```
> restart: with (linalg):
Warning, new definition for norm
Warning, new definition for trace
```

A MAPLE PROGRAM to construct the Repere Mobile, or moving Frame Matrix,
the Cartan matrix, the Shape Matrix,
the Mean Curvature, and Gauss Curvature for a parametric 2-surface in R3

```
> with(difforms):
> with(liesymm): setup(u,v):
Warning, new definition for `&^`
Warning, new definition for close
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for wdegree
```

The Position Vector in R3 parametrized with (u,v). The example is for a Monge Surface z=g(u,v)

```
> XX:=u;YY:=v;
```

$$XX := u$$

$$YY := v$$

```
> ZZ:=g(u,v);rho:=scale(u,v);
```

$$ZZ := g(u, v)$$

$$\rho := \mathrm{scale}(u, v)$$

Now you can specify formats for g(u,v); scale(u,v) and evaluate specific examples. You could also plot the surfaces. For example just uncheck the line below, or change XX and YY to be specific functions of u and v.

```
> rho:=1;
>
```

$$\rho := 1$$

The position vector in R3

```
> RR:=[XX,YY,ZZ];
```

$$RR := [u, v, g(u, v)]$$

```
> Yu:=diff(RR,u);
```

$$Yu := \left[1, 0, \frac{\partial}{\partial u} g(u, v)\right]$$

```
> Yv:=diff(RR,v);
```

$$Yv := \left[0, 1, \frac{\partial}{\partial v} g(u, v)\right]$$

```
> NNU:=crossprod(Yu,Yv);
```

$$NNU := \left[-\left(\frac{\partial}{\partial u} g(u, v)\right), -\left(\frac{\partial}{\partial v} g(u, v)\right), 1\right]$$

Scale the adjoint normal field here by rho

```
> rho:=innerprod(NNU,NNU)^(1/2);
```

$$\rho := \sqrt{\left(\frac{\partial}{\partial u} g(u, v)\right)^2 + \left(\frac{\partial}{\partial v} g(u, v)\right)^2 + 1}$$

```
> #rho:=1;
>
```

This vector (surface normal) NNU can be computed from the Adjoint Matrix operation on the two tangent vectors Yu and Yv. The basis frame utilizes this surface normal with arbitrary scaling

```
> NN:=([factor(NNU[1]),factor(NNU[2]),simplify(factor(NNU[3]))]);
```

$$NN := \left[-\left(\frac{\partial}{\partial u} g(u,v)\right), -\left(\frac{\partial}{\partial v} g(u,v)\right), 1 \right]$$

```
> FF:=array([[Yu[1],Yv[1],NN[1]/rho],[Yu[2],Yv[2],NN[2]/rho],[Yu[3],Yv[3],NN[3]/rho]  ] );
```

$$FF := \begin{bmatrix} 1 & 0 & -\frac{\frac{\partial}{\partial u} g(u,v)}{\sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}} \\ 0 & 1 & -\frac{\frac{\partial}{\partial v} g(u,v)}{\sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}} \\ \frac{\partial}{\partial u} g(u,v) & \frac{\partial}{\partial v} g(u,v) & \frac{1}{\sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}} \end{bmatrix}$$

The Repere Mobile or FRAME MATRIX, FF. note that the frame matrix is not orthonormal.!!

```
> detFF:=simplify((det(FF)));
```

$$detFF := \sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}$$

For the Monge casel the deteminant is non-zero globally, hence an inverse always exists.

```
> FFINVD:=evalm(FF^(-1));
```

$$FFINVD := \begin{bmatrix} \frac{1 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2}{\%1} & -\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)}{\%1} & \frac{\frac{\partial}{\partial u} g(u,v)}{\%1} \\ -\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)}{\%1} & \frac{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + 1}{\%1} & \frac{\frac{\partial}{\partial v} g(u,v)}{\%1} \\ -\frac{\frac{\partial}{\partial u} g(u,v)}{\sqrt{\%1}} & -\frac{\frac{\partial}{\partial v} g(u,v)}{\sqrt{\%1}} & \frac{1}{\sqrt{\%1}} \end{bmatrix}$$

