CARTAN'S METHODS of EXTERIOR CALCULUS

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1. Introduction

Cartan's methods utilize (what might be unfamiliar) techniques that are described as the "exterior product" with an algebraic symbol, ˆ, and the "exterior differential" with a differential symbol, d, acting on objects, ω^p, defined as exterior differential forms of degree (not power) p. These basic concepts will be discussed below, briefly, but are best studied in detail from texts such as that by Harley Flanders ("Exterior Differential Forms"), Bamberg and Sternberg ("A course in Advanced Calculus ", Vols 1 and 2) and by Gockeler and Schucker ("Differential Geometry, Gauge theories, and Gravity"). I believe the best way to learn about these "new" operations and the objects upon they act is to try a few examples. Flanders is the best text with which to start. About the only thing missing in the Flander's presentation is a discussion of the Lie derivative acting on p-forms.

There are a number of other texts available that discuss exterior differential forms, but many are a bit pedantic or too pompous to be useful at the applied engineering level. Most applications that have appeared in the literature are in the field of general relativity or super-symmetry or super-gravity or string types of theories. Very few texts are to be found which present the Cartan methods as applied to hydrodynamics, electrodynamics, thermodynamics, and other engineering sciences. (domage!) To reiterate previous statements, recall that the methods of exterior differential forms are important because they carry topological information, and can be used to study topological evolution. My interest in the Cartan methods was cemented when I realized how natural it was to write Maxwell's equations in terms of differential forms. Moreover it became apparent the Maxwell theory was independent from metric or connection constraints. From this point of view (perhaps initiated by VanDantzig) electromagnetism is not a geometrical theory, but instead is a topological theory. The PDE's of the Maxwell - Faraday induction equations, form a nested set in every dimension greater than 3. Experience with electromagnetic theory is very useful, for you can use EM theory to check your developing skills with the Cartan methods. If your application of the Cartan techniques does not replicate well known results in electromagnetism, you have made a mistake. These concepts will be detailed in that which follows.

There are two other important operations, besides the exterior product and the exterior differential, which act on differential forms, but these other operations require the specification of a vector field, V, in addition to the differential form, ω. From a physical point of view, differential form(s) may be used to define a physical system, and the vector field may be used to define a thermodynamic evolutionary process. What is remarkable is that this point of view can be used to justify the topological basis of thermodynamics, and to give a non-statistical description of irreversible processes. The two additional operations (with respect to V) are called the "interior product" with the symbol $i(V)$, and the Lie differential, $L_{(V)} = i(V)d + di(V)$, which combines two operations, the interior product and the exterior derivative. The Lie differential is an alternative to the "covariant" differential of tensor analysis, and, like the covaraint derivative, will produce tensors from other tensors by means of a differential process. Examples and definitions will be given below. The Lie differential will become the most important tool from a topological perspective, for it permits computations to be made which will distinguish those objects which are topological invariants of a process and those objects which are not. It is remarkable that the Lie differential operating on a 1-form of Action is equivalent to the cohological statement that defines the first law of thermodynamics.

The exterior differential forms (differential forms for short) are objects that are built from functions defined on a vector bundle. What this means is that starting from the assumption that there exists an n dimensional variety of independent variables, $\{y^a\}$, often called coordinates, it is possible to construct two other vector spaces. These two vector spaces consist of a 1 dimensional vectorspace $\Lambda^0\{1\}$ and an m dimensional vector space $\Lambda^1\{\sigma^k\}$. The one dimensional vector space $\Lambda^0\{1\}$ consists of all functions that can be constructed from $\{y^a\}$, and has the unit 1 as a basis element. The m dimensional vector space $\Lambda^1\{\sigma^k\}$ is endowed with a "differential" basis elements, σ^k, $k = \{1, 2, ...m\}$ The basis set σ^k is presumed to be linearly related to the differentials dy^a of the independent variables via the formula

$$|dy^a\rangle \Rightarrow \left|\sigma^k\right\rangle = \left[F_a^k\right] \circ |dy^a\rangle . \tag{1.1}$$

(The dimension of the set σ^k may be different from the dimension of $\{y^a\}$.) An alternative point of view is that a linear combination of the differentials on the initial state $|\sigma^a\rangle$ such that the linear mapping $\left[F_a^k\right]$ acting on the $|\sigma^a\rangle$ produces perfect differentials $\left|dx^k\right\rangle$ on the final state (This point of view is assumed in the Flander's book).

$$|\varpi^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[F_a^k\right] \circ |\varpi^a\rangle . \tag{1.2}$$

The two vector sets of linear combinations of differentials, $|\varpi^a\rangle$ and $\left|\sigma^k\right\rangle$ on the initial state are not the same, even when the dimension of the initial and final states are the same. Given the initial state $|dy^a\rangle$, the $\left|\sigma^k\right\rangle$ are determined by the linear map $\left[F_a^k\right]$. Given the final state $\left|dx^k\right\rangle$, the $|\varpi^a\rangle$ are determined by the inverse of the linear map, $\left[F_a^k\right]^{-1} = [G_k^a]$. If the two spaces are not of the same dimension then the inverse of the linear map need not exist. It is important for generalizations that the concept of the initial state or domain (and its coordinate functions, y^a) be kept distinct from the final state or range (and its coordinate functions, x^k). The format $\left|\sigma^k\right\rangle = \left[F_a^k(y)\right] |dy^a\rangle$ will be emphasized herein.

All of this had its origins in the theory of differentiable coordinate mappings from an initial to a final state, where, given a (more than likely) non-linear map ϕ from a domain $\{y^a\}$ to a range $\{x^k\}$, a linear map $d\phi$ between differentials could be generated to establish the (linear) differential vector space ideas. In symbols

$$Nonlinear \quad \phi : \{y^a\} \Rightarrow \{x^k\} = \phi^k(y) \tag{1.3}$$

$$Linear \quad d\phi : |dy^a\rangle \Rightarrow \left|dx^k\right\rangle = \left[\partial\phi^k(y)/\partial y^a\right] |dy^a\rangle = \left[F_a^k(y)\right] |dy^a\rangle . \tag{1.4}$$

The Linear mapping $\left[F_a^k(y)\right]$ so generated is the Jacobian matrix of partial differentials of the coordinate (or vector space) mappings. As an example, review the concept of spherical or cylindrical coordinates mapped into cartesian space. (Maple programs have been developed giving most of the details in terms of symbolic mathematics. See http://www22.pair.com/csdc/pdf/mtpertu5.pdf)

At first, in this presentation, the dimension of the vector space $\left|\sigma^k\right\rangle = \left|dx^k\right\rangle$ will be assumed to be n, the same dimension as the space of independent variables. Then the ranges of the index a is: $a = \{1...n\}$, and the range of the index k is : $k = \{1...n\}$. This restriction will be relaxed later during the development of a more general theory. The $n\,x\,n$ Jacobian matrix of the n mapping functions establishes the vector space ideas as a linear mapping, and gives the primitive realization of what is to be known as a Frame matrix (of functions on the initial state).

Suppose that another function, say $\Phi(y^a)$, is given in terms of the initial variety, $\{y^a\}$. Then its total differential is given by the expression,

$$d\Phi(y^a) = \{\partial\Phi(y^a)/\partial y^b\}\, dy^b = \sum_b A_b(y^a) dy^b = \langle A_b(y^a)| \circ \left|dy^b\right\rangle. \tag{1.5}$$

The object on the right is an example of an exterior differential 1-form, ω^1, with coefficient functions $A_b(y^a)$ and basis elements, dy^b. (From here on the sum convention on up-down symbols - the index b in the formula above - will be presumed, without the use of the $\sum$ symbol). The coefficients, by construction in this example,

$$A_b(y^a) = \{\partial\Phi(y^a)/\partial y^b\}, \tag{1.6}$$

form the components of a covariant gradient vector field.

