Topological Torsion, Pfaff Dimension an Coherent structures.

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Abstract

A coherent structure is viewed as a deformable connected domain of velocity space with certain similar topological properties. The topology of interest is the topology induced by the constraint on the variety {x,y,z,t} such that the vector field of flow satisfies a kinematic system of ordinary differential equations, as well as a dynamical partial differential equation of evolution. The Pfaff dimension or class of a domain is a topological property whose evolution may be computed. Of the four Pfaff classes of coherent structures admitted over space time, potential flow and integrable vorticial flows make up the first two Pfaff classes. The third and fourth Pfaff classes lead to the notion of Topological Torsion and Topological Parity. A concept of Helicity current is introduced, a current whose non-zero divergence signals the production of 3-dimensional defects internal to the flow field. Invariant surfaces of separation associated with the Jacobian of the unit tangent field may be used to define topological domains. An example is given demonstrating that these domains, as deformable domains of similar topology in the flow velocity field, may be put into correspondence with coherent hermodynamic phases.

1 INTRODUCTION

Topological properties of an evolutionary process are properties that stay the same under continuous and reversible transformations . From

this viewpoint, a coherent structure is defined to be a connected, convectively deformable domain with certain topological properties

that remain dynamically invariant. Examples of topological properties are: Connectivity, Boundary, Orientation, Pfaff Dimension, and Limit points, which are to be distinguished from geometrical properties of Size and Shape. The fundamental problem is how to extract, both theoretically and experimentally, topological information from a hydrodynamic flow. The kinematic velocity field is viewed as a set of differential constraints on R4, prescribing a topology on the variety {x,y,z,t}.

Herein, two related - but distinct - methods are suggested for extracting such topological information. Both methods have the same point of departure: both use the concept of a normalized, or unit speed, tangent velocity field:

$$\mathbf{t} = \mathbf{V}(x, y, z, t)/(\mathbf{V}.\mathbf{V})\mathbf{1/2} \tag{1}$$

which to a differential geometer is a conformal Gauss map of the velocity space. The first method utilizes the Cartan calculus based on the 1 form of action, or unit arclength, $A = \overline{d}s = t_\mu dx^\mu$ (an imperfect differential constructed from the unit tangent field) to define the topological property of Pfaff dimension and Topological Torsion.

The second method of extracting topological information considers the null sets of the six invariant scalar functions associated with the Jacobian of the unit tangent vector field. As deformation invariants, these surfaces separate domains of different topological properties. It will be demonstrated that these domains can be put into correspondence with topological or thermodynamic phases.

2 TOPOLOGICAL TORSION

The traditional hydrodynamic vectors of velocity and vorticity are tensors of the first and second rank. The Cartan analysis focuses attention on the utility of a third rank tensor field, called Topological Torsion, and its divergence, which is a fourth rank tensor field, called Topological Parity. Although these fields are of third and fourth rank, they are set intersections of the velocity and the vorticity, and require no more than first derivatives of the velocity for their construction. Therefore, these fields are as amenable to measurement as are the traditional fields of velocity and vorticity, but they carry different kinds of information. Their components, in principle, can be measured directly with a 9-wire probe.

Following the methods of Cartan, a certain amount of topological information can be obtained by the construction of the Pfaff sequence based on the 1-form of Action, A . (Note that the closed integral of the action is related to the hydrodynamic concept of circulation.)

$$\begin{array}{ccc} TOPOLOGICAL\ ACTION & A = & A\mu dx^\mu \\ TOPOLOGICAL\ VORTICITY & F = dA = & F_{\mu\nu} dx^\mu \hat{}\, dx^\nu \\ TOPOLOGICAL\ TORSION & H = A\hat{}\, dA = & H_{\mu\nu\rho} dx^\mu \hat{}\, dx^\nu \hat{}\, dx^\rho \\ TOPOLOGICAL\ PARITY & K = dA\hat{}\, dA = & K_{\mu\nu\rho\sigma} dx^\mu \hat{}\, dx^\nu \hat{}\, dx^\rho \hat{}\, dx^\sigma \end{array} \tag{2}$$