$$\%1 := \left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1$$

The 1-form components of the differential position vector with respect to the Basis Frame, F.

```
> dR:=innerprod(FFINVD,[d(XX),d(YY),d(ZZ)]);
```

$$dR := \left[\left(d(u) + d(u)\left(\frac{\partial}{\partial v} g(u,v)\right)^2 - \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right) d(v) + \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right) d(u) \right. \right.$$
$$\left. + \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial}{\partial v} g(u,v)\right) d(v) \right) \Big/ \left(\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1 \right), -\left(\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right) d(u) \right.$$
$$- d(v)\left(\frac{\partial}{\partial u} g(u,v)\right)^2 - d(v) - \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right) d(u) - \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial v} g(u,v)\right) d(v) \Big) \Big/ \Big($$

$$\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1\Bigg),$$

$$-\frac{\left(\frac{\partial}{\partial u} g(u,v)\right) d(u) + \left(\frac{\partial}{\partial v} g(u,v)\right) d(v) - \left(\frac{\partial}{\partial u} g(u,v)\right) d(u) - \left(\frac{\partial}{\partial v} g(u,v)\right) d(v)}{\sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}}\Bigg]$$

```
> sigma1:=(wcollect(dR[1]));
```

$$\sigma 1 := \frac{\left(1 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)\right) d(u)}{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}$$
$$+ \frac{\left(-\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right) + \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial}{\partial v} g(u,v)\right)\right) d(v)}{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}$$

```
> sigma2:=wcollect(dR[2]);
```

$$\sigma 2 := -\frac{\left(\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right) - \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)\right) d(u)}{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}$$
$$- \frac{\left(-\left(\frac{\partial}{\partial u} g(u,v)\right)^2 - 1 - \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial v} g(u,v)\right)\right) d(v)}{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}$$

Note that sigma1 is du and sigma2 is dv for a parametric Monge surfaces!!

```
> omega:=(wcollect(dR[3]));
```

$$\omega := -\frac{\left(\left(\frac{\partial}{\partial u} g(u,v)\right) - \left(\frac{\partial}{\partial u} g(u,v)\right)\right) d(u)}{\sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}} - \frac{\left(\left(\frac{\partial}{\partial v} g(u,v)\right) - \left(\frac{\partial}{\partial v} g(u,v)\right)\right) d(v)}{\sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}}$$

Note that this term vanishes for a parametric Monge surface, hence parametric Monge surfaces exhibit no TORSION!! of the Affine type (that is there is no translational shear defects!)

```
>
>
```

Compute the Cartan Matrix of connection forms from C=[F(inverse)] times d[F]

```
> dFF:=array([[d(FF[1,1]),d(FF[1,2]),d(FF[1,3])],[d(FF[2,1]),d(FF[2,2]),d(FF[2,3])],[d(FF[3,1]),d(FF[3,2]),d(FF[3,3])]]);
```

$$
dFF := \begin{bmatrix} 0\,, & 0\,, & -\dfrac{\left(\dfrac{\partial^2}{\partial u^2}\,\mathrm{g}(u,v)\right)\mathrm{d}(u) + \%4\ \mathrm{d}(v)}{\sqrt{\%2}} - \left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\%3 \\ 0\,, & 0\,, & -\dfrac{\%4\ \mathrm{d}(u) + \left(\dfrac{\partial^2}{\partial v^2}\,\mathrm{g}(u,v)\right)\mathrm{d}(v)}{\sqrt{\%2}} - \left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\%3 \\ \left(\dfrac{\partial^2}{\partial u^2}\,\mathrm{g}(u,v)\right)\mathrm{d}(u) + \%4\ \mathrm{d}(v)\,, & \%4\ \mathrm{d}(u) + \left(\dfrac{\partial^2}{\partial v^2}\,\mathrm{g}(u,v)\right)\mathrm{d}(v)\,, & \%3 \end{bmatrix}
$$

$$
\%1 := \frac{\partial^2}{\partial v\,\partial u}\,\mathrm{g}(u,v)
$$

$$
\%2 := \left(\frac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)^2 + \left(\frac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)^2 + 1
$$