In the Cartan theory of differential forms these concepts are extended to situations where the differential basis elements σ^k, of the vector space $\Lambda^1\{\sigma^k\}$ can be written in terms of some arbitrary matrix (of functions on the initial state) acting on the differentials of the independent variables in a linear way:

$$\left|\sigma^k\right\rangle = \left[F_a^k(y)\right] \, |dy^a\rangle. \tag{1.7}$$

In other words, in the Cartan extension, it is not assumed that the linear map $\left[F_a^k(y)\right]$ is necessarily a Jacobian matrix of some non-linear coordinate mapping, nor is it assumed that the matrix is even similar to the Jacobian matrix of a coordinate mapping. This more general matrix of functions, $\left[F_a^k(y)\right]$, will be defined as a basis Frame and is the cornerstone of Cartan's development of the Repere Mobile. Given such a matrix of functions a key question revolves about the determination of the solubility of the Frame. Given a Frame, does there exist a unique set of mapping functions ϕ from which the Frame is determined to be the Jacobian matrix $d\phi$ of the mapping? If not, is it possible that there exists a non-unique solution set to the problem? The question of non-unique integrability of the Frame matrix is the basis of what is called Affine and Topological Torsion. Torsion appears when the basis Frame (or its equivalence class) is NOT integrable.

For applications, how the Frame may be related to specific physical problems is of key importance. The early development of the Frenet-Serret-Cartan Frame for a point moving along a space curve indicates that it is possible to construct the Frame from differentials of the mapping function with respect to a parameter along a space curve. That is, the velocity, acceleration and the rate of change of acceleration can be used to build a Frame matrix of a point moving along a space curve in three dimensions. These things are physical, measureable, and applicable quantities. The same idea can be generated for continuous media such as a fluid. The velocity field, the vorticity field, and the helicity field of the fluid become the analogs of the Frenet - Serret differentiations.

For those ubiquitous cases (or better said, on those restricted domains) where the Frame has an inverse, then the Frame matrix is an element of the General Linear group. Often in particular applications the Frame matrix is constrained to be an element of an equivalence class of "admissable" Frames by assuming the Frame belongs to some sub-group of the GL group. In the Frenet-Serret case, a usual restriction constrains the 3D Frame matrix elements such that the Frame is a member of the special othogonal (orthonormal) group. The columns of the basis Frame matrix are orthogonal unit vectors. This constraint is used to create the concepts of arc length, curvature, and torsion of the 3D space curve. These "intrinsic" properties of the space curve are the similarity invariants of all equivalent Frames (that is, all Frames that are members of the orthonormal group). These intrinsic (often called invariant) properties of the equivalence class are computed by means of the coefficients of the Cayley-Hamilton theorem.

From a physics point of view, all observers who may use different elements or representations of the orthonormal group for reference systems will be able to express their views in terms of a common set of qualities, the similarity invariants. All equivalent observers will agree that the values of the similarity invariants are the same. Restrictions to paarticular subgroups are often called "gauge theories". It is important to note that certain (normal) subgroups (such as the orthonormal subgroup) cannot distinguish between left and right handedness (chirality) , but other equivalence classes of subgroups can. It would seem that this ability to distinguish a chiral property is of value to the study of biological systems, where most biological molecules appear to be left or right handed. The moral (or warning) of this paragraph is that the common orthonormal system of basis vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of engineering practice must be modified to handle chiral distinctions.

It is important to be reminded of the idea of a similarity transformation. Given a matrix $[M]$ and a transformation matrix $[F]$, the matrix $[N]$ is said to be similar to $[M]$ if

$$[M] \Rightarrow [N] = [F]^{-1} \circ [M] \circ [F]\,. \tag{1.8}$$

When the Cayley-Hamilton polynomial is constructed for $[M]$ and $[N]$ the coefficients of the polynomials are the same (if the matrices are "similar"). Two of the important similarity invariants are the trace of $[M]$ and its determinant. In differential geometry, these ideas will be used to define curvature properties of manifolds. In the Frenet-Serret-Cartan theory of the orthonormal subgroup, the similarity invariants lead to the concepts of arc length, curvature and torsion. The zero sets of the similarity invariants have particular physical importance. In the thermodynamics of a VanderWaals gas, the Cayley-Hamilton polynomial based upon the Gibbs function is a cubic polynomial with the surface shape of a swallow-tail. The critical point is where all three similarity invariants vanish. The spinodal line of phase instability is where the quadratic similarity coefficient (the Gauss curvature of the swallow-tail surface) vanishes.

The similarity equation can be rewritten in a manner that does not require the immediate computation of an inverse:

$$[F] \circ [N] = [M] \circ [F] . \tag{1.9}$$

This equation can be used to test if $[N]$ is similar to $[M]$. A special situation occurs if the matrix $[N]$ is the same as $[M]$. This situation places a constraint on the equivalence class of matrices that can be used for the transformations $[F]$. Suppose that $[N] = [M] \mp d[M]$, then the differential similarity equation becomes

$$d[M] = [M] \circ [F] \pm [F] \circ [M] , \tag{1.10}$$

and is suggestive of the Heisenberg matrix operator format (the transformation matrix $[F]$ plays the role of the "Hamiltonian" operator). These similarity formats will reappear below when the matrix of connection 1-forms is discussed.

Note that the column vector array of differential basis elements, $|dy^a\rangle$, transforms as a contravariant tensor in the Jacobian case, where a coordinate mapping is available. This property can be extended if the Frame matrix of functions has a non-zero determinant, for then the columns of the Frame matrix can be used as a basis set for contravariant vectors in the initial space (domain or state). This basis argument does not depend upon the fact that matrix elements $\left[F_a^k(y)\right]$ form a Jacobian (i.e., integrable) system. It will be demonstrated that it is this lack of unique integrability for the $\left|\sigma^k\right\rangle = \left[F_a^k(y)\right] |dy^a\rangle$ that leads to the concepts of affine Torsion and Topological Torsion , two topics that will be discussed in great detail in subsequent sections.

At present, given a basis of 1-forms, construct arbitrary exterior differential 1- forms from the matrix product of arbitrary coefficient functions arranged as a row vector, $\left\langle \widehat{A}_k(y^a)\right|$ and the basis set arranged as a column vector, $\left|\sigma^k\right\rangle$

$$\omega^1 = \left\langle \widehat{A}_k(y^a)\right| \circ \left|\sigma^k\right\rangle = \widehat{A}_k(y)\, \sigma^k. \tag{1.11}$$

Note that if the coefficient functions are chosen to be a covariant vector array (and that is why the index is a lower index on the $A_k(y)$), then the differential 1-form ω is a scalar invariant of "coordinate transformations". The coefficient functions, however, do not have to be a gradient array. The covariant constraint implies that if (as assumed)

$$\left|\sigma^k\right\rangle = \left[F_a^k(y)\right] \, |dy^a\rangle \tag{1.12}$$

then

$$\left\langle \widehat{A}_k(y^a)\right| = \langle A_b| \circ \left[G_k^b\right] \tag{1.13}$$

where $\left[G_k^b(y)\right]$ is the inverse matrix of functions to the frame matrix, $\left[F_a^k(y)\right]$. These are the rules of classical tensor analysis defining what is meant by contravariant and covariant vectors of ordered sets of components with respect to special transformations (defined as diffeomorphisms).

