent to laminar flows in fluid dynamics and completely integrable, non-chaotic, Hamiltonian systems. It is known that if a Lagrangian system is not chaotic, then the action, A, is reducible to two variables (or less), and the 3-form H is necessarily zero. However when there exists a sense of helicity in the evolutionary process, or chaos is present, then the formula for H describes the appropriate topological evolution.

The first expression (3.80) may be put into correspondence with the evolution of energy, while the second fundamental equation (3.81) may be described as the evolution of complexity, or perhaps better as the evolution of defects, links, knots, or in abstract terms, the evolution of an entropic concept. If the heat 1-form Q is zero, then the evolutionary process is adiabatic, and topology is preserved. However, as the Cartan topology is not connected when $H \neq 0$, then continuous evolution of H can be accomplished only between connected subsets. The transition from a connected topology with $H = 0$ to a disconnected topology with $H \neq 0$ can only take place via a discontinuous transformation. The idea is that the continuous rate of change of H is definite (and arbitrarily taken to be positive). This feature is one of the key properties of entropy. Entropy can never change its sign. The creation of topological torsion, H, is a discontinuous process from a state of zero topological torsion, but once created, the growth (or decay) of H can be described by a continuous process (relative to the Cartan topology). These entropic features of the topological torsion 3-form will be useful in the description of the transition to turbulence.

3.8.4 The Hamiltonian Sub-Category

It should be remarked, that Cartan has proved, on a domain of dimension 2n+1, that if

$$i(\mathbf{V})F = W = 0, \qquad Q = dU \tag{3.82}$$

for any reparametrization, ρ, then $\mathbf{V}$ generates a Hamiltonian system, and visa versa [14]. This remarkable result indicates that Hamiltonian flows are not only continuous, but preserve many topological properties. The 1-form Q must be exact for Hamiltonian flows. Hence the observable holes and handles are topological invariants of Hamiltonian flows, as the ρ terms vanish. However, the fact that Q is exact for Hamiltonian flows does not completely establish a proof that Hamiltonian systems preserve all topological properties of the Cartan topology.

In the calculus of variations, vector fields that satisfy (21) are defined as extremal vector fields. Characteristic vector fields are a subclass of extremal fields that satisfy the equations

$$L_{(\mathbf{V})}A = 0 \textit{ and} \quad (3.83)$$
$$L_{(\mathbf{V})}F = 0. \quad (3.84)$$

In other words, continuous characteristics preserve the Cartan topology ($Q = 0$ and $dQ = 0$). Characteristic Hamiltonian vector fields generate waves in systems that can be endowed with the additional structure of a metric.

3.8.5 The Bernoulli-Euler subcategory

The Bernoulli-Euler category is not quite Hamiltonian. W is not zero, but must be a perfect differential, $W = d\Theta$. However, this perfect differential must be a first integral in order to satisfy the transversality condition, $i(\rho\mathbf{V})W = 0$. The 1 form Q is not necessarily so constrained. The abstract flows of this category are to be compared with the equations of motion of a compressible Eulerian fluid in which there may be stratification. If the pressure, P, is a function of the density, ρ, alone, then the Eulerian flow can be reduced to a Hamiltonian system [15]. If the exists some anisotropy due to stratification, then the Hamiltonian reduction is not perfect. Note that the first integral, Θ, acts as a Bernoulli constant along a given streamline, but the constant can vary from streamline to streamline because the function is transversal.

3.8.6 The Stokes subcategory

The Stokes category admits topological evolution in the sense that the harmonic contributions to W are not null, and therefore the "hole and handle" count of the Cartan topology is changing in an evolutionary manner. Such closed flows are not reversible. Note that all closed flows preserve topological vorticity and topological parity, and so if the flow is without vorticity in the initial state, then the flow is without vorticity in the final state. The Pfaff dimension [16] remains less than 2. However, if the initial state has vorticity, that vorticity will be preserved, but the Topological Torsion 3-form can change. In fact the Topological Torsion 3-form could be non-zero in the initial state, and zero in the final state, for the decay rate of topological torsion is proportional to $Q\hat{}F$ (See Figure 6). Both the 1-form of action and its hole count, and the 3-form of Topological Torsion, and its twisted handle count, are not necessarily invariants of a Stokes flow.

A method of distinguishing between "holes and twisted handles" is of some interest. Note that physically a handle can be constructed by deforming the rims of two holes in a surface into tubes and pasting the tubular ends together. If the rims are twisted by half integer or integer multiples of pi before the ends are glued together, then the handles have torsion (see Figure 7). Note that a handle cannot be constructed in the plane, so it is an intrinsically 3-dimensional thing. If the 3-form H vanishes, then there are no handles in the initial state, and as the Hamiltonian evolution produces no more new holes, there can be no more new handles in a Hamiltonian flow. However, existing handles may become twisted or knotted, because $Q\hat{}F \neq 0$, even for Hamiltonian flows. These facts correspond to the physical result that Hamiltonian systems are not dissipative and preserve energy, but that does not mean that entropy must be conserved.

It should be noted that for all closed flows, $dW = 0$. It follows that for closed flows, the transversality condition $i(\rho\mathbf{V})W = 0$ implies that the 1-form of virtual work W is an absolute invariant of the flow :

$$Closed\ Flows: \quad L_{(\rho\mathbf{V})}W = 0. \tag{3.85}$$

3.8.7 The Navier-Stokes category of open flows

It should be noted that the 1-form Q may be use to construct the Pfaff sequence, $\{Q, dQ, Q\hat{}dQ, dQ\hat{}dQ\}$,and generates another Pfaff dimension depending upon the rank or class of the elements of the Pfaff sequence for Q. For closed flows, $dQ = 0$ and the Pfaff dimension generated by Q is 1. The bulk of this article is devoted to closed flows. For open flows, $dQ \neq 0$, but the Pfaff sequence demonstrates that the topological features of open flows can have various levels of complexity. For example, the criteria that the Pfaff dimension of Q be 2 or less is equivalent to the Frobenious integrability constraint, $Q\hat{}dQ = 0$. This is precisely the Caratheodory condition that there exist "inaccessible paths" [17], and that (on a simply connected neighborhood) the 1-form of heat be representable as, $Q = TdS$. The topological evolution theory presented herein permits an analysis to be made for non-equilibrium processes, where the heat 1-form is not of the equilibrium monomial format, $Q \neq TdS$.

For the Navier-Stokes flow, the key feature is that $dQ = dW \neq 0$, but it still must be true that W is transversal. Therefore the 1-form W must be constructed from fluctuations, in the format,

$$W = f(dx - \mathbf{V}dt) + \text{closed additions transversal to}\mathbf{V}. \tag{3.86}$$

For open flows W is no longer a flow invariant. In the examples below, a particular choice is made for f which will generate the Navier-Stokes equations, which may have equilibrium or non-equilibrium solutions.

3.8.8 The Kinematic Topological Base

For continuous evolution in space-time, the key idea is that the exterior differential system consists of a Pfaff sequence constructed from a single 1-form of Action A, plus (perhaps) some additional constraints defining a domain of support and its boundary. The work of Arnold (and others) [18] has established that the singular points (zero's) of a global 1-form carry topological information. This idea is to be extended to the singular points of all elements of the Pfaff sequence, or topological base. In Appendix A, the idea of how a global 1-form of Action, A, existing on a space of dimension N+1 can be put into correspondence with a line bundle on a variety of dimension N is worked out in detail. The key features are that the Jacobian matrix of the projectivized 1-form of Action carries most of the information about the subspace. The trace and determinant of the Jacobian matrix determine the mean and Gaussian curvature of the subspace. The anti-symmetric components of the Jacobian are the functions that make up the 2-form, $F = dA$. The polynomial powers of F form the Chern classes for the line bundle.

For continuous transformations on a variety of $\{x, y, z, t\}$, the Cartan Action, A, can be defined kinematically as:

$$A = \sum_1^3 \mathbf{v}_k dx^k - \mathcal{H}dt, \tag{3.87}$$

where the "Hamiltonian" function, $\mathcal{H}$, is defined as,

$$\mathcal{H} = \mathbf{v} \bullet \mathbf{v}/2 + \int dP/\rho \tag{3.88}$$

Substitute this 1-form into the constraint equation given by **??**. Carry out the indicated operations of exterior differentiation and exterior multiplication to yield a system of necessary partial differential equations yields of the form,

$$\partial \mathbf{v}/\partial t + grad(\mathbf{v} \bullet \mathbf{v}/2) - \mathbf{v} \times curl\mathbf{v} = -gradP/\rho. \tag{3.89}$$

These equations are exactly the Euler partial differential equations for the evolution of a perfect fluid.

By direct computation, the 2-form $F = dA$ has components,

$$F = dA = \omega_z dx\hat{}dy + \omega_x dy\hat{}dz + \omega_y dz\hat{}dx + a_x dx\hat{}dt + a_y dy\hat{}dt + a_z dz\hat{}dt, \tag{3.90}$$

where by definition

$$\boldsymbol{\omega} = curl\ \mathbf{v}, \quad \mathbf{a} = -\partial\mathbf{v}/\partial t - grad\mathcal{H} \tag{3.91}$$

These vector fields always satisfy the Poincare-Faraday induction equations, $dF = ddA = 0$ for C2 functions, or,

$$curl\ \mathbf{a} - \partial\boldsymbol{\omega}/\partial t = 0, \quad div\boldsymbol{\omega} = 0. \tag{3.92}$$

The 3-form of Helicity or Topological Torsion, H, is constructed from the exterior product of A and dA as,

$$H = A\hat{}dA = H_{ijk}dxi\hat{}dxj\hat{}dxk \tag{3.93}$$
$$= -\mathbf{T}_x dy\hat{}dz\hat{}dt - \mathbf{T}_y dz\hat{}dx\hat{}dt - \mathbf{T}_z dx\hat{}dy\hat{}dt + hdx\hat{}dy\hat{}dz, \tag{3.94}$$

where $\mathbf{T}$ is the fluidic Torsion axial vector current, and h is the torsion (helicity) density:

$$\mathbf{T} = \mathbf{a} \times \mathbf{v} + \mathcal{H}\boldsymbol{\omega}, \qquad h = \mathbf{v} \bullet \boldsymbol{\omega} \tag{3.95}$$

The Torsion current, $\mathbf{T}$, consists of two parts. The first term represents the shear of translational accelerations, and the second part represents the shear of rotational accelerations. The topological torsion tensor, H_{ijk} , is a third rank completely anti-symmetric covariant tensor field, with four components on the variety $\{x, y, z, t\}$.

The Topological Parity becomes

$$K = dH = dA\hat{}dA = -2(\mathbf{a} \bullet \boldsymbol{\omega})dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.96}$$

This equation is in the form of a divergence when expressed on $\{x, y, z, t\}$,

$$div\mathbf{T} + \partial h/\partial t = -2(\mathbf{a} \bullet \boldsymbol{\omega}), \tag{3.97}$$

and yields the helicity-torsion current conservation law if the anomaly, $-2(\mathbf{a} \bullet \boldsymbol{\omega})$, on the RHS vanishes. It is to be observed that when $K = 0$, the integral of K vanishes, which implies that the Euler index, χ, is zero. It follows that the integral of H over a boundary of support vanishes by Stokes theorem. This idea is the generalization of the conservation of the integral of helicity density in an Eulerian flow. Note the result is independent from viscosity, subject to the constraint of zero Euler index, $\chi = 0$.

The torsion vector, $\mathbf{T}$, consists of two parts. The first term represents the shear of translational accelerations, and the second part represents the shear of rotational accelerations. The pseudo scalar function, K, acts as the source for the divergence of the torsion vector, T, and the torsion or helicity density, h. When $K = 0$, the evolutionary "lines" associated with the torsion tensor never cross, implying that the system is free of defects in space time. If K is positive or negative, the defects in the system are either growing or decaying. Equation (3.97) is the fundamental new law of topological physics that governs the specific realizations of controlled processes that minimize or maximize defect evolution.

Recall that if $H = A\hat{}dA = 0$, the 1-form of action satisfies the complete integrability condition of Frobenius. Similar to the Caratheodory equilibrium result for Q, the flow can be described then in terms of two variables; i.e., the flow is laminar. Turbulent flow is not laminar, and the transition from the laminar to the turbulent state must involve the topological evolution of H. It was the evolution of the 3-form of topological torsion as displayed in Figure 6 that galvanized the author's interest in topological evolution. The 3-form, H, and its evolution is intuitively related to the thermodynamic property of entropy. The fact that the Cartan topology is disconnected if the topological torsion, H, is not zero implies that the turbulent state cannot be created from the laminar state by means of a continuous transformation. Turbulence must be created by a discontinuous process. However, the decay of turbulence can be described by means of continuous process.

3.9 Global Conservation Laws

3.9.1 First Variation

Extremal (or Hamiltonian) flows and Eulerian flows induce a set of global conservation laws in the sense that the closed integrals of all odd dimensional Pfaff classes of the fundamental forms are relative integral invariants of uniformly continuous evolution. The result follows from the fact that the evolutionary rates, Q and $Q\hat{}F$ respect to such flows are zero. Integrals of exact forms evaluated over closed cycles, whether the cycle ($Z1$ *or* $Z3$) is a boundary or not, vanish. Hence all closed integrals of odd dimensional sets, $\int_{z1} A$ and $\int_{z3} H$, are evolutionary invariants of Hamiltonian and Eulerian flows.

For the closed flows of the Stokes category, the evolutionary rates of all odd Pfaff classes are closed, but not necessarily exact. That is,

$$dQ = 0, \;\; and \;\; d(Q\hat{}F) = 0, \tag{3.98}$$

implying closure, but Q and $Q\hat{}F$ are not exact. The DeRham classes are not empty and are not flow invariants. Topology changes during such evolutionary processes.

Hence a global set of conservation laws in terms of closed integrals of A and H can be devised only for those closed chains that satisfy Stokes theorem, and those chains must be boundaries (of support). Arbitrary closed integrals are not evolutionary invariants. This lack of relative integral invariance [19] for $\int_{z3} H$ corresponds to the production or destruction of 3 dimensional defects, and these new defects are indications of changing topology and changing inhomogeneity. Formally, a closed integral over a closed form is a period integral whose value, by Brouwer's theorem [20], is an integer multiple of some smallest value. A variation of a period integral signals a change in a Betti number and hence a change in topology. Such flows can produce three dimensional defects.

These results point out the limitations of Moffatt's and Gaffet's claims [21] that the volume integral of helicity density, $\mathbf{v}\bullet curl\mathbf{v}$, is an evolutionary invariant. Helicity is NOT necessarily an invariant of a continuous flow. Moreover, open or closed integrals of Helicity are not necessarily integral invariants of continuous evolution. In particular, the closed volume integral of helicity density, the fourth component of the Helicity four current, is not an invariant of continuous flows for which there is a torsion current .

A theorem depending on only the first variation can be stated for the continuous evolution of flows restricted to Hamiltonian or Eulerian flows:

Theorem 7 ***III***: *The (uniformly) continuous evolution of all odd dimensional Pfaff classes of the Cartan base with respect to Hamiltonian or Eulerian flows ($dQ = 0$, Q exact) are exact. Hence, the closed integrals of A and $H = A\hat{\ }dA$ over closed cycles or boundaries are relative integral invariants with respect to Hamiltonian or Eulerian flows.*

The proof of the theorem is as follows:

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V}))A) = d[P + i(\mathbf{V}))A] \; = Q \text{ and is exact.}$$

Therefore $L_{(\mathbf{V})}\int_{z1} A = \int_{z1} Q = \int_{z1} d[P + i(\mathbf{V}))A] = 0 \supset$ invariance of $\int_{z1} A$.

Similarly,

$$L_{(\mathbf{V})}H = L_{(\mathbf{V})}(A\hat{\ }F) = (L_{(\mathbf{V})}A)\hat{\ }F = Q\hat{\ }F = d[P + i(\mathbf{V}))A]\hat{\ }F \text{ is}$$

exact such that

$$L_{(\mathbf{V})}\int_{z3} H = \int_{z3} d[P + i(\mathbf{V}))A]\hat{\ }F \supset \text{invariance of } \int_{z3} H. \text{ Q.E.D.}$$

In the hydrodynamic case of a compressible Eulerian fluid, this theorem is the generalization of the "invariance of Helicity theorem" often stated for a barotropic domain or isentropic constraints. Closed flows therefore exhibit global conservation laws based on relative integral invariants of A and H, as well as absolute integral invariants of F and K. As will be demonstrated below, the integral of the 3-form of topological torsion, not the helicity density, over a boundary is an invariant of all flows that satisfy the Navier-Stokes equations and for which the vorticity vector field satisfies the Frobenius complete integrability conditions. This result is independent from the magnitude of the viscosity coefficient. On the other hand, the continuous destruction of 3-dimensional defects can be associated with closed flows of the Stokes category. Helicity is NOT necessarily a relative integral invariant of Stokes flows.

Remarkably, such flows also admit a set of relative integral invariants, but these are determined only in terms of a second variational process.

3.9.2 Second Variation

It should be noted that the second Lie differential of the odd dimensional Pfaff classes (represented by A and H) does produce a set of global conservation laws for uniformly continuous processes. The result follows from the fact that the second Lie differential of the Action with respect to closed flows is exact, where the first Lie differential is closed!

The fundamental theorem is then:

Theorem 8 ***IV***: *The (uniformly) continuous evolution of all odd dimensional Pfaff classes of the Cartan base with respect to closed flows ($dQ = 0$) are closed , but not necessarily exact. The second Lie differential is always exact so that $\int_{z1} Q$ and $\int_{z3} Q\hat{}F$ are relative integral invariants of (uniformly) continuous ($dQ = 0$) evolution.*

The proof of the fundamental theorem is as follows:

$L_{(\rho\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V}))A) = Q$

$L_{(\rho\mathbf{V})}L_{(\rho\mathbf{V})}A = L_{(\rho\mathbf{V})}Q = R$

$= i(\rho\mathbf{V})d(i(\rho\mathbf{V})dA) + di(\rho\mathbf{V})di(\rho\mathbf{V})A = i(\rho\mathbf{V})d(Q) + di(\rho\mathbf{V})di(\rho\mathbf{V})A = 0 + d(\Lambda)$

which is exact.

Similarly,

$L_{(\rho\mathbf{V})}L_{(\rho\mathbf{V})}A\hat{}dA = L_{(\rho\mathbf{V})}Q\hat{}F = d(\Delta\ F)$

which is exact. It follows that

$L_{(\rho\mathbf{V})}\int_{z3} Q\hat{}F = \int_{z3} d(\Delta\ F) = 0$

such that $\int_{z3} Q\hat{}F$ is a relative integral invariant. Q.E.D.

Uniform continuity requires that $d(L_{(\rho\mathbf{V})}A) = L_{(\rho\mathbf{V})}dA = dQ = 0$, which insures that Q and $Q\hat{}F$ are closed. Hence closed integrals of the odd dimensional p-forms of Q and $Q\hat{}F$ (and not necessarily A and H) are relative integral invariants of uniformly continuous evolution. The integrals $\int_{z1} Q$ and $\int_{z3} Q\hat{}F$ generate global conservation laws for uniformly continuous processes in which $dQ = 0$. In elementary terms, on a space time variety, the fundamental theorem of uniformly continuous evolution states that the Lorentz force has zero curl, and the torsion defect production rate has zero divergence ($K = 0$), whether the system is dissipative or not.

The successive Lie derivation with respect to a uniformly continuous vector field $J = \rho V$ produces an exact sequence, starting from the concept of action-angular momentum, A, evolving to a closed set, Q, which under continued Lie derivation evolves to an exact kernel of radiation-power, R [20]. A similar exact sequence can be constructed for all odd dimensional Pfaff classes, $A, A\hat{}dA, A\hat{}dA\hat{}dA, ...$

3.9.3 Continuity and the Integers

A most remarkable feature of the fundamental theorem of uniformly continuous evolution is that the integral of any radiation 1-form, R, through a container which is a maximal cycle is in relation to the integers. This concept is another application of the Brouwer degree of a map theorem, that says that all period integrals are integer multiples of some smallest value. The maximal cycle is a closed set that is not a boundary but can contain a system with internal defects, hence the name, the "container". As a simple example consider a disc with several internal holes; the maximal cycle is the cycle which would be the boundary if the disc had no holes. The global conservation laws stated above imply that radiation through the maximal cycle must be compensated by a change in the cohomology class, or the production of a defect of inhomogeneity in the interior. Radiation defects ("holes and torsion handles") are quantized, for it is impossible to create half a hole.

It would appear from the above argument that Planck's hypothesis of quantized radiation oscillators may be considered a consequence of theorem IV and Uniformly CONTINUOUS evolution as defined by equation (17).

3.9.4 The Navier-Stokes Fluid

Although the bulk of this article is limited to the study of uniformly continuous evolution ($dQ = 0$), some remarks should be made about continuous evolution of the Navier-Stokes category ($dQ \neq 0$). The kinematic topology is often too course for direct application to a typical physical system. Additional topological constraints must be applied. For a Navier-Stokes fluid, the additional topological constraints on the admissible flow fields, $V = \{\mathbf{v}, 1\}$ implies a specific format is required for the dissipative force, f. Let f take the form $\nu\ curl\boldsymbol{\omega}$ such that upon dividing through by μ, the equation for the Work 1-form becomes:

$$W = i(V)dA = -\sum_i \{(\nu\ curl\boldsymbol{\omega})_i(dx^i - V^i dt)\}. \tag{3.99}$$

Evaluating both sides explicity and comparing coefficients of the terms dx^i yields the Navier-Stokes partial differential equations,

$$\partial\mathbf{v}/\partial t + grad(\mathbf{v} \circ \mathbf{v}/2) - \mathbf{v} \times curl\mathbf{v} = -gradP/\rho + \nu\ curlcurl\mathbf{v} \tag{3.100}$$

This process is typical of the Cartan method, where by the coefficients of a system of differential forms are equivalent to a system of partial differential equations. For the kinematic Action, A, the equation above expressing and constraining the 1-form of Work is the differential form equivalent to the Navier-Stokes equations. The constraint limits the class of all V to those V that are solutions to the Navier-Stokes partial differential equations.

The constraint given by (39) may be used evaluate the behavior of the topological base with respect to the evolution described by V. For example, the evolution of the Action is given by the expression,

$$L(V)A = i(V)dA + d\{i(V)A\} = -\{(\nu\ curl\boldsymbol{\omega}) \circ (d\mathbf{x}^i - \mathbf{v}^i dt)\} + d\{(\mathbf{v} \circ \mathbf{v}/2) + H\} \tag{3.101}$$

The evolution of the limit sets is given by

$$L(V)dA = -d\{(\nu\ curl\boldsymbol{\omega}) \circ (d\mathbf{x}^i - \mathbf{v}^i dt)\}. \tag{3.102}$$

If the flow V is uniformly continuous, then the RHS of 3.9.4 must vanish, making $F = dA$ a flow invariant. The Navier-Stokes equations have $C2$ solutions that belong to the Stokes category of closed flows. This result is an extension of the Helmholtz theorem on the conservation of vorticity. It would follow that the 4-form, $K = dA\hat{}dA$ is also a flow invariant, for uniformly continuous flows. A remarkable result is that even for dissipative Navier Stokes flows where $\nu\ curl\boldsymbol{\omega} \neq \mathbf{0}$, it is still possible that the RHS of 3.9.4 may vanish, and the flow is uniformly continuous. Examples of such harmonic solutions to the Navier Stokes equations were presented by this author at the Permb conference on Large Scale Structures [3]. One such harmonic closed form solution was shown to develop a tertiary Hopf bifurcation in terms of the parameter of mean flow. The surface of null helicity density, $h = \mathbf{v} \circ \boldsymbol{\omega} = 0$ went through a topological phase change as the bifurcation took place similar to that presented by soap films initially forming a single sheeted surface between two rings, and then with increased ring displacement, forming a double sheeted surface.