$$
\%3 := -\frac{1}{2}\,\frac{\left(2\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\left(\dfrac{\partial^2}{\partial u^2}\,\mathrm{g}(u,v)\right) + 2\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\%1\right)\mathrm{d}(u)}{\%2^{3/2}}
$$

$$
-\frac{1}{2}\,\frac{\left(2\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\%1 + 2\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\left(\dfrac{\partial^2}{\partial v^2}\,\mathrm{g}(u,v)\right)\right)\mathrm{d}(v)}{\%2^{3/2}}
$$

$$
\%4 := \frac{\partial^2}{\partial v\,\partial u}\,\mathrm{g}(u,v)
$$

```
> cartan:=evalm(FFINVD&*dFF);
```

cartan :=

$$
\left[\frac{\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\%2}{\%3},\ \frac{\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\%4}{\%3},\ \frac{\left(1+\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)^2\right)\left(-\dfrac{\%2}{\sqrt{\%3}} - \left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\%6\right)}{\%3}\right.
$$

$$
\left. -\frac{\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\left(-\dfrac{\%4}{\sqrt{\%3}} - \left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\%6\right)}{\%3} + \frac{\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\%6}{\%3}\right]
$$

$$
\left[\frac{\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\%2}{\%3},\ \frac{\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\%4}{\%3},\ -\frac{\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\left(-\dfrac{\%2}{\sqrt{\%3}} - \left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\%6\right)}{\%3}\right.
$$

$$
\left. +\frac{\left(\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)^2+1\right)\left(-\dfrac{\%4}{\sqrt{\%3}} - \left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\%6\right)}{\%3} + \frac{\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\%6}{\%3}\right]
$$

$$
\left[\frac{\%2}{\sqrt{\%3}},\ \frac{\%4}{\sqrt{\%3}},\right.
$$

$$
\left. -\frac{\left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\left(-\dfrac{\%2}{\sqrt{\%3}} - \left(\dfrac{\partial}{\partial u}\,\mathrm{g}(u,v)\right)\%6\right)}{\sqrt{\%3}} - \frac{\left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\left(-\dfrac{\%4}{\sqrt{\%3}} - \left(\dfrac{\partial}{\partial v}\,\mathrm{g}(u,v)\right)\%6\right)}{\sqrt{\%3}} + \frac{\%6}{\sqrt{\%3}}\right]
$$

$$\%1 := \frac{\partial^2}{\partial v\,\partial u} \mathrm{g}(u, v)$$

$$\%2 := \left(\frac{\partial^2}{\partial u^2} \mathrm{g}(u, v)\right) \mathrm{d}(u) + \%1\ \mathrm{d}(v)$$

$$\%3 := \left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1$$

$$\%4 := \%1\ \mathrm{d}(u) + \left(\frac{\partial^2}{\partial v^2} \mathrm{g}(u, v)\right) \mathrm{d}(v)$$

$$\%5 := \frac{\partial^2}{\partial v\,\partial u} \mathrm{g}(u, v)$$

$$\%6 := -\frac{1}{2} \frac{\left(2 \left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial u^2} \mathrm{g}(u, v)\right) + 2 \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right) \%5\right) \mathrm{d}(u)}{\%3^{3/2}} - \frac{1}{2} \frac{\left(2 \left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right) \%5 + 2 \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial v^2} \mathrm{g}(u, v)\right)\right) \mathrm{d}(v)}{\%3^{3/2}}$$

The interior connection coefficients (can be Christoffel symbols on the parameter space

```
> Gamma11:=(wcollect(cartan[1,1]));
```

$$\Gamma 11 := \frac{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial u^2} \mathrm{g}(u, v)\right) \mathrm{d}(u)}{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1} + \frac{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial v\,\partial u} \mathrm{g}(u, v)\right) \mathrm{d}(v)}{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1}$$

```
> Gamma12:=(wcollect(cartan[1,2]));
```

$$\Gamma 12 := \frac{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial v\,\partial u} \mathrm{g}(u, v)\right) \mathrm{d}(u)}{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1} + \frac{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial v^2} \mathrm{g}(u, v)\right) \mathrm{d}(v)}{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1}$$

```
> Gamma21:=(wcollect(cartan[2,1]));
```