The differential form so constructed in terms of tensor coefficients is then independent from a "choice of coordinate system".

$$\omega^1 = \left\langle \widehat{A}_k(y^a)\right| \circ \left|\sigma^k\right\rangle = \langle A_b| \circ \left[G_k^b\right] \circ \left[F_a^k\right] \circ |dy^a\rangle . \tag{1.14}$$

For physicists and engineers what this implies is that laws of physics written in terms of differential forms are independent of the observer's choice of a "reference system". A "reference system" is defined as an element of an equivalence class of differentiable mappings. The most common equivalence class usually accepted is the class of diffeomorphisms, which implies that the mapping, ϕ, and the linear mapping, $d\phi$,have inverses, and the inverse mapping is differentiable. Such diffeomorphic mappings are constrained subsets of other mappings known as homeomorphisms. Homeomorphisms (and therefor diffeomorphisms) preserve topology from initial to final state, and therefor cannot be used to describe topological evolution. (Bummer.) Sometimes the equivalence class of reference systems is even further constrained. For example, the acceptable class of reference systems known as inertial frames of reference in the physics of special relativity is constrained to be the Lorentz equivalence class. Sometimes such constraints throw the baby out with the wash. For example, General Relativity is designed to admit all diffeomorphisms as the equivalence class of frames of reference; Special Relativity admits only elements of the Lorentz equivalence class, which is a subset of all diffeomorphisms.

The Lorentz equivalence class consists of those matrices, $[L]$, for which the Minkowski line metric is preserved. That is

$$\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} = [\eta] = \left[L^{-1}\right] \circ [\eta] \circ [L] \tag{1.15}$$

There is a further subclass of Lorentz matrices, with matrix elements which are constants, and which are use in special relativistic (non-accelerated) applications. (This special subclass turns out to be "affine" torsion free, so that left handed and right handed chirality species evolve in the same way). However, there are Lorentz matrices that preserve the Minkowski metric that are not composed of constant elements. Such matrices admit accelerations, and also admit Affine Torsion coefficients. (See http://www22.pair.com/csdc/pdf/lorentz.pdf). That is, the system of 1-forms generated by a Lorentz transformation of non-constant elements is not necessarily uniquely integrable, and therefor admit different behavior for chiral systems. (This difference in behavior distinguishes Optical Activity from Faraday Rotation in electromagnetic systems.)

. A major thrust of the work that appears on Cartan's Corner is that the reference systems are extended to include topological change, so that non-diffeomorphic transformations will be investigated. A closer look a Cartan's concepts yields the result that, unlike tensors which are well behaved with respect to diffeomorphisms, exterior differential forms are well behaved in a functional sense with respect to a class of transformations even wider than the class of diffeomorphisms – in fact, wider than the class of homeomorphisms! Hence differential forms are useful to the study of topological evolution, which is the main theme of Cartan's Corner and these lectures. This result will be exploited in the following chapters. (See "Retrodictive Determinism" http://www22.pair.com/csdc/pdf/retrodic.pdf).

It should be realized that differential forms have the tensor like property that if the differential form is zero in one coordinate system of reference, then it is zero in all other diffeomorphically equivalent systems, no matter what constraints are applied to limit the elements of the equivalence class of diffeomorphisms. In addition, if a differential form is zero on the final state, then its pullback to the initial state is also zero with respect to continuous but not homeomorphic, and therefor not diffeomorphic maps. (See http://www22.pair.com/csdc/ed3/ed3fre1.htm)

2. The exterior algebra

The exterior algebra of Cartan is based upon an associative, but not commutative, multiplication rule defined as the exterior (wedge or hook) product of objects defined as exterior differential forms. The symbol for the product is ˆ. The structure of the algebra can be built starting from 1-forms on the n-dimensional vector space $\Lambda^1\{\sigma^a\}$, in terms of a basis of 1-forms denoted by $\{\sigma^a\}$. The arbitrary 1-form is constructed from the basis elements according to the formula given above, $\omega^1 = A_k(y)\ \sigma^k$. The addition rule of the algebra is that of vector space addition: add the coefficients of the same basis elements.

$$\begin{aligned} \omega 1 &= A_k(y)\ \sigma^k \\ \omega 2 &= B_k(y)\ \sigma^k \\ \omega 1 + \omega 2 &= \{A_k(y) + B_k(y)\}\ \sigma^k. \end{aligned} \tag{2.1}$$

Example 2.1. *Add* $(3x\,dx + 4xz\,dy)$ *and* $(2y\,dy + 17y\,dz)$:

$$basis = (dx, dy, dz)$$

$$(3x\,dx + 4xz\,dy) + (2y\,dy + 17y\,dz) = 3x\,dx + (4xy + 2y)\,dy + 17y\,dz$$

It is the multiplication rule that is perhaps unfamiliar. The multiplication rules are defined in terms of elements of the basis set.

$$\begin{aligned} \sigma^a\hat{}\sigma^b &= -\sigma^b\hat{}\sigma^a \\ \sigma^b\hat{}\sigma^b &= 0 \end{aligned} \tag{2.2}$$

$$\begin{aligned} dy^a\hat{}dy^b &= -dy^b\hat{}dy^a \\ dy^b\hat{}dy^b &= 0 \end{aligned} \tag{2.3}$$

These rules are similar to the cross product of Gibbs 3D vector analysis, but the difference is that the exterior product rule extends to n dimensions (the Gibbs cross product does not) and is associative (Gibbs product is not). Associative means $(\sigma^a\hat{}\sigma^b)\hat{}\sigma^c = \sigma^a\hat{}(\sigma^b\hat{}\sigma^c)$. In 3D, the Gibbs cross product yields $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

Example 2.2. *Multiply* $A = \sum A_k\sigma^k$ *times* $B = \sum B_m\sigma^m$ $\qquad A\hat{}B = C$

$$
\begin{aligned}
A\hat{}B &= \{A_k\sigma^k\}\hat{}\{B_m\sigma^m\} \\
&= ...A_kB_k\sigma^k\hat{}\sigma^k + ...A_kB_m\sigma^k\hat{}\sigma^m + ...A_mB_k\sigma^m\hat{}\sigma^k \\
&= ... \quad 0 \qquad + ...A_kB_m\sigma^k\hat{}\sigma^m - ...A_mB_k\sigma^k\hat{}\sigma^m \\
&= ... \ \{A_kB_m - A_mB_k\}\sigma^k\hat{}\sigma^m... \\
&= ...C_{[km]}\sigma^k\hat{}\sigma^m.. \\
&= ...C_{[km]}\sigma^{[km]}... \\
&= ...C_H\sigma^H
\end{aligned} \tag{2.4}
$$

Note that $A\hat{}B \neq B\hat{}A$ and that the exterior product of two elements of the vector space $\Lambda^1(\sigma^k)$ produce a linear combination of paired basis elements of the form $(\sigma^k\hat{}\sigma^m)$. Moreover the coefficients of these elements are always anti-symmetric under interchange of the paired indices. The anti-symmetry and the rules of 1-form mltiplication permit the writing of the product of two 1-forms in terms of another vector space, whose basis elements are the anti-symmetric pairs $(\sigma^k\hat{}\sigma^m)$. This object is defined as a 2-form. It is conventional to rewrite the 2-form so constructed without repeating basis element pairs that are of different sign. That is, $\Lambda^2(\sigma^k\hat{}\sigma^m) \Rightarrow \Lambda^2(\sigma^{[km]}) = \Lambda^2(\sigma^{[H]})$.

