Notes on CLASSICAL FIELD THEORY - A MODERN VIEW

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Abstract

Classical Field Theory is described in the language of exterior differential forms.

1. The Exterior Differential Form Method.

Classical Field theory had its foundations in the Calculus of Variations. See (Wentzel, Handbuch. der Physik, Aharoni, etc.) However, a more transparent development may be obtained using the theory of differential forms and exterior differential systems. To this end, first consider a space M of m variables $\{x^k\}$ and an immersive map, Φ, into a space N of $n + n \times m$ variables $\{ \Psi^k, \Psi^\mu_k \}$. This mapping includes those special (canonical) cases where the variables Ψ^μ_k are defined as partial derivatives: $\Psi^\mu_k = \partial\Psi^\mu/\partial x^k$. The special cases imply that the new variables of interest on N are the components of a vector and its Jacobian coefficients. However, the dimension n need not equal the dimension m, and the Ψ^μ_k may be functions different from the partial derivatives, $\Psi^\mu_k \neq \partial\Psi^\mu/\partial x^k$.

Consider a Lagrange function, $L(\mathbf{\Psi}^\mu, \mathbf{\Psi}^\mu_k)$, (ultimately chosen to describe some physical system) on N, and its differential dL :

$$dL(\mathbf{\Psi}^\mu, \mathbf{\Psi}^\mu_k) = (\partial L/\partial \mathbf{\Psi}^\mu)d\mathbf{\Psi}^\mu + (\partial L/\partial \mathbf{\Psi}^\mu_k)d\mathbf{\Psi}^\mu_k. \tag{1.1}$$

As the arguments of $L(\mathbf{\Psi}^\mu, \mathbf{\Psi}^\mu_k)$ on N are well defined as functions on M, by functional substitution, a new function $L(\mathbf{\Psi}^\mu, \mathbf{\Psi}^\mu_k) \Rightarrow \underline{L}(x^k)$ can be defined on M. The basic idea is to compute the differential of $\underline{L}(x^k)$ on M and equate the two differentials.

$$d\underline{L}(x^k) = dL(\mathbf{\Psi}^\mu, \mathbf{\Psi}^\mu_k) \tag{1.2}$$

The objective will be to relate this differential identity to a power theorem, or to the first law of thermodynamics.

Consider the system of n 1-forms which are not necessarily zero on N:

$$\omega^\mu = d\mathbf{\Psi}^\mu - \mathbf{\Psi}^\mu_k dx^k \tag{1.3}$$

It is a topological constraint to assume that the 1-forms, ω^μ, vanish on the space N. It is possible that they vanish on the space M, and not on the space N. In such cases, the map is said to define M as an integral sub-manifold of N. If the 1-forms ω^μ vanish on the space N, then the coefficients must be such that $\Psi^\mu_k \Rightarrow \partial\Psi^\mu/\partial x^k$. In this note such an assumption is **not** presumed, a priori. The deviations of ω^μ from zero will be of use in physical theories that employ the concept of thermodynamic fluctuations. (Perhaps these possible deviations from zero is what Newton had in mind when he invented the calculus of fluxions)

A topological constraint that will be subsumed in this note is the condition that the system of 1-forms ω^μ are differentially closed. There are two possible versions of the concept of closure. The weaker condition implies that the exterior derivative of any form ω^μ resides in the set of forms composed of linear combinations of the original set $\{\omega^\mu\}$.

$$d\omega^\mu = \omega^\sigma C^\mu_\sigma \tag{1.4}$$

This assumption defines a connection (or a gauge field). Note that on a subspace where the fluctuation forms vanish, then their exterior derivative also vanishes. This idea leads to a stronger topological assumption: That is, assume that the exterior derivative of each fluctuation 1-form vanishes. This stronger assumption leads to a somewhat simpler theory, and will be taken up first in this note.

$$\text{Topo log ical Assumption I:} \qquad d\omega^\mu = 0 \tag{1.5}$$

This strong closure condition implies that the deviations from zero for the set of fluctuations, $\omega^\mu = d\mathbf{\Psi}^\mu - \mathbf{\Psi}^\mu_k dx^k$, consists, at most, as a set of perfect differentials, and a set of harmonic forms. The system of fluctuations may have non-zero period integrals, for integrations over a cycle, but integrals over a boundary (which can have cyclic parts) must always vanish, subject to the strong closure condition. Physically the idea is that over some suitable bounded integration domain, the fluctuations integrate to zero on average.