$$\Gamma 21 := \frac{\left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial u^2} \mathrm{g}(u, v)\right) \mathrm{d}(u)}{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1} + \frac{\left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial v\,\partial u} \mathrm{g}(u, v)\right) \mathrm{d}(v)}{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1}$$

```
> Gamma22:=(wcollect(cartan[2,2]));
```

$$\Gamma 22 := \frac{\left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial v\,\partial u} \mathrm{g}(u, v)\right) \mathrm{d}(u)}{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1} + \frac{\left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)\left(\frac{\partial^2}{\partial v^2} \mathrm{g}(u, v)\right) \mathrm{d}(v)}{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1}$$

The second fundamental form or shape matrix comes from the third row of the Cartan matrix

```
> h1:=wcollect(cartan[3,1]);
```

$$h1 := \frac{\left(\frac{\partial^2}{\partial u^2} \mathrm{g}(u, v)\right) \mathrm{d}(u)}{\sqrt{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1}} + \frac{\left(\frac{\partial^2}{\partial v\,\partial u} \mathrm{g}(u, v)\right) \mathrm{d}(v)}{\sqrt{\left(\frac{\partial}{\partial u} \mathrm{g}(u, v)\right)^2 + \left(\frac{\partial}{\partial v} \mathrm{g}(u, v)\right)^2 + 1}}$$

```
> gamma1:=wcollect(cartan[1,3]);
```

$$\gamma 1 := \left(\frac{\left(1+\left(\frac{\partial}{\partial v} g(u,v)\right)^2\right)\left(-\frac{\frac{\partial^2}{\partial u^2} g(u,v)}{\sqrt{\%3}}+\frac{1}{2}\frac{\left(\frac{\partial}{\partial u} g(u,v)\right)\%4}{\%3^{3/2}}\right)}{\%3} - \frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)\left(-\frac{\frac{\partial^2}{\partial v \partial u} g(u,v)}{\sqrt{\%3}}+\frac{1}{2}\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\%4}{\%3^{3/2}}\right)}{\%3} - \frac{1}{2}\frac{\left(\frac{\partial}{\partial u} g(u,v)\right)\%4}{\%3^{5/2}} \right) d(u) + \left(\frac{\left(1+\left(\frac{\partial}{\partial v} g(u,v)\right)^2\right)\left(-\frac{\frac{\partial^2}{\partial v \partial u} g(u,v)}{\sqrt{\%3}}+\frac{1}{2}\frac{\left(\frac{\partial}{\partial u} g(u,v)\right)\%2}{\%3^{3/2}}\right)}{\%3} - \frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)\left(-\frac{\frac{\partial^2}{\partial v^2} g(u,v)}{\sqrt{\%3}}+\frac{1}{2}\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\%2}{\%3^{3/2}}\right)}{\%3} - \frac{1}{2}\frac{\left(\frac{\partial}{\partial u} g(u,v)\right)\%2}{\%3^{5/2}} \right) d(v)$$

$$\%1 := \frac{\partial^2}{\partial v \partial u} g(u,v)$$

$$\%2 := 2\left(\frac{\partial}{\partial u} g(u,v)\right)\%1 + 2\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial^2}{\partial v^2} g(u,v)\right)$$

$$\%3 := \left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1$$

$$\%4 := 2\left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial^2}{\partial u^2} g(u,v)\right) + 2\left(\frac{\partial}{\partial v} g(u,v)\right)\%1$$

```
> h2:=(wcollect(cartan[3,2]));
```

$$h2 := \frac{\left(\frac{\partial^2}{\partial v \partial u} g(u,v)\right) d(u)}{\sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}} + \frac{\left(\frac{\partial^2}{\partial v^2} g(u,v)\right) d(v)}{\sqrt{\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1}}$$

```
> gamma2:=(wcollect(cartan[2,3]));
```