The symbol $H = [km]$ stands for all order pairs where $k < m$, and is equivalent to the set $(12, 13, 14.....23, 24.....34...)$. It is apparent that the number of ordered singlet basis elements in 3 dimensions is the same as the number of ordered anti-symmetric pairs; but this is only true in 3D. For 4D, the number of singlet basis elements is 4 and the number of ordered anti-symmetric pairs is 6. The result of the exterior product is to produce from elements of one vector space of dimension n, another element of a different vector space of dimension $n(n-1)/2$. In this limited sense the exterior product is not closed. The exterior multiplicative combination of two objects of the same type (1-forms) does not produce an object of the same type, but instead produces a 2-form. The process of exterior multiplication can be repeated where 2 -forms are multiplied by 1-forms to produce 3-forms, and 3-forms are multiplied by 1-forms to produce 4-forms, ultimately building a "closed" algebra. The elements of the closed algebra will consist of classes, or vector spaces with basis element doublets, $\sigma^a\hat{}\sigma^b$, classes of triplets, $\sigma^a\hat{}\sigma^b\hat{}\sigma^c$, ... and even n-tuplets of basis vectors, $\Omega = \sigma^a\hat{}\sigma^b\hat{}\sigma^c\hat{}...\hat{}\sigma^n$. However, the rules of multiplication are such that the exterior multiplicative combination of more that n basis vectors must vanish. Hence any element of the algebra times another element of the algebra is an element of the algebra, or zero. In this sense, the exterior algebra is closed.

The doublets are called 2-forms, the triplets are called 3-forms, and the n-tuplets are called n-forms. The multiplication rules demonstrate that each of the p-tuplets has a number of linearly independent elements equal to the possible combinations of n things take p at a time. Each of the p-tuplets forms a vector space basis of dimension equal to the appropriate combinatorial number of Pascal's triangle.

$$\text{Pascal's Triangle:}\quad \begin{array}{lccccccccc} n=1: & & & & & 1 & & & & \\ n=2: & & & & 1 & & 1 & & & \\ n=3: & & & 1 & & 3 & & 3 & 1 & \\ n=4: & 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The vector subspace dimension of the 0-forms, ω^0, is 1; the dimension of the 1-forms, ω^1, is n, the dimension of the 2-forms, ω^2, is equal to the number of combinations of n things taken 2 at a time (n(n-1)/2 is equal to 3 for n =3, equal to 6 for $n = 4$, etc.); the dimension of the n -1 forms, ω^{n-1}, is n, the dimension of the n-forms, ω^n, is 1. The dimension of the exterior algebra is the sum of the dimensions of all vector spaces produced by the exterior product; this dimension is equal to 2^n.

As an example consider the exterior algebras up to $n = 4$. The elements of Pascal's triangle yield a 1-dimensional (scalar) vector space Λ^0 for the 0-forms, a 4 dimensional vector space Λ^1 for the 1-forms, a 6 dimensional space Λ^2 for the 2-forms, a 4 dimensional vector space Λ^3 for the 3-forms, and a 1 dimensional vector space Λ^4 for the n=4 forms.

From geometical studies in 3 dimensions, n=3, the elements of the 4 different vector spaces are called points, lines, surfaces and volumes. From applications to 3D mechanics, the position vector and the momentum vector are from the vector subspace, Λ^1, and their exterior (cross) product is an element of the vector subspace, Λ^2. The fact that angular momentum is an element of a vector space Λ^2 is why one never sees angular momentum added to linear momentum (which is an element of a different vector space Λ^1) in the elementary mechanics text books. From applications to special relativistic physics in 4 dimensions, n=4, the coefficients of the elements of these five different vector subspaces of the exterior algebra are known as scalars, vectors, tensors, pseudo-vectors, and pseudo scalars.

$p-forms\ in\ 4D$	0	1	2	3	4
$\#\ of\ basis\ elements$	1	4	6	4	1
$name$	Scalars	Vectors	Tensors	Pseudo-Vectors	Pseudo-Scalars

(2.5)

The Cartan-Grassman exterior algebra in consists of a vector space of 2^n (= 16 in 4D) components with n+1 (= 5 in 4D) different vector subspaces. The algebra is technically called a graded algebra. The exterior algebra is closed with respect to multiplication, for all possible products of the algebra reside within the algebra of 2^n dimensions (or are zero). The exterior product of a p form and a q-form produces a p+q form, or zero if p+q > n. In every case, the higher p-forms can be constructed from sums of products of the singlets. All elements of the algebra can be constructed from linear combinations of the primitive n basis elements, σ^k and their products.

A nice feature of the exterior algebra (besides being closed) is that the definitions of symbolic operations can be described entirely in terms of 0-forms and 1-forms, when the collective index is used. Every p-form can be rewritten in terms of symbolic coefficients (0-forms) and basis elements (p-forms from vector spaces of different dimensionality of course) with a format similar to that of a 1-form. For example, for n = 4, there are 6 elements of the vector subspace of two forms. That is, there are 6 independent non-zero pairs of of the 4 basis 1-forms σ^k from Λ^1 that can be used as the basis elements of Λ^2 : namely the set $\{\sigma^1\hat{}\sigma^2, \sigma^1\hat{}\sigma^3, \sigma^1\hat{}\sigma^4, \sigma^2\hat{}\sigma^3, \sigma^2\hat{}\sigma^4, \sigma^3\hat{}\sigma^4\}$. If these basis pairs are given a new symbolism as $\{\sigma^{12}, \sigma^{13}, \sigma^{14}, \sigma^{23}, \sigma^{24}, \sigma^{34}\}$,then the general 2-form (for n=4) can have the expansion coefficient - basis representation given by the formula:

$$F = A_{12}\sigma^{12} + A_{13}\sigma^{13} + ... + A_{34}\sigma^{34} = A_H\sigma^H, \tag{2.6}$$

where H is the collective index described above for ordered pairs. This formula for the 2-form F in 4 dimensions looks like a vector formula for a 1-form, but in space of 6, not 4, dimensions; it is just that the index labels are different. All of the distinct basis combinations will be completely antisymmetric in their indices $(p > 1)$. For example, H could be the set of triples $[i1, i2, i3]$ with $i1 < i2 < i3,$ for the vector space of 3 forms, H could be the set of quadruples $[i1, i2, i3, i4]$ with $i1 < i2 < i3 < i4,$ for the vector space of 4 forms; etc. With this collective index notation, combinatorial rules of multiplication and differentiation developed for 1 forms can be applied directly to higher order p-forms.

This technique, where a 2-form expansion in 4D was used as a 6 - vector, was applied (intuitively?) more than 60 years ago by Arnold Sommerfeld in his studies of electromagnetic systems. The 3 components of **E** and the 3 components of **B** formed the components of the 6-vector. (See his volumes on Lectures on Theoretical Physics.) A similar but little used 6 vector composed of the acceleration and vorticity can be developed for a fluid. It is not clear whether

Sommerfeld knew the theory of exterior differential systems at the time of his 6 - vector development.