According to theorem II, the even dimensional topological properties $\{F, K\}$ are invariants of a uniformly continuous flow. If topology is to change in a uniformly continuous manner, the only possible candidates for topological evolution must be the 1-dimensional circulation, A, and the 3-dimensional torsion, H. For incompressible flows ($div\mathbf{v} = 0$) circulation defects must be associated with boundaries; however, if $K \neq 0$, then torsion defects can occur within the bulk media. It is the author's perception that the production of torsion defects is the key to the understanding of large scale structures in continuous media, and the transition to turbulence.

In general, as has been stated above, if the flow is continuous, then the limit sets $d\Sigma$ must remain within the closure of Σ. Abstractly this idea can be written as,

$$L(V)d\Sigma = d\Sigma + \Sigma\hat{}\Sigma. \tag{3.103}$$

Uniform continuity is the stronger constraint,

$$L(V)d\Sigma = 0. \tag{3.104}$$

For the Navier-Stokes flows, where the evolution is not necessarily uniformly continuous, the Navier-Stokes constraint may be used to express the acceleration term, a, dynamically; i.e.,

$$\mathbf{a} = -gradH - \partial v/\partial t = -\mathbf{v} \times curl\mathbf{v} + \nu\{curl\ curl\mathbf{v}\}. \tag{3.105}$$

By substituting this expression for **a** into equation 3.95 a simple engineering representation is obtained for the torsion vector current, T, of a Navier-Stokes fluid:

$$\mathbf{T} = \{h\,\mathbf{v} - L\,curl\,\mathbf{v}\} - \nu\{\mathbf{v} \times (curl\ curl\mathbf{v})\}(45) \tag{3.106}$$

Note that the torsion axial vector current persists even for Euler flows, where $\nu = 0$. When $h = 0$, the torsion axial vector is proportional to the vorticity of the flow. It is the opinion of this author that many of the visual phenomena of fluid dynamics which have been associated with "vortices" are actually representations of torsion defects. In fact, a closed form solution to the Navier-Stokes equations was presented at the Perm conference [3] which indicates that the experimental phenomena of "vortex" bursting can be emulated by the streamlines of a flow for which there is no parametric evolutionary change of vorticity, but for which there is a parametric evolution and topological phase change of the 3-form of topological torsion. As the critical value of flow is achieved, a re-entrant compact torsion bubble is produced in what was originally a unidirectional flow. The measurement of the components of the Torsion vector have been completely ignored by experimentalists (and theorists) in hydrodynamics (and other dynamical systems).

The measurement of the components of the Torsion vector have been completely ignored by experimentalists in hydrodynamics.

By a similar substitution using the value of a given by the constraint 3.9.4, the topological parity pseudo-scalar becomes expressible in terms of engineering quantities as,

$$K = dH = dA\hat{}dA = -2\nu(\boldsymbol{\omega} \circ curl\,\boldsymbol{\omega})dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.107}$$

From this expression it is apparent that if the vorticity field is integrable in the sense of Frobenius, then viscosity does NOT contribute to the creation of torsion defects. As described below, the integral of K over $\{x, y, z, t\}$ gives the Euler index induced by the flow on the space time variety. If $K = 0$, the flow lines never intersect.

3.10 Pfaff's Problem, Characteristics, and the Torsion Current.

Closely related to the concept of topological torsion is the Pfaff problem that asks about the solubility of the system of differential equations defined by setting each element of the Cartan closure to zero. The problem is equivalent to finding characteristic vector fields which, if continuous, generate an evolutionary flow that preserves the Cartan topology. The key idea of Pfaff's problem is to find maps from spaces of q dimensions into the variety, X, such that when these maps and their differentials are substituted into the system of forms that make up the Cartan closure, then the new forms are equal to zero. In this sense, the pullback of the forms of the Cartan closure to the spaces of dimension q are zero. In the case of usual interest to physics, the maps are of a single parameter which almost always is associated with the concept of time. However, they may exist higher dimensional solutions of say two parameters or more.

The question arises as to the largest dimension of such a "solution" and is determined in terms of the "characters" and "genus" of the Pfaff system [22]. It is the objective of this section to demonstrate that the genus of the Pfaff system built from a single 1-form of action is 3 if the Torsion current, **T**, vanishes, and can be 2 only if $\mathbf{T} \neq 0$. The genus is an arithmetic invariant and a topological property. A change of genus implies topological evolution. However for the special Pfaff system described, the characters are such that only 1-parameter solutions are possible, when $\mathbf{T} = 0$, and a unique 2 parameter solution is admissible only when $\mathbf{T} \neq 0$. In other words the Pfaff problem admits a "string" solution (a two parameter solution) only when the Torsion current is not zero.

Consider an electromagnetic format. For the electromagnetic case, the Cartan 1-form may be defined in terms of the vector and scalar potentials,

$$A = \mathbf{A} \bullet d\mathbf{r} - \varphi dt. \tag{3.108}$$

Using the classical notation of Sommerfeld, define the **E** and **B** field intensities as

$$\mathbf{B} = curl\mathbf{A}, \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - grad\varphi. \tag{3.109}$$

Then the components of the Darboux-Cartan-Maxwell field, $F_{\mu\nu}$, may be written as an anti-symmetric matrix (or as a Sommerfeld six-vector) of components :

$$F_{12} = B_z,\ F_{13} = -B_y,\ F_{23} = B_x,\ F_{14} = E_x,\ F_{24} = E_y,\ F_{34} = E_z \tag{3.110}$$

such that the components of $dA = F = F_{\mu\nu} dx^{\mu} \hat{}\, dx^{\nu}$

The Topological torsion, H, becomes

$$H = A\hat{}dA = -i\{\mathbf{E} \times \mathbf{A} + \varphi\mathbf{B}, \mathbf{A} \bullet \mathbf{B}\} dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.111}$$

with the torsion current defined as,

$$\mathbf{T} = \mathbf{E} \times \mathbf{A} + \varphi\mathbf{B} \tag{3.112}$$

and the helicity density,

$$h = \mathbf{A} \bullet \mathbf{B}. \tag{3.113}$$

The Topological Parity 4-form becomes the global top Pfaffian on the 4 dimensional space-time variety, and is equal to

$$K = dA\hat{}dA = -2\mathbf{E} \bullet \mathbf{B} dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.114}$$

Note that

$$div\mathbf{T} + \partial h/\partial t = -2\mathbf{E} \bullet \mathbf{B} \tag{3.115}$$

. The 3-form of axial current, H, is NOT conserved when $K \neq 0$. This result has been observed by Berger [23]. Following Chern, the Euler index on a compact manifold would be the integral

$$\chi = \int_{z4} 2\mathbf{E} \bullet \mathbf{B} dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.116}$$

Now the Pfaff problem is determined by the equations

$$A = 0, \quad F = 0. \tag{3.117}$$

Following Slebodzinsky, as there is only one 1-form in the Pfaff system, the first character, $s0$, of the Pfaff system is equal to 1. Multiply F by φ, and use $A = 0$ to eliminate φdt in the equation $F = 0$. The result is given by the equation,

$$\{\mathbf{E} \times \mathbf{A} + \varphi \mathbf{B}\}_{\mu\nu} dx^{\mu}\hat{}dx^{\nu} = \{\mathbf{T}\}_{\mu\nu} dx^{\mu}\hat{}dx^{\nu} = 0, \tag{3.118}$$

which is an expression that does not contain dt. The polar system of these resultant equations determines the genus of the Pfaff system. In particular, if $\mathbf{T}$, the torsion current vanishes, then (3.118) vanishes, the second character, $s1$ is zero and the genus of the Pfaff system is 3. All higher characters vanish, so the Pfaff system is special. Only 1-parameter homeomorphic evolutionary solutions are possible for the Pfaff system in 4 dimensions, when $\mathbf{T} = 0$.

On the other hand, for any arbitrary vector field, $\mathbf{V}$, such that the two 1-forms $\{\mathbf{T} \times \mathbf{V}\}_{\mu} dx^{\mu}$ and A, are linearly independent, then the second character, $s1$, equals 1, and the genus is 2. There then exists a two parameter characteristic evolutionary system (a string). In other words, the presence of the torsion current is necessary for the existence of a two parameter solution to the Pfaff problem. There are no 3 parameter solutions to this Pfaff problem in 4-dimensions. This extraordinary connection between the concept of the Torsion current and the solubility of Pfaff's problem serves to further emphasize the content of the often neglected quantity of topological torsion.

3.10.1 The Euler index

The coefficients of the Action 1-form globally define a covariant vector field on the variety. This vector field need not be a section without singularities. As mentioned in section 13 Arnold has shown how the singular points (zeros) of the Action 1-form, A, can be used to define the Euler index of the topology induced on the variety. Another method for evaluating this key topological property has been devised by Chern [24]. Following Chern, the Euler index becomes the integral

$$\chi = \int_{z4} K = \int_{z4} 2\mathbf{E} \bullet \mathbf{B} dx\hat{}dy\hat{}dz\hat{}dt. \tag{3.119}$$

In Lagrangian field theories, a non-zero value for K implies that the second Chern class is not empty and signals the demise of time reversal and parity symmetry [25] (hence, the name Topological Parity 4-form). It should be remarked that K is the exterior derivative of the 3-form of topological torsion, H, and that this 3-form can be put into correspondence with the Chern-Simons 3-form of differential geometry. In effect the evolutionary law for the 3-form of Topological Torsion given by (10) is a Lagrangian field theory built on a Chern-Simons action. In this article, no constraint of self dualism is imposed, as is usually the case in current string theories.

When the electric field is orthogonal to the magnetic field, then the Euler index is zero. The idea that this Poincare invariant might have deeper meaning led Eddington [26] to state: "It is somewhat curious that the scalar-product of the electric and magnetic forces is of so little importance in classical theory, for ..(eq (53)) .. would seem to be the most fundamental invariant of the field. Apart from the fact that it vanishes for electromagnetic waves propagated in the absence of any bound electric field (i.e., remote from electrons), this invariant seems to have no significant properties. Perhaps it may turn out to have greater importance when the study of electron-structure is more advanced."

A non-zero value of the Topological Parity 4-form, K, implies that the divergence of $\mathbf{T}$ is not zero. Therefore, torsion lines can stop or start within the variety even though the evolution is C2 continuous. The torsion current is not necessarily conserved and 3-dimensional defects can be produced internally. String theorists describe this effect as an anomaly of the axial (Torsion) current. In the same sense that the closed but not exact 1-form leads to a complex representation involving ordered pair of variables, a closed but not exact 3-form leads to a quaternionic representation.

The concept of a domain of non-null Euler index ($K \neq 0$) now appears to be useful to the theory of magnetic reconnection in the electromagnetic case [27] and to vortex reconnection [28] in the hydrodynamic case. The correspondence between the bridging and rib structures produced in numerical simulations of turbulent fluid flows and the 4-string interaction of superstring theory is remarkable [29]. The concept ($K \neq 0$) appears to be applicable to the understanding of the stretching of lines and surfaces in turbulent flows where time-reversal symmetry is violated [30]. The appearance of large scale structures in certain flows has been associated with the lack of parity invariance [31]. The concepts of macroscopic violations of P and T symmetries appear to have application to the theory of the quantum Hall effect [32].

With regards to hydrodynamic systems, the evolution of a flow from a laminar flow to a turbulent flow involves topological evolution. For the Navier-Stokes system, the Euler index depends upon the viscosity and the lack of Frobenius integrability of the vorticity field (see equation 36). Such a term yields a local source for the creation of Torsion currents. The lack of reversibility of such flows, and the irreducible time dependent, 3 dimensional features of such flows, implies that K can not be zero for the turbulent state. It is conjectured that the Euler index of the flow (the integral of K over the domain) is not zero during the transition to turbulence. That is, K is not a last multiplier of the spatial volume element, $dx\hat{}dy\hat{}dz$ for the flow describing the continuous (relative to the Cartan C2 topology) transition to turbulence. If $dQ\hat{}F = 0$ then the function K defines an integrating actor in the sense of a mass density such that

$$div(K\mathbf{V}) + \partial K/\partial t = 0. \tag{3.120}$$

If K were a mass density, this equation is often called the "equation of continuity", but it is more accurately described as the "conservation of mass". Relative to the Cartan topology all C2 vector fields are continuous. The transition to the turbulent state, however, must be discontinuous, for the Cartan topology in the turbulent state is disconnected.

3.11 SUMMARY

To review, a topology has been constructed on a variety in terms of the elements of closure of a Cartan system of C2 differential forms and their intersections. The associated topological structure indicates that all processes generated by the Lie convective derivative (relative to a C2 vector field, $\mathbf{V}$) are continuous relative to the Cartan topology. However, the processes so generated are not necessarily homeomorphisms for they need not be reversible; i.e., the topology of the initial state can evolve continuously into a different topology on the final state. The method for constructing the Cartan topology is the same on both the initial and the final state, but, for example, the "hole and handle" count on the initial state can be different from the "hole and handle" count in the final state.

In terms of a single 1-form of Action, A, a Cartan topological base was constructed in terms of a set of distinct elements, defined as a Pfaff sequence, and their closures. The fundamental laws of evolution of each of the elements of the topological base was formulated relative to an arbitrary vector field. It was determined that there are two categories of continuous flows, those which are "closed" and those which are "open". A special sub-category of closed flows describe a Hamiltonian evolution, an evolutionary process which preserves the number of "holes and handles".

Relative to the closed category of continuous processes, all even dimension elements of the Cartan topological base are evolutionary invariants. For closed flows, topological evolution takes place only in terms of the odd elements of the topological base. The first odd element of the topological base is the Action, and its law of evolution is equivalent to the evolution of energy. The next odd element (and the only other odd element on space-time) of the Cartan topological base is formulated as the novel 3-form of Topological Torsion. The evolution of this 3-form is studied, for although it does not necessarily satisfy a local conservation law, the anomalous source term, defined as topological parity, can be computed. It is a source of system evolutionary defects. However, it is still possible to establish a set of global conservation laws for the category of closed, (uniformly) continuous but irreversible evolutionary flows. Although the evolution of topooloigcal torsion may be described by a continuous process, the creation of topological torsion from a state without topological torsion is not described by a continuous process. As the Cartan topology is not connected, the creation of topological torsion must involve discontinuous processes or shocks.

The fundamental equation of topological evolution, $L_{(\rho\mathbf{V})}A = Q$, is equivalent to cohomological format of the first law of thermodynamics, $W + dU = Q$. The heat 1-form Q may be used to form a Pfaff sequence whose Pfaff dimension may be used to further classify evolutionary flows. For example, if the Pfaff dimension of Q is 2 or less, then Q can be written in the equilibrium format, $Q = TdS$. An example of an open system of flows (defined as $dQ \neq 0$) was presented in terms of the Navier-Stokes equations, for which the anomalous source term, can be computed. In effect it was demonstrated that C2 irreversible flows are among the solution set to the Navier-Stokes system. An abstract example was also given for an electromagnetic Action, in which the concept of time reversal and parity symmetry breaking was associated with a non-null Euler characteristic of the Cartan topology.

3.12 Applications

3.12.1 Frozen-in Fields, the Master Equation

A starting point for many discussions of the magnetic dynamo and allied problems in hydrodynamics starts with what has been called the "master equation" [21],

$$Curl(\mathbf{V} \times \mathbf{B}) = \partial \mathbf{B}/\partial t. \tag{3.121}$$

Using the Cartan methods it may be shown that this equation is equivalent to the constraint of "uniform" continuity relative to the Cartan topology. Moreover, it is easy to show these constraints generate symplectic processes which include Hamiltonian evolutionary systems, such as Euler flows, as well as a number of other evolutionary processes which are continuous, but not homeomorphic. In addition a criteria can be formulated to develop an extension of the "helicity" conservation law to a more general setting.

The proof of these results produces a nice exercise in use of the Cartan theory. Consider a 1-form A that satisfies the exterior differential system

$$F - dA = 0, \tag{3.122}$$

where A is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, F, of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials.

Now search for all vector fields that leave the 2-form F an absolute invariant of the flow; that is, search for all vectors that satisfy Cartan's magic formula

$$L_{(\mathbf{V})}F = i(V)dF + di(V)F = 0 + di(V)F = 0. \tag{3.123}$$

For C2 functions, the term involving dF vanishes, leaving the expression,

$$L_{(\mathbf{V})}F = di(V)F \tag{3.124}$$
$$= d\{(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot d\mathbf{r} - (\mathbf{E} \cdot \mathbf{V})dt\} \tag{3.125}$$
$$= \{curl(\mathbf{E} + \mathbf{V} \times \mathbf{B})\}_z dy\hat{\ } dz... \tag{3.126}$$
$$+ \{\partial(\mathbf{E} + \mathbf{V} \times \mathbf{B})/\partial t + grad(\mathbf{E} \cdot \mathbf{V})\} \cdot d\mathbf{r}\hat{\ } dt \tag{3.127}$$
$$= 0. \tag{3.128}$$

Setting the first three factors to zero yields

$$curl(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = 0 \tag{3.129}$$

From the Maxwell Faraday equations for C2 functions, $curl\mathbf{E} = -\partial\mathbf{B}/\partial t$, and when this expression is substituted into the above equation, the "master" equation given above is the result. Now recall that dF generates the limit points of A, and if $F = dA$ is a flow invariant, then all limit points are flow invariants relative to the Cartan topology. This result implies that the vector fields, $\mathbf{V}$, that satisfy the constraints of the "master equation" are uniformly continuous evolutionary processes, as the limit points, $F = dA$, of the 1-form A are flow invariants, and the lines of vorticity are "frozen-in" the flow. Non-uniform continuity would imply that the limit points are not invariants of the process, but that the closure of the limit points of the target range includes the limit points of the initial domain. Such processes would correspond to a folding of the "lines" of vorticity, which preserve the limit points, but not their sequential order. A second criteria for limit point invariance is given by the equation,

$$\{\partial(\mathbf{E}+\mathbf{V}\times\mathbf{B})/\partial t + grad(\mathbf{E}\cdot\mathbf{V})\} = 0. \tag{3.130}$$

The formula indicates that limit point invariance can occur in the presence of input-output power, $\mathbf{E}\cdot\mathbf{V}\neq 0$.

The criteria for frozen-in fields is established as a constraint of uniform continuity on the admissable vector fields,

$$\text{Uniform Continuity:} \quad di(V)dA = di(V)F = 0. \tag{3.131}$$

The solution vector fields, V, subject to this constraint can be put into three global categories:

1.	Extremal (Hamiltonian)	$i(V)F = 0.$
2.	Bernoulli-Casimir (Hamiltonian)	$i(V)F = d\Theta.$
3.	Symplectic	$i(V)F = d\Phi + \gamma_{harmonic}$

The first category can exist only on domains of support of F which are of odd Pfaff dimension, but then the solution vector is unique to within a factor. In the other categories, the solution vector need not be unique. Vector fields that satisfy the equation for uniform continuity are said to be symplectic relative to the 1-form, A. Vector fields that belong to categories 1 and 2 have a Hamiltonian representation. Vector fields that belong to category 1, are said to be "extremal" relative to the 1-form, A.

When the concepts are applied to the integrals of the 2-form F, then the criteria for invariance of the flux integral depends on the topology of the integration domain. If the integration area of the 2-form is a boundary or a cycle of a 3 dimensional domain, the flux integral over the closed boundary or cycle is always a flow invariant. If the integration area is bounded, then by Stokes theorem the flux integral depends only on the boundary conditions: F or $i(V)F$ must vanish on the boundary, or when integrated over the boundary.

3.12.2 Euler flows and Hamiltonian systems.

In 1922 Cartan established the idea that the necessary and sufficient conditions for a system to admit a unique Hamiltonian representation for its evolution, $\mathbf{V}$, on a space of 2n+1 dimensions is given by the category 1 constraint,

$$\text{Extremal process } V: \quad W = i(V)dA = i(V)F = 0. \tag{3.132}$$

In the language of mechanics on state space $\{p, q, t\}$, the 1-form of Action is given by an expression of the form,

$$A = p_\mu dq^\mu - H(p,q,t)dt. \tag{3.133}$$

For a given 1-form, those vector direction fields, $V = [\dot{p}_\mu, \dot{q}^\mu, 1]$,that satisfy this constraint are said to be extremal vector fields. The word extremal comes from the theory of the calculus of variations. By direct computation,

$$dA = dp_\mu \hat{} dq^\mu - dH(p,q,t) \hat{} dt, \tag{3.134}$$

$$i(V)dA = \dot{p}_\mu dq^\mu - \dot{q}^\mu dp_\mu + dH(p,q,t) - (\partial H/\partial p_\mu \dot{p}_\mu - \partial H/\partial q^\mu \dot{q}^\mu - \partial H/\partial t)dt, \tag{3.135}$$

$$= \dot{p}_\mu \{dq^\mu - (\partial H/\partial p_\mu)dt\} - \dot{q}^\mu \{dp_\mu + (\partial H/\partial q^\mu)dt\} - (\partial H/\partial p_\mu \dot{p}_\mu - \partial H/\partial q^\mu \dot{q}^\mu)dt \tag{3.136}$$

It follows that the extremal condition $i(V)dA = i(V)F = 0$, if the extremal vector exists, is satisfied by

$$\text{Hamiltonian equations} \quad \dot{p}_\mu = -\partial H/\partial q^\mu, \quad \dot{q}^\mu = +\partial H/\partial p_\mu. \tag{3.137}$$

When the space is of odd Pfaff topological dimension, then the extremal field is the unique null eigen vector of the antisymmetric matrix of functions that make up the 2-form, $F = dA$. For an even dimensional space of maximum topological dimension, the anti-symmetric matrix, F, has no null eigen vectors - the extremal field does not exist as there are no null eigenvalues. However, for category 2 situations, a function can be found, $H' = H + \theta$, such that the Hamiltonian equations are valid using the variable, H' instead of H.

It is apparent that this extremal condition is more stringent than that given in the preceding section for uniform continuity, $di(V)F = 0$. Such extremal vector fields are independent of parameterization. That is, for extremal processes, $i(\rho V)dA = 0$ if $i(V)dA = 0$, for any function, ρ. Extremal vector fields do not exist on domains where the Pfaff dimension of the Cartan 1-form is even. In classical mechanics, the 1-form W is defined as the 1-form of Virtual Work, and the Cartan constraint is typical of problems in the variational calculus where it is presumed that the Virtual Work vanishes.

As an example,consider a 1-form of Action defined as

$$A = \mathbf{v}\ d\mathbf{r} - (\mathbf{v} \cdot \mathbf{v}/2 + \Psi)dt, \tag{3.138}$$

where $d\Psi = dP/\rho$. Application of the extremal constraint yields the resulting necessary system of partial differential equations is given by known as the Euler equations of hydrodynamics.

$$\partial \mathbf{v}/\partial t + grad(\mathbf{v} \cdot \mathbf{v}/2) - \mathbf{v} \times \mathbf{w} = -gradP/\rho, \tag{3.139}$$

It also follows that the Master equation is valid, with the only difference being that $curl\mathbf{v}$ is defined as $\boldsymbol{\omega}$, the vorticity of the hydrodynamic flow. The master equation becomes,

$$curl(\mathbf{v} \times \boldsymbol{\omega}) = \partial\boldsymbol{\omega}/\partial t, \tag{3.140}$$

and this equation is to be recognized as Helmoltz' equation for the conservation of vorticity. In the hydrodynamic sense, conservation of vorticity implies uniform continuity. In other words, the Eulerian flow is not only Hamiltonian, it is also uniformly continuous, and satisfies the master equation and the conservation of vorticity constraints. In addition, it may be demonstrated that such systems are at most of Pfaff dimension 3, and admit a relative integral invariant which generalizes the hydrodynamic concept of invariant helicity. In the electromagnetic topology, the Hamiltonian constraint is equivalent to the statement that the Lorentz force vanishes, a condition that has been used to define the "ideal" plasma or "force-free" plasma state.