The Pfaff dimension of a domain is the rank of the largest non-zero element of the above sequence. The topological torsion tensor of an arbitrary 1-form of Action has four components in space-time, the first three of which form a Helicity or Torsion current, $\mathbf{T}$. The fourth component, Ω, is related to the helicity density, h, introduced to hydrodynamics by Moffatt [1969]. The differential topology of the "torsionlines" of the Helicity current, $dx/T_x = dy/T_y = dz/T_z = dt/\Omega$, can be evaluated and studied in a manner similar to the non-linear analysis applied to the "flowlines".

A first topological result is that if the four components of this third rank tensor of Helicity current and helicity density vanish over a domain, then the hydrodynamic flow satisfies the Frobenius complete integrability theorem [Flanders 1963], and the flow is never chaotic, nor braided, nor knotted over the domain. It follows that the concept of Topological Torsion is necessary for the understanding of turbulent and chaotic flow.

The Pfaff sequences, and the Topological Torsion tensor, can be constructed for arbitrary fields, but the importance of the Gauss map and the unit tangent field to global issues cannot be overemphasized. Sometimes, the technique is surprising. For example, rigid body rotation and the potential vortex have the same Gauss map! The two flows are Cremona duals of one another, being projections exterior to and interior to the unit sphere, respectively. In terms of the unit tangent field, the four components of topological torsion may be written as:

$$\mathbf{T} \;= -(\partial\mathbf{t}/\partial t)\times\mathbf{t}, \qquad and \qquad \Omega = \mathbf{t}\cdot curl\,\mathbf{t} = \mathbf{v}\cdot curl\,\mathbf{v}/(\mathbf{v}\cdot\mathbf{v} = h/(\mathbf{v}\cdot\mathbf{v}). \quad (3)$$

An important result is the fact that the divergence of the Topological Torsion tensor is not necessarily zero, and does not depend explicitly upon viscosity:

$$div\mathbf{T} \; + \partial\Omega/\partial t = 2(\partial\mathbf{t}/\partial t)\times\mathbf{t}\cdot curl\;\mathbf{t} = K. \quad (4)$$

The implication is that 3 dimensional defects of Helicity current can be created or destroyed spontaneously within the bulk medium of a fluid. The production of defects is an entropy increasing process, and therefore these processes, if continuous, must be irreversible. It is conjectured that these processes are part of the turbulent process. A visual signature of such hydrodynamic defects will be irreducibly 3 dimensional helical contrails and scroll waves.

The vector, $curl\;\mathbf{t}$ is related to the Darboux vector, $\mathbf{D}$, of the system of streamlines associated with $\mathbf{t}$. It may be shown that the norm of $\mathbf{D}$ is exactly equal to the sum of the squares of the Frenet torsion and the Frenet curvature of these streamlines. When the unit normal field is integrable, the Frenet torsion, τ, is precisely the 4th component, Ω, of the Topological Torsion tensor of the unit tangent field! In this sense, the helicity density divided by the square of the velocity, $\mathbf{v}\cdot curl\;\mathbf{v}/(\mathbf{v}\cdot\mathbf{v}$, is an intrinsic property of the flow, independent from the observer's frame of reference. Similarly, the intrinsic property of the square of the Frenet curvature may be evaluated as

$$\kappa^2 = (curl\;\mathbf{t}\times\mathbf{t})^2, \qquad with \qquad \tau^2 = \Omega^2 = (\mathbf{t}\cdot curl\;\mathbf{t})^2. \quad (5)$$