Example 2.3. *in 3D, exterior multiply the 1-form, A, times 2-form, B to produce the 3-form $C = A\hat{}B$.*

$$\begin{aligned} & (A_x dx + A_y dy + A_z dz)\hat{}(B_x dy\hat{}dz + B_y dz\hat{}dx + B_z dx\hat{}dy) \\ = & (A_x B_x + A_y B_y + A_z B_z) dx\hat{}dy\hat{}dz = C \end{aligned}$$

Note that this product of a 1-form and and n-1=3-1=2 - form produces a n - form with a coefficent that looks the same as the euclidean inner product of two ordinary vectors $\mathbf{A} \circ \mathbf{B}$. Recall that the 1-form has n components and the n-1 form has n components. The euclidean inner product result is valid in all dimensions, n.

Example 2.4. *in 3D, exterior multiply the 1-form, A, times 1-form, B to produce the 2-form $D = A\hat{}B$.*

$$\begin{aligned} D &= (A_x dx + A_y dy + A_z dz)\hat{}(B_x dx + B_y dy + B_z dz) \\ &= (A_x B_y - A_y B_x) dx\hat{}dy \\ &\quad +(A_y B_z - A_z B_y) dy\hat{}dz \\ &\quad +(A_z B_x - A_x B_z) dz\hat{}dx \end{aligned}$$

Note that the result has coefficients equivalent to the Gibbs cross product of two vectors, $\mathbf{A} \times \mathbf{B}$.

2.1. The Exterior Differential.

The exterior differential is a definition of a differential process acting on p-forms, ω^p. The operation takes the p-form into a p+1 form. Hence, like the exterior product, the exterior differential generates a vector in a different vector subspace of the exterior algebra.

$$d(\omega^p) \Rightarrow \omega^{p+1}. \tag{2.7}$$

Other properties of the exterior differential will be described by the rules for distributing the operator over a product of 1-forms (note the order of factors and the minus sign modification of the Liebniz rule for the differential of a product of scalars)

$$d(A\hat{}B) \ = dA\hat{}B - A\hat{}dB, \tag{2.8}$$

and

$$d(d(\omega^p)) \Rightarrow 0. \tag{2.9}$$

For a product of a p-form and a q-form, it follows that

$$d(\omega^p\hat{}\omega^q) \ = d\omega^p\hat{}\omega^q - (-1)^p\omega^p\hat{}d\omega^q, \tag{2.10}$$

The epitome of the exterior differential is the concept of the total differential of a scalar function, which is a familiar operation that takes the 0-form, or function $\omega^0 = \theta(y^a)$, into the 1-form, $\omega^1 = A_b\, dy^b$.

$$d(\omega^0) = d\{\theta(y^a)\} \Rightarrow \{\partial\theta(y^a)/\partial y^b\}dy^b = A_b\, dy^b = \omega^1, \tag{2.11}$$

The function could be constrained such that the set $\theta(y^a) =$ constant defines an implicit surface. It follows that in the constrained case, the total differential is also zero: $d\{\theta(y^a)\} = 0$. An object that has zero for the value of its exterior differential is said to be closed (in an exterior differential - not algebraic - sense). For consistency reasons note that the differential basis elements symbolized by dy^b are defined to be closed. That is, $d(dy^b) = 0$. The exterior derivatives of an arbitrary basis, σ^k, are NOT necessarily zero, but $d(\sigma^k)$ is closed for $dd(\sigma^k) = 0$.

The exterior differential of a 1-form is defined as

$$\begin{aligned} d\omega^1 &= d(A_b\, dy^b) = (dA_b)\hat{}dy^b + A_b\, d(dy^b) \\ &= (\partial A_b/\partial y^e\, dy^e)\hat{}dy^b + 0 \\ &= (\partial A_b/\partial y^e - \partial A_e/\partial y^b)\, dy^e\, \hat{}\, dy^b \\ &= F_{[eb]}\, dy^{[eb]} = F_{[H]}dy^{[H]}. \end{aligned} \tag{2.12}$$

The collective index notation permits the formula defining exteriordifferentiation to be generalized:

$$d\omega^p = d(A_H\, dy^H) = (dA_H)\hat{}dy^H \tag{2.13}$$

So to compute the exterior derivative of any p -form, first compute the exterior (= total differential) of the scalar coefficients dA_H and then exterior multiply the result into the remaining base elements of the form dy^H, component by component.

Also note that the special 1-form with gradient coefficients, $\omega^1 = d\{\theta(y^a)\}$, has an exterior differential equal to

$$\begin{aligned} d\omega &= d(d\{\theta(y^a)\}) \qquad (2.14) \\ &= ...\{\partial^2\theta(y^a)/\partial y^b\partial y^c\}dy^c\hat{}\ dy^b + ... + \{\partial^2\theta(y^a)/\partial y^c\partial y^b\}dy^b\hat{}\ dy^c \\ &= ...\{\partial^2\theta(y^a)/\partial y^b\partial y^c - \partial^2\theta(y^a)/\partial y^c\partial y^b\}dy^c\hat{}\ dy^b \\ &= 0, \end{aligned}$$

for C2 functions. Hence this special 1-form ω^1 is a closed 1-form, assuming the coefficient functions are twice differentiable. However, as the example 1-form ω^1 has a unique primitive function, $\theta(y^a)$, whose exterior derivative creates $\omega^1 = d\theta$, the 1-form ω is said to be not only closed, but also exact. The same concepts hold for all p forms. A p-form is closed if its exterior differential vanishes, and the p-form is exact if it is constructed by means of the exterior differential operation acting on some p-1 form. There are differential forms that are closed but not exact, and those that are neither exact nor closed. The importance of closed and exact, or closed but not exact, p-forms is that they carry topological information about the domain of definition. For example, in a two dimensional surface every hole is associated with a unique 1-form that is closed, but not exact. The number closed but not exact 1-forms on a domain counts the topological number of holes. This fact is the basis of the Bohm-Aharanov idea in EM theory, and is at the foundation of the theory of flight in terms of the Joukowski transformation.

Suppose that the given exterior differential p-form is expressed in terms of a non-integrable basis set σ^H. Then the exterior differential formula becomes

$$d\omega^p = d(A_H\, \sigma^H) = (dA_H)\hat{}\sigma^H + A_H(d\sigma^H) \tag{2.15}$$

Now it must be recognised that the second term $(d\sigma^H)$ is not necessarily zero. Such complications arise when the Frame matrix generates 1-forms (σ^k) which are not closed. The basis Frame in that case is not uniquely integrable.

Remember that the exterior derivative has to be applied to products of 1-forms in terms of a modified Leibniz rule, that alternates in sign for every other odd factor. For example, the exterior derivative of the product of two 1-forms is

$$d(\sigma^1\hat{}\sigma^2) = d\sigma^1\hat{}\sigma^2 - \sigma^1\hat{}d\sigma^2. \tag{2.16}$$

Example 2.5. *Compute the exterior differential of the function* $\theta(y^a) = (y1)^2 + (y2)^2 + (y3)^2 - 1$.

$$\omega = d\theta(y^a) = 2(y^1 dy^1 + y^2\, dy^2 + y^3\, dy^3)$$

Note that the zero set of the function describes a spherical 2-surface, and the coefficients of the deduced 1-form descibe the normal field orthogonal to the tangent vectors on the surface. Direct computation demonstrates that ω is closed, as

$$d\omega = 2(dy^1\hat{}dy^1 + dy^2\hat{}\, dy^2 + dy^3\hat{}\, dy^3) = 0.$$

Example 2.6. *Compute the exterior differential of* $(A_x dx + A_y dy + A_z dz)\hat{}(B_x dx + B_y dy + B_z dz)$

$$d(A\hat{}B) = dA\hat{}B - A\hat{}dB = d(C_{[mn]}dy^{[mn]}) = d((C_{[mn]})\hat{}dy^{[mn]}$$

The object which is the result of the exterior differentiation of the 2-form constructed by the product is a 3-form with completely antisymmetric indices. The modified Leibniz rule for products is required to make the two ways of computing the resultant 3-form compatible.