3.12.3 Conservation of Topological Torsion

A slightly more general class of evolutionary processes (flows) is given by the constraints which are gauge equivalent to the Hamiltonian extremal case; a search is made for those flows that satisfy the (non-extremal, but Hamiltonian) constraint:

$$i(\rho V)dA = i(\rho V)F = d\Theta. \tag{3.141}$$

Such flows admit two topological invariants of the relative integral invariant form. The first integral invariant is 1-dimensional:

$$L_{(\rho\mathbf{V})} \oint_{1d\_closed} A = \oint_{1d\_closed} i(\rho V)dA + di(\rho V)A = \tag{3.142}$$

$$\oint_{1d\_closed} d\Theta + di(\rho V)A = \oint_{1d\_closed} d\{\Theta - i(\rho V)A\} \Rightarrow 0, \tag{3.143}$$

expressing the relative integral invariance of circulation (Kelvin's theorem). The second integral invariant is 3-dimensional:

$$L_{(\rho\mathbf{V})} \oint_{3d\_closed} A\hat{\ } dA = \oint_{3d\_closed} d\{\Theta - i(\rho V)A\}\hat{\ } dA \Rightarrow 0, \tag{3.144}$$

a result expressing the generalization of the law which in hydrodynamics is called the conservation of Helicity. The integrations are over closed 1 and 3 dimensional domains. These closed integration domains can be either cycles or boundaries. For exampled the 1-dimensional closed curve in the punctured disc that encircles the central hole is a cycle but not a boundary. As the integrands are exact differentials, the closed integrals vanish.

Note that on the domain $\{x, y, z, t\}$, the 3-form of topological torsion, $A\hat{}dA$, has the general representation with coefficients, $Z_{\mu\nu\sigma}$, that transform as a covariant tensor field of third rank. On a 4 dimensional space, the components of $A\hat{}dA$ are proportional to a contravariant tensor density of rank 1, whose four components have a vector part defined as, $\mathbf{T}$, the torsion (pseudo) current, and a (pseudo) density part, h. The 3-form $A\hat{}dA$ is not an impair form (density). In electromagnetic engineering language, the general formula for the torsion 3-form has a component expression given by:

$$T = [\mathbf{T}, h] = [\mathbf{E} \times \mathbf{A} + \phi\mathbf{B}, \mathbf{A} \cdot \mathbf{B}]. \tag{3.145}$$

For the constraints of an Eulerian flow, the 4 components of the Torsion three form reduce to

$$T = [\mathbf{T}, h] = [(\mathbf{v} \cdot \boldsymbol{\omega})\mathbf{v} - (\mathbf{v} \cdot \ \mathbf{v}/2 + \Psi)\boldsymbol{\omega}, \mathbf{v} \cdot \boldsymbol{\omega}]. \tag{3.146}$$

Recall that the closed integration domain used to evaluate the relative integral invariant is *not* necessarily restricted to a spatial volume integral with a boundary upon which the normal component of $\mathbf{v}$ vanishes. Also note that the helicity density of hydrodynamic fame is the fourth component, $h = \mathbf{v} \cdot \boldsymbol{\omega}$, of a contravariant vector density, equivalent to a covariant tensor of third rank. Care must be used in its transformation with respect to diffeomorphisms, such as the Galilean transformation. Furthermore, for the constraints of an Eulerian flow (an extremal field) described above, the topological parity 4-form vanishes globally, such that there exists a pointwise conservation law of the 3-form, equivalent to the expression,

$$div_3\mathbf{T} + \partial h/\partial t = 0. \tag{3.147}$$

Chapter 4

TOPOLOGICAL DEFECTS

4.1

Part II

Appendix

Chapter 5
APPENDIX 1. POINT SET TOPOLOGY

Perhaps the best way to learn basic ideas about topology is through the study of point set topology. The concepts and definitions can be illuminated by measns of examples over a discrete and small set of elements. The early champions of point set topology were Kuratowski in Poland and Moore at UT-Austin. For a long time Point Set topologists were isolated from the Combinatorial Topologists. In fact the name topology, I am told, was introduced about 1925, about the time that it was recognized that topology had many equivalent expositions.

One of the best books for rapid assimilation of point set topology is the Schaums Outline Series, "General Topology" by S Lipschutz. I have an old copy (1965). I advise you to review Chapter 1, then skip to chapters 5, 6, and 7.

In that which follows, four different topologies will be defined over a set of five elements, {a,b,c,d,e}. Then the topological definitions of Open sets, Closed Sets, Limit Points, Closure, Boundary, Interior, Exterior , and the concept of Continuity will be defined and exemplified for each of the four point set topologies. It should become apparent that the same sets of points can have different topologies imposed as a set of constraints on the same elements. It is the topology that allows the concepts of boundary, closure and limit points to be defined. In that which follows, the applications of these ideas will done for systems of differential forms, rather than systems of points.

5.0.1 Closed and Open Sets

Consider a set of elements $\{a, b, c, d, e\}$ and a combinatorial process which is symbolized, for example, by the brackets (ab) or (ade). Construct all possible combinations, and include the null set, 0. Define $X = (abcde)$.

Now from the set of all possible combinations, it is possible to select many subset collections. Certain of these subset collections have the remarkable property of logical closure. As an example, consider the subset collection, or class of subsets, given by:

$$T1(closed) = \{X, 0, a, b, (ab), (bcd), (abcd)\} \tag{5.1}$$

Note that the intersection of (a) with (ab) is a, $(a \cap ab = a)$, which is an element of the collection, and the intersection of (ab) with (bcd) is b which is also an element of the collection. In fact, the intersection of every element of $T1(closed)$ with every other element of $T1(closed)$ produces one of the original seven elements of the collection, $T1closed$. In other words, the process of set intersection acting on any number of elements is closed with respect to $T1(closed)$.

Now also note that the union of any two elements of the set is also contained within the set. The idea that a closed algebra can be built upon the notions of union and intersection, and that this algebra be a division algebra, is at the heart of the theory of logic. This idea of logical closure with respect to arbitrary intersection and finite union is said to define a topology, $T1(closed)$, of closed sets.

Definition: A topology $T1(closed)$ on a set X is a collection or class of subsets that obeys the following axioms:

1. **A1(closed):** X and the null set 0 are elements of the collection.
2. **A2(closed)**: The arbitrary intersection of any number of elements of the collection belongs to the collection.
3. **A3(closed):** The arbitrary union of any pair of elements of thecollection belongs to the collection.

The elements of the collection, $T1(closed)$, are defined to be "closed" sets. The compliments of the closed sets are defined as "open" sets. The open sets of the topology are the collection of subsets given by

$$T1(open) = \{0, X, (bcde), (acde), (cde), (ae), e\}. \tag{5.2}$$

It is important to note that the same set of all combinations of subsets can support many topologies. For example, the subsets of the collection,

$$T2(closed) = \{X, 0, (bcde), (cde), (de)\}, \tag{5.3}$$

are closed with respect to both logical intersection and union. Hence $T2(closed)$ is a different topology built on the same set of points, X. The open sets of this topology are

$$T2(open) = \{0, X, a, (ab), (abc)\} \tag{5.4}$$

The compliments of closed sets are defined to be "open" sets and they too can be used to define a topology.

A subset can be both open, or closed, or both, or neither, relative to a specified topology. For example, with respect to the topology given by the closed sets,

$$T3 = \{X, 0, a, (bcde)\}, \tag{5.5}$$

$(bcde)$ is both open and closed, and the set (bc) is neither open nor closed.

The topology of closed sets given by the collection,

$$T4(closed) = \{X, 0, (bcde), (abe), (be), (a)\}, \tag{5.6}$$

has its dual as the topology of open sets

$$T4(open) = \{0, X, a, (cd), (acd), (bcde)\}. \tag{5.7}$$

Note that this topology, $T4$, is a refinement of the topology, $T3$, in that it contains additonal closed (or open) sets.

Remarks: In the definition of a topology when the number of elements of the set is not finite, the logical intersection of open sets is restricted to any pair, and the logical union of closed sets to restricted to any pair. There are many other ways to define a topology, but the concepts always come back to the idea of logical closure.

5.0.2 Limit Points

The next idea to be presented is the concept of a limit point. A standard definition states that a point p is a limit point of a subset, A, iff every open set that contains p contains another point of A. Note that p is an element of X and need not be an element of A. Given a subset, A, each point of X must be tested to see if it is a limit point of A relative to the topology specified on the points. If A is a singleton, it can have no limit points, for there are no other points of A. It follows that the limit points of a limit point (a singleton) is the null set. If the limit point of A consists of the singletons or points symbolized by dA, then $d(dA) = 0$. The set of limit points as a collection of singletons, $\{a, b, c..\}$ will be denoted by dA, where the union of all limit points will be denoted by A‘. The symbol d may be viewed as a limit point operator; the symbol d when applied to a set, A, means that each point p of the domain is tested against the specified topology to see if another point of A is included in each open set of the topology.

Consider the subset $A = (ab)$ and the topology given by $T1(open)$. Now test each point relative to the collection $T1(open)$:

$$T1(open) = \{0, X, (bcde), (acde), (cde), (ae), e\} \tag{5.8}$$

The point a is not a limit point of (ab) because the open set $(acde)$ which contains a does

not contain b;

$dA = 0$ at a.

The point b is not a limit point of (ab) because the open set $(bcde)$ does not contain a;

$dA = 0$ at b.

The point c is not a limit point of (ab) because the open set (cde) does not contain either a

or b;

$dA = 0$ at c.

The point d is not a limit point of (ab) because the open set (cde) does not contain either a

or b;

$dA = 0$ at d.

The point e is not a limit point of (ab) because the open set (cde) does not contain either a

or b;

$dA = 0$ at e.

In other words, the subset (ab) has no limit points in the topology given by $T1$. The limit point set of ab, designated in this monograph as $A^{\mathrm{lim}}(ab)$, is given by

$$A^{\mathrm{lim}}(ab) = \{0\}, \tag{5.9}$$

the empty set, as $dA = 0$ at all points of X.

Now make the same tests with regard to the same subset $A = (ab)$, but this time relative to the topology given by $T2(open)$.

$$T2(open) = \{0, X, a, (ab), (abc)\} \tag{5.10}$$

The point a is not a limit point of (ab) because the open set (a) is a singleton;

$dA = 0$ at a.

The point b is a limit point of (ab) because both the open sets that contain b also contain a;

$dA \neq 0$ at b.

The point c is a limit point of (ab) because the open set X and the open set (abc) that contains

c, contains a and b which are points of A;

$dA \neq 0$ at c.

The point d is a limit point of (ab) because the open set X , the only open set that contains

d, contains another point of A;

$dA \neq 0$ at d.

The point e is a limit point of (ab) because the open set X, the only open set that contains

e, another point of A;

$dA \neq 0$ at e.

Hence, the points b, c, d and e are limit points of $A = (ab)$ relative to the topology $T2(open)$.

$$\text{The limit set of } A^{\text{lim}}(ab) = (bcde) \tag{5.11}$$

With respect to the topology of $T4(open)$,

$$T4(open) = \{0, X, a, (cd), (acd), (bcde)\} \tag{5.12}$$

test for limit points of the set (ab):

The point a is not a limit point of (ab) because the open set a is a singleton;

$dA = 0$ at a.

The point b is not a limit point of (ab) because the open set $(bcde)$ does not contain a;

$dA = 0$ at b.

The point c is not a limit point of (ab) because the open set (cd) which contains c does not contain a or b;

$dA = 0$ at c.

The point d is not a limit point of (ab) because the open set (cd) which contains d does not contain a or b; $dA = 0$ at d.

The point e is a limit point of (ab) because the open set $(bcde)$ which contains e contains another point b of A; $dA \neq 0$ at e.

The limit set of (ab) relative to $T4(open)$ becomes $A^{\text{lim}} = \{e\}$.

Note that the set of limit points as a collection, or a class of sets, may or may not have limit points. If the limit set is a singleton, then the limit points of the set of limit points is the null set. However, consider the limit set $A^{\text{lim}} = \{bcde\}$ of the set (ab) relative to the topology $T2$. Then the limit points of A^{lim} are the points (b, c, d, e). In other words $dA^{\text{lim}} \neq 0$ necessarily, but $ddA^{\text{lim}} = 0$.

5.0.3 Closure

The closure of a set is defined to be the union of the set and its limit points. Note that a closed set contains its limit points, if any exist. In the examples given above the closure of the set $A = (ab)$ relative to the topology $T1$ is equal to the union of $A = (ab)$ and its limit points, which is the null set.

$$\overline{A} = A \cup A^{\text{lim}} = (ab) \tag{5.13}$$

Note that $A = (ab)$ is a closed set, and has no limit points relative to the topology $T1(open)$. However, the closure of A relative to the topology $T2(open)$ is

$$\overline{A} = A \cup A^{\lim} = (ab) \cup (bcde) = (abcde). \tag{5.14}$$

which is the whole set. When the closure of a subset is the whole set X, the subset is said to be dense in X relative to the specified topology.

The closure of (ab) relative to the topology $T4(open)$ is

$$\overline{A} = A \cup A^{\lim} = (ab) \cup (e) = (abe). \tag{5.15}$$

Note that the closure of a subset is equal to the smallest closed set that contains the subset. Every closed set is its own closure. A closed set may or may not have limit points, but if it does have limit points they are contained within the (closed) set.

5.0.4 Continuity

Now comes a major issue of this appendix. Continuity of a transformation is defined relative to the topologies that may exist on the initial and final states. Let the set of points X with topology $T1(open)$ be mapped into the set of points Y with the topology $T2(open)$. Then the map is continuous iff the closure of every subset of the initial state relative to $T1$ is included in the closure of the image of the final state relative to the topology $T2$.

Another test for continuity is given by the statement that the inverse image of every open set of Y relative to $T2$ is an open set of X relative to the topology $T1$.

Later on, the first definition will be used to prove that any topology built on subsets of exterior forms with C2 (twice differentiable) coefficients will be continuously transformed by evolutionary processes that are generated by the Lie convective derivative with respect to C2 vector fields. For the present, the second definition will be used in terms of simple point set topological systems.

5.0.5 Continuous Non-Homeomorphic Example

As a first example consider the transformations given on X to Y by the following diagram:

Let the open sets of the two topologies be $T1 = \{X, 0, (a), (ab), (abc)\}$ and $T2 = \{Y, 0, (x), (y), (xy), (yzt)\}$.

The open set (y) has a preimage (a) which is open. The open set (yzt) has a preimage (abc) which is open. Hence the map is continuous. However, the inverse image mapping is not continuous for the open set (ab) has a preimage as (yz) but (yz) is not an open set of $T2$. Hence the map, although continuous, is not a homeomorphism.

5.0.6 Discontinuous Non-Homeomorphic Example

Let the open sets of the two topologies be $T1 = \{X, 0, (a), (ab), (abc)\}$ and $T2 = \{Y, 0, (x), (y), (xy), (yzt)\}$.

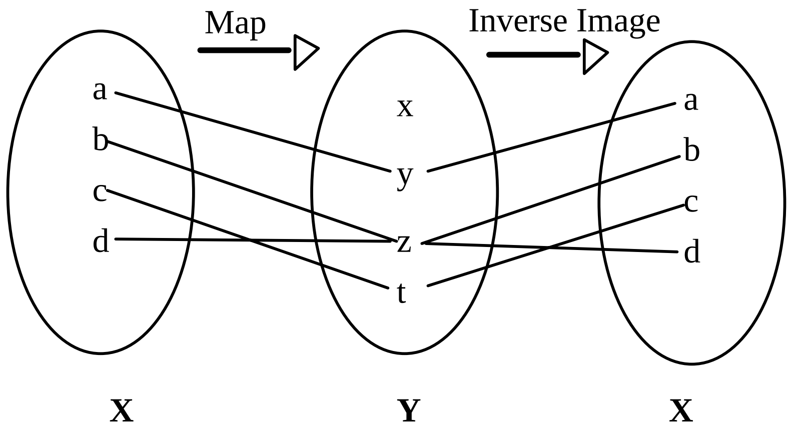

Figure 1

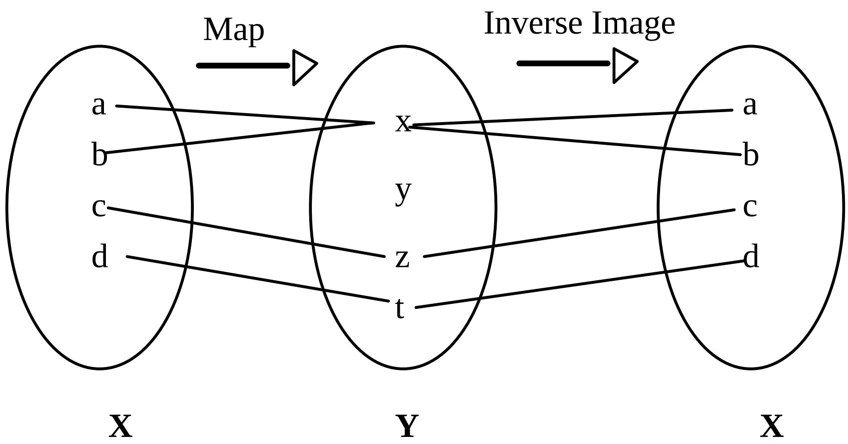

Figure 2

The open set (x) of Y has a preimage (ab) which is open in $T1$. The open set (yzt) of Y does not have a preimage as an open set in $T1$. Hence the map is not continuous. Note that the open set (y) on Y does not enter into the test for it is not connected to points of X.

Also note that the inverse image map is not continuous either, for every open set $T1$ of the inverse image does not have a preimage which is an open set in $T2$. The open set (abc) of X does not have a preimage as an open set of $T2$. The open sets (a) and the open sets (ab) of X do have a preimage as the open set (x) of Y. Hence the inverse image is NOT continuous.

5.0.7 Homeomorphic example (equivalent topologies)

5.0.8 Interior

When emphasis is placed on open sets rather that closed sets, other ideas come to the forefront. In particular, the concept dual to the notion of closure is the concept of interior. While closure asks for the smallest closed set that covers any specified subset, the idea of interior asks for the largest open set included in the specified subset. The interior of a set can be empty (for there may be no open sets other than the null set contained within the specified set)!

For example, the subset (ab) has no interior relative to the topology $T1(open)$. However, the interior of (ab) is itself, (ab), relative to the topology $T2(open)$, because (ab) is an open set in this topology! Relative to the topology $T4(open)$, the interior of (ab) is the singleton, (a):

$IntA = \{0\}$ relative to $T1$

$IntA = (ab)$ relative to $T2$

$IntA = (a)$ relative to $T4$.

The set (abe) has an interior (ae) relative to $T1(open)$ and an interior (ab) relative to the topology $T2(open)$.

5.0.9 Exterior

The exterior of a specified set is the interior of the compliment of the specified set. The compliment of (ab) is the set (cde) which has the interior (cde) relative to $T1(open)$ and has no interior relative to the topology $T2(open)$. Relative to the topology $T4(open)$, the exterior of (ab) is the set (cd).

$ExtA = \{cde\}$ relative to $T1$

$ExtA = (0)$ relative to $T2$

$ExtA = (cd)$ relative to $T4$.

5.0.10 The Boundary

The points that make up the boundary of a subset are union of those points that are not included in the interior or the exterior. However, the union of the points that make up the boundary may have subsets that are not connected. Consider a solid disk. The points that make up the rim of the disk forms its boundary. Now punch a hole in the disk. The collection of points that make up the outer rim and the inner hole now form the boundary of the disk. The two sets of boundary points are not connected.

Similar to the limit point operator, d, a boundary operator, ∂ (some books use δ) , may be defined in terms of a procedure, such that when ∂ is applied to the set A, it implies that a test is made at each point p to see if p is an element of the interior or of the exterior of the selected subset A. If the test fails then $\partial A \neq 0$ and the point is a boundary point. It the point p is an element of the exterior or interior of A, then $\partial A = 0$ at the point p. The boundary of A, or bA, of the set A, is defined as the union of all boundary points.

As a first example, consider again the set $A = (ab)$ and the $T1$ topology. The set $A = (ab)$ has no interior, but the exterior of (ab) is the set (cde), and therefore the boundary set, bA, consists of the union of the points (a, b). In this first example, then

$A^{\lim} = \{0\}$, but $bA = (ab) \subset \overline{A}$

It follows that

$A \cap A^{\lim} = 0$,

$A \cap bA \neq 0$.

The boundary set exists even though the limit set does not!

Relative to $T2(open)$, the set (ab) has an interior set (ab), no exterior set, but a boundary set is $bA = (cde)$. In this case the boundary is included in the closure,

$$A^{\lim} = (bcde), bA = (cde) \subset \overline{A} = (abcde), \tag{5.16}$$

and

$A \cap A^{\lim} \neq 0$,

$A \cap bA = 0$.

Although all boundary points are limit points, there exist limit points that are not elements of the boundary.

Relative to $T4(open)$, the set (ab) has an interior set (a), and exterior set (cd) and a boundary set (be),

$A^{\lim} = (e), bA = (be)$,

$A \cap A^{\lim} = 0$,

$A \cap bA \neq 0$.

It is apparent that the boundary points contain limit points, but there are boundary points which are not limit points!.

In all cases, note that the union of the interior and the boundary is equal to the union of the set and its limit points. The boundary is always included in the closure, but the boundary may contain points which are not limit points.

$$\overline{A} = IntA \cup bA = A \cup A^{\lim}. \tag{5.17}$$

These examples point out that there exist certain correspondences between limit points and boundaries, but they are not necessarily the same concept. Much of current physical theory has emphasized the boundary and open set point of view, while in this monograph the emphasis is on the limit points and closure point of view. It will become evident that these concepts are at the heart of the differences between contravariant and covariant concepts in physical theories, an idea that ultimately expresses itself in the differences between the particle or wave perspective of physics. In topology, these notions are at the heart of the differences between Homology and Cohomology (which will be discussed in detail later). In this monograph, the cohomological point of view is emphasized.

If a set has the property that its intersection with its limit set is empty, then the set is said to be isolated. This idea of isolation, whereby $A \cap A^{\lim} = 0$, will be translated into the Cartan statement, $A\hat{\ }dA = 0$, in the next chapter. The physical significance ot the topological concept of isolation will be correlated with the Caratheodory statement of the existence of inaccessible states in a thermodynamic system, and to the notion of Frobenius complete integrability for a laminar, non-chaotic flow. The concept of isolation is a topological property. and its compliment is a necessary condition for chaos. The observation of a flow transforming from a laminar state (isolated) to a turbulent state (non-isolated) is an observation of topological evolution.

It should be mentioned that with respect to diffeomorphic transformations, or more simply those transformations that preserve pure geometrical properties, the differences between contravariant and covariant concepts cannot be distinguished. Further note that the existence of a metric implies that the contravariant concepts can be converted into covariant concepts, and their possible differences are masked into an alias-alibi format; that is, there are no measurable differences between the two concepts. However, with respect to an aging process, the behavior of the two concepts is observably different. The differences between the behavior of contravariant and covariant concepts may be interpreted as the existence of topological evolution.

5.0.11 Examples of Topologies on 4 elements

Example 1: The T4 topology

The set of 4 elements consists is:

$$X : \{a, b, c, d\}. \tag{5.18}$$

All possible subsets:

$$\emptyset, \tag{5.19}$$
$$\{a\}, \{b\}, \{c\}, \{d\}, \tag{5.20}$$
$$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \tag{5.21}$$
$$\{a,b,c\}, \{a,c,d\}, \{b,c,d\}, \{a,b,d\}, \tag{5.22}$$
$$\{a,b,c,d\} = X \tag{5.23}$$

Several topologies may be constructed from the subsets of X. In Part 1, Section 1.1, the open sets of what was defined as the T4 topology were chosen as:

$$T4\{open\} : \emptyset, \{a\}, \{c\}, \{a,b\}, \{c,d\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}, \{a,b,c,d\}. \tag{5.24}$$

The standard definitions for certain topological concepts are:

1. A limit point of a subset A is a point p such that all open sets that contain p also contain a point of A not equal to p.

2. The closure of a subset A is the union of the subset and its limit points, and is the smallest closed set that contains A.