$$\gamma 2 := \left(-\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)\left(-\frac{\frac{\partial^2}{\partial u^2} g(u,v)}{\sqrt{\%3}}+\frac{1}{2}\frac{\left(\frac{\partial}{\partial u} g(u,v)\right)\%4}{\%3^{3/2}}\right)}{\%3} + \frac{\left(\left(\frac{\partial}{\partial u} g(u,v)\right)^2+1\right)\left(-\frac{\frac{\partial^2}{\partial v \partial u} g(u,v)}{\sqrt{\%3}}+\frac{1}{2}\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\%4}{\%3^{3/2}}\right)}{\%3} - \frac{1}{2}\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\%4}{\%3^{5/2}} \right) d(u) + \Bigg($$

$$-\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)\left(-\frac{\frac{\partial^2}{\partial v\,\partial u} g(u,v)}{\sqrt{\%3}} + \frac{1}{2}\,\frac{\left(\frac{\partial}{\partial u} g(u,v)\right)\%2}{\%3^{3/2}}\right)}{\%3}$$

$$+\frac{\left(\left(\frac{\partial}{\partial u} g(u,v)\right)^2+1\right)\left(-\frac{\frac{\partial^2}{\partial v^2} g(u,v)}{\sqrt{\%3}}+\frac{1}{2}\,\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\%2}{\%3^{3/2}}\right)}{\%3}-\frac{1}{2}\,\frac{\left(\frac{\partial}{\partial v} g(u,v)\right)\%2}{\%3^{5/2}}\Bigg)\,\mathrm{d}(v)$$

$$\%1 := \frac{\partial^2}{\partial v\,\partial u} g(u,v)$$

$$\%2 := 2\left(\frac{\partial}{\partial u} g(u,v)\right)\%1 + 2\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial^2}{\partial v^2} g(u,v)\right)$$

$$\%3 := \left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1$$

$$\%4 := 2\left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial^2}{\partial u^2} g(u,v)\right) + 2\left(\frac{\partial}{\partial v} g(u,v)\right)\%1$$

The abnormality for the parametric surface will show up as a non-zero entry in the [3,3] slot of the Cartan Matrix. Always an exact differental for parametric and Monge surfaces. Therefore implicit Monge surfaces will admit disclination defects (Torsion of the second kind due to rotations)

```
> Omega:=(wcollect(factor(simpform(cartan[3,3]))));
```

$$\Omega := \Bigg(\left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial^2}{\partial u^2} g(u,v)\right) - \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial^2}{\partial v\,\partial u} g(u,v)\right) + \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial^2}{\partial v\,\partial u} g(u,v)\right) - \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial^2}{\partial u^2} g(u,v)\right)\Bigg)\mathrm{d}(u) \Bigg/ \left(\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1\right) + \Bigg(\left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial^2}{\partial v^2} g(u,v)\right) + \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial^2}{\partial v\,\partial u} g(u,v)\right) - \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial^2}{\partial v\,\partial u} g(u,v)\right) - \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial^2}{\partial v^2} g(u,v)\right)\Bigg)\mathrm{d}(v) \Bigg/ \left(\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1\right)$$

Omega vanishes for a given normalization.

```
> wcollect(factor(simpform(d(Omega))));
```

$$((\mathrm{d}(u))\ \&\wedge\ (\mathrm{d}(v)))\Bigg(\%2\,\%5\left(\frac{\partial}{\partial u} g(u,v)\right)^2 - \%2\,\%5\left(\frac{\partial}{\partial v} g(u,v)\right)^2 + \%2\,\%5 - \%4\,\%1\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \%4\,\%1\left(\frac{\partial}{\partial v} g(u,v)\right)^2 + \%4\,\%1 + \left(\frac{\partial}{\partial u} g(u,v)\right)^2\%6\,\%5 - \%6\,\%5 + 2\left(\frac{\partial}{\partial u} g(u,v)\right)\%6\left(\frac{\partial}{\partial v} g(u,v)\right)\%3 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2\%4\,\%3 - \%4\,\%3 - \%6\,\%5\left(\frac{\partial}{\partial v} g(u,v)\right)^2 - \%4\,\%3\left(\frac{\partial}{\partial u} g(u,v)\right)^2 - 2\left(\frac{\partial}{\partial v} g(u,v)\right)\%2\left(\frac{\partial}{\partial u} g(u,v)\right)\%1\Bigg) \Bigg/ \left(\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1\right)^2$$