A somewhat more general situation could be described by the constraint that the 1-forms ω^μ be integrable, in the sense that for each fluctuation 1-form there exists a set of integrating factors such that $d(\beta\omega^\mu) = 0$.

2. Field Theory based on Assumption I.

The strong closure condition (Assumption I) imposes a "null curl" constraint on the functions, $\mathbf{\Psi}_k^\mu$. As the mapping is differentiable it follows that

$$d\Psi_k^\mu = (\partial\Psi_k^\mu/\partial x^j)dx^j. \tag{2.1}$$

and from the constraint of closure,

$$d\omega^\mu = d\mathbf{\Psi}_k^\mu \hat{}\, dx^k = (\partial\mathbf{\Psi}_k^\mu/\partial x^j - \partial\mathbf{\Psi}_j^\mu/\partial x^k)dx^j \hat{}\, dx^k \Rightarrow 0. \tag{2.2}$$

It follows that

$$\partial\mathbf{\Psi}_k^\mu/\partial x^j = \partial\mathbf{\Psi}_j^\mu/\partial x^k, \tag{2.3}$$

such that by a different arrangement a second expression is obtained for $d\Psi_k^\mu$:

$$d\Psi_k^\mu = (\partial\Psi_k^\mu/\partial x^j)dx^j = (\partial\Psi_j^\mu/\partial x^k)dx^j. \tag{2.4}$$

(My intuition tells me that this result, due to the strong closure assumption, is somehow related to isotropy, but I don't know how to express this exactly.) Note that if the additional - and not required - (canonical) constraint is made, such that $\Psi_k^\mu = \partial\Psi^\mu/\partial x^k$, then the strong closure condition is equivalent to the assumption that the order of partial differentiation is commutative.

By substituting this last relation into the first equation for the total differential of the Lagrange function on N, and using the Leibniz rule for differentiation yields

$$\partial\{\partial L/\partial\mathbf{\Psi}_k^\mu)\mathbf{\Psi}_j^\mu\}/\partial x^k = (\partial L/\partial\mathbf{\Psi}_k^\mu)\partial\mathbf{\Psi}_j^\mu/\partial x^k + \{\partial(\partial L/\partial\mathbf{\Psi}_k^\mu)/\partial x^k\}\mathbf{\Psi}_j^\mu, \tag{2.5}$$

(an expression which is equivalent to the trick of integrating by parts in the calculus of variations). The last term on the RHS of the expression above for $dL(\mathbf{\Psi}^\mu, \mathbf{\Psi}_k^\mu)$ becomes,

$$(\partial L/\partial \mathbf{\Psi}^\mu_k)d\mathbf{\Psi}^\mu_k = (\partial L/\partial \mathbf{\Psi}^\mu_k)(\partial \Psi^\mu_j/\partial x^k)dx^j \tag{2.6}$$

$$= \partial\{\partial L/\partial \mathbf{\Psi}^\mu_k)\mathbf{\Psi}^\mu_j\}/\partial x^k dx^j - \{\partial(\partial L/\partial \mathbf{\Psi}^\mu_k)/\partial x^k\}\mathbf{\Psi}^\mu_j dx^j \tag{2.7}$$

and the first term on the RHS becomes,

$$(\partial L/\partial \mathbf{\Psi}^\mu)d\mathbf{\Psi}^\mu = (\partial L/\partial \mathbf{\Psi}^\mu)\mathbf{\Psi}^\mu_k dx^k - (\partial L/\partial \mathbf{\Psi}^\mu)\omega^\mu \tag{2.8}$$