3. The interior of a subset is the largest open set contained by the subset.

4. The exterior of a subset is the interior of its compliment.

5. A boundary of a subset is the set of points not contained in the interior or exterior.

6. The closure of a subset is also equal to the union of its interior and its boundary.

7. A subset A consists of disconnected parts, if there exist open sets P and Q whose intersections with A are not empty, but are disjoint, and if the union of these intersections is equal to the subset.

The results of applying these definitions to the $T4$ topology of 4 points were given in the Section 1.1, and are repeated here::

The **T4 Topology of 4 points**

$X = \{a, b, c, d\}$

Basis subsets $\{a\}, \{a, b\}, \{c\}, \{c, d\}$

$T4\{open\} : \emptyset, \{a\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X$

$T4\{closed\} : X, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \emptyset$

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{c\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c, d\}$
$\{d\}$	$\emptyset$	$\emptyset$	$\{d\}$	$\{d\}$
$\{a, b\}$	$\{b\}$	$\{a, b\}$	$\emptyset$	$\{a, b\}$
$\{a, c\}$	$\{b\}, \{d\}$	$\{a, c\}$	$\{b, d\}$	X
$\{a, d\}$	$\{b\}$	$\{a\}$	$\{b, d\}$	$\{a, b, d\}$
$\{b, c\}$	$\{d\}$	$\{c\}$	$\{b, d\}$	$\{b, c, d\}$
$\{b, d\}$	$\emptyset$	$\emptyset$	$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{d\}$	$\{c, d\}$	$\emptyset$	$\{c, d\}$
$\{a, b, c\}$	$\{b\}, \{d\}$	$\{a, b, d\}$	$\{d\}$	X
$\{b, c, d\}$	$\{d\}$	$\{c, d\}$	$\{b\}$	$\{b, c, d\}$
$\{a, c, d\}$	$\{b\}, \{d\}$	$\{a, c, d\}$	$\{b\}$	X
$\{a, b, d\}$	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{b, c, d\}$
$\{a, b, c, d\}$	$\{b\}, \{d\}$	X	$\emptyset$	X

(5.25)

This $T4$ topology is quite interesting for many demonstrable reasons. First note that the all of the singletons of the topology are not closed. This implies that the topology is NOT a metric topology, NOT a Hausdorf topology, and even does NOT satisfy the separation axioms to be a T_1 topology. Note that all closed sets contain all of their limit points. Some open sets can contain limit points, but some open sets do not contain their limit points. Some subsets have boundaries that are composed of their limit points. Some subsets have limit points which are not boundary points. Certain subsets have a boundary, but do not have limit points, and in other cases there are subsets that have limit points, but do not have a boundary. There are certain subsets with a boundary, but without an interior. There are certain subsets with an interior, but without a boundary. Although the closure of a subset is the union of the subset and its limit points, or the union of the interior of a subset and its boundary, the intersection of a subset and its limit points may be empty or non-empty, while the intersection of the interior of a subset and the boundary of the subset is always empty. These situations, though topologically correct, are not always intuitive to those accustomed to metric based topological concepts, which impose a number of additional constraints on the sets of interest. Yet all of these topological ideas, including the non-intuitive ones, are easy to grasp from a study of the simple example of the $T4$ point set topology.

Example 2: The T4A topology (WARNING THIS TABLE IS YET TO BE WORKED OUT)

In the T4A topology, the open sets are chosen as

$$T4A\{open\} : \emptyset, \{c\}, \{c, d\}, \{a, c, d\}, X. \tag{5.26}$$

The truth table for the T4A topology is:

The T4A Topology of 4 points

$X = \{a, b, c, d\}$

$T4\{open\} : \emptyset, \{c\}, \{c, d\}, \{a, b, c\}, X$

$T4A\{closed\} : X, \{a, b, d\}, \{b, c\}, \{d\}, \emptyset$

Subset	Limit Pts	Interior	Boundary	Closure
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{b\}$	$\emptyset$	$\{a, b\}$	$\{a, b\}$
$\{b\}$	$\{a\}$	$\emptyset$	$\{a, b\}$	$\{a, b\}$
$\{c\}$	$\{a, b, d\}$	$\{c\}$	$\{a, b, d\}$	$\{a, b, c, d\}$
$\{d\}$	$\emptyset$	$\emptyset$	$\{d\}$	$\{d\}$
$\{a, b\}$	$\{b\}$	$\{a, b\}$	$\emptyset$	$\{a, b\}$
$\{a, c\}$	$\{b\}, \{d\}$	$\{a, c\}$	$\{b, d\}$	X
$\{a, d\}$	$\{b\}$	$\{a\}$	$\{b, d\}$	$\{a, b, d\}$
$\{b, c\}$	$\{d\}$	$\{c\}$	$\{b, d\}$	$\{b, c, d\}$
$\{b, d\}$	$\emptyset$	$\emptyset$	$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{d\}$	$\{c, d\}$	$\emptyset$	$\{c, d\}$
$\{a, b, c\}$	$\{b\}, \{d\}$	$\{a, b, d\}$	$\{d\}$	X
$\{b, c, d\}$	$\{d\}$	$\{c, d\}$	$\{b\}$	$\{b, c, d\}$
$\{a, c, d\}$	$\{b\}, \{d\}$	$\{a, c, d\}$	$\{b\}$	X
$\{a, b, d\}$	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{b, c, d\}$
$\{a, b, c, d\}$	$\{b\}, \{d\}$	X	$\emptyset$	X

(5.27)

The T4A is a connected topology. However, there are subsets (for example the subset $\{a, d\}$) which are disconnected.

Example 3: The T4B topology (WARNING THIS TABLE IS YET TO BE WORKED OUT)

In the T4B topology, the open sets are chosen as

$$T4B\{open\} : \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X. \tag{5.28}$$

The truth table for the T4A topology is:

The T4B Topology of 4 points

$$X = \{a, b, c, d\}$$
$$T4B\{open\} : \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X$$
$$T4B\{closed\} : X, \{b, c, d\}, \{a, b\}, \{b\}, \{a\}, X$$

Subset	**Limit Pts**	**Interior**	**Boundary**	**Closure**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{c\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c, d\}$
$\{d\}$	$\emptyset$	$\emptyset$	$\{d\}$	$\{d\}$
$\{a, b\}$	$\{b\}$	$\{a, b\}$	$\emptyset$	$\{a, b\}$
$\{a, c\}$	$\{b\}, \{d\}$	$\{a, c\}$	$\{b, d\}$	X
$\{a, d\}$	$\{b\}$	$\{a\}$	$\{b, d\}$	$\{a, b, d\}$
$\{b, c\}$	$\{d\}$	$\{c\}$	$\{b, d\}$	$\{b, c, d\}$
$\{b, d\}$	$\emptyset$	$\emptyset$	$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{d\}$	$\{c, d\}$	$\emptyset$	$\{c, d\}$
$\{a, b, c\}$	$\{b\}, \{d\}$	$\{a, b, d\}$	$\{d\}$	X
$\{b, c, d\}$	$\{d\}$	$\{c, d\}$	$\{b\}$	$\{b, c, d\}$
$\{a, c, d\}$	$\{b\}, \{d\}$	$\{a, c, d\}$	$\{b\}$	X
$\{a, b, d\}$	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{b, c, d\}$
$\{a, b, c, d\}$	$\{b\}, \{d\}$	X	$\emptyset$	X

(5.29)

The T4B topology is a disconnected topology, as $\{a\}$ and $\{b, c, d\}$ are both open and closed.

5.0.12 Problems

1. Determine which of the following classes of subsets form a topology on X = (abcde). Are the sets selected open or closed?
   a. $T1 = \{X, 0, (a), (cd), (acd), (bcde)\}$
   b. $T2 = \{X, 0, (a), (cd), (acd), (bcd)\}$
   c. $T3 = \{X, 0, (c), (be), (abe), (bcde)\}$
   d. $T4 = \{X, 0, (a), (be), (abe), (bcde)\}$
2. For those collections above find the limit sets if the subsets form a topology
3. Let $X = \{A, F, H, K\}$ be four points.
   Show that
   $$CT := \{X, 0, A, H, A \cup F, H \cup K, A \cup H, A \cup H \cup K, A \cup F \cup H\}$$
   defines a topology of open sets.
   Find all limit points, closures, interiors, and boundaries of all subsets.

This topology will be defined as the Cartan Topology, and is important for the study of relativistic dynamical systems. The "point" A will be put into correspondence with the 1-form of Action for the physical system. In electromagnetism it will be defined in terms of the potentials. The set $F = dA$ turns out to be the "limit points" for A. In electromagnetism, the coefficients of the 2-form F turns out to be the collection of electromagnetic **E** and **B** fields. Hence, the **E** and **B** fields can be considered as the "limit points" of the potentials (relative to the Cartan Topology).

Remarks

The concept of continuity based on open sets exemplified above is much too hard to apply in physical systems, because it presupposes knowledge of the inverse image. The inverse image often is either hard to define, or it does not exist. The equivalent definition of continuity which is much more useful is based upon the concept of closure. To repeat, a process is said to be continuous if and only if for every subset A of the initial state, X, the closure of the subset, Ac , relative to the topology of X is included in the closure of the image of the subset, with respect to the topology of the final state, Y. In physical evolution, the process mapping can be determined by experiment (or theory), but whether or not it is continuous is a problem to be solved. With the definition based on closure, all that has to be determined is what the closures are on the initial state for every subset relative to the topology of the initial state (which is usually given information). Next construct the forward image of the closures. Then test the forward (not inverse) images of the subsets for their closures with respect to the final state. If the closures are included in the closures of the forward images, the map is continuous. Surprisingly this method turns out to be not too difficult when the evolutionary mapping is at least 2 times differentiable.

Now if it is known that topology of the final state is not the same as the topology of the initial state, and if can be shown that the forward mapping is continuous, then the conclusion is reached that either the inverse image mapping is NOT continuous, or it does not exist. In either case it could be said that the continuous evolutionary process is irreversible. Such is the basis of a theory of the aging process.

All of the concepts defined above, such as the limit set, a boundary set, closure, etc., are what are called topological properties. The list of topological properties is not exhausted by the above collection of ideas. Topological properties are preserved with respect to continuous and reversibly continuous mappings between an initial state and a final state (homeomorphisms). Some of the properties above are preserved if the mapping is only continuous; that is, the inverse mapping does not have to be continuous and yet the property is preserved.)

The concept of invariance with respect to mappings is at the heart of almost every physical theory. The conservation laws (of energy, momentum, angular momentum, etc.) are concepts of evolutionary invariance.

Much of current physical theory has historically emphasized the boundary and open set point of view. It turns out that these ideas are natural with respect to what in tensor analysis is called a contravariant point of view. Contravariant things are like velocity and acceleration of a point. In physics, these contravariant notions correspond to the concept of particles. In topology, these ideas lead to a specialization called homology theory.

Another point of view places emphasis on the concepts of limit points and closure. It may be shown that these ideas are natural with respect to what in tensor analysis is called the covariant point of view. In physics, these notions correspond to the concept of waves and fields. In topology these ideas lead to a specialization called co-homology. It can be shown that the first law of thermodynamics is essentially a statement about cohomology theory. All of these ideas above are independent from the concept of distance, which is often an elementary building block of physical measurement.

When such geometrical constraints (such as a metric, or things involving size and shape) are placed on a physical system, then the particle and the wave point of view, or more precisely, the contravariant or covariant point of view become an alias and an alibi. That is, the ideas of one point of view can be converted one into the other; Moreover, it is then impossible to measure intrinsic differences between the two points of view. The evolution of the contravariant set can be compared to the evolution of the covariant set, with identical comparisons found in both the initial and the final states, if the topological properties are preserved during the evolution.

However, if the evolutionary process does not preserve the topological properties, then differences between the two points of view lead to measurable differences. The concept of aging is one of these differences.

Realize that the concepts of isolated, closed and open sets form the basis of the theory of thermodynamics. It is therefor apparent that thermodynamics is a study of topological properties of matter. It is unfortunate that almost no engineering literature, and very little physical literature, emphasizes this fact

Chapter 6
APPENDIX 2. CARTAN'S METHODS OF EXTERIOR CALCULUS

6.1 Introduction

Cartan's methods utilize (what might be unfamiliar) techniques that are described as the "exterior product" with an algebraic symbol, ˆ, and the "exterior differential" with a differential symbol, d, acting on objects, ω^p, defined as exterior differential forms of degree (not power) p. These basic concepts will be discussed below, briefly, but are best studied in detail from texts such as that by Harley Flanders ("Exterior Differential Forms"), Bamberg and Sternberg ("A course in Advanced Calculus ", Vols 1 and 2) and by Gockeler and Schucker ("Differential Geometry, Gauge theories, and Gravity"). I believe the best way to learn about these "new" operations and the objects upon they act is to try a few examples. Flanders is the best text with which to start. About the only thing missing in the Flander's presentation is a discussion of the Lie derivative acting on p-forms.

There are a number of other texts available that discuss exterior differential forms, but many are a bit pedantic or too pompous to be useful at the applied engineering level. Most applications that have appeared in the literature are in the field of general relativity or super-symmetry or super-gravity or string types of theories. Very few texts are to be found which present the Cartan methods as applied to hydrodynamics, electrodynamics, thermodynamics, and other engineering sciences. (domage!) To reiterate previous statements, recall that the methods of exterior differential forms are important because they carry topological information, and can be used to study topological evolution. My interest in the Cartan methods was cemented when I realized how natural it was to write Maxwell's equations in terms of differential forms. Moreover it became apparent the Maxwell theory was independent from metric or connection constraints. From this point of view (perhaps initiated by VanDantzig) electromagnetism is not a geometrical theory, but instead is a topological theory. The PDE's of the Maxwell - Faraday induction equations, form a nested set in every dimension greater than 3. Experience with electromagnetic theory is very useful, for you can use EM theory to check your developing skills with the Cartan methods. If your application of the Cartan techniques does not replicate well known results in electromagnetism, you have made a mistake. These concepts will be detailed in that which follows.

There are two other important operations, besides the exterior product and the exterior differential, which act on differential forms, but these other operations require the specification of a vector field, V, in addition to the differential form, ω. From a physical point of view, differential form(s) may be used to define a physical system, and the vector field may be used to define a thermodynamic evolutionary process. What is remarkable is that this point of view can be used to justify the topological basis of thermodynamics, and to give a non-statistical description of irreversible processes. The two additional operations (with respect to V) are called the "interior product" with the symbol $i(V)$, and the Lie differential, $L_{(V)} = i(V)d + di(V)$, which combines two operations, the interior product and the exterior derivative. The Lie differential is an alternative to the "covariant" differential of tensor analysis, and, like the covaraint derivative, will produce tensors from other tensors by means of a differential process. Examples and definitions will be given below. The Lie differential will become the most important tool from a topological perspective, for it permits computations to be made which will distinguish those objects which are topological invariants of a process and those objects which are not. It is remarkable that the Lie differential operating on a 1-form of Action is equivalent to the cohological statement that defines the first law of thermodynamics.

The exterior differential forms (differential forms for short) are objects that are built from functions defined on a vector bundle. What this means is that starting from the assumption that there exists an n dimensional variety of independent variables, $\{y^a\}$, often called coordinates, it is possible to construct two other vector spaces. These two vector spaces consist of a 1 dimensional vectorspace $\Lambda^0\{1\}$ and an m dimensional vector space $\Lambda^1\{\sigma^k\}$. The one dimensional vector space $\Lambda^0\{1\}$ consists of all functions that can be constructed from $\{y^a\}$, and has the unit 1 as a basis element. The m dimensional vector space $\Lambda^1\{\sigma^k\}$ is endowed with a "differential" basis elements, σ^k, $k = \{1, 2, ...m\}$ The basis set σ^k is presumed to be linearly related to the differentials dy^a of the independent variables via the formula

$$|dy^a\rangle \Rightarrow \left|\sigma^k\right\rangle = \left[F_a^k\right] \circ |dy^a\rangle . \tag{6.1}$$

(The dimension of the set σ^k may be different from the dimension of $\{y^a\}$.) An alternative point of view is that a linear combination of the differentials on the initial state $|\sigma^a\rangle$ such that the linear mapping $\left[F_a^k\right]$ acting on the $|\sigma^a\rangle$ produces perfect differentials $\left|dx^k\right\rangle$ on the final state (This point of view is assumed in the Flander's book).

$$|\varpi^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[F_a^k\right] \circ |\varpi^a\rangle . \tag{6.2}$$

The two vector sets of linear combinations of differentials, $|\varpi^a\rangle$ and $\left|\sigma^k\right\rangle$ on the initial state are not the same, even when the dimension of the initial and final states are the same. Given the initial state $|dy^a\rangle$, the $|\sigma^k\rangle$ are determined by the linear map $[F_a^k]$. Given the final state $|dx^k\rangle$, the $|\varpi^a\rangle$ are determined by the inverse of the linear map, $[F_a^k]^{-1} = [G_k^a]$. If the two spaces are not of the same dimension then the inverse of the linear map need not exist. It is important for generalizations that the concept of the initial state or domain (and its coordinate functions, y^a) be kept distinct from the final state or range (and its coordinate functions, x^k). The format $|\sigma^k\rangle = [F_a^k(y)]\ |dy^a\rangle$ will be emphasized herein.

All of this had its origins in the theory of differentiable coordinate mappings from an initial to a final state, where, given a (more than likely) non-linear map ϕ from a domain $\{y^a\}$ to a range $\{x^k\}$, a linear map $d\phi$ between differentials could be generated to establish the (linear) differential vector space ideas. In symbols

$$Nonlinear \quad \phi : \{y^a\} \Rightarrow \{x^k\} = \phi^k(y) \tag{6.3}$$

$$Linear \quad d\phi : |dy^a\rangle \Rightarrow |dx^k\rangle = [\partial\phi^k(y)/\partial y^a]\ |dy^a\rangle = [F_a^k(y)]\ |dy^a\rangle . \tag{6.4}$$

The Linear mapping $[F_a^k(y)]$ so generated is the Jacobian matrix of partial differentials of the coordinate (or vector space) mappings. As an example, review the concept of spherical or cylindrical coordinates mapped into cartesian space. (Maple programs have been developed giving most of the details in terms of symbolic mathematics. See http://www22.pair.com/csdc/pdf/mtpertu5.pdf)

At first, in this presentation, the dimension of the vector space $|\sigma^k\rangle = |dx^k\rangle$ will be assumed to be n, the same dimension as the space of independent variables. Then the ranges of the index a is: $a = \{1...n\}$, and the range of the index k is : $k = \{1...n\}$. This restriction will be relaxed later during the development of a more general theory. The $n\,x\,n$ Jacobian matrix of the n mapping functions establishes the vector space ideas as a linear mapping, and gives the primitive realization of what is to be known as a Frame matrix (of functions on the initial state).

Suppose that another function, say $\Phi(y^a)$, is given in terms of the initial variety, $\{y^a\}$. Then its total differential is given by the expression,

$$d\Phi(y^a) = \{\partial\Phi(y^a)/\partial y^b\}\, dy^b = \sum_b A_b(y^a) dy^b = \langle A_b(y^a)| \circ |dy^b\rangle . \tag{6.5}$$

The object on the right is an example of an exterior differential 1-form, ω^1, with coefficient functions $A_b(y^a)$ and basis elements, dy^b. (From here on the sum convention on up-down symbols - the index b in the formula above - will be presumed, without the use of the $\sum$ symbol). The coefficients, by construction in this example,

$$A_b(y^a) = \{\partial\Phi(y^a)/\partial y^b\}, \tag{6.6}$$

form the components of a covariant gradient vector field.

In the Cartan theory of differential forms these concepts are extended to situations where the differential basis elements σ^k, of the vector space $\Lambda^1\{\sigma^k\}$ can be written in terms of some arbitrary matrix (of functions on the initial state) acting on the differentials of the independent variables in a linear way:

$$\left|\sigma^k\right\rangle = \left[F_a^k(y)\right] \left|dy^a\right\rangle . \tag{6.7}$$

In other words, in the Cartan extension, it is not assumed that the linear map $\left[F_a^k(y)\right]$ is necessarily a Jacobian matrix of some non-linear coordinate mapping, nor is it assumed that the matrix is even similar to the Jacobian matrix of a coordinate mapping. This more general matrix of functions, $\left[F_a^k(y)\right]$, will be defined as a basis Frame and is the cornerstone of Cartan's development of the Repere Mobile. Given such a matrix of functions a key question revolves about the determination of the solubility of the Frame. Given a Frame, does there exist a unique set of mapping functions ϕ from which the Frame is determined to be the Jacobian matrix $d\phi$ of the mapping? If not, is it possible that there exists a non-unique solution set to the problem? The question of non-unique integrability of the Frame matrix is the basis of what is called Affine and Topological Torsion. Torsion appears when the basis Frame (or its equivalence class) is NOT integrable.

For applications, how the Frame may be related to specific physical problems is of key importance. The early development of the Frenet-Serret-Cartan Frame for a point moving along a space curve indicates that it is possible to construct the Frame from differentials of the mapping function with respect to a parameter along a space curve. That is, the velocity, acceleration and the rate of change of acceleration can be used to build a Frame matrix of a point moving along a space curve in three dimensions. These things are physical, measureable, and applicable quantities. The same idea can be generated for continuous media such as a fluid. The velocity field, the vorticity field, and the helicity field of the fluid become the analogs of the Frenet - Serret differentiations.

For those ubiquitous cases (or better said, on those restricted domains) where the Frame has an inverse, then the Frame matrix is an element of the General Linear group. Often in particular applications the Frame matrix is constrained to be an element of an equivalence class of "admissable" Frames by assuming the Frame belongs to some sub-group of the GL group. In the Frenet-Serret case, a usual restriction constrains the 3D Frame matrix elements such that the Frame is a member of the special othogonal (orthonormal) group. The columns of the basis Frame matrix are orthogonal unit vectors. This constraint is used to create the concepts of arc length, curvature, and torsion of the 3D space curve. These "intrinsic" properties of the space curve are the similarity invariants of all equivalent Frames (that is, all Frames that are members of the orthonormal group). These intrinsic (often called invariant) properties of the equivalence class are computed by means of the coefficients of the Cayley-Hamilton theorem.

From a physics point of view, all observers who may use different elements or representations of the orthonormal group for reference systems will be able to express their views in terms of a common set of qualities, the similarity invariants. All equivalent observers will agree that the values of the similarity invariants are the same. Restrictions to paarticular subgroups are often called "gauge theories". It is important to note that certain (normal) subgroups (such as the orthonormal subgroup) cannot distinguish between left and right handedness (chirality) , but other equivalence classes of subgroups can. It would seem that this ability to distinguish a chiral property is of value to the study of biological systems, where most biological molecules appear to be left or right handed. The moral (or warning) of this paragraph is that the common orthonormal system of basis vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of engineering practice must be modified to handle chiral distinctions.

It is important to be reminded of the idea of a similarity transformation. Given a matrix $[M]$ and a transformation matrix $[F]$, the matrix $[N]$ is said to be similar to $[M]$ if

$$[M] \Rightarrow [N] = [F]^{-1} \circ [M] \circ [F] . \tag{6.8}$$

When the Cayley-Hamilton polynomial is constructed for $[M]$ and $[N]$ the coefficients of the polynomials are the same (if the matrices are "similar"). Two of the important similarity invariants are the trace of $[M]$ and its determinant. In differential geometry, these ideas will be used to define curvature properties of manifolds. In the Frenet-Serret-Cartan theory of the orthonormal subgroup, the similarity invariants lead to the concepts of arc length, curvature and torsion. The zero sets of the similarity invariants have particular physical importance. In the thermodynamics of a VanderWaals gas, the Cayley-Hamilton polynomial based upon the Gibbs function is a cubic polynomial with the surface shape of a swallow-tail. The critical point is where all three similarity invariants vanish. The spinodal line of phase instability is where the quadratic similarity coefficient (the Gauss curvature of the swallow-tail surface) vanishes.