$$\%1 := \frac{\partial^2}{\partial u^2} g(u,v)$$

$$\%2 := \frac{\partial^2}{\partial v^2} g(u, v)$$

$$\%3 := \frac{\partial^2}{\partial v^2} g(u, v)$$

$$\%4 := \frac{\partial^2}{\partial v\, \partial u} g(u, v)$$

$$\%5 := \frac{\partial^2}{\partial v\, \partial u} g(u, v)$$

$$\%6 := \frac{\partial^2}{\partial u^2} g(u, v)$$

```
> FROBOMEGA:=simpform(Omega&^d(Omega));
```

$$FROBOMEGA := 0$$

The coefficients of the shape matrix determined from the Cartan matrix.

```
> factor(simpform(Omega&^gamma1));
```

$$-((d(u))\,\&\wedge\,(d(v)))\left(-\left(\frac{\partial}{\partial u} g(u,v)\right)^2 \%2 \left(\frac{\partial}{\partial v} g(u,v)\right)\%1 + \left(\frac{\partial}{\partial u} g(u,v)\right)^2 \left(\frac{\partial^2}{\partial u^2} g(u,v)\right)\left(\frac{\partial}{\partial v} g(u,v)\right)\%4 - \left(\frac{\partial}{\partial v} g(u,v)\right)\%4\,\%3 \left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial u} g(u,v)\right)^2 \%1^2 \left(\frac{\partial}{\partial v} g(u,v)\right) + \left(\frac{\partial}{\partial u} g(u,v)\right)\%2\,\%3 \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + \left(\frac{\partial}{\partial u} g(u,v)\right)\%2\,\%3 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 \%2 \left(\frac{\partial}{\partial u} g(u,v)\right)\%4 - \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial^2}{\partial u^2} g(u,v)\right)\%1 \left(\frac{\partial}{\partial v} g(u,v)\right)^2 - \left(\frac{\partial}{\partial u} g(u,v)\right)\left(\frac{\partial^2}{\partial u^2} g(u,v)\right)\%1 - \left(\frac{\partial}{\partial v} g(u,v)\right)^2 \left(\frac{\partial^2}{\partial v^2} g(u,v)\right)\left(\frac{\partial}{\partial u} g(u,v)\right)\%1 + \%1^2 \left(\frac{\partial}{\partial v} g(u,v)\right) - \%2 \left(\frac{\partial}{\partial v} g(u,v)\right)\%1 - \left(\frac{\partial}{\partial v} g(u,v)\right)\%4\,\%3 + \left(\frac{\partial}{\partial v} g(u,v)\right)^3 \left(\frac{\partial^2}{\partial v^2} g(u,v)\right)\%3 + \%1^2 \left(\frac{\partial}{\partial v} g(u,v)\right)^3 + \left(\frac{\partial}{\partial v} g(u,v)\right)\left(\frac{\partial^2}{\partial v^2} g(u,v)\right)\%3 - \left(\frac{\partial}{\partial v} g(u,v)\right)^3 \%4\,\%3 - \%2 \left(\frac{\partial}{\partial v} g(u,v)\right)^3 \%1\right) \Bigg/ \left(\left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial v} g(u,v)\right)^2 + 1\right)^{5/2}$$

$$\%1 := \frac{\partial^2}{\partial v\, \partial u} g(u, v)$$

$$\%2 := \frac{\partial^2}{\partial v\, \partial u} g(u, v)$$

$$\%3 := \frac{\partial^2}{\partial u^2} g(u, v)$$

$$\%4 := \frac{\partial^2}{\partial v^2} g(u, v)$$

```
> simplify(Omega&^gamma1);
```