It is important to recall that the topology of interest is the topology induced on the underlying 4 dimensional variety by a system of kinematic constraints. These constraints are specified by the condition that the velocity field, which in this case is a solution of a partial differential system (the Navier-Stokes equations), form a kinematical system of ordinary differential equations

$$dx - udt = 0, \quad dy - vdt = 0, \quad dz - wdt = 0. \tag{6}$$

3 PFAFF DIMENSION

The Pfaff dimension of the unit tangent field is an invariant of a continuous deformation of the domain. The primitive idea is: what is the minimum number of functions that are required to describe a vector field over a domain? Gradient fields are adequately described by 1 function, hence a potential flow is of Pfaff dimension 1. Non-chaotic, or completely integrable vorticial flows are of Pfaff Dimension 2. Flows with domains of Helicity current and/or helicity density are of Pfaff dimension 3. The point of departure beyond Clebsch is that in space time the flows may be of Pfaff dimension 4!

Connected domains of the same Pfaff dimension can serve as candidates for coherent structures. In this sense, a vortex ring or a vortex tube is a coherent structure of Pfaff dimension 2 embedded in a larger domain of Pfaff dimension 1. Connected toroidal tubes such as those found in the ABC flow would be candidates for a coherent structure of Pfaff dimension 3. These domains or structures of different Pfaff dimension can evolve with deformation, but their connectivity and the sense of coherence instilled by their topological sameness, will be preserved with respect to homeomorphic flows. Domains of Pfaff dimension 4 form coherent structures that are irreducibly space-time dependent. The domains of Pfaff dimension 3 or greater may or may not be chaotic. For example, the toroidal bundles of ergodic trajectories suggested by Moffatt as candidates for coherent structures would fall into domains of Pfaff dimension 3.

4 DETERMINISTIC CHAOS vs. NON DETERMINISTIC VELOCITY FIELDS

Vector fields that define a hydrodynamic flow may be classified into flows that are deterministic and separable, and those that are not. Separability implies that the time dependent flow vector field factors into a vector function of coordinates and a common factor of time and space: $V^{\mu} = \mathbf{v}(x, y, z)T(x, y, z, t)$. According to Eisenhart [1963], the separability condition is necessary and sufficient for the vector field to be the generator of a single parameter semi-group of transformations. The physical interpretation of the single parameter is that it represents time. The remarkable feature of the separable flows is that the associated unit tangent field, $\mathbf{t}$, is explicitly *independent* from time! The Helicity current, $\mathbf{T}$, of the unit tangent field vanishes, and the Pfaff sequence terminates at dimension 3. The solution to the system of unit tangent ODE's form a set of streamlines! The concept of Frenet torsion and Topological Torsion are, in this case, the same. The Rott solution [Rott ,1958] is an example of a non-separable solution to the Navier-Stokes equations that has a non-null Helicity current. The integrable flows (Pfaff dimension 2 or less) demonstrate a 1-1 correspondence between an increment of arclength and an increment of time; the non-integrable flows do not.

It is suggested that turbulent flows are non-separable flows of irreducible Pfaff dimension 4, and are to be distinguished from deterministic chaotic flows which are of Pfaff dimension 3. The remarkable feature of the concept of Pfaff dimension is that it simultaneously offers a precise definition of a coherent structure (a domain of the same Pfaff dimension), and distinguishes between a chaotic (Pfaff dimension 3) and a turbulent state (Pfaff dimension 4).

5 TOPOLOGICAL SURFACES AND THERMODYNAMIC PHASES

Any vector field, V (x,y,z,t), including those which are solutions to the Navier Stokes equations, may be normalized to 1 everywhere except at its fixed points, by the Gauss map, given by Eq. 1. This unit tangent field may be used to construct the Jacobian matrix, [**J**],

$$[\mathbf{J}] = [\partial \mathbf{t}_\mu / \partial x^\nu], \tag{7}$$

as well as the one form of action, $A = \overline{d}s = t_\mu dx^\mu$ based on the unit arc length. There are six primary invariant functions of [**J**], which are to be associated with its complex eigenspectra. The Jacobian is to be viewed as a dyadic of functions rather than as a matrix of values. No linearization process is subsumed.