The operation of the exterior differential acting on an arbitrary 1-form is defined as

$$\begin{aligned} d(A_k(y^a)\,\sigma^k) &= \{d(A_k(y^a)\}\,\hat{}\,\sigma^k + A_k(y^a)\hat{}\{d\sigma^k\} \\ &= \{\partial A_k(y^a)/\partial y^b\}dy^b\}\,\hat{}\,\sigma^k + A_k(y^a)\hat{}\{d\sigma^k\}. \end{aligned} \tag{2.17}$$

Consider the case where the arbitrary 1-forms are known to be linearly related to the differentials by means of the linear (Frame) formulas, $|\sigma^k\rangle = [F^k_a(y)]\ |dy^a\rangle$. Then

$$
\begin{aligned}
d(A_k(y^a)\,\sigma^k) &= \{\partial(A_k F^k_a)/\partial y^b\}dy^b\,\hat{}\,dy^k \\
&= \{\partial\widehat{A}_a/\partial y^b - \partial\widehat{A}_b/\partial y^a\}dy^b\,\hat{}\,dy^k \\
&= \widehat{F}_{[ab]}(y)dy^b\,\hat{}\,dy^k, \qquad (2.18)
\end{aligned}
$$

where

$$\widehat{A}_a(y) = (A_k(y)F^k_a(y)) \qquad (2.19)$$

This formula for $d(\omega^1) = d(\widehat{A}_a dy^a) = \widehat{F}_{[ab]}(y)dy^b\,\hat{}\,dy^k = \widehat{F}_{[ab]}(y)dy^{[ab]} = \widehat{F}_{[H]}(y)dy^{[H]}$ is valid on the initial state, or variety $\{y^a, dy^a\}$, whether the Frame matrix has an inverse or not. The coefficients of the 2-form correspond to the antisymmetric components of a "curl" (when n = 3). However, the exterior differnential procedure generalizes to spaces of higher dimension.

For differential forms expanded in terms of non-closed basis 1-forms, the exterior differential has two terms. The first term is just the exterior product of the total differential of the coefficient function(s) and the remaining factor of non-closed basis forms, while the second term is the exterior product of the functions and the exterior differentials of the non-closed basis forms.

The operation of the exterior differential acting on a p-form follows that same formulas, using the collective ordered index, H:

$$d(A_H(y^a)\,\sigma^H) = \{d(A_H(y^a)\}\,\hat{}\,\sigma^H + A_H(y^a)\hat{}\{d\sigma^H\}. \qquad (2.20)$$

Lets examine more carefully the situation for the exterior differential of a 1-form expanded in terms on non-closed basis 1-forms. The outcome of the exterior differential process is to produce a 2-form, which can be expanded in terms of products of 1-forms. For any particular basis 1-form, σ^k, the differential is a 2-form, and as such it can be expanded in terms of the paired basis elements, $\sigma^{[mn]}$. That is

$$
\begin{aligned}
d\sigma^k &= \Lambda^k_{[mn]}(y^e)\sigma^{[mn]} \qquad (2.21) \\
&= \Lambda^k_{[12]}\sigma^{[12]} + ... + \Lambda^k_{[34]}\sigma^{[34]} \\
&= \Lambda^k_{[12]}\sigma^1\hat{}\sigma^2 + ... + \Lambda^k_{[34]}\sigma^3\hat{}\sigma^4
\end{aligned}
$$

Hence the exterior differential of a 1-form where the basis σ^k is not integrable is given by the formula

$$d(A_k\sigma^k) = dA_k \,\hat{}\, \sigma^k + A_k\Lambda^k_{[mn]}\,\sigma^{[mn]} \tag{2.22}$$

When the basis forms σ^k are closed in a differential sense, then the coefficients $\Lambda^k_{[mn]}$ vanish. How this relates to Affine and Topological torsion will be discussed below, along with the topic of anholonomic coordinates.

The n basis 1-forms must be linearly independent otherwise the dimension of the vector space $\Lambda^1\{\sigma^k\}$ is not n. The implies that the exterior product of the n 1-forms σ^k are such that the n-form so constructed is not zero. For basis 1-forms σ^k constructed from a Frame matrix according to the formula $\left|\sigma^k\right\rangle = \left[F^k_a(y)\right]\,|dy^a\rangle$, the non-zero property for the n-fold product implies that the Frame matrix has a non-zero determinant.

$$\omega^n = \sigma^1\hat{}\sigma^2\hat{}...\hat{}\sigma^n = \det[F]dy^1\hat{}dy^2....\hat{}dy^n \neq 0. \tag{2.23}$$

Such domains are either positive or negative and are therefor said to be orientable.

Compute the exterior differential of the 1-form $A = dz + ydx - xdy$

$$dA = ddz + dy\hat{}dx - dx\hat{}dy = 0 - 2(dx\hat{}dy)$$

Example 2.7. ***The Gradient:*** *Compute the exterior differential of the general 0-form* $\theta(x, y, z)$

$$d\theta(x, y, z) = \partial\theta/dx\,dx + \partial\theta/dy\,dy + \partial\theta/dz\,dz =$$

The coefficients form the gradient of the scalar function (3D)

Example 2.8. ***The Curl:*** *Compute the exterior differential of a general 1-form* $A = (A_xdx + A_ydy + A_zdz)$

$$dA = \{\partial A_y/\partial x - \partial A_x/\partial y\}dx\hat{}dy + \{\partial A_z/\partial y - \partial A_y/\partial z\}dy\hat{}dz + \{\partial A_x/\partial z - \partial A_z/\partial x\}dz\hat{}dx$$

The coefficients form the components of the "curl **A**" in 3D.

Example 2.9. ***The Divergence:*** *Compute the exterior differential of the 2-form*

$V = Udy\hat{}dz - Vdz\hat{}dx + Wdx\hat{}dy$

$$dV = dU\hat{}dy\hat{}dz - dV\hat{}dz\hat{}dx + dW\hat{}dx\hat{}dy$$

If $\mathbf{V} = [U(x,y,z), V(x,y,z), W(x,y,z)$, then

$$\begin{aligned} dV &= \partial U/\partial x\, dx\hat{}dy\hat{}dz + \partial U/\partial y\, dy\hat{}dy\hat{}dz + \partial U/\partial z\, dz\hat{}dy\hat{}dz \\ &\quad -\partial V/\partial x\, dx\hat{}dz\hat{}dx - \partial V/\partial y\, dy\hat{}dz\hat{}dx - \partial V/\partial z\, dz\hat{}dz\hat{}dx \\ &\quad +\partial W/\partial x\, dx\hat{}dx\hat{}dy + \partial W/\partial y\, dy\hat{}dx\hat{}dy + \partial W/\partial z\, dz\hat{}dx\hat{}dy \\ &= \partial U/\partial x\, dx\hat{}dy\hat{}dz + 0 + 0 \\ &\quad -0 - \partial V/\partial y\, dy\hat{}dz\hat{}dx - 0 \\ &\quad +0 + 0 + \partial W/\partial z\, dz\hat{}dx\hat{}dy \\ &= \{\partial U/\partial x + \partial V/\partial y + \partial W/\partial z\} dx\hat{}dy\hat{}dz = div(\mathbf{V}) dx\hat{}dy\hat{}dz \end{aligned}$$

Note that these algebraic ideas do not depend upon the existence of a norm or a metric.