The similarity equation can be rewritten in a manner that does not require the immediate computation of an inverse:

$$[F] \circ [N] = [M] \circ [F] . \tag{6.9}$$

This equation can be used to test if $[N]$ is similar to $[M]$. A special situation occurs if the matrix $[N]$ is the same as $[M]$. This situation places a constraint on the equivalence class of matrices that can be used for the transformations $[F]$. Suppose that $[N] = [M] \mp d[M]$, then the differential similarity equation becomes

$$d[M] = [M] \circ [F] \pm [F] \circ [M] , \tag{6.10}$$

and is suggestive of the Heisenberg matrix operator format (the transformation matrix $[F]$ plays the role of the "Hamiltonian" operator). These similarity formats will reappear below when the matrix of connection 1-forms is discussed.

Note that the column vector array of differential basis elements, $|dy^a\rangle$, transforms as a contravariant tensor in the Jacobian case, where a coordinate mapping is available. This property can be extended if the Frame matrix of functions has a non-zero determinant, for then the columns of the Frame matrix can be used as a basis set for contravariant vectors in the initial space (domain or state). This basis argument does not depend upon the fact that matrix elements $\left[F_a^k(y)\right]$ form a Jacobian (i.e., integrable) system. It will be demonstrated that it is this lack of unique integrability for the $\left|\sigma^k\right\rangle = \left[F_a^k(y)\right] |dy^a\rangle$ that leads to the concepts of affine Torsion and Topological Torsion , two topics that will be discussed in great detail in subsequent sections.

At present, given a basis of 1-forms, construct arbitrary exterior differential 1- forms from the matrix product of arbitrary coefficient functions arranged as a row vector, $\left\langle \widehat{A}_k(y^a)\right|$ and the basis set arranged as a column vector, $\left|\sigma^k\right\rangle$

$$\omega^1 = \left\langle \widehat{A}_k(y^a)\right| \circ \left|\sigma^k\right\rangle = \widehat{A}_k(y)\ \sigma^k. \tag{6.11}$$

Note that if the coefficient functions are chosen to be a covariant vector array (and that is why the index is a lower index on the $A_k(y)$), then the differential 1-form ω is a scalar invariant of "coordinate transformations". The coefficient functions, however, do not have to be a gradient array. The covariant constraint implies that if (as assumed)

$$\left|\sigma^k\right\rangle = \left[F_a^k(y)\right]\ |dy^a\rangle \tag{6.12}$$

then

$$\left\langle \widehat{A}_k(y^a)\right| = \langle A_b| \circ \left[G_k^b\right] \tag{6.13}$$

where $\left[G_k^b(y)\right]$ is the inverse matrix of functions to the frame matrix, $\left[F_a^k(y)\right]$. These are the rules of classical tensor analysis defining what is meant by contravariant and covariant vectors of ordered sets of components with respect to special transformations (defined as diffeomorphisms).

The differential form so constructed in terms of tensor coefficients is then independent from a "choice of coordinate system".

$$\omega^1 = \left\langle \widehat{A}_k(y^a)\right| \circ \left|\sigma^k\right\rangle = \langle A_b| \circ \left[G_k^b\right] \circ \left[F_a^k\right] \circ |dy^a\rangle\,. \tag{6.14}$$

For physicists and engineers what this implies is that laws of physics written in terms of differential forms are independent of the observer's choice of a "reference system". A "reference system" is defined as an element of an equivalence class of differentiable mappings. The most common equivalence class usually accepted is the class of diffeomorphisms, which implies that the mapping, $\phi,$ and the linear mapping, $d\phi$,have inverses, and the inverse mapping is differentiable. Such diffeomorphic mappings are constrained subsets of other mappings known as homeomorphisms. Homeomorphisms (and therefor diffeomorphisms) preserve topology from initial to final state, and therefor cannot be used to describe topological evolution. (Bummer.) Sometimes the equivalence class of reference systems is even further constrained. For example, the acceptable class of reference systems known as inertial frames of reference in the physics of special relativity is constrained to be the Lorentz equivalence class. Sometimes such constraints throw the baby out with the wash. For example, General Relativity is designed to admit all diffeomorphisms as the equivalence class of frames of reference; Special Relativity admits only elements of the Lorentz equivalence class, which is a subset of all diffeomorphisms.

The Lorentz equivalence class consists of those matrices, $[L]$, for which the Minkowski line metric is preserved. That is

$$\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} = [\eta] = \left[L^{tranpose}\right] \circ [\eta] \circ [L] \tag{6.15}$$

There is a further subclass of Lorentz matrices, with matrix elements which are constants, and which are use in special relativistic (non-accelerated) applications. (This special subclass turns out to be "affine" torsion free, so that left handed and right handed chirality species evolve in the same way). However, there are Lorentz matrices that preserve the Minkowski metric that are not composed of constant elements. Such matrices admit accelerations, and also admit Affine Torsion coefficients. (See http://www22.pair.com/csdc/pdf/lorentz.pdf). That is, the system of 1-forms generated by a Lorentz transformation of non-constant elements is not necessarily uniquely integrable, and therefor admit different behavior for chiral systems. (This difference in behavior distinguishes Optical Activity from Faraday Rotation in electromagnetic systems.)

. A major thrust of the work that appears on Cartan's Corner is that the reference systems are extended to include topological change, so that non-diffeomorphic transformations will be investigated. A closer look a Cartan's concepts yields the result that, unlike tensors which are well behaved with respect to diffeomorphisms, exterior differential forms are well behaved in a functional sense with respect to a class of transformations even wider than the class of diffeomorphisms – in fact, wider than the class of homeomorphisms! Hence differential forms are useful to the study of topological evolution, which is the main theme of Cartan's Corner and these lectures. This result will be exploited in the following chapters. (See "Retrodictive Determinism" http://www22.pair.com/csdc/pdf/retrodic.pdf).

It should be realized that differential forms have the tensor like property that if the differential form is zero in one coordinate system of reference, then it is zero in all other diffeomorphically equivalent systems, no matter what constraints are applied to limit the elements of the equivalence class of diffeomorphisms. In addition, if a differential form is zero on the final state, then its pullback to the initial state is also zero with respect to continuous but not homeomorphic, and therefor not diffeomorphic maps. (See http://www22.pair.com/csdc/ed3/ed3fre1.htm)

6.2 The exterior algebra

The exterior algebra of Cartan is based upon an associative, but not commutative, multiplication rule defined as the exterior (wedge or hook) product of objects defined as exterior differential forms. The symbol for the product is ˆ. The structure of the algebra can be built starting from 1-forms on the n-dimensional vector space $\Lambda^1\{\sigma^a\}$, in terms of a basis of 1-forms denoted by $\{\sigma^a\}$. The arbitrary 1-form is constructed from the basis elements according to the formula given above, $\omega^1 = A_k(y)\,\sigma^k$. The addition rule of the algebra is that of vector space addition: add the coefficients of the same basis elements.

$$\begin{aligned} \omega 1 &= A_k(y)\,\sigma^k \\ \omega 2 &= B_k(y)\,\sigma^k \\ \omega 1 + \omega 2 &= \{A_k(y) + B_k(y)\}\,\sigma^k. \end{aligned} \tag{6.16}$$

Example 9 *Add* $(3x\,dx + 4xz\,dy)$ *and* $(2y\,dy + 17y\,dz)$:

$$basis = (dx, dy, dz)$$

$$(3x\,dx + 4xz\,dy) + (2y\,dy + 17y\,dz) = 3x\,dx + (4xy + 2y)\,dy + 17y\,dz$$

It is the multiplication rule that is perhaps unfamiliar. The multiplication rules are defined in terms of elements of the basis set.

$$\begin{aligned} \sigma^a\hat{}\sigma^b &= -\sigma^b\hat{}\sigma^a \\ \sigma^b\hat{}\sigma^b &= 0 \end{aligned} \tag{6.17}$$

$$\begin{aligned} dy^a\hat{}dy^b &= -dy^b\hat{}dy^a \\ dy^b\hat{}dy^b &= 0 \end{aligned} \tag{6.18}$$

These rules are similar to the cross product of Gibbs 3D vector analysis, but the difference is that the exterior product rule extends to n dimensions (the Gibbs cross product does not) and is associative (Gibbs product is not). Associative means $(\sigma^a\hat{}\sigma^b)\hat{}\sigma^c = \sigma^a\hat{}(\sigma^b\hat{}\sigma^c)$. In 3D, the Gibbs cross product yields $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

Example 10 *Multiply* $A = \sum A_k\sigma^k$ *times* $B = \sum B_m\sigma^m$ $\qquad A\hat{}B = C$

$$\begin{aligned} A\hat{}B &= \{A_k\sigma^k\}\hat{}\{B_m\sigma^m\} \qquad (6.19) \\ &= ...A_kB_k\sigma^k\hat{}\sigma^k + ...A_kB_m\sigma^k\hat{}\sigma^m + ...A_mB_k\sigma^m\hat{}\sigma^k \\ &= ... \quad 0 \qquad\qquad + ...A_kB_m\sigma^k\hat{}\sigma^m - ...A_mB_k\sigma^k\hat{}\sigma^m \\ &= ... \ \{A_kB_m - A_mB_k\}\sigma^k\hat{}\sigma^m... \\ &= ...C_{[km]}\sigma^k\hat{}\sigma^m.. \\ &= ...C_{[km]}\sigma^{[km]}... \\ &= ...C_H\sigma^H \end{aligned}$$

Note that $A\hat{}B \neq B\hat{}A$ and that the exterior product of two elements of the vector space $\Lambda^1(\sigma^k)$ produce a linear combination of paired basis elements of the form $(\sigma^k\hat{}\sigma^m)$. Moreover the coefficients of these elements are always anti-symmetric under interchange of the paired indices. The anti-symmetry and the rules of 1-form mltiplication permit the writing of the product of two 1-forms in terms of another vector space, whose basis elements are the anti-symmetric pairs $(\sigma^k\hat{}\sigma^m)$. This object is defined as a 2-form. It is conventional to rewrite the 2-form so constructed without repeating basis element pairs that are of different sign. That is, $\Lambda^2(\sigma^k\hat{}\sigma^m) \Rightarrow \Lambda^2(\sigma^{[km]}) = \Lambda^2(\sigma^{[H]})$.

The symbol $H = [km]$ stands for all order pairs where $k < m$, and is equivalent to the set $(12, 13, 14.....23, 24.....34...)$. It is apparent that the number of ordered singlet basis elements in 3 dimensions is the same as the number of ordered anti-symmetric pairs; but this is only true in 3D. For 4D, the number of singlet basis elements is 4 and the number of ordered anti-symmetric pairs is 6. The result of the exterior product is to produce from elements of one vector space of dimension n, another element of a different vector space of dimension $n(n-1)/2$. In this limited sense the exterior product is not closed. The exterior multiplicative combination of two objects of the same type (1-forms) does not produce an object of the same type, but instead produces a 2-form. The process of exterior multiplication can be repeated where 2 -forms are multiplied by 1-forms to produce 3-forms, and 3-forms are multiplied by 1-forms to produce 4-forms, ultimately building a "closed" algebra. The elements of the closed algebra will consist of classes, or vector spaces with basis element doublets, $\sigma^a\hat{}\sigma^b$, classes of triplets, $\sigma^a\hat{}\sigma^b\hat{}\sigma^c$, ... and even n-tuplets of basis vectors, $\Omega = \sigma^a\hat{}\sigma^b\hat{}\sigma^c\hat{}...\hat{}\sigma^n$. However, the rules of multiplication are such that the exterior multiplicative combination of more that n basis vectors must vanish. Hence any element of the algebra times another element of the algebra is an element of the algebra, or zero. In this sense, the exterior algebra is closed.

The doublets are called 2-forms, the triplets are called 3-forms, and the n-tuplets are called n-forms. The multiplication rules demonstrate that each of the p-tuplets has a number of linearly independent elements equal to the possible combinations of n things take p at a time. Each of the p-tuplets forms a vector space basis of dimension equal to the appropriate combinatorial number of Pascal's triangle.

$$\text{Pascal's Triangle}: \begin{array}{lccccccc} n=1: & & & & 1 & & & \\ n=2: & & & 1 & 2 & 1 & & \\ n=3: & & 1 & 3 & & 3 & 1 & \\ n=4: & 1 & 4 & & 6 & & 4 & 1 \end{array} \tag{6.20}$$

The vector subspace dimension of the 0-forms, ω^0, is 1; the dimension of the 1-forms, ω^1, is n, the dimension of the 2-forms, ω^2, is equal to the number of combinations of n things taken 2 at a time (n(n-1)/2 is equal to 3 for n =3, equal to 6 for n = 4, etc.); the dimension of the n -1 forms, ω^{n-1}, is n, the dimension of the n-forms, ω^n, is 1. The dimension of the exterior algebra is the sum of the dimensions of all vector spaces produced by the exterior product; this dimension is equal to 2^n.

As an example consider the exterior algebras up to $n = 4$. The elements of Pascal's triangle yield a 1-dimensional (scalar) vector space Λ^0 for the 0-forms, a 4 dimensional vector space Λ^1 for the 1-forms, a 6 dimensional space Λ^2 for the 2-forms, a 4 dimensional vector space Λ^3 for the 3-forms, and a 1 dimensional vector space Λ^4 for the n=4 forms.

From geometical studies in 3 dimensions, n=3, the elements of the 4 different vector spaces are called points, lines, surfaces and volumes. From applications to 3D mechanics, the position vector and the momentum vector are from the vector subspace, Λ^1, and their exterior (cross) product is an element of the vector subspace, Λ^2. The fact that angular momentum is an element of a vector space Λ^2 is why one never sees angular momentum added to linear momentum (which is an element of a different vector space Λ^1) in the elementary mechanics text books. From applications to special relativistic physics in 4 dimensions, n=4, the coefficients of the elements of these five different vector subspaces of the exterior algebra are known as scalars, vectors, tensors, pseudo-vectors, and pseudo scalars.

$p-forms\ in\ 4D$	0	1	2	3	4
$\#\ of\ basis\ elements$	1	4	6	4	1
name	Scalars	Vectors	Tensors	Pseudo-Vectors	Pseudo-Scalars

(6.21)

The Cartan-Grassman exterior algebra in consists of a vector space of 2^n (= 16 in 4D) components with n+1 (= 5 in 4D) different vector subspaces. The algebra is technically called a graded algebra. The exterior algebra is closed with respect to multiplication, for all possible products of the algebra reside within the algebra of 2^n dimensions (or are zero). The exterior product of a p form and a q-form produces a p+q form, or zero if p+q > n. In every case, the higher p-forms can be constructed from sums of products of the singlets. All elements of the algebra can be constructed from linear combinations of the primitive n basis elements, σ^k and their products.

A nice feature of the exterior algebra (besides being closed) is that the definitions of symbolic operations can be described entirely in terms of 0-forms and 1-forms, when the collective index is used. Every p-form can be rewritten in terms of symbolic coefficients (0-forms) and basis elements (p-forms from vector spaces of different dimensionality of course) with a format similar to that of a 1-form. For example, for n = 4, there are 6 elements of the vector subspace of two forms. That is, there are 6 independent non-zero pairs of of the 4 basis 1-forms σ^k from Λ^1 that can be used as the basis elements of Λ^2 : namely the set $\{\sigma^1\hat{}\sigma^2, \sigma^1\hat{}\sigma^3, \sigma^1\hat{}\sigma^4, \sigma^2\hat{}\sigma^3, \sigma^2\hat{}\sigma^4, \sigma^3\hat{}\sigma^4\}$. If these basis pairs are given a new symbolism as $\{\sigma^{12}, \sigma^{13}, \sigma^{14}, \sigma^{23}, \sigma^{24}, \sigma^{34}\}$,then the general 2-form (for n=4) can have the expansion coefficient - basis representation given by the formula:

$$F = A_{12}\sigma^{12} + A_{13}\sigma^{13} + ... + A_{34}\sigma^{34} = A_H\sigma^H, \tag{6.22}$$

where H is the collective index described above for ordered pairs. This formula for the 2-form F in 4 dimensions looks like a vector formula for a 1-form, but in space of 6, not 4, dimensions; it is just that the index labels are different. All of the distinct basis combinations will be completely antisymmetric in their indices ($p > 1$). For example, H could be the set of triples $[i1, i2, i3]$ with $i1 < i2 < i3,$ for the vector space of 3 forms, H could be the set of quadruples $[i1, i2, i3, i4]$ with $i1 < i2 < i3 < i4,$ for the vector space of 4 forms; etc. With this collective index notation, combinatorial rules of multiplication and differentiation developed for 1 forms can be applied directly to higher order p-forms.

This technique, where a 2-form expansion in 4D was used as a 6 - vector, was applied (intuitively?) more than 60 years ago by Arnold Sommerfeld in his studies of electromagnetic systems. The 3 components of **E** and the 3 components of **B** formed the components of the 6-vector. (See his volumes on Lectures on Theoretical Physics.) A similar but little used 6 vector composed of the acceleration and vorticity can be developed for a fluid. It is not clear whether Sommerfeld knew the theory of exterior differential systems at the time of his 6 - vector development.

Example 11 *in 3D, exterior multiply the 1-form, A, times 2-form, B to produce the 3-form $C = A\hat{}B$.*

$$\begin{aligned} & (A_x dx + A_y dy + A_z dz)\hat{}(B_x dy\hat{}dz + B_y dz\hat{}dx + B_z dx\hat{}dy) \\ = \; & (A_x B_x + A_y B_y + A_z B_z) dx\hat{}dy\hat{}dz = C \end{aligned}$$

Note that this product of a 1-form and and n-1=3-1=2 - form produces a n - form with a coefficent that looks the same as the euclidean inner product of two ordinary vectors $\mathbf{A} \circ \mathbf{B}$. Recall that the 1-form has n components and the n-1 form has n components. The euclidean inner product result is valid in all dimensions, n.

Example 12 *in 3D, exterior multiply the 1-form, A, times 1-form, B to produce the 2-form $D = A\hat{}B$.*

$$\begin{aligned} D &= (A_x dx + A_y dy + A_z dz)\hat{}(B_x dx + B_y dy + B_z dz) \\ &= (A_x B_y - A_y B_x) dx\hat{}dy \\ &\quad + (A_y B_z - A_z B_y) dy\hat{}dz \\ &\quad + (A_z B_x - A_x B_z) dz\hat{}dx \end{aligned}$$

Note that the result has coefficients equivalent to the Gibbs cross product of two vectors, $\mathbf{A} \times \mathbf{B}$.

6.2.1 The Exterior Differential.

The exterior differential is a definition of a differential process acting on p-forms, ω^p. The operation takes the p-form into a p+1 form. Hence, like the exterior product, the exterior differential generates a vector in a different vector subspace of the exterior algebra.

$$d(\omega^p) \Rightarrow \omega^{p+1}. \tag{6.23}$$

Other properties of the exterior differential will be described by the rules for distributing the operator over a product of 1-forms (note the order of factors and the minus sign modification of the Liebniz rule for the differential of a product of scalars)

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB, \tag{6.24}$$

and

$$d(d(\omega^p)) \Rightarrow 0. \tag{6.25}$$

For a product of a p-form and a q-form, it follows that

$$d(\omega^p\hat{}\omega^q) \ = d\omega^p\hat{}\omega^q - (-1)^p\omega^p\hat{}d\omega^q, \tag{6.26}$$

The epitome of the exterior differential is the concept of the total differential of a scalar function, which is a familiar operation that takes the 0-form, or function $\omega^0 = \theta(y^a)$, into the 1-form, $\omega^1 = A_b\, dy^b$.

$$d(\omega^0) = d\{\theta(y^a)\} \Rightarrow \{\partial\theta(y^a)/\partial y^b\}dy^b = A_b\, dy^b = \omega^1, \tag{6.27}$$

The function could be constrained such that the set $\theta(y^a)$ = constant defines an implicit surface. It follows that in the constrained case, the total differential is also zero: $d\{\theta(y^a)\} = 0$. An object that has zero for the value of its exterior differential is said to be closed (in an exterior differential - not algebraic - sense). For consistency reasons note that the differential basis elements symbolized by dy^b are defined to be closed. That is, $d(dy^b) = 0$. The exterior derivatives of an arbitrary basis, σ^k, are NOT necessarily zero, but $d(\sigma^k)$ is closed for $dd(\sigma^k) = 0$.

The exterior differential of a 1-form is defined as

$$\begin{aligned} d\omega^1 &= d(A_b\, dy^b) = (dA_b)\hat{}dy^b + A_b\, d(dy^b) \\ &= (\partial A_b/\partial y^e\, dy^e)\hat{}dy^b + 0 \\ &= (\partial A_b/\partial y^e - \partial A_e/\partial y^b)\, dy^e\ \hat{}\ dy^b \\ &= F_{[eb]}\, dy^{[eb]} = F_{[H]}dy^{[H]}. \end{aligned} \tag{6.28}$$

The collective index notation permits the formula defining exteriordifferentiation to be generalized:

$$d\omega^p = d(A_H\, dy^H) = (dA_H)\hat{}dy^H \tag{6.29}$$

So to compute the exterior derivative of any p -form, first compute the exterior (= total differential) of the scalar coefficients dA_H and then exterior multiply the result into the remaining base elements of the form dy^H, component by component.

Also note that the special 1-form with gradient coefficients, $\omega^1 = d\{\theta(y^a)\}$, has an exterior differential equal to

$$\begin{aligned} d\omega &= d(d\{\theta(y^a)\}) \\ &= ...\{\partial^2\theta(y^a)/\partial y^b\partial y^c\}dy^c\ \hat{}\ dy^b + ... + \{\partial^2\theta(y^a)/\partial y^c\partial y^b\}dy^b\ \hat{}\ dy^c \\ &= ...\{\partial^2\theta(y^a)/\partial y^b\partial y^c - \partial^2\theta(y^a)/\partial y^c\partial y^b\}dy^c\ \hat{}\ dy^b \\ &= 0, \end{aligned} \tag{6.30}$$

for C2 functions. Hence this special 1-form ω^1 is a closed 1-form, assuming the coefficient functions are twice differentiable. However, as the example 1-form ω^1 has a unique primitive function, $\theta(y^a)$, whose exterior derivative creates $\omega^1 = d\theta$, the 1-form ω is said to be not only closed, but also exact. The same concepts hold for all p forms. A p-form is closed if its exterior differential vanishes, and the p-form is exact if it is constructed by means of the exterior differential operation acting on some p-1 form. There are differential forms that are closed but not exact, and those that are neither exact nor closed. The importance of closed and exact, or closed but not exact, p-forms is that they carry topological information about the domain of definition. For example, in a two dimensional surface every hole is associated with a unique 1-form that is closed, but not exact. The number closed but not exact 1-forms on a domain counts the topological number of holes. This fact is the basis of the Bohm-Aharanov idea in EM theory, and is at the foundation of the theory of flight in terms of the Joukowski transformation.

Suppose that the given exterior differential p-form is expressed in terms of a non-integrable basis set σ^H. Then the exterior differential formula becomes

$$d\omega^p = d(A_H\, \sigma^H) = (dA_H)\hat{}\sigma^H + A_H(d\sigma^H) \tag{6.31}$$

Now it must be recognised that the second term $(d\sigma^H)$ is not necessarily zero. Such complications arise when the Frame matrix generates 1-forms (σ^k) which are not closed. The basis Frame in that case is not uniquely integrable.