Recall that by functional substitution (the pullback) the function $L(\mathbf{\Psi}^\mu, \mathbf{\Psi}^\mu_k)$ on N can be expressed explicitly in terms of the m variables, $\overline{L}(x^k)$ on M. Then the differential of $\overline{L}(x^k)$ on M with respect to the m variables becomes

$$d\overline{L}(x^m) = \{\partial\overline{L}(x^m)/\partial x^k\}dx^k = \{\partial\overline{L}(x^m)/\partial x^k\}\delta^k_j dx^j, \tag{2.9}$$

where δ^k_j is the Kronecker delta function.

Substituting, combining and rearranging terms in the desired equation $d\overline{L} = dL$ leads to a "power theorem" for fields

$$[\{\partial(\partial L/\partial \mathbf{\Psi}^\mu_k)/\partial x^k - \partial L/\partial \mathbf{\Psi}^\mu\}\mathbf{\Psi}^\mu_j]dx^j \tag{2.10}$$

$$= [\partial\{(\partial L/\partial \mathbf{\Psi}^\mu_k)\mathbf{\Psi}^\mu_j - (\overline{L})\delta^k_j\}/\partial x^k]dx^j + (\partial L/\partial \mathbf{\Psi}^\mu)\omega^\mu \tag{2.11}$$

The tautology is described as a power theorem, for if the direction fields (proportional to the dx^j) are describable in terms of a singly parametrized vector field, then the coefficients of the single parameter on the LHS of the theorem have the format of the classical definition: power is force times velocity.

The term $(\partial L/\partial \mathbf{\Psi}^\mu)\omega^\mu$ represents the "fluctuations" in the fields. The **Stress Energy Tensor** W^j_k (a mixed second rank non-symmetric tensor) is defined as

$$W^k_j = [(\partial L/\partial \mathbf{\Psi}^\mu_k)\mathbf{\Psi}^\mu_j - \overline{L}\delta^k_j]. \tag{2.12}$$

The Lagrange-Eulerian (covariant) generalized **force** field, $f^{LaGrange}_\mu$, on the target domain

$$f^{LaGrange}_\mu = \{\partial(\partial L/\partial \mathbf{\Psi}^\mu_k)/\partial x^k - \partial L/\partial \mathbf{\Psi}^\mu\} \tag{2.13}$$

has a covariant pre-image as a Lagrangian force $F^{Lagrange}_j$ on the initial space of parameters defined by the pullback formula similar to that for a covariant tensor field,

$$F^{LaGrange}_j = f^{LaGrange}_\mu \mathbf{\Psi}^\mu_j \tag{2.14}$$

If the $\mathbf{\Psi}_j^\mu$ were functions representing an invertible Jacobian map, then the dissipative force on M is exactly the tensor pull back of a classical covariant tensor field on N. However, in this analysis, the $\mathbf{\Psi}_j^\mu$ need not be elements of an invertible Jacobian.

With these definitions, the power theorem becomes a tautology for the virtual work:

$$W = F_j^{LaGrange} dx^j = [\partial W_j^k / \partial x^k] dx^j + (\partial L / \partial \mathbf{\Psi}^\mu) \omega^\mu \tag{2.15}$$

The notations above equation make the assertion that the tautology is power theorem more transparent, for if the equation is divided by dt, then the expression $F_j^{LaGrange} dx^j / dt$ represents the innerproduct of force times velocity; i.e., the standard definition of power. As it stands, the equation defines the virtual work 1-form W as the sum of a contribution from divergence of the stress-energy tensor, and a set of fluctuations.