$$-((d(u))\,\&\wedge\,(d(v)))\left(-\left(\frac{\partial}{\partial u} g(u,v)\right)^2 \%2 \left(\frac{\partial}{\partial v} g(u,v)\right)\%1 + \left(\frac{\partial}{\partial u} g(u,v)\right)^2 \left(\frac{\partial^2}{\partial u^2} g(u,v)\right)\left(\frac{\partial}{\partial v} g(u,v)\right)\%4 - \left(\frac{\partial}{\partial v} g(u,v)\right)\%4\,\%3 \left(\frac{\partial}{\partial u} g(u,v)\right)^2 + \left(\frac{\partial}{\partial u} g(u,v)\right)^2 \%1^2 \left(\frac{\partial}{\partial v} g(u,v)\right) + \left(\frac{\partial}{\partial u} g(u,v)\right)\%2\,\%3 \left(\frac{\partial}{\partial v} g(u,v)\right)^2\right.$$

$$+\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\%2\ \%3+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2\%2\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\%4-\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)\right)\%1\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2$$

$$-\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)\right)\%1-\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\%1+\%1^2\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)$$

$$-\%2\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\%1-\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\%4\ \%3+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^3\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)\%3+\%1^2\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^3$$

$$\left.+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)\%3-\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^3\%4\ \%3-\%2\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^3\%1\right)\Bigg/$$

$$\left(\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2+1\right)^{5/2}$$

$$\%1 := \frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)$$

$$\%2 := \frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)$$

$$\%3 := \frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)$$

$$\%4 := \frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)$$

The components of the disclination 2-form are given above. Note that they are proportional to the Square Root of the Gauss Curvature (for scaling = 1) and form the "Stream" vector relative to the gradient of the Monge function g -- a symplectic rotation

```
> shape11:=-factor(gamma1&^d(v)/d(u)&^d(v));
```

$$shape11 := -\frac{\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)\right)-\left(\frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)\right)-\left(\frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2}{\left(\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2+1\right)^{3/2}}$$

```
> shape12:=-factor(gamma1&^d(u)/d(v)&^d(u));
```

$$shape12 := -\frac{\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)-\left(\frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)\right)-\left(\frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2}{\left(\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2+1\right)^{3/2}}$$

```
> shape21:=-factor(gamma2&^d(v)/d(u)&^d(v));
```

$$shape21 := \frac{\left(\frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2-\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)\right)+\left(\frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)\right)}{\left(\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2+1\right)^{3/2}}$$

```
> shape22:=-factor(gamma2&^d(u)/d(v)&^d(u));
```

$$shape22 := \frac{\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)-\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)\right)+\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)}{\left(\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2+1\right)^{3/2}}$$

```
>
> SHAPE:=array([[shape11,shape12],[shape21,shape22]]):
```

```
> HH:=simplify(trace(SHAPE)/2):
> print(`Mean Curvature is `,HH);
```

$$\mathit{Mean\ Curvature\ is\ },\ \frac{1}{2}\left(-2\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)\right)+\left(\frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)\right)+\left(\frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)+\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)\right)\Bigg/\left(\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2+1\right)^{3/2}$$

```
> KK:=simplify(det(SHAPE)):
> print(`Gauss Curvature is `,KK);
```

$$\mathit{Gauss\ Curvature\ is\ },\ \frac{-\left(\frac{\partial^2}{\partial v\,\partial u}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial^2}{\partial u^2}\mathrm{g}(u,v)\right)\left(\frac{\partial^2}{\partial v^2}\mathrm{g}(u,v)\right)}{\left(\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2+1\right)^2}$$

```
>
```

Note that the scaling of the normal or adjoint vector is a common factor of the formulas for the mean curvature and the Gauss curvature. Note the appearance of the Hessian of the Monge function. The induce metric appears below

```
> GUN:=innerprod(transpose(FF),FF);
```

$$GUN := \begin{bmatrix} \left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right)^2+1 & \left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right) & 0 \\ \left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)\left(\frac{\partial}{\partial u}\mathrm{g}(u,v)\right) & 1+\left(\frac{\partial}{\partial v}\mathrm{g}(u,v)\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
>
> a:=1;b:=1;
```

You could plot the surface here if you wanted too.

```
> plot3d(RR(u,v),u=-1*Pi..1*Pi,v=-1*Pi..1*Pi,axes=BOXED,shading=ZGREYSCALE,style=PATCH);
>
```