The six invariant functions are defined as: ($\mathbf{D} = curl\ \mathbf{t}$)

$P(x, y, z) = \text{trace } [\mathbf{J}] = \text{div } \mathbf{t}$, $\quad L(x, y, z) = \langle \mathbf{D}| \circ \lfloor \mathbf{D} \rangle$,

$Q(x, y, z) = \text{trace } [adjoint \mathbf{J}]$ $\quad M(x, y, z) = \langle \mathbf{D}| \circ [\mathbf{J}] \circ \lfloor \mathbf{D} \rangle$

$R(x, y, z) = \det [\mathbf{J}]$ $\quad N(x, y, z) = \langle \mathbf{D}| \circ [\mathbf{J}] \circ [\mathbf{J}] \circ \lfloor \mathbf{D} \rangle$.

The null sets of these functions, or combinations of these functions, form invariant surfaces of separation, which globally separate domains of different topology. The idea to be developed is that these surfaces of separation (in velocity space) define domains of thermodynamic phase! What better way to think of a coherent structure, than as domains of pure or mixed thermodynamic phase.

The invariant surfaces may also be viewed as sets of partial differential equations whose solutions have interesting properties:

P(x,y,z) =0 defines a minimal surface (soap film of zero mean curvature)

L(x,y,z) =0 defines an asymptotic surface of zero enstrophy.

Q(x,y,z)=0 defines a surface of zero Gauss sectional curvature

M(x,y,z)=0 defines a surface of null vortex stretching rate

The three dimensionality of the Jacobian immediately focuses attention on the Cardano function, $C(x, y, z)$, whose null set generates a surface that separates domains of complex versus real cubic roots. In terms of the invariant scalars, the Cardano function may be constructed as $C = -27R^2 - P(P^2 - 18Q)R - (4Q - P^2)Q^2$.

When $C > 0$ there are 3 distinct real solutions to the Hamilton characteristic equation for $[\mathbf{J}]$. As shown in appendix A, this condition corresponds to the mixed phase region of a Van der Waals gas.

When C = 0 the eigen values of the Jacobian are real, but at least 2 eigenvalues are equal. If, in addition, the rank of the Jacobian is less than 3, then the Cardano condition corresponds to the binodal and spinodal lines on the equilibrium surface of a Van der Waals gas. The criteria of rank less than 3 implies that $R = 0$, and indicates that the solution set is satisfied by the spinodal constraint $Q = 0$, or the binodal constraint, $4Q - P^2 = 0$.

When $R = P^3/27$ *and* $Q = P^2/3$, all three eigen values are real and equal. For $R = 0$, then $P = 0$ and $Q = 0$. This condition corresponds to the critical point of a Van der Waals gas. See Appendix A. When $C < 0$ there is 1 real eigen value and two conjugate complex solutions. This condition corresponds to the pure phase domains of the Van der Waals gas.

Define two functions, A and B, such that $3B = Q - P^2/3$ and $2A = -R + QP/3 - 2P^3/27$. Use A and B as the coordinates of the bifurcation space. Then the functions P, Q, or R can be chosen as the bifurcation parameter, for constant values of the other scalar functions. This third order system fits a GLOBAL bifurcation scheme which is : Hysteretic in the variable R; a Pitchfork in the variable Q, and a winged cusp in the variable P [Golubitsky 1985].

When $R = 0$ the system can be put into correspondence with an equilibrium thermodynamic system, which is not hysteretic. When $R \neq 0$, the system is irreducibly 3-dimensional, and the non-equilibrium thermodynamic system admits irreversible hysteresis as R varies. It may be shown that tertiary bifurcations, such as the Hysteretic Hopf bifurcation [Langford 1983] can lead to intermittency. Such flows have non-zero Topological Torsion. An example of an exact solution to the Navier-Stokes equations that yields an intermittent transverse torsion wave packet is given in Appendix B.