Example 2.10. *Derivation of the Maxwell Faraday induction equations*

The Maxwell-Faraday induction equations are a set of partial differential equations that are logically deducible starting with the ordered sequence $[1, 2, 3, 4]$. Next assume the existence of "ordered coordinate" vaiables given the symbols $[x, y, z, t]$. Next assume the existence of an ordered set of functions of the coordinate variables, with symbols $[A_x, A_y, A_z, \phi]$. From these beginnings the Maxwell - Faraday equations follow as a consequence of the Exterior Calculus of Cartan.

Construct the 1-form from the ordered set of functions and variables:

$$A = A_x dx + A_y dy + A_z dz - \phi dt. \tag{2.24}$$

Next construct the 2-form $F = dA$. Then construct the 3-form ddA which must vanish: $ddA = dF \Rightarrow 0$. In 4D the 3-form has 4 coefficient functions of partial derivatives that must vanish. These PDE's correspond in format to the 4 Maxwell - Faraday equations, with 3D symbols

$$curl\mathbf{E} + \partial \mathbf{B}/\partial t = 0 \quad div\mathbf{B} = 0, \tag{2.25}$$

where the symbols are defined in terms of the coefficient functions of the 1-form (of potentials) as,

$$\mathbf{E} = -grad\,\phi - \partial\mathbf{A}/t \qquad \mathbf{B} = curl\,\mathbf{A}. \tag{2.26}$$

Now the choice of symbol functions and coordinate functions was completely arbitrary, but the format of the PDE's that satisfy $ddA = dF \Rightarrow 0$ are always the same relative to the ordering process. Experimentally, the logical equations of Maxwell - Faraday have been exploited in electromagnetic applications. However, the SAME formulas (different symbols) are applicable to hydrodynamics (as well as other physical systems of interest). Surprisingly, little has been done with the induction equations in hydrodynamics. These are not analogies. These are consequences of the logic of the Exterior Calculus and have universal applicability.

2.2. The Interior Product

The interior product is an operation on p forms that requires a direction Vector field, V. The interior product lowers the degree of a p-form, changing a p-form into a p-1 form. The interior product of a Vector direction field and a zero form (function) is defined to be zero. The symbol for the interior product herein is taken to be $i(V)$. The interior product of a Vector field and an exact basis element equal to the differential of a coordinate dy^a is not zero, but is defined to equal to the a^{th} component of V. Hence the fundamental definitions can be written as

$$i(V)\theta(y^a) = 0 \qquad , \qquad i(V)dy^a = V^a. \tag{2.27}$$

It follows that the inner product with respect to the vector field V acting on a 1-form, $A = (A_x dx + A_y dy + A_z dz)$ is given by the expression:

$$i(V)A = i(V)(A_x dx + A_y dy + A_z dz) = (A_x V^x + A_y V^y + A_z V^z) \tag{2.28}$$

An additional rule is required to take care of the anti symmetries of differential forms. That is for the product $A\hat{}B$ of two 1-forms, the interior product with respect to V becomes

$$i(V)\{A\hat{}B\} = (i(V)A)\hat{}B\} - A\hat{}(i(V)B)$$

and

$$i(V)i(V)A = 0 \tag{2.29}$$

similar to the modified Leibniz rule for the exterior differential.. Other expressions can be worked out for higher p-forms can be worked out using these rules,

$$i(V)i(V)\omega^p = 0 \qquad i(V)i(W)\omega^p \neq \ i(W)i(V)\omega^p$$

Example 2.11. *Compute the interior product of* $J = [J^x, J^y, J^z]$ *in 3D with the 3-form vol element* $Vol = dx\hat{}dy\hat{}dz$

$$i(J)Vol = J^x dy\hat{}dz - J^y dx\hat{}dz + J^z dx\hat{}dy.$$

Example 2.12. *Compute the interior product of* $V = [V^x, V^y, V^z]$ *with the 2-form* $i(J)Vol$

$$\begin{aligned} i(V)\{i(J)Vol\} &= i(V)\{J^x dy\hat{}dz - J^y dx\hat{}dz + J^z dx\hat{}dy\} \\ &= (J^y V^z - J^z V^y)dx + (J^z V^x - J^x V^z)dy + (J^x V^y - J^y V^x)dz. \end{aligned}$$

Note that the construction (in 3D) of the double interior product generates coefficients equal to the cross product of the two different vector fields, J and V, and the double interior product with the same vector is zero.

2.3. The Lie Derivative

The Lie derivative with respect to a vector field generates a p-form ϑ^p from a p-form ω^p. It is constructed from the raising operator d and the lowering operator $i(V)$. The general formula is

$$\omega^p \Rightarrow \vartheta^p : \qquad L_{(V)}\omega^p = i(V)d\omega^p + d(i(V)\omega^p = \vartheta^p. \tag{2.30}$$

Marsden has called this Cartan's Magic formula. The reason is that most of the equations of mechanics can be put into this form or derived from its construction. For example, those processes V which are "Hamiltonian" processes are those V such that $i(V)d\omega^p$ is exact. It is also remarkable that this formula is equivalent to the first law of thermodynamics. Consider a 1-form of Action, A, that presents a physical system (this will be done in detail in later sections). Then consider a vector field V that represents an evolutionary process. Define the 0-form (scalar function) of internal energy as $U = i(V)A$, the 1-form of Work as $W = i(V)dA$, and the output 1-form ϑ^p as Q. Then Cartan's Magic formula becomes

$$L_{(V)}A = i(V)dA + d(i(V)A = W + dU = Q \tag{2.31}$$

which is to be recognized as the first law of thermodynamics for a physical system A undergoing an evolutionary process V. This result will be exploited in later sections.

Example 2.13. *Compute the Lie derivative with respect to $V = [F, V, 1]$ acting on the 1–form*
$A = pdq - H(p, q, t)dt$ in 3 dimensions. The basis elements are $[dp, dq, dt]$.

$$\begin{aligned}
L_{(V)}A &= i(V)dA + d(i(V)A) \\
&= i(V)\{dp\hat{\ }dq - dH\hat{\ }dt\} + d(pV - H) \\
&= i(V)\{dp\hat{\ }dq - \partial H/\partial p\, dp\hat{\ }dt - \partial H/\partial q\, dq\hat{\ }dt\} + d(pV - H) \\
&= Fdq - Vdp - F(\partial H/\partial p)dt - V(\partial H/\partial q)dt + dH + d(pV - H) \\
&= F(dq - \partial H/\partial pdt) - V(dp + \partial H/\partial qdt) + d(pV)
\end{aligned}$$

Note that the RHS of the equation above is a perfect differential for all evolutionary vector fields with components $V = [F, V, 1]$, if the two bracket factors vanish. Therefor, vector fields that are generated from the partial derivatives of $H(p, q, t)$ according to the formulas,

$$\begin{aligned}
(dq - \partial H/\partial pdt) &\Rightarrow 0 \supset V = \partial H/\partial p \\
(dp - \partial H/\partial qdt) &\Rightarrow 0 \supset F = -\partial H/\partial p,
\end{aligned} \tag{2.32}$$

which produce a 1-form of heat Q which is closed, $dQ \Rightarrow 0$. Such processes (vector fields) are defined to be Hamiltonian vector fields (processes). Hamiltonian dynamics is the (constrained) domain of much of theoretical mechanics. The domain is constrained, as the 3-form $Q\hat{\ }dQ \Rightarrow 0$. Such processes are then always thermodynamically reversible. Later on, irreversible processes for which $Q\hat{\ }dQ \neq 0$ will be studied.

