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The Ideal Gas

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Abstract

Notes for Physics 1312

I. The Ideal Thermodynamic Substance

The ideal thermodynamic substance (often called the "ideal gas") is a physical system that is characterized by the following constraints.

There is a quantity (an extensive variable) called the Internal Energy, U, that depends on the number, n, of components that make up the system, and an intensity (or intensive variable) defined as the Temperature, T.

$$U = nC_vT:$$

In addition the system has another quantity called the Volume, V, that depends on the number of parts and the ratio of the Temperature, T; and another intensive variable, the Pressure, P.

$$V = nRT=P$$

Experimentally, the coefficient R turns out to be universal, while the ratio of C_v to R is a rational ratio of integers - another indicator of the topological foundation of thermodynamics.

Although often called an ideal gas, these two properties approximately describe systems that often do not have the visual perspective of a "gas". In fact these equations, to a certain extent, describe such diverse things such as the interior of an atomic nucleus, as well as the universe of galactic domains. The usual sophomoric presentation defines the equation of "state" as $PV = nRT$. Note that "the single equation of state" is not sufficient to describe all of the thermodynamic properties of the ideal substance. The second equation - of internal energy - is also needed.

These representations come from an analysis of the fundamental equation of thermodynamics, the first law:

$$W + dU = Q:$$

This equation is a statement of cohomology (a topological idea), and relates the inexact differential if Heat, Q; and the inexact differential of Work, W: The concept of differential work comes from the mechanical idea of a force acting through a differential distance, $W = F \cdot dr$. The concept of Heat comes from the empirical result that temperature changes and internal energy changes can be accomplished without doing mechanical work.

When the Work 1-form, W; is "integrable" the differential expression on the right can be expressed globally (over the whole domain of the substance) in terms of only two independent functions, and not more than two are necessary. (The idea is that more functions can be used to describe the work process, but there exist functional relations between these superfluous functions such that the number can be reduced to 2 at most). The "direction of the process" is uniquelly defined by the "gradient" of one of the necessary functions. This concept of "directional uniqueness" leads to the definition of what is meant by P and V. When integrable, W is said to be of the form

$$W = PdV.$$

The direction of the process is given by the "gradient" of the Volume function. Of course, this idea neatly fits in with experiments conducted on dilute gases.

When the work one form is not integrable, then it takes more than two functions to express the properties of the differential of work. The substance can no longer said to be an ideal thermodynamic substance.

The first law demonstrates that there are processes that can change the internal energy of a substance without doing mechanical work. It is this notion that distinguishes heat from work. For example, if $W = PdV = 0$ for the ideal substance, then the constraint(s) that the volume is constant ($dV = 0$) and that the number of parts stays the same ($dn = 0$) - lead to the concept that the change in heat (at constant volume and constant number) is related to a temperature change, dT. The idea that $dV = 0$ is why the coefficient C_v has the subscript, v.

$$dU = dn \cdot C_v T + nC_v \cdot dT = Q.$$

For the ideal substance with constant parts, the change in internal energy depends only on the temperature differences between any two configurations, and not on how one configuration evolves into another configuration. (process path independence of internal energy) .

$$dU = nC_v \cdot dT \text{ no matter what the process path}$$

So that for a constant volume, no work process, the heat 1-form is given by the exact expression:

$$nC_v \cdot dT = Q$$

Now from the equation of state for an ideal substance,

$$PdV + VdP = dn \cdot RT + nR \cdot dT.$$

This equation permits a solution for the Work 1-form

$$W = PdV = dn \cdot RT + nR \cdot dT - VdP$$

Then the first law becomes

$$PdV + dU = dn \cdot RT + nR \cdot dT - VdP + dn \cdot C_v T + nC_v \cdot dT = Q$$

or combining terms,

$$dn\cdot(C_v + R)T - V\,dP + n(C_v + R)\cdot dT = Q$$

Now assume that the number of parts stays the same, and that the Pressure is held constant. Then (at constant pressure and number)

$$n(C_v + R)\cdot dT = nC_p\cdot dT = Q$$

where C_p is defined as the heat capacity per part at constant pressure and constant number of parts. Again, these features match experimental findings on gases such that $C_p - C_v = R$ a universal constant for all gases. The ratio of heat capacities is a rational ratio of integers, and is conveniently defined as

$$C_p/C_v = \gamma = 1 + R/C_v$$

Going back to the first law another manipulation gives

$$P\,dV + dU = (nRT/V)dV + dn\cdot C_vT + nC_v\cdot dT = Q$$

In this case. assume that Q and dn are zero (the Adiabatic constraints). Then the first law becomes a separable equation which is immediately integrable:

$$n(R\cdot dV/V + C_v\cdot dT/T) = 0$$

The integral is

$$T\cdot V^{\gamma-1} = \text{some adiabatic constant}$$

Again using the equation of state to eliminate T, an alternate expression for the adiabatic constraint becomes

$$PV^{\gamma} = \text{some adiabatic constant}$$

These fundamental results lead to two types of processes that can be described on the PV plane for an Ideal Substance.

The first process is an **adiabatic process** ($Q = 0$) described by the curve $PV^{\gamma} = P_0V_0^{\gamma}$ (where P_0 and V_0 are intial points on the curve).

The second process is an **isothermal process** ($dT = 0$) described by the curve $PV = P_0V_0$:

By alternating an adiabatic process by an isothermal process in 2 or more pairs, a cyclic process on the PV plane can be created. Such a cyclic process is called **Carnot Cycle.**

Any real cycle can be approximated by a Carnot Cycle.

The work done along a segment of an isothermal path can be computed from the formula

$$\int_a^b W = \int_a^b P\,dV = \int_a^b nRT \cdot dV/V = nRT \int_a^b dV/V = nRT \cdot \ln(V_b/V_a)$$

The work done along a segment of an adiabatic path that connects two isotherms can be computed from the formula

$$\int_a^b W = \int_a^b P\,dV = -\int_a^b dU = nC_v \int_b^a dT = nC_v(T_a - T_b)$$