The above analysis indicates that kinematic domains have properties that are topologically equivalent to thermodynamic phases. Borrowing from thermodynamic experience, cooperative and coherent behavior is to be expected in complex kinematic flows, along with kinematic phase transitions, depending on initial conditions and parameters.

6 TOPOLOGICAL EVOLUTION AND PFAFF DIMENSION

A fundamental question of turbulence is: How do systems originally of Pfaff dimension 3 or less evolve into systems of Pfaff dimension 4? Such evolutionary flows involve changing topology, and are either irreversible, discontinuous, or both.

The decomposition theorems of deRham [Goldberg 1982] may be used to form 4 equivalence classes of flows, V , relative to the 1-form of Action, A. Consider all flows that satisfy either:

1.	i(V) dA = 0	Hamiltonian flows
2.	i(V) dA = dP	Eulerian flows
3.	i(V) dA = dP + G	Stokes flows
4.	i(V) dA = dP + G + *∂*Z	Open flows

where G is harmonic (dG = 0 , *d*G=0). The first three equivalence classes are closed with respect to exterior differentiation, $d(i(V)dA = 0$, while the 4th is not.

To test for topological evolution, form the integral over a closed domain of one of the 4 Pfaff classes. Then, construct the Lie derivative with respect to V of the integral. If the Lie (convective) derivative vanishes, then the topological property represented by the integral is an invariant of the evolution. Conversely, a non-null value of the Lie derivative of the integral indicates topological change.

An extraordinary result is that for all closed flows of the deRham categories (1,2,3 above), the Lie variation of all even dimensional Pfaff sequences vanishes, L(V) dA^dA^... = 0. However, it may be demonstrated that relative to closed flows, the odd dimensional sets, that is, the 1 and 3 dimensional Pfaff structures of Circulation and Helicity, may undergo topological evolution with the production of defects, while the even dimensional Pfaff structures of Vorticity and Parity remain invariant. These defects may be associated with the production of topological "holes and handles". Moreover, according to the Brouwer theorem, the evolutionary change of topology is quantized to the integers (you cannot create half a hole).

If the view is taken that the limit points of the Pfaff topology are given in terms of the exterior derivative operator, then the invariance of the even dimensional sets relative to closed flows implies invariance of the limit points, and it follows that such closed flows are continuous relative to the Pfaff topology! Hence, when topology changes by variation of the odd dimensional sets, the closed process must be irreversible.

The topological evolution of the 1 dimensional Pfaff structures may be put into 1 to 1 correspondence with Newton's laws for the evolution of energy, (if the internal energy is constant – note added 3/26/2003)

$$L(V)A = W \quad \tilde{} \quad F.dr = Work. \tag{8}$$

The corresponding evolutionary law for the third rank field of topological torsion corresonds to the evolution of entropy:

$$L(V)H = S. \tag{9}$$

7 APPENDIX

7.1 A : TOPOLOGICAL AND THERMODYNAMIC PHASES

Consider the non-linear flow:

$$u = -(3x/8)^2/(3y-1), \quad v = +3x/(3y-1)^2 - 3/y^2, \quad w = 1. \tag{10}$$

Transform the variables by means of the equations: $x = exp3/8s,\ y = V,\ z = U$, redefining $ds/dt = 8/3d(lnx)/dt$ as $-T$, dy/dt as p , and $\rho = 1/V$. The solution to the system is given by the expression

$$\Psi == U - \rho\ exp(3s/8)/(3-\rho) + 3\rho. \tag{11}$$

The unit tangent field may be constructed as:

$$\mathbf{t}_s = T/(1+T^2+p^2)^{1/2}, \mathbf{t}_v = -p/(1+T^2+p^2)^{1/2},\ \mathbf{t}_u = -1/(1+T^2+p^2)^{1/2} \tag{12}$$

from which it is possible to compute the Jacobian Dyadic. As the system depends only on two variables, it is possible to show that $R = 0$ identically. A plot of the functions p vs ρ> for various T yields a pressure density relationship typical of Van der Waals gas (See Figure 1). The intersection of this surface and the surface $Q = 0$ defines the Spinodal line of absolute phase instability; it is the line of zero Gauss curvature for the fundamental surface, $\Psi = 0$

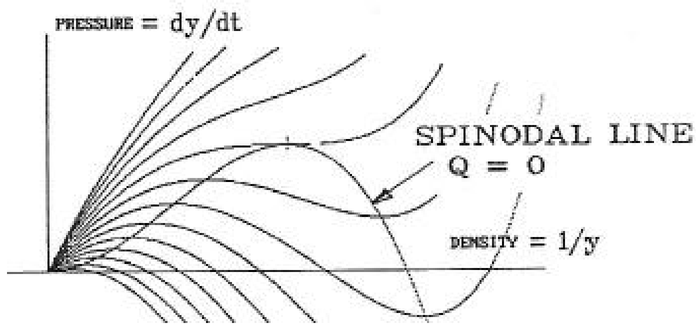

Figure 1 The Kinematic Surface Topology is similar to a Van der Waals Gas.

A more dramatic result is given for the Gibbs surface $g = u - Ts + pV$ which is plotted in Figure 2. The intersection of the Gibbs surface, $g(T, \rho) = 0$ and the surface $Q = 0$ of zero Gauss sectional curvature forms a line in space which is the envelope, or tac locus, of the cuspoids of the Gibbs surface. It is apparent that the cuspoidal edge corresponds to the Cardano condition $C = 0$

where the three eigenfunctions are real, but two are degenerate. Moreover, the point $R = 0$, $Q = 0$, and $P = 0$ defines the thermodynamic critical point.

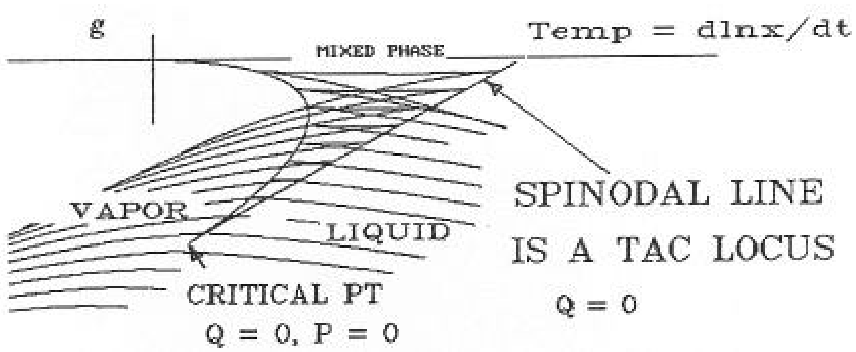

Figure 2 The Kinematic Gibbs Surface Topology of a non-linear flow.

7.2 B : INTERMITTENT TORSION BURSTS

Consider the non-linear flow:

$$dx/dt = -\Omega y + xz \quad (13)$$
$$dy/dt = +\Omega x + yz \quad (14)$$
$$dz/dt = F + Az - Dz^3 - B(x^2 + y^2) \quad (15)$$

which as a Velocity field, V , is not only an exact solution of the Navier-Stokes equations in a rotating frame of reference, but also is a model of the tertiary hysteretic bifurcation studied by Langford. A plot of the transverse torsion wave packet burst generated by such a solution is presented in Figure 3. The time between bursts is not periodic.

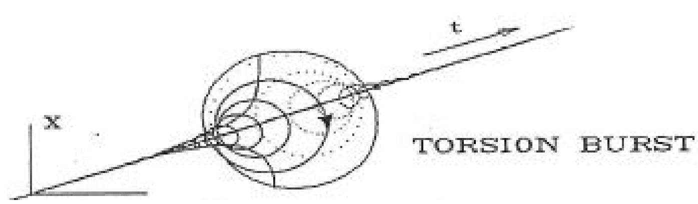

Figure 3. An intermittent torsion wave packet in a Navier-Stokes fluid.

8 ACKNOWLEDGMENTS

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