Remember that the exterior derivative has to be applied to products of 1-forms in terms of a modified Leibniz rule, that alternates in sign for every other odd factor. For example, the exterior derivative of the product of two 1-forms is

$$d(\sigma^1\hat{}\sigma^2) = d\sigma^1\hat{}\sigma^2 - \sigma^1\hat{}d\sigma^2. \tag{6.32}$$

Example 13 *Compute the exterior differential of the function* $\theta(y^a) = (y1)^2 + (y2)^2 + (y3)^2 - 1$.

$$\omega = d\theta(y^a) = 2(y^1 dy^1 + y^2\, dy^2 + y^3\, dy^3)$$

Note that the zero set of the function describes a spherical 2-surface, and the coefficients of the deduced 1-form descibe the normal field orthogonal to the tangent vectors on the surface. Direct computation demonstrates that ω is closed, as

$$d\omega = 2(dy^1\hat{}dy^1 + dy^2\hat{}\, dy^2 + dy^3\hat{}\, dy^3) = 0.$$

Example 14 *Compute the exterior differential of* $(A_x dx + A_y dy + A_z dz)\hat{}(B_x dx + B_y dy + B_z dz)$

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB = d(C_{[mn]}dy^{[mn]}) = d((C_{[mn]})\hat{}dy^{[mn]}$$

The object which is the result of the exterior differentiation of the 2-form constructed by the product is a 3-form with completely antisymmetric indices. The modified Leibniz rule for products is required to make the two ways of computing the resultant 3-form compatible.

The operation of the exterior differential acting on an arbitrary 1-form is defined as

$$\begin{aligned} d(A_k(y^a)\,\sigma^k) &= \{d(A_k(y^a)\}\,\hat{}\,\sigma^k + A_k(y^a)\hat{}\{d\sigma^k\} \\ &= \{\partial A_k(y^a)/\partial y^b\}dy^b\}\,\hat{}\,\sigma^k + A_k(y^a)\hat{}\{d\sigma^k\}. \end{aligned} \tag{6.33}$$

Consider the case where the arbitrary 1-forms are known to be linearly related to the differentials by means of the linear (Frame) formulas, $|\sigma^k\rangle = [F_a^k(y)]\ |dy^a\rangle$. Then

$$\begin{aligned} d(A_k(y^a)\,\sigma^k) &= \{\partial(A_k F_a^k)/\partial y^b\}dy^b\,\hat{}\,dy^k \\ &= \{\partial\widehat{A}_a/\partial y^b - \partial\widehat{A}_b/\partial y^a\}dy^b\,\hat{}\,dy^k \\ &= \widehat{F}_{[ab]}(y)dy^b\,\hat{}\,dy^k, \end{aligned} \tag{6.34}$$

where

$$\widehat{A}_a(y) = (A_k(y)F_a^k(y)) \tag{6.35}$$

This formula for $d(\omega^1) = d(\widehat{A}_a dy^a) = \widehat{F}_{[ab]}(y)dy^b\,\hat{}\,dy^k = \widehat{F}_{[ab]}(y)dy^{[ab]} = \widehat{F}_{[H]}(y)dy^{[H]}$ is valid on the initial state, or variety $\{y^a, dy^a)$, whether the Frame matrix has an inverse or not. The coefficients of the 2-form correspond to the anti-symmetric components of a "curl" (when n = 3). However, the exterior differential procedure generalizes to spaces of higher dimension.

For differential forms expanded in terms of non-closed basis 1-forms, the exterior differential has two terms. The first term is just the exterior product of the total differential of the coefficient function(s) and the remaining factor of non-closed basis forms, while the second term is the exterior product of the functions and the exterior differentials of the non-closed basis forms.

The operation of the exterior differential acting on a p-form follows that same formulas, using the collective ordered index, H:

$$d(A_H(y^a)\,\sigma^H) = \{d(A_H(y^a)\}\,\hat{}\,\sigma^H + A_H(y^a)\hat{}\{d\sigma^H\}. \tag{6.36}$$

Lets examine more carefully the situation for the exterior differential of a 1-form expanded in terms on non-closed basis 1-forms. The outcome of the exterior differential process is to produce a 2-form, which can be expanded in terms of products of 1-forms. For any particular basis 1-form, σ^k, the differential is a 2-form, and as such it can be expanded in terms of the paired basis elements, $\sigma^{[mn]}$. That is

$$\begin{aligned} d\sigma^k &= \Lambda^k_{[mn]}(y^e)\sigma^{[mn]} \\ &= \Lambda^k_{[12]}\sigma^{[12]} + ... + \Lambda^k_{[34]}\sigma^{[34]} \\ &= \Lambda^k_{[12]}\sigma^1\hat{}\sigma^2 + ... + \Lambda^k_{[34]}\sigma^3\hat{}\sigma^4 \end{aligned} \tag{6.37}$$

Hence the exterior differential of a 1-form where the basis σ^k is not integrable is given by the formula

$$d(A_k\sigma^k) = dA_k\ \hat{}\ \ \sigma^k + A_k\Lambda^k_{[mn]}\,\sigma^{[mn]} \tag{6.38}$$

When the basis forms σ^k are closed in a differential sense, then the coefficients $\Lambda^k_{[mn]}$ vanish. How this relates to Affine and Topological torsion will be discussed below, along with the topic of anholonomic coordinates.

The n basis 1-forms must be linearly independent otherwise the dimension of the vector space $\Lambda^1\{\sigma^k\}$ is not n. The implies that the exterior product of the n 1-forms σ^k are such that the n-form so constructed is not zero. For basis 1-forms σ^k constructed from a Frame matrix according to the formula $\left|\sigma^k\right\rangle = \left[F^k_a(y)\right]\ |dy^a\rangle$, the non-zero property for the n-fold product implies that the Frame matrix has a non-zero determinant.

$$\omega^n = \sigma^1\hat{}\sigma^2\hat{}...\hat{}\sigma^n = \det[F]dy^1\hat{}dy^2....\hat{}dy^n \neq 0. \tag{6.39}$$

Such domains are either positive or negative and are therefor said to be orientable.

Compute the exterior differential of the 1-form $A = dz + ydx - xdy$

$$dA = ddz + dy\hat{}dx - dx\hat{}dy = 0 - 2(dx\hat{}dy)$$

Example 15 ***The Gradient:*** *Compute the exterior differential of the general 0-form $\theta(x, y, z)$*

$$d\theta(x, y, z) = \partial\theta/dx\,dx + \partial\theta/dy\,dy + \partial\theta/dz\,dz =$$

The coefficients form the gradient of the scalar function (3D)

Example 16 ***The Curl:*** *Compute the exterior differential of a general 1-form $A = (A_x dx + A_y dy + A_z dz)$*

$$dA = \{\partial A_y/\partial x - \partial A_x/\partial y\}dx\hat{}dy + \{\partial A_z/\partial y - \partial A_y/\partial z\}dy\hat{}dz + \{\partial A_x/\partial z - \partial A_z/\partial x\}dz\hat{}dx$$

The coefficients form the components of the "curl **A**" in 3D.

Example 17 ***The Divergence:*** *Compute the exterior differential of the 2-form* $V = Udy\hat{}dz - Vdz\hat{}dx + Wdx\hat{}dy$

$$dV = dU\hat{}dy\hat{}dz - dV\hat{}dz\hat{}dx + dW\hat{}dx\hat{}dy$$

If $\mathbf{V} = [U(x,y,z), V(x,y,z), W(x,y,z)$, then

$$\begin{aligned} dV &= \partial U/\partial x\, dx\hat{}dy\hat{}dz + \partial U/\partial y\, dy\hat{}dy\hat{}dz + \partial U/\partial z\, dz\hat{}dy\hat{}dz \\ &\quad -\partial V/\partial x\, dx\hat{}dz\hat{}dx - \partial V/\partial y\, dy\hat{}dz\hat{}dx - \partial V/\partial z\, dz\hat{}dz\hat{}dx \\ &\quad +\partial W/\partial x\, dx\hat{}dx\hat{}dy + \partial W/\partial y\, dy\hat{}dx\hat{}dy + \partial W/\partial z\, dz\hat{}dx\hat{}dy \\ &= \partial U/\partial x\, dx\hat{}dy\hat{}dz + 0 + 0 \\ &\quad -0 - \partial V/\partial y\, dy\hat{}dz\hat{}dx - 0 \\ &\quad +0 + 0 + \partial W/\partial z\, dz\hat{}dx\hat{}dy \\ &= \{\partial U/\partial x + \partial V/\partial y + \partial W/\partial z\}dx\hat{}dy\hat{}dz = div(\mathbf{V})dx\hat{}dy\hat{}dz \end{aligned}$$

Note that these algebraic ideas do not depend upon the existence of a norm or a metric.

Example 18 *Derivation of the Maxwell Faraday induction equations*

The Maxwell-Faraday induction equations are a set of partial differential equations that are logically deducible starting with the ordered sequence $[1, 2, 3, 4]$. Next assume the existence of "ordered coordinate" vaiables given the symbols $[x, y, z, t]$. Next assume the existence of an ordered set of functions of the coordinate variables, with symbols $[A_x, A_y, A_z, \phi]$. From these beginnings the Maxwell - Faraday equations follow as a consequence of the Exterior Calculus of Cartan.

Construct the 1-form from the ordered set of functions and variables:

$$A = A_x dx + A_y dy + A_z dz - \phi dt. \tag{6.40}$$

Next construct the 2-form $F = dA$. Then construct the 3-form ddA which must vanish: $ddA = dF \Rightarrow 0$. In 4D the 3-form has 4 coefficient functions of partial derivatives that must vanish. These PDE's correspond in format to the 4 Maxwell - Faraday equations, with 3D symbols

$$curl\mathbf{E} + \partial\mathbf{B}/\partial t = 0 \quad div\mathbf{B} = 0, \tag{6.41}$$

where the symbols are defined in terms of the coefficient functions of the 1-form (of potentials) as,

$$\mathbf{E} = -grad\,\phi - \partial\mathbf{A}/t \qquad \mathbf{B} = curl\,\mathbf{A}. \tag{6.42}$$

Now the choice of symbol functions and coordinate functions was completely arbitrary, but the format of the PDE's that satisfy $ddA = dF \Rightarrow 0$ are always the same relative to the ordering process. Experimentally, the logical equations of Maxwell - Faraday have been exploited in electromagnetic applications. However, the SAME formulas (different symbols) are applicable to hydrodynamics (as well as other physical systems of interest). Surprisingly, little has been done with the induction equations in hydrodynamics. These are not analogies. These are consequences of the logic of the Exterior Calculus and have universal applicability.

6.2.2 The Interior Product

The interior product is an operation on p forms that requires a direction Vector field, V. The interior product lowers the degree of a p-form, changing a p-form into a p-1 form. The interior product of a Vector direction field and a zero form (function) is defined to be zero. The symbol for the interior product herein is taken to be $i(V)$. The interior product of a Vector field and an exact basis element equal to the differential of a coordinate dy^a is not zero, but is defined to equal to the a^{th} component of V. Hence the fundamental definitions can be written as

$$i(V)\theta(y^a) = 0 \qquad , \qquad i(V)dy^a = V^a. \tag{6.43}$$

It follows that the inner product with respect to the vector field V acting on a 1-form, $A = (A_x dx + A_y dy + A_z dz)$ is given by the expression:

$$i(V)A = i(V)(A_x dx + A_y dy + A_z dz) = (A_x V^x + A_y V^y + A_z V^z) \tag{6.44}$$

An additional rule is required to take care of the anti symmetries of differential forms. That is for the product $A\hat{}B$ of two 1-forms, the interior product with respect to V becomes

$$i(V)\{A\hat{}B\} = (i(V)A)\hat{}B\} - A\hat{}(i(V)B)$$

and

$$i(V)i(V)A = 0 \tag{6.45}$$

similar to the modified Leibniz rule for the exterior differential.. Other expressions can be worked out for higher p-forms can be worked out using these rules,

$$i(V)i(V)\omega^p = 0 \qquad i(V)i(W)\omega^p \neq \; i(W)i(V)\omega^p$$

Example 19 *Compute the interior product of* $J = [J^x, J^y, J^z]$ *in 3D with the 3-form vol element* $Vol = dx\hat{\ }dy\hat{\ }dz$

$$i(J)Vol = J^x dy\hat{\ }dz - J^y dx\hat{\ }dz + J^z dx\hat{\ }dy.$$

Example 20 *Compute the interior product of* $V = [V^x, V^y, V^z]$ *with the 2-form* $i(J)Vol$

$$\begin{aligned} i(V)\{i(J)Vol\} &= i(V)\{J^x dy\hat{\ }dz - J^y dx\hat{\ }dz + J^z dx\hat{\ }dy\} \\ &= (J^y V^z - J^z V^y)dx + (J^z V^x - J^x V^z)dy + (J^x V^y - J^y V^x)dz. \end{aligned}$$

Note that the construction (in 3D) of the double interior product generates coefficients equal to the cross product of the two different vector fields, J and V, and the double interior product with the same vector is zero.

6.2.3 The Lie Derivative

The Lie derivative with respect to a vector field generates a p-form ϑ^p from a p-form ω^p. It is constructed from the raising operator d and the lowering operator $i(V)$. The general formula is

$$\omega^p \Rightarrow \vartheta^p : \qquad L_{(V)}\omega^p = i(V)d\omega^p + d(i(V)\omega^p = \vartheta^p. \tag{6.46}$$

Marsden has called this Cartan's Magic formula. The reason is that most of the equations of mechanics can be put into this form or derived from its construction. For example, those processes V which are "Hamiltonian" processes are those V such that $i(V)d\omega^p$ is exact. It is also remarkable that this formula is equivalent to the first law of thermodynamics. Consider a 1-form of Action, A, that presents a physical system (this will be done in detail in later sections). Then consider a vector field V that represents an evolutionary process. Define the 0-form (scalar function) of internal energy as $U = i(V)A$, the 1-form of Work as $W = i(V)dA$, and the output 1-form ϑ^p as Q. Then Cartan's Magic formula becomes

$$L_{(V)}A = i(V)dA + d(i(V)A = W + dU = Q \tag{6.47}$$

which is to be recognized as the first law of thermodynamics for a physical system A undergoing an evolutionary process V. This result will be exploited in later sections.

Example 21 *Compute the Lie derivative with respect to $V = [F, V, 1]$ acting on the 1–form*
$A = pdq - H(p, q, t)dt$ in 3 dimensions. The basis elements are $[dp, dq, dt]$.

$$\begin{aligned}
L_{(V)}A &= i(V)dA + d(i(V)A) \\
&= i(V)\{dp\hat{}dq - dH\hat{}dt\} + d(pV - H) \\
&= i(V)\{dp\hat{}dq - \partial H/\partial p\, dp\hat{}dt - \partial H/\partial q\, dq\hat{}dt\} + d(pV - H) \\
&= Fdq - Vdp - F(\partial H/\partial p)dt - V(\partial H/\partial q)dt + dH + d(pV - H) \\
&= F(dq - \partial H/\partial pdt) - V(dp + \partial H/\partial qdt) + d(pV)
\end{aligned}$$

Note that the RHS of the equation above is a perfect differential for all evolutionary vector fields with components $V = [F, V, 1]$, if the two bracket factors vanish. Therefor, vector fields that are generated from the partial derivatives of $H(p, q, t)$ according to the formulas,

$$\begin{aligned}
(dq - \partial H/\partial pdt) &\Rightarrow 0 \supset V = \partial H/\partial p \\
(dp - \partial H/\partial qdt) &\Rightarrow 0 \supset F = -\partial H/\partial p,
\end{aligned} \tag{6.48}$$

which produce a 1-form of heat Q which is closed, $dQ \Rightarrow 0$. Such processes (vector fields) are defined to be Hamiltonian vector fields (processes). Hamiltonian dynamics is the (constrained) domain of much of theoretical mechanics. The domain is constrained, as the 3-form $Q\hat{}dQ \Rightarrow 0$. Such processes are then always thermodynamically reversible. Later on, irreversible processes for which $Q\hat{}dQ \neq 0$ will be studied.

One notes for Hamiltonian processes,

$$\begin{aligned}
L_{(V)}H &= i(V)dH \\
&= F\partial H/\partial p + V\partial H/\partial q + \partial H/\partial t \\
&= FV - VF + \partial H/\partial t \\
&= \partial H/\partial t,
\end{aligned} \tag{6.49}$$

so that if H is independent from time, then H is an evolutionary invariant. In mechanics, the function H is typically defined to be equal to be the sum of kinetic and potential energy, $H = p^2/2m + \varphi(x)$, so that time independent Hamiltonian processes "conserve energy". Even if the Hamiltonian is a function of time, Hamiltonian processes are thermodynamically reversible, as $Q\hat{}dQ = 0$.

6.2.4 Some Topological Features

The concepts of intersections, closure, and limit points are fundamental topological concepts that have a relationships to the Cartan Calculus. In certain situations the exterior product exhibits properties of intersection operator, and the exterior derivative exhibits properties of a limit point operator. More formally, given a domain with two exact 1-forms in 3D, the exterior product of the two exact 1-forms (if not zero) represents the points of intersection of the two implicit surfaces generated by the two functions whose gradient coefficients make up the components of the two exact 1-forms.

Example 22 *The exterior product and the concept of intersection.*

Consider two 1-forms created by applying the exterior differential to two distinct functions $\alpha(x, y, z)$ and $\beta(x, y, z)$. The coefficients of $d\alpha = grad(\alpha) \circ d\mathbf{r}$ form the gradient field $grad(\alpha)$ which is perpendicular to the implicit surface $\alpha(x, y, z) = 0$. Similarly, $d\beta = grad(\beta) \circ d\mathbf{r}$ implies that the gradient coefficients $grad(\beta)$ are perpendicular to the implicit surface $\beta(x, y, z) = 0$. If the two implicit surfaces intersect, then exterior product of the two 1-forms create a 2-form, $J = d\alpha\hat{}d\beta$, which is not zero. The components of the 2 form, J, can be interpreted as a contravariant vector in 3D, which is tangent to the points in common (intersections) that make up the intersection of the two surfaces. For $n = 3$, the number of components of a 2-form are 3, and are in agreement with the 3D cross-product formulas of Gibbs.

Example 23 *The exterior derivative is a limit point generator.*

From another point of view, it is possible to deduce a topological structure from a given 1-form A on the domain. The it is possible to show that the exterior derivative, relative to this Cartan topology, acts as a generator of the limit points of the given topology. This is given further credence from the physical idea that the divergence (an application of the exterior derivative) of the $\mathbf{D}$ field in electromagnetism has finite values that terminate on charges. That is, the Faraday lines of $\mathbf{D}$ come from limit points of postive charge and wind up on limit points of negative charge. However, the concept that the exterior differential d is a limit point operator is more formal, and has a basis in Kuratowski's closure operator.

Example 24 *The Lie derivative can be used to select topological invariants of a process.*

The Lie derivative with respect to a vector field V may be construed as a convective propagator describing the flow of the points of a p-form down the flow lines generated by V. If the p-form is integrated over a domain of such flowing points, then it is possible to ask if the integral is an invariant of the flow. Moreover it is possible to ask if the flowing points are distorted and deformed, does the integral over the deformed points equal the integral over the undeformed points. If is true that the value of the integral is unchanged by continuous deformation, then the integral must represent a topological property.

To deform the flowing points is easy enough; just multiply the original vector field V by a function of (say) $\lambda(x, y, z, t)$. The function λ does not change the flow lines generated by V, but it does deform the points that make up the flowlines by stretching or compression along the flow lines. Then $V \Rightarrow \lambda V$ and $L_{(V)} \Rightarrow L_{(\lambda V)}$, and the Lie derivative becomes a deformation operator. If it can be shown that if

$$L_{(\lambda V)} \oint A = \oint i(\lambda V)dA + \oint d(i(\lambda V)A) = 0 \tag{6.50}$$

for any function λ then the closed integral $\oint A$ is a deformation invariant of the process. Note that the second integral always vanishes, $\oint d(i(\lambda V)A) = 0$, as the integrand is an exact perfect differential. For the first integral to vanish for arbitrary deformation parameter, λ, the integrand must be zero. This leads to the conclusion that

$$\begin{aligned} if \quad i(\lambda V)dA &= \lambda i(V)dA = 0 \quad \text{any } \lambda, \\ then \quad \oint A &= \text{deformation invariant} \end{aligned} \tag{6.51}$$

Hence if the Work 1-form is zero, $W = i(V)dA \Rightarrow 0$ then the closed integral of the Action $\oint A$ is a topological property (of that process). Cartan has shown that a necessary and sufficient condition for a process to be a Hamiltonian process, is that the closed integral of the Action should be a topological invariant of the process.

Example 25 *The first law of thermodynamics is a topological statement of Cohomology.*

A non-exact p form Q is defined to be Cohomologous to another non-exact p-form, W, if the difference between the two p-forms is exact. This means that the integrals of the two different p-forms over any closed integration path (cycle or boundary) are the same. For non-exact 1-forms of Heat, Q, and Work, W, the cohological statement is the First Law:

$$Q - W = dU.$$

It was noted above that the Lie derivative with respect to a vector field (process) acting on a physical system described by a 1-form of Action, is essentially a Cohomological statement of the first law. The Lie derivative is a Cohomological generator.

Example 26 *Thermodynamic Isolation and Frobenius integrability.*

In thermodynamics, it is recognized that there are isolated, closed, and open systems. These words are also used to describe topological properties. A set is topologically isolated if it has no intersection with its limit points. This result translates to $A\hat{}dA = 0$ for a given 1-form and its induced Cartan topology. The constraint of isolation is also equivalent to the Frobenius idea of unique integrability. That is when $A\hat{}dA = 0,$ there exists a unique function whose gradient (or surface normal) is proportional to the given coefficients of the given 1-form. Caratheodory's statements about inaccessible states is a statement related to the concept of isolation and connectivity to an equilibrium system. When $A\hat{}dA = 0$ (no matter what the dimension of the coordinate space happens to be) there exists a transformation to a domain of two independent functions that will describe the properties of the 1-form. That is, the 1-form can be written as $\phi d\chi$, and its coefficient functions are proportional to a gradient, $d\chi$. The problem becomes essentially a two dimensional problem.

The property of isolation is a topological property, hence if a process causes $A\hat{}dA \neq 0$ to change to $A\hat{}dA = 0$, or from a state where $A\hat{}dA = 0$ to a state where $A\hat{}dA \neq 0$, a topological change has take place. In hydrodynamics, all streamline flows satisfy $A\hat{}dA = 0$. Hence turbulent flows must involve domains where $A\hat{}dA \neq 0$. The transition to (from) turbulence from (to) a state of non-turbulence must involve topological change.

It should be mentioned that with respect to diffeomorphic transformations, or more simply those transformations that preserve pure geometrical properties, the differences between contravariant and covariant concepts cannot be distinguished. But with respect to an aging process involving topological change, the behavior of the two concepts is observably different.

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$$\begin{aligned} &\text{Isolated}: H = A\hat{}dA = 0 \\ &\text{Closed}: dH = 0, \textit{ but } H \neq 0. \\ &\text{Open}: \quad dH \neq 0 \end{aligned}$$

Chapter 7
APPENDIX 3. INTERSECTION, ENVELOPES AND TOPOLOGICAL TORSION

7.1 Introduction:

In physical systems the existence of an envelope has its most well-known example in the form of Huygens principle: A wave front (in 3D) is the envelope of multiple expanding spherical surfaces whose multiple origins reside on some initial surface. Herein attention is focused on the fact that the envelope is to be associated with the concept of non-uniqueness: at each point on the wave front, there exists not only the wave front surface but also the spherical wavelet surface. The concept of non-uniqueness implies that a parametric point of view of a surface with its unique range is not applicable. This observation focuses attention on implicit representations of curves and surfaces, where non-uniqueness is admissable.

In the theory of implicit surfaces, the criteria of uniqueness - and therefor the existence of a parametric representation - is related to a differential constraint on the neighborhoods in the form of a Pfaffian equation (a 1-form set equal to zero) defining the surface. If the 1-form, A, satisfies the Frobenious criteria of unique integrability, $A\hat{}dA = 0$, then the surface can be uniquely established in the sense that a normal field can be defined by at most two functions, one giving its scale, and the other its direction field. The direction field is a vector with components defined in terms of the partial derivatives of a unique function. That is, $N = \phi d\psi$. In these cases the Pfaff dimension of the 1-form, A,is 2, and the Topological Torsion 3-form $A\hat{}dA$ is null.

On the otherhand, if $A\hat{}dA \neq 0$, the Pfaff dimension is 3 or greater, and non-uniqueness is to be expected. Topological torsion is not exactly the same as the Frenet torsion of space curve, (which is a parametric, not implicit, concept) nor the more subtle Affine torsion of a connection, but like these concepts Topological Torsion is an artifact of three dimensions or more.