The field momentum tensor is defined by the expression

$$\Pi_\mu^k = (\partial L / \partial \mathbf{\Psi}_k^\mu). \tag{2.16}$$

When the differentials dx^j are presumed to be arbitrary, and when the fluctuation term is ignored, then the Power Theorem becomes the classical stress-energy theorem,

$$F_j^{LaGrange} = \partial [W_j^k] / \partial x^k = f_\mu^{LaGrange} \mathbf{\Psi}_j^\mu, \tag{2.17}$$

which reads "The divergence of the Stress Energy Tensor = the Lagrangian Force". This fundamental formalism was obtained without metric, without a connection and is in accurate tensor format. The stress energy tensor is a mixed 2nd rank tensor field, and the Lagrangian force is a covariant vector.

Note that $W = F_j^{LaGrange} dx^j$ is a 1-form whose closed loop integral represents the cyclic work of thermodynamics. If $dW = 0$ (when the curl of $F_j^{LaGrange}$ is zero) then the system is thermodynamically reversible. In the more stringent extremal situation the virtual work vanishes, $W = 0$ (where the "Lagrange generalized force" $F_j^{LaGrange}$ vanishes), and the "equations of motion" or "Field Equations" are the ubiquitous Lagrange-Euler equations of classical mechanics:

$$\{\partial(\partial L / \partial \mathbf{\Psi}_k^\mu) / \partial x^k - \partial L / \partial \mathbf{\Psi}^\mu\} = 0. \tag{2.18}$$

2.1. Examples

2.1.1. Example 1. Single parameter evolution in 3-space; the conservative particle.

Let

$$\Phi : \{t\} \Rightarrow \{\mathbf{x}, \mathbf{v}\}; \tag{2.19}$$

where in the notation above

$$\mathbf{\Psi}^{\mu} \approx \mathbf{x}, \quad and \quad \mathbf{\Psi}_t^{\mu} \approx \mathbf{v}. \tag{2.20}$$

$$\omega^{\mu} \approx d\mathbf{x} - \mathbf{v}dt. \tag{2.21}$$

Note that differential closure, $d\omega^{\mu} = 0$ implies $d\mathbf{v}\hat{\ }dt = 0$ and therefore $\mathbf{v} = \mathbf{v}(t)$.

If $\omega^{\mu} = 0$ then $\mathbf{v} = \partial\mathbf{x}(t)/\partial t$, and is equivalent to the assumption of a kinematic topology without fluctuations.

Suppose the Lagrange function

$$L(\mathbf{x}, \mathbf{v}) = m\mathbf{v}^2/2 - U(x) = KE - PE \tag{2.22}$$

Then

$$\partial L/\partial\mathbf{\Psi}^{\mu} \approx \partial L/\partial\mathbf{x} = -gradU; \tag{2.23}$$

$$\partial L/\partial\mathbf{\Psi}_k^{\mu} \approx \partial L/\partial\mathbf{v} = m\mathbf{v}, \ the\ momentum, \ \mathbf{p}(t). \tag{2.24}$$

The Stress Energy tensor has a single component

$$[W_j^k] = [(\partial L/\partial\mathbf{\Psi}_k^{\mu})\mathbf{\Psi}_j^{\mu} - \overline{L}\delta_j^k] \approx [\mathbf{p} \circ \mathbf{v} - m\mathbf{v}^2/2 + U(x)] = KE + PE = TE \tag{2.25}$$

The single component of W_j^k, is recognized as the total energy, TE, which is the sum of the kinetic and potential energy. As the assumed form of L is not an explicit function of time, it follows that the divergence of the stress energy tensor vanishes, $\partial[TE]/\partial t = 0$. This is a classic result that corresponds to the concept of the conservation of total energy.

The Lagrange generalized (perhaps irreversible) force expression becomes

$$f_{\mu}^{Lagrange} = \{\partial(\partial L/\partial\mathbf{\Psi}_k^{\mu})/\partial x^k - \partial L/\partial\mathbf{\Psi}^{\mu}\} \approx [\partial\mathbf{p}/