One notes for Hamiltonian processes,

$$
\begin{aligned}
L_{(V)}H &= i(V)dH \\
&= F\partial H/\partial p + V\partial H/\partial q + \partial H/\partial t \\
&= FV - VF + \partial H/\partial t \\
&= \partial H/\partial t,
\end{aligned} \tag{2.33}
$$

so that if H is independent from time, then H is an evolutionary invariant. In mechanics, the function H is typically defined to be equal to be the sum of kinetic and potential energy, $H = p^2/2m + \varphi(x)$, so that time independent Hamiltonian processes "conserve energy". Even if the Hamiltonian is a function of time, Hamiltonian processes are thermodynamically reversible, as $Q\hat{}dQ = 0$.

2.4. Some Topological Features

The concepts of intersections, closure, and limit points are fundamental topological concepts that have a relationships to the Cartan Calculus. In certain situations the exterior product exhibits properties of intersection operator, and the exterior derivative exhibits properties of a limit point operator. More formally, given a domain with two exact 1-forms in 3D, the exterior product of the two exact 1-forms (if not zero) represents the points of intersection of the two implicit surfaces generated by the two functions whose gradient coefficients make up the components of the two exact 1-forms.

Example 2.14. *The exterior product and the concept of intersection.*

Consider two 1-forms created by applying the exterior differential to two distinct functions $\alpha(x, y, z)$ and $\beta(x, y, z)$. The coefficients of $d\alpha = grad(\alpha) \circ d\mathbf{r}$ form the gradient field $grad(\alpha)$ which is perpendicular to the implicit surface $\alpha(x, y, z) = 0$. Similarly, $d\beta = grad(\beta) \circ d\mathbf{r}$ implies that the gradient coefficients $grad(\beta)$ are perpendicular to the implicit surface $\beta(x, y, z) = 0$. If the two implicit surfaces intersect, then exterior product of the two 1-forms create a 2-form, $J = d\alpha\hat{}d\beta$, which is not zero. The components of the 2 form, J, can be interpreted as a contravariant vector in 3D, which is tangent to the points in common (intersections) that make up the intersection of the two surfaces. For $n = 3$, the number of components of a 2-form are 3, and are in agreement with the 3D cross-product formulas of Gibbs.

Example 2.15. *The exterior derivative is a limit point generator.*

From another point of view, it is possible to deduce a topological structure from a given 1-form A on the domain. The it is possible to show that the exterior derivative, relative to this Cartan topology, acts as a generator of the limit points of the given topology. This is given further credence from the physical idea that the divergence (an application of the exterior derivative) of the **D** field in electromagnetism has finite values that terminate on charges. That is, the Faraday lines of **D** come from limit points of postive charge and wind up on limit points of negative charge. However, the concept that the exterior differential d is a limit point operator is more formal, and has a basis in Kuratowski's closure operator.

Example 2.16. *The Lie derivative can be used to select topological invariants of a process.*

The Lie derivative with respect to a vector field V may be construed as a convective propagator describing the flow of the points of a p-form down the flow lines generated by V. If the p-form is integrated over a domain of such flowing points, then it is possible to ask if the integral is an invariant of the flow. Moreover it is possible to ask if the flowing points are distorted and deformed, does the integral over the deformed points equal the integral over the undeformed points. If is true that the value of the integral is unchanged by continuous deformation, then the integral must represent a topological property.

To deform the flowing points is easy enough; just multiply the original vector field V by a function of (say) $\lambda(x, y, z, t)$. The function λ does not change the flow lines generated by V, but it does deform the points that make up the flowlines by stretching or compression along the flow lines. Then $V \Rightarrow \lambda V$ and $L_{(V)} \Rightarrow L_{(\lambda V)}$, and the Lie derivative becomes a deformation operator. If it can be shown that if

$$L_{(\lambda V)} \oint A = \oint i(\lambda V) dA + \oint d(i(\lambda V)A) = 0 \tag{2.34}$$

for any function λ then the closed integral $\oint A$ is a deformation invariant of the process. Note that the second integral always vanishes, $\oint d(i(\lambda V)A) = 0$, as the integrand is an exact perfect differential. For the first integral to vanish for arbitrary deformation parameter, λ, the integrand must be zero. This leads to the conclusion that

$$\begin{aligned} if \quad i(\lambda V)dA &= \lambda i(V)dA = 0 \quad \text{any } \lambda, \\ then \quad \oint A &= \text{deformation invariant} \end{aligned} \tag{2.35}$$

Hence if the Work 1-form is zero, $W = i(V)dA \Rightarrow 0$ then the closed integral of the Action $\oint A$ is a topological property (of that process). Cartan has shown that a necessary and sufficient condition for a process to be a Hamiltonian process, is that the closed integral of the Action should be a topological invariant of the process.

Example 2.17. *The first law of thermodynamics is a topological statement of Cohomology.*

A non-exact p form Q is defined to be Cohomologous to another non-exact p-form, W, if the difference between the two p-forms is exact. This means that the integrals of the two different p-forms over any closed integration path (cycle or boundary) are the same. For non-exact 1-forms of Heat, Q, and Work, W, the cohological statement is the First Law:

$$Q - W = dU.$$

It was noted above that the Lie derivative with respect to a vector field (process) acting on a physical system described by a 1-form of Action, is essentially a Cohomological statement of the first law. The Lie derivative is a Cohomological generator.

Example 2.18. *Thermodynamic Isolation and Frobenius integrability.*

In thermodynamics, it is recognized that there are isolated, closed, and open systems. These words are also used to describe topological properties. A set is topologically isolated if it has no intersection with its limit points. This result translates to $A\hat{}dA = 0$ for a given 1-form and its induced Cartan topology. The constraint of isolation is also equivalent to the Frobenius idea of unique integrability. That is when $A\hat{}dA = 0,$ there exists a unique function whose gradient (or surface normal) is proportional to the given coefficients of the given 1-form. Caratheodory's statements about inaccessible states is a statement related to the concept of isolation and connectivity to an equilibrium system. When $A\hat{}dA = 0$ (no matter what the dimension of the coordinate space happens to be) there exists a transformation to a domain of two independent functions that will describe the properties of the 1-form. That is, the 1-form can be written as $\phi d\chi$, and its coefficient functions are proportional to a gradient, $d\chi$. The problem becomes essentially a two dimensional problem.

The property of isolation is a topological property, hence if a process causes $A\hat{}dA \neq 0$ to change to $A\hat{}dA = 0$, or from a state where $A\hat{}dA = 0$ to a state where $A\hat{}dA \neq 0$, a topological change has take place. In hydrodynamics, all streamline flows satisfy $A\hat{}dA = 0$. Hence turbulent flows must involve domains where $A\hat{}dA \neq 0$. The transition to (from) turbulence from (to) a state of non-turbulence must involve topological change.

It should be mentioned that with respect to diffeomorphic transformations, or more simply those transformations that preserve pure geometrical properties, the differences between contravariant and covariant concepts cannot be distinguished. But with respect to an aging process involving topological change, the behavior of the two concepts is observably different.

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$$\begin{array}{ll} \text{Isolated}: & H = A\hat{}dA = 0 \\ \text{Closed}: & dH = 0,\ but\ H \neq 0. \\ \text{Open}: & dH \neq 0 \end{array}$$