First, a few examples of envelopes will be given to demonstrate how the existence of topological torsion is related to the concept of non-uniqueness.

7.2 A Family of Curves in the Plane (2+1 space)

As mentioned above, the basic idea of an envelope is that there is a *non-uniqueness* criteria lurking somewhere. First consider the concept of a implicit *curve* in the plane given as the "global" zero set of a function $F(x, y)$ of two variables, (x, y). It is important to note that the curve itself is not necessarily a parameterized set, and can consist of multiple components and branches. No direction (of motion) is defined a priori on any particular curve component by the implicit function equation. In order to define a parameterization of the curve (that is, a direction along a curve component), it is necessary to introduce some third variable, or parameter, say s. This parameter s will be defined as the parameter of *orientation* or *directed* arc length, but such a parameter is not of immediate interest.

A family of non-directed (non-oriented, but orientable) curves may be constructed if the implicit function is a function of one or more *other* parameters, such as σ, λ,.... Then, for example, in the case of a single (family) parameter, σ, the global zero set of $F(x, y..\sigma) = 0$ defines an implicit 2-surface in the 2+1 space of variables $\{x, y..\sigma\}$, with an induced differential Pfaffian equation, or 1-form set equal to zero.

$$dF \equiv (\partial F/\partial x)dx + (\partial F/\partial y)dy + (\partial F/\partial \sigma)d\sigma = F_x dx + F_y dy + F_\sigma d\sigma \Rightarrow 0. \quad (7.1)$$

For an explicit choice of the family parameter, σ, the implicit function defines a curve which may be viewed as the intersection of the surface $F(x, y..\sigma) = 0$ and the plane defined by the value of σ in the space $\{x, y..\sigma\}$. The differential, dF, has components which may be viewed as the direction field normal to the implicit surface in the space, $\{x, y..\sigma\}$. At certain points all components of the direction field vanish. Such a singular point is defined as a critical point of the implicit function. Critical points will be determined by the points of intersection and contact between two implicit surfaces: the selected surface of the family, $F(x, y..\sigma) = 0$, and the surface where all components of the direction field vanish.

7.2.1 Singular points

It is important to distinguish between the singular point sets of the implicit function and the stationary points that may exist on the selected surface of the family, $F(x, y..\sigma) = 0$. The singular critical points are where the induced differential form, dF, is identically zero, a constraint which implies that for every differential directional displacement, $d\mathbf{R}$, the differential form vanishes. Each partial derivative of F as a function of $(x, y..\sigma)$ must vanish identically. For points which are not singular points, it is possible to find n-1 (=2 in the example) differential directions for which the 1-form dF vanishes. These n-1 directions define the tangent space to the implicit surface. If all components of the direction field vanish simultaneously, then Cramer's rule implies that the determinant of the Jacobian matrix of the direction field (F_x, F_y, F_σ) must vanish at certain points, $\mathbf{R}_{critical}(x, y, \sigma)$. The condition can be expressed as by the fact that n=3 form must vanish,

$$\Theta = dF_x\hat{}dF_y\hat{}dF_\sigma = \beta(x,y,\sigma)dx\hat{}dy\hat{}d\sigma \Rightarrow 0 \text{ at a singular point.} \tag{7.2}$$

Note that the zero set of the function

$$\beta(x,y,\sigma) = \det[\mathbb{J}(gradF)] = \det \begin{bmatrix} F_{xx} & F_{xy} & F_{x\sigma} \\ F_{yx} & F_{yy} & F_{y\sigma} \\ F_{\sigma x} & F_{\sigma y} & F_{\sigma\sigma} \end{bmatrix} \Rightarrow 0 \tag{7.3}$$

defines the "surface of singular points" of the function, $F(x,y,s)$. The two (possibly multiple component) surfaces, $\beta(x,y,\sigma) = 0$, and the selected surface, $F(x,y,\sigma) = 0$, have intersections at points when $dF\hat{}d\beta \neq 0$. In three dimensions this object (a 2-form) has components proportional to the Gibbs cross product of the direction field normal to the implicit surface and the direction field normal to the surface of critical points. At points where the intersection 2-form vanishes, the two surfaces can either be disjoint, or have a point of contact. Hence logical intersection of the critical points includes points of surface intersection and points of surface contact. The problem of finding the critical points is a global issue, but given a point it is possible to test to see if it is a critical point using (local) differential methods. If the 3-form Θ never changes sign, there is no implicit surface of critical points. If Θ is zero, then a surface of critical points exists, but this surface may not have intersection with the surface $F(x,y,\sigma) = 0$. An intersection exists producing a curve of critical points when the 2-form $dF\hat{}d\beta$ is not zero.

7.3 Envelopes

Envelopes are also related to the intersection of two surfaces generated by the family. To find an envelope is more difficult than to determine whether or not the envelope exists.

There exist neighborhood directions constraining the displacements dx, dy, ds such that the differential form dF vanishes in those selected (not all) directions. The covariant components of the 1-form dF define the normal direction field to the implicit 2-surface $F(x,y,s) = 0$. Displacements orthogonal to the normal field satisfy the equation $dF = 0$. Note that the zero set of the implicit function creates a surface in 3 dimension space, $\{x,y,s\}$, not a curve. In order to determine a space curve, a second surface must be described, and the intersection of this second surface and the first surface yields the "space curve". For example, the intersection of the surface $F(x,y,s) = 0$ with the plane $s = 0$ determines a curve in the x,y plane. As s varies a family of such curves are produced which may have multiple components. It is assumed that the members of the family can be projected to the $\{x,y\}$ plane.

For a point p on the surface $F(x,y,s) = 0$ (in $\{x,y,s\}$ space), the neighborhood directions that cause dF to vanish are directions orthogonal to the surface normal at the point p. These directions determine the tangent plane to the surface at p. The differential of the surface function has the format

$$dF = (\partial F/\partial x)dx + (\partial F/\partial y)dy + (\partial F/\partial s)ds = F_x dx + F_y dy + F_s ds. \tag{7.4}$$

A second surface in 2+1 space can be determined from the original implicit function by differentiation with respect to the family parameter and setting the resulting function to zero: $F_s(x, y, s) = \partial F/\partial s = 0$. The surface constraint induces the Pfaffian equation derived from the differential 1-form,

$$dF_s = (\partial^2 F/\partial s\partial x)dx + (\partial^2 F/\partial s\partial y)dy + (\partial^2 F/\partial s\partial s)ds = F_{sx}dx + F_{sy}dy + F_{ss}ds. \tag{7.5}$$

The intersection of these two surfaces produces a tortuous curve of perhaps several segments (components) in the space $\{x, y, s\}$. A necessary condition that the two surfaces $F(x, y, s) = 0$ and $F_s(x, y, s) = 0$ have an intersection (simultaneous solution) is established by the requirement that the exterior product of the two 1 forms dF and dF_s does not vanish. On the sets $F(x, y, s) = 0$ and $F_s(x, y, s) = 0$, this requirement reduces to the constraint

$$dF\hat{}dF_s = \{F_x F_{sy} - F_y F_{sx}\}dx\hat{}dy + F_{ss}\{F_x dx + F_y dy\}\hat{}ds \neq 0. \tag{7.6}$$

This intersection has a component in the $\{x, y\}$ plane if the first factor does not vanish. Note that (for fixed s) the critical points of $(F = 0) \cap (F_x = 0) \cap (F_y = 0)$ must be excluded. In more simple language, the critical points are where the tangent vector to the surface vanishes, and the points of interest for self intersection and envelopes is where the normal vector to the surface is zero or for then such a point is

If $F_{ss} \neq 0$, then it is possible to solve for s from the equation of the second surface, $F_s(x, y, s) = 0$. Use this value to eliminate the family parameter in the first equation of the surface. The result is a Function of $\{x, y\}$ only, that defines a curve in the $\{x, y\}$ plane independent from the family parameter. This curve is the envelope.

The three components of this 2-form on 2+1 space form the components of a contravariant vector, **J**, which is tangent to the curve of intersection. If all three components vanish, then the two surfaces do not intersect. In particular, if

$$\{F_x F_{sy} - F_y F_{sx}\} = 0 \quad and \quad F_{ss} = 0, \tag{7.7}$$

there is no intersection and no singularity. If

$$[\{F_x F_{sy} - F_y F_{sx}\} = 0 \; and \; F_{ss} \neq 0] \tag{7.8}$$

or

$$[\{F_x F_{sy} - F_y F_{sx}\} \neq 0 \;\; and \;\; F_{ss} = 0], \tag{7.9}$$

there is a singularity, but no envelope.

If both

$$[\{F_x F_{sy} - F_y F_{sx}\} \neq 0 \;\; and \;\; F_{ss} \neq 0] \tag{7.10}$$

then there is a curve which is an envelope of the family of curves. Note that the envelope condition implies that the primitive function, F, is non-linear in the family parameter, s.

The process can be continued. A cuspoidal point of regression can be determined when the three functions $F, F_s, \; and \; F_{ss}$ satisfy the equation,

$$dF\hat{\ }dF_s\hat{\ }dF_{ss} \neq 0. \tag{7.11}$$

7.4 A Family of Surfaces in 3+1 space

The basic idea extends to higher dimensions. An implicit function $\Phi(x, y, z, \sigma) = 0$,does not determine a surface in 3-space, but instead determines a hypersurface in 4 space. For a family of surfaces in three dimensions $\{x, y, z\}$, with a family parameter, σ, the criteria for intersection of $\Phi(x, y, z, \sigma) = 0$ and $\partial\Phi(x, y, z, \sigma)/\partial\sigma = \Phi_\sigma(x, y, z, \sigma) = 0$ becomes

$$\begin{aligned} d\Phi\hat{\ }d\Phi_\sigma &= \{\Phi_x \Phi_{\sigma y} - \Phi_y \Phi_{\sigma x}\} dx\hat{\ }dy + \\ &\quad \{\Phi_y \Phi_{\sigma z} - \Phi_z \Phi_{\sigma y}\} dy\hat{\ }dz + \\ &\quad \{\Phi_z \Phi_{\sigma x} - \Phi_x \Phi_{\sigma z}\} dz\hat{\ }dx + \\ &\quad \Phi_{\sigma\sigma} d\Phi\hat{\ }d\sigma \\ &\neq 0 \end{aligned} \tag{7.12}$$

The first three terms are to be recognized as the components of the cross product,

$$\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma. \tag{7.13}$$

The argument is that when either

$$\{\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma \neq 0 \qquad \text{and} \;\; \Phi_{\sigma\sigma} = 0\}, \tag{7.14}$$

or

$$\{\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma = 0 \qquad \text{and} \quad \Phi_{\sigma\sigma} \neq 0\}, \tag{7.15}$$

then the family has an intersection singularity.

When both

$$\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma \neq 0 \ \ \textit{and} \ \ \Phi_{\sigma\sigma} \neq 0 \tag{7.16}$$

then there is a surface envelope. Only non-linear family parameters produce envelopes.

7.4.1 The edge of regression

The process can be extended to find an edge of regression. In this case it is assumed that the three zero sets $\Phi(x,y,z,\sigma) = 0$, $\Phi_\sigma(x,y,z,\sigma) = 0$ and $\Phi_{\sigma\sigma}(x,y,z,\sigma) = 0$ have a common solution. The criteria for solubility for an edge of regression requires that the three form, which is the exterior product of all three differentials, does not vanish:

$$d\Phi\hat{}d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma} \neq 0. \tag{7.17}$$

The spatial components of this expression require that

$$(\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma) \bullet (\nabla_{(x,y,z)}\Phi_{\sigma\sigma}) \neq 0 \tag{7.18}$$

for the existence of an (cuspoidal) edge of regression.

7.5 Examples of Envelopes of families of surfaces.

7.5.1 Spheres moving along x axis: The cylindrical canal surface.

Consider the function

$$\Phi = (x-\sigma)^2 + y^2 + z^2 - 1 \tag{7.19}$$

with a zero set which represents a family of unit spheres with centers at points $x = \sigma$. If $\sigma = ct$, then the centers of the spheres can be interpreted as moving along the x axis.

$$\Phi_\sigma = \partial\Phi/\partial\sigma = -2(x-\sigma), \qquad \Phi_{\sigma\sigma} = +2. \tag{7.20}$$

The 2-form $dA = -d\Phi\hat{}d\Phi_\sigma \Rightarrow \{4zdx\hat{}dz + 4ydx\hat{}dy\}$ at $d\sigma = 0$, is non-zero, and $\Phi_{\sigma\sigma} \neq 0$. From another point of view, $\nabla_{(x,y,z)}\Phi\times\nabla_{(x,y,z)}\Phi_\sigma = 0\mathbf{i}-4z\mathbf{j}+4y\mathbf{k} \neq 0$. Therefor the necessary conditions for the existence of an envelope are valid. Solving for σ from $\Phi_\sigma = 0$ and substituting in $\Phi = 0$, leads to the equation of the envelope,

$$y^2 + z^2 - 1 = 0 \tag{7.21}$$

The envelope is a cylinder of radius 1, with the x axis as the axis of rotational symmetry. The 3-form $d\Phi\hat{}d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}$ vanishes so there is no edge of regression..

7.5.2 Expanding spheres moving along the x-axis: The Mach cone.

Consider the function

$$\Phi = (x - k\sigma)^2 + y^2 + z^2 - \sigma^2 \tag{7.22}$$

with a zero set which represents a family of expanding spheres of radius σ with centers at $k\sigma$ moving along the x axis. When $k > 1$ the translational speed exceeds the expansion speed (of, say, sound, where $\sigma = ct$)

$$\Phi_\sigma = \partial\Phi/\partial\sigma = -2k(x) + 2(k^2 - 1)\sigma, \qquad \Phi_{\sigma\sigma} = +2(k^2 - 1). \tag{7.23}$$

The 2-form $dA = -d\Phi\hat{}d\Phi_\sigma \Rightarrow \{4zkdx\hat{}dz + 4ykdx\hat{}dy\}$ for $d\sigma = 0$, and is non-zero, and $\Phi_{\sigma\sigma} \neq 0$.. Therefor the necessary conditions for the existence of an envelope are valid. Solving for σ from $\Phi_\sigma = 0$ and substituting in $\Phi = 0$, leads to the equation of the envelope,

$$(k^2 - 1)(y^2 + z^2) - x^2 = 0 \tag{7.24}$$

which is a cone (the Mach cone), with a symmetry axis as the x axis, and an aperature

$$tan\theta = \sqrt{1/(k^2 - 1)}. \tag{7.25}$$

The 3-form $d\Phi\hat{}d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}$ vanishes so there is no edge of regression.

7.5.3 Concentric Spheres

Consider the function

$$\Phi = x^2 + y^2 + z^2 - \sigma^2 \tag{7.26}$$

with a zero set which represents a family of unit spheres with variable radii, $\sigma = ct$, and centered on the origin.

$$\Phi_\sigma = -2\sigma, \qquad \Phi_{\sigma\sigma} = -2. \tag{7.27}$$

The 2-form $dA = -d\Phi\hat{\ }d\Phi_\sigma = 0$ for $d\sigma = 0$. Therefor the necessary conditions for the existence of an envelope are not valid. The family of surfaces do not intersect as

$$\nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma = 0 \qquad . \tag{7.28}$$

The 1-form A is integrable. The 3-form $d\Phi\hat{\ }d\Phi_\sigma\hat{\ }d\Phi_{\sigma\sigma}$ vanishes so there is no edge of regression.

7.5.4 Spheres with a common point of tangency on the x axis.

Consider the function

$$\Phi = (x-\sigma)^2 + y^2 + z^2 - \sigma^2 \tag{7.29}$$

with a zero set which represents a family of spheres of various radii and with centers along the x axis.

$$\Phi_\sigma = \partial\Phi/\partial\sigma = -2x, \qquad \Phi_{\sigma\sigma} = 0. \tag{7.30}$$

The 2-form $dA = -d\Phi\hat{\ }d\Phi_\sigma \neq 0$ for $d\sigma = 0$. Therefor the *necessary* condition for the intersection singularity exists, but the subsidiary condition $\Phi_{\sigma\sigma} \neq 0$ is not satisfied. The singularity is the point where all the spheres have a common tangent, $\{x = 0,\ y^2 + z^2 = 0\}$. The envelope **does not exist** because the subsidiary condition $\Phi_{\sigma\sigma} \neq 0$ is not valid.

7.5.5 Spheres with a common circle of intersection.

Consider the function

$$\Phi = (x-\sigma)^2 + y^2 + z^2 - (a^2 + \sigma^2) \tag{7.31}$$

with a zero set which represents a family of spheres with centers along the x axis.

$$\Phi_\sigma = \partial\Phi/\partial\sigma = -2(x), \qquad \Phi_{\sigma\sigma} = 0. \tag{7.32}$$

The 2-form $dA = -d\Phi\hat{\ }d\Phi_\sigma \neq 0$ for $d\sigma = 0$. Therefor the *necessary* condition for the intersection singularity exists, but the subsidiary condition $\Phi_{\sigma\sigma} \neq 0$ for an envelope is not satisfied. The envelope **does not exist** because of the subsidiary condition $\Phi_{\sigma\sigma} \neq 0$ is not valid. However, the intersection exists as the circle of radius a in the x=0 plane: $\{x = 0,\ y^2 + z^2 = a^2\}$.

7.5.6 The Jacobian cubic characteristic polynomial.

Consider the cubic polynomial

$$\Phi(X, Y, Z; \sigma) = \sigma^3 - X\sigma^2 + Y\sigma - Z \tag{7.33}$$

with

$$\Phi_\sigma(X, Y, Z; \sigma) = \partial\Phi/\partial\sigma = 3\sigma^2 - X2\sigma + Y \tag{7.34}$$

and similarly,

$$\Phi_{\sigma\sigma}(X, Y, Z; \sigma) = 6\sigma - 2X \tag{7.35}$$

For real σ, the zero set, $\Phi(X, Y, Z; \sigma) = 0$, may be viewed as a family of 3 dimensional hypersurfaces in the space $(X, Y, Z; \sigma)$ with the family parameter, σ. It is useful to determine if the family of surfaces has an envelope. The envelope will be determined by the intersection of the two hypersurfaces, $\Phi(X, Y, Z; \sigma) = 0$, and $\Phi_\sigma(X, Y, Z; \sigma) = 0$. The envelope is a two dimensional surface independent from the parametrization, and is regular if $\Phi_{\sigma\sigma}(X, Y, Z; \sigma) \neq 0$.

For a given vector field, $\mathbf{V}(x, y, z)$, the similarity invariants of the Jacobian matrix of $\mathbf{V}$ with respect to (x, y, z) can be used to determine the explicit form for the functions, $\{X(x, y, z), Y(x, y, z), Z(x, y, z)\}$. The similarity invariants of a given Jacobian matrix, $[\mathbb{J}]$, are given by formulas:

$$X(x, y, z) = trace\,[\mathbb{J}]\,, \tag{7.36}$$

$$Y(x, y, z) = trace\,[\mathbb{J}]^{adjoint}\,, \tag{7.37}$$

$$Z(x, y, z) = det\,[\mathbb{J}]\,. \tag{7.38}$$

The cubic polynomial is then the Cayley-Hamilton characteristic polynomial of the Jacobian matrix, which always exists for the 3x3 matrix,

$$Characteristic\ Polynomial\ of\ \ [\mathbb{J}] \Rightarrow \sigma^3 - X\sigma^2 + Y\sigma - Z = 0, \tag{7.39}$$

with the "family parameter" $\sigma(X, Y, Z)$ playing the role of the eigenvalues for the matrix, $[\mathbb{J}]$. From classic theory,

$$X = \sigma_1 + \sigma_2 + \sigma_3 \tag{7.40}$$
$$Y = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \tag{7.41}$$
$$Z = \sigma_1\sigma_2\sigma_3, \tag{7.42}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the eigenvalues of $[\mathbb{J}]$, and can be viewed as functions on (x, y, z).

Solving for σ from $\Phi_\sigma = 0$ leads to two roots, σ_+ and σ_-. Substitution into the primary cubic $\Phi = 0$ leads to two envelop equations, $\Phi_+ = 0$, and $\Phi_- = 0$, depending on which root is chosen for the substitution. There are two envelope sheets with each envelope function independent of the parameter, σ. The product of the two functions leads to a new function which is precisely the historical Cardano function, with values that determine the root structure of the polynomial, and whose zero set determines the degenerate root structure, or the edge of regression of the two envelope sheets.

$$Cardano\ = \Phi_+ \cdot \Phi_- = -4(X^2 - 3Y)^3 + (2X^3 - 9XY + 27Z)^2. \tag{7.43}$$

This derivation of the Cardano formula is remarkable in that it is based upon the theory of envelopes in a space of 3 coordinates and 1 family parameter.

The intersection of the two gradient surfaces is given by the expression,

Another way to specify the envelope is to construct the 1-form A in a space of 3+1 variables,

$$A = \Phi_x dx + \Phi_y dy + \Phi_z dz... = d\Phi - \Phi_\sigma d\sigma = d(\Phi - \sigma\Phi_\sigma) + \sigma d\Phi_\sigma, \tag{7.44}$$

(which by construction is not explicitly dependent only upon displacement, $d\sigma$). This 1-form may not be globally exact, as $dA = -d\Phi_\sigma\hat{}d\sigma \neq 0$,necessarily. In fact, this 1-form, A, need not be uniquely integrable, for globally $A\hat{}dA = -d\Phi\hat{}d\Phi_\sigma\hat{}d\sigma \neq 0$, necessarily. If the 2-form $d\Phi\hat{}d\Phi_\sigma = 0 \mod d\sigma$, then no envelope exists, and the Topological Torsion of the 1-form vanishes, $A\hat{}dA = 0$. In other words, the 1-form A defined above does not satisfy the Frobenius criteria of unique integrability, when an envelope exists. Moreover, the space exhibits Topological Torsion. This result is pleasing, for the concept of an envelope intuitively implies non-uniqueness.

For the cubic polynomial it is known that the Cardano function not only separates the domains for which the eigenvalues are real or complex, but also the zero set of the Cardano function, when it has an intersection with the envelope determines a curve upon which there can exist repeated roots. The edge of regression for the two sheets of the envelope is precisely such a curve of repeated roots. The tangent vector to the curve which is the edge of regression is given by solving for $\sigma = X/3$ from the equation, $\Phi_{\sigma\sigma}(X, Y, Z; \sigma) = 0$. Substitution of this value for sigma into the equation for the position vector in $[X, Y, Z]$ space, $\mathbf{J} =[X, Y, Z]$ leads to tangent vector at the edge of regression, $\mathbf{J} =[1, 2X/3, X^2/9]$. A plot of the double sheeted Cardano envelope appears in Figure 1.

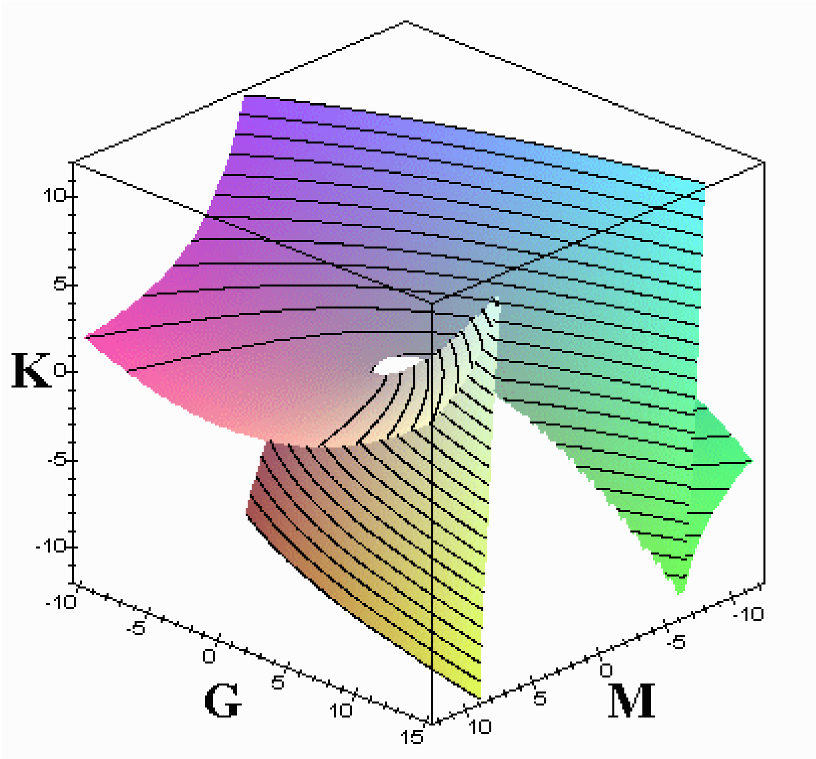

Figure 1 **Cardano surface. Note edge of regression**

The 2-form $J = -d\Phi\hat{}d\Phi_\sigma \Rightarrow \{dy\hat{}dz - 2\sigma dx\hat{}dz + \sigma^2 dx\hat{}dz\}$ for $d\sigma = 0$, and is non-zero, and $\Phi_{\sigma\sigma} \neq 0$. Therefor the conditions for the existence of an envelope are valid. The tangent vector to the characteristic curve is given by $\mathbf{J} = [1, 2\sigma, \sigma^2]$. The envelope consists of two sheets which join at the edge of regression. As the 3-form, $d\Phi\hat{}d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma} \Rightarrow -2dx\hat{}dy\hat{}dz$ is non-zero for $d\sigma = 0$, and $\Phi_{\sigma\sigma\sigma} \neq 0$, an edge of regression exists. For the cubic polynomial it is known that the Cardano function not only separates the domains for which the eigenvalues are real or complex, but is also the surface upon which there can exist repeated roots. The edge of regression is precisely such a curve of repeated roots. The tangent vector to the curve which is the edge of regression is given by solving for $\sigma = x/3$ from the equation, $\Phi_{\sigma\sigma}(x, y, z; \sigma) = 0$. Substitution of this value into the equation for $\mathbf{J}$ leads to tangent vector at the edge of regression, $\mathbf{J} = [1, 2x/3, x^2/9]$. A plot of the Cardano envelope appears in Figure 1. Note the edge of regression.

The plot is a universal plot, where the coordinates are the similarity invariants, the Mean curvature (M), the Gauss curvature (G) , and the Determinant (K), of the Jacobian matrix of the associated cubic polynomial.

The Cardano function (or envelope) can be constructed as a tangential developable based on the curve whose tangent vector is given by $\mathbf{J} = [1, 2x/3, x^2/9]$. A point on the Cardano surface is given by $\mathbf{X} = \mathbf{R} \pm \lambda \mathbf{J}$ with 1 sheet of the envelope determined by positive motion along the edge of regression and the other sheet of the envelope determined by motion in the opposite direction. It is important to note that the neighborhoods are not "time reversal invariant", although the edge of regression is "time reversal invariant". This property of the trajectory neighborhoods is due to the fact the edge of regression has torsion in the sense of Frenet.

These concepts have utility in thermodynamics, for the Gibbs equilibrium surface of a Van der Waals gas is a function which is cubic in its family parameter. The Spinodal Line of the Gibbs surface is an edge of regression (and is determined by the condition that the Gauss curvature vanish). The Binodal line is a line of self intersection. The critical point is where both the mean curvature and the Gauss curvature of the surface vanish.

Thermodynamically, the Spinodal line is the edge of regression of the Gibbs surface for a Van der Waals gas. The observation above regarding time reversal invariance implies that motion along the spinodal line in the direction of the crictical point is stable in one direction, but unstable in the other.

7.6 Summary

In each of the examples above, 2-form $d\Phi\hat{}d\Phi_\sigma \neq 0$ is a necessary and sufficient condition for the existence of an intersection, and a necessary but not sufficient condition for the existence of an envelope, for the family of surfaces given by the equation, $\Phi(x, y, z...\sigma) = 0$. Consider the 1-form A (which by construction is not explicitly dependent only upon displacement, $d\sigma$) in a space of 3+1 variables, and its Pfaff sequence:

$$A = \Phi_x dx + \Phi_y dy + \Phi_z dz... = d\Phi - \Phi_\sigma d\sigma = d(\Phi - \sigma\Phi_\sigma) + \sigma d\Phi_\sigma, \quad (7.45)$$

$$dA = d\sigma\hat{}d\Phi_\sigma, \quad (7.46)$$

$$A\hat{}dA = d\sigma\hat{}d\Phi\hat{}d\Phi_\sigma, \quad (7.47)$$

$$dA\hat{}dA = 0. \quad (7.48)$$

This 1-form, A, may not be globally exact, as $dA = -d\Phi_\sigma\hat{}d\sigma \neq 0$,necessarily. In fact, this 1-form, A, need not be uniquely integrable, for globally $A\hat{}dA = -d\Phi\hat{}d\Phi_\sigma\hat{}d\sigma \neq 0$, necessarily. If the 2-form $d\Phi\hat{}d\Phi_\sigma = 0$, then no intersections and no envelope exist, and the Topological Torsion of the 1-form vanishes, $A\hat{}dA = 0$.

When the Topological Torison does not vanish, $A\hat{}dA \neq 0$,then there exists more than one solution function to the equation $A = 0$ (non-uniqueness), and therefore the failure of the Frobenius unique integrability criteria leads to the possibility of an envelope. The conclusion to be reached is that:

The existence of topological torsion is necessary for the existence of an envelope of a family of hypersurfaces.

7.6.1 The General Theory

The necessary and sufficient conditions for an envelope of a family of functions parameterized by σ are given by the exterior diferential system:

$$d\Phi\hat{}d\Phi_\sigma \neq 0 \ \ and \ \ \Phi_{\sigma\sigma} \neq 0. \tag{7.49}$$

Consider once again the 1-form A in a space of 3+1 variables,

$$A = \Phi_x dx + \Phi_y dy + \Phi_z dz ... = d\Phi - \Phi_\sigma d\sigma, \tag{7.50}$$

If A satisfies the Frobenious integrability theorem, $A\hat{}dA = 0$, and then there exits a globally unique function, $\Theta(x, y, z; \sigma)$ such that the zero set of $\Theta(x, y, z; \sigma)$ defines a hypersurface of dimension 3 and is a solution to the Pfaffian equation, $\lambda A = 0$. For such integrable cases, A is reducible to the format, $A = \beta d\Theta$.

When $H = A\hat{}dA \neq 0$,then there exists more than one solution function to the equation $A = 0$ (non-uniqueness), and therefore the failure of the Frobenius criteria uniqueness leads to the possibility of an envelope. On the otherhand, if the Frobenious condition is valid, the topological torsion, H, is zero; the criteria for the existence of an envelope is not satisfied.

In 2+1 space, the first criteria for an envelope depends upon the possibility that the two surfaces $\Phi = 0$ and $\Phi_\sigma = \partial\Phi/\partial\sigma = 0$, for fixed values of σ, have an intersection. The criteria that an intersection exists is given by the differential form statement that $d\Phi\hat{}d\Phi_\sigma \neq 0$. The curve that represents this intersection of the two surfaces is usually called the "characteristic" curve. This characteristic curve in the plane is obtained from the solutions to the subsidiary equations $d\mathbf{R} - \mathbf{J}ds = 0$ where $\mathbf{J} = \{\nabla\Phi \times \nabla\Phi_\sigma\}$ is the tangent vector to the curve of intersection projected to the x,y plane. The initial conditions of this characteristic curve are not arbitrary; they must be adjusted such that the tangent vector resides on the intersection of the two surfaces, $\Phi = 0$ and $\Phi_\sigma = 0$. The characteristic curve is a very special curve selected out of the vector field, $\mathbf{J}$. The gradient operations are with respect to the three variables $\{x, y, \sigma\}$

$$\mathbf{J} = +\{\Phi_y\Phi_{\sigma\sigma} - \Phi_\sigma\Phi_{\sigma y}\}\mathbf{i} - \{\Phi_x\Phi_{\sigma\sigma} - \Phi_\sigma\Phi_{\sigma x}\}\mathbf{j} + \{\Phi_x\Phi_{\sigma y} - \Phi_y\Phi_{\sigma x}\}\mathbf{k} \tag{7.51}$$

As $\Phi_\sigma = 0$, the tangent vector to the curve of intersection becomes

$$\mathbf{J} = +\{\Phi_y\Phi_{\sigma\sigma}\}\mathbf{i} - \{\Phi_x\Phi_{\sigma\sigma}\}\mathbf{j} + \{\Phi_x\Phi_{\sigma y} - \Phi_y\Phi_{\sigma x}\}\mathbf{k} \tag{7.52}$$

such that if $\Phi_{\sigma\sigma} = 0$, then the tangent vector to the enveloping curve has no components in the two dimensional subspace of {x,y}. In this situation, the envelope is not "visible" and has no extension when projected to the x,y plane.

The same argument works in higher dimensions. The basic idea is that if for a singly parametrized function, $\Phi(x, y, z...; \sigma)$ on a space of N+1 dimensions, the 1-form $A = d\Phi - \Phi_\sigma d\sigma$ is not necessarily globally integrable, a fact which implies non-uniqueness of the solution to the Paffian equation, $A = 0$. The concept of non-uniqueness admits to the possibility of finding an envelope of dimension N-2 which is independent from the parameter, σ. For suppose $\Phi_\sigma = 0$ defines a set of dimension N-1 which intersects with the N-1 set $\Phi = 0$, to produce a set of dimension N-2. In order for an envelope to exist, the non-uniqueness argument implies as a necessary condition that the 2 form $d\Phi\hat{}d\Phi_\sigma$ cannot vanish, and a sufficient condition for non-uniqueness as $A\hat{}dA = -d\Phi\hat{}d\Phi_\sigma\hat{}d\sigma \neq 0$. This result implies that the condition for the existence of an envelope in three spatial dimensions and one parametric dimension requires that

$$A\hat{}dA \neq 0 \Rightarrow \nabla_{(x,y,z)}\Phi \times \nabla_{(x,y,z)}\Phi_\sigma \neq 0. \tag{7.53}$$

In 3+1 space, the "envelope" is the 2 dimensional *surface* of intersection of the two 3 dimensional sets, $\Phi = 0$ and $\Phi_\sigma = 0$, subject to the constraint that $\Phi_{\sigma\sigma} \neq 0$.

7.6.2 The edge of regression

The surface function may be non-linear in the parameter σ, such that it is possible to compute $\Phi = 0$, $\Phi_\sigma = 0$, and $\Phi_{\sigma\sigma} = 0$ to find a simultaneous intersection of the three N-1 sets to produce in this case a 1 dimensional line. For the intersection to be non empty it is necessary that the three form $d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}\hat{}d\Phi \neq 0$. As described above, the 1-form $A = d\Phi - \Phi_\sigma d\sigma$ constructed to be independent of the differential displacements $d\sigma$ plays an important role.

The second function $\Phi_\sigma(x, y, \sigma) = \partial\Phi/\partial\sigma$ induces second 1-form given by the expression,

$$d\Phi_\sigma = \Phi_{\sigma x}dx + \Phi_{\sigma y}dy + \Phi_{\sigma\sigma}d\sigma. \tag{7.54}$$

Following the procedure established above, a second 1-form $B = d\Phi_\sigma - \Phi_{\sigma\sigma}d\sigma$ can be constructed to be independent from the differential displacements of the family parameter. The intersection of the two 1-forms, A and B, becomes

$$\begin{aligned} A\hat{}B &= (d\Phi - \Phi_\sigma d\sigma)\hat{}(d\Phi_\sigma - \Phi_{\sigma\sigma}d\sigma) && (7.55)\\ &= d\Phi\hat{}d\Phi_\sigma + (\Phi_\sigma d\Phi_\sigma - \Phi_{\sigma\sigma}d\Phi)\hat{}d\sigma && (7.56)\\ &= d\Phi\hat{}d\Phi_\sigma + \omega\hat{}d\sigma. && (7.57)\end{aligned}$$

Then $d\omega = d\Phi_{\sigma\sigma}\hat{}d\Phi$, and $\omega\hat{}d\omega = -\Phi_\sigma d\Phi_\sigma\hat{}d\Phi_{\sigma\sigma}\hat{}d\Phi$ is a three form in 4 dimensional space. When $\omega\hat{}d\omega \neq 0$, it is possible to find a common intersection of the three equations, $\Phi = 0$, $\Phi_\sigma = 0$, and $\Phi_{\sigma\sigma} = 0$, represented as a non-zero three form in $\{x, y, z, \sigma\}$ space. The components of this contravariant vector density may be used to compute a tortuous curve which projects to 3 dimensional space as the 1-dimensional curve representing the edge of regression of the two dimensional surface envelope.

7.6.3 Dynamical Systems

Consider a dynamical system in 3 dimensional space, defined as a vector field, $V^k(x, y, z; t)$, of 3 components and a single parameter of time. Construct the 3 x 3 Jacobian matrix of the vector field:

$$[\mathbb{J}_j^k(V^n)] = [\partial V^k(x, y, z; t)/\partial x^j]. \tag{7.58}$$

From the Cayley Hamilton theorem there always exists a cubic polynomial for this 3x3 matrix in terms of a complex eigenvalue parameter, λ:

$$\Theta(x, y, z; t, \lambda) = \lambda^3 - M(x, y, z, t)\lambda^2 + G(x, y, z, t)\lambda - K(x, y, z, t). \tag{7.59}$$

The functions $\{M, G, K\}$ are the similarity invariants of the Jacobian matrix. The zero set of the cubic polynomial forms a hypersurface in the 5D space $(x, y, z;\ t, \lambda)$, or it may be considered as a family of surfaces in 3D space $\{x, y, z\}$ with two family parameters, (t, λ). For fixed t, the cubic polynomial has been analyzed above. What is remarkable is that all 3D dynamical systems have such a cubic representation equivalent to a thermodynamic equation of state, and that equation of state is in fact a projective equivalent of a van der Waals gas. It can be shown that the renormalized eigenvalue parameter has the properties of a complex thermodynamic “molar density”. It follows that every 3 dimensional dynamical system has an equivalent representation as a Van der Waals gas. The concept of a Van der Waals gas is a universal topological concept.

Chapter 8

REFERENCES (TO BE FIXED LATER)

[1] E. Cartan "Sur certaines expressions differentielles et le systeme de Pfaff" Ann Ec. Norm. **16** 329 (1899)

[2] E. Cartan, "Systems Differentials Exterieurs et leurs Applications Geometriques", Actualites sci. et industrielles 944 (1945)

[3] E. Cartan, "Lecons sur la theorie des spineurs" (Hermann, Paris 1938)

[4] E. Cartan, "La Theorie des Spaces a Connexion Projective", (Hermann, Paris, 1937)

[5] S.S.Chern, Annais of Math. 45, 747- 752 (1944).

[6] R. M. Kiehn, "Retrodictive Determinism" Int. Journ. Eng Sci (1976)

[7] R.M. Kiehn, 'Topological Torsion, Pfaff Dimension and Coherent Structures" in Topological Fluid Mechanics, H. K. Moffatt and A. Tsinober, editors, (Cambridge University Press, 1990), p. 225.

[8] H.Flanders, "Differential Forms", (Academic Press, N. Y. 1963).

[9] R.M. Kiehn and J. F. Pierce, The Physics of Fluids **9** 1941 (1969)

[10] R. M. Kiehn, J. Math Phys. 9 1975

[11] R. M. Kiehn, "Are there three kinds of superconductivity" INt J. of Mod. Phys.10 1779 (1991)

[12] W. Slebodzinsky, "Exterior Forms and their Applications

[13] Van der Kulk and Schouten "Pfaffs Problem Oxford University Press.

[14] S. Lipschutz, "General Topology", (Schaums Publishing Co.,New York, 1965) p.97

[15] R. Hermann, "Differential Geometry and the Calculus of Variations", (Academic Press, New York, 1968).

[16] R. M. Kiehn, Lett al Nuovo Cimento **14**, 308 (1975) Submersive Equivalence Classes for Metric Fields"

[17] W. Gellert, et.al. Editors 'The VNR Concise Encyclopedia of Mathematics", (Van Nostrand, New York 1977), p.686.

[18] J. G. Hocking, "Topology", (Addison Wesley, N. Y., 1961), p.2.

[19] R. L Bishop and S. l. Goldberg, "Tensor Analysis on Manifolds", (Dover, N. Y., 1968).

[20] R. M. Kiehn, 'Topological Parity and the Turbulent State" submitted to Jap. J. of Fluid Res.

[21] N. E. Kochin, l. A. Kibel, and N. V. Roze Theoretical Hydrodynamics" (Inter-science, New York 1965)

[22] H. W. Turnbull, "The Theory of determinants, matrices and invariants" (Dover, New York 1960)

[23] D. Struik, "Differential Geometry" , Addison Wesley, (Reading, Mass 1961)

[24] J. Yorke and T.U, Amercan Mathematical Monthly **82**, 985 (1975).

[25] R. M. Kiehn, Lett al Nuovo Cimento **12**, 300 (1975); Lett al Nuovo Cimento **22**, 308 (1978)

8.1 REFERENCES

[1] J. G. Hocking, "Topology" (Addison Wesley, N. Y. , 1961), p.2.

[2] R. L. Bishop and S. I. Goldberg, "Tensor Analysis on Manifolds" (Dover, N. Y., 1968), p. 199.

[3] R. M. Kiehn, "Compact Dissipative Flow Structures with Topological Coherence Embedded in Eulerian Environments" in "The Generation of Large Scale Structures in Continuous Media", (Singapore World Press, 1991).

[4] W. Gellert, et.al. Editors "The VNR Concise Encyclopedia of Mathematics" , (Van Nostrand, New York 1977), p.686."

[5] S. Lipschutz, "General Topology" , (Schaums Publishing Co.,New York, 1965) p.97

[6] C. Nash and S. Sen, "Topology and Geometry for Physicists" , (Academic Press, San Diego 1989).

[7] P. T. Landsberg, "Thermodynamics" (Interscience 1961).

[8] J. Kestin, "A Course in Thermodynamics" (Blaisdel, Waltham, Mass. 1966).

[9] L. Tisza, "Generalized Thermodynamics" (MIT press, Cambridge 1966), p .125.

[10] J. Klein, Ann. Inst. AIFUB Fourier, Grenoble 12, 1 (1962).

[11] R. M. Kiehn, Int. J. of Eng. Sci. 13, 941 (1975).

[12] E.T. Whittaker, "Analytical Dynamics" ,(Dover, New York 1944).

[13] G. DeRham, "Varietes Differentables" (Hermann, Paris, 1960).

S. I. Goldberg, "Curvature and Homology" (Dover, N. Y., 1982).

[14] E. Cartan, "Systems Differentials Exterieurs et leurs Applications Geometriques" (Hermann, Paris, 1922).

[15] H. Lamb, "Hydrodynamics" (Dover, New York ,1945), p.245.

[16] R. M. Kiehn, "Topological Torsion, Pfaff Dimension and Coherent Structures" in "Topological Fluid Mechanics" H. K. Moffatt and A. Tsinober, editors, (Cambridge University Press, 1990), p. 225.

[17] R. Hermann, "Differential Geometry and the Calculus of Variations" (Academic Press, New York, 1968).

[18] V. I. Arnold, London Math. Soc. Lect. Note Series 53, (Cambridge University Press, 1981), p. 225.

[19] R. M. Kiehn, J. Math Phys. 5, 9 (1974).

[20] R. M. Kiehn, J. Math Phys. 18, 614 (1977).

[21] H. K. Moffatt, J. Fluid Mech. 35,117 (1969).

B. Gaffet, J. Fluid Mech. 156, 141 (1985).

[22] W. Slebodzinsky, "Exterior Forms and their Applications" (PWN Warsaw, 1970).

[23] M. A. Berger, and G. B. Field, J Fluid Mech. 147, 133-148 (1984).

[24] S.S.Chern, Annals of Math. 45, 747- 752 (1944).

W. Greub, S.Halperin and R. Vanstone (Academic Press, New York , 1973).

Chern, S.S. MSRI 04808-88 (Berkeley, CA. 1988).

[25] C. G. Callan, Jr., P. B. Gilkey and A. J. Hanson, (1976) Phys. Lett 63B 334 (1976) Note the phrase "This E.B^{1}0 in the Lagrangian) clearly breaks P and T invariance.. " Also see A. Schultz, et. al.Phys Lett 74A, 384 (1979).

[26] A. S. Eddington,. "The Mathematical Theory of Relativity" (Cambridge University Press 1963), p. 194.

[27] A. Otto, M. Hesse, and K. Schindler (1990) General Magnetic Reconnection in 3D Systems, in "Topological Fluid Mechanics", H. K. Moffatt and A. Tsinober, editors, (Cambridge University Press, 1990), p. 225-234.

[28] R. W. Metcalfe and F. Hussain, "Topology of Coherent structures and Flame Sheets in Reacting Mixing Layers", in "Topological Fluid Mechanics" H. K. Moffatt and A. Tsinober, editors, (Cambridge University Press, 1990) 659-678. Also see: M. V. Melander, and F. Hussain, "Topological Aspects of Vortex Reconnection" , in: loc. cit. 485-499.

[29] M. Kaku, "Introduction to Superstrings"(Springer-Verlag, New York 1988) 386.

[30] I. T. Drummond, and W. H. P. Munch, "Turbulent Stretching of Lines and Surfaces", in "Topological Fluid Mechanics" H. K. Moffatt and A. Tsinober, editors, (Cambridge University Press, 1990), p. 628-636.

[31] P. L. Sulem, Z. S. She, H. Scholl, and U. Frisch, J. Fluid Mech 205 341 (1989).

[32] X.G.Wen, F. Wilczek, and A.Zee, Phys Rev B 39 11413 (1989).

[33] R. M. Kiehn, NASA AMES NCA-2-OR-295-502 (1976)

[34] P. R. Baldwin and R. M. Kiehn "Cartan's Topological Structure"(1989) http://www22.pair.com/csdc/pdf/topstruc.pdf

[35] R. M. Kiehn "Falaco Solitons" (1987)

http://www22.pair.com/csdc/pdf/falaco01.pdf

[36] R. M. Kiehn, "Topology and Turbulence" (1985-1993)

http://www22.pair.com/csdc/pdf/topturb.pdf

[37] R. M. Kiehn,"Coherent Structures in Fluids are Deformable Topological

Torsion Defects" presented at the IUTAM-SIMFLO Conference at DTU, Denmark, May
25-29, (1997)
http://www22.pair.com/csdc/pdf/copen5.pdf
[38] R. M. Kiehn, "2D turbulence is a Myth" (2000)
http://www22.pair.com/csdc/pdf/hague6.pdf
[39] R. M. Kiehn, "Topological evolution of classical electromagnetic fields and the photon" (1999)
http://www22.pair.com/csdc/pdf/photon5.pdf
[40] R. M. Kiehn, "The Photon Spin and other features of classical electromagnetism" (2000)
http://www22.pair.com/csdc/pdf/Vig2000.pdf
[41] R.M.Kiehn, " An Intrinsic Transport Theorem " (1967)
http://www22.pair.com/csdc/pdf/inttrans.pdf
R. M. Kiehn, "Periods on Manifolds, Quantization and Gauge" (1977)
http://www22.pair.com/csdc/pdf/periods.pdf
R. M. Kiehn, "Are there three kinds of superconductivity" (1991)
http://www22.pair.com/csdc/pdf/sc3.pdf
[42] R. M. Kiehn, "Dissipation, Irreversibility and Symplectic Lagrangian systems on Thermodynamic space of dimension 2n+2" (1996)
http://www22.pair.com/csdc/pdf/irrev1.pdf
[43] Marsden, J.E. and Riatu, T. S. (1994) "Introduction to Mechanics and Symmetry", Springer-Verlag, NY p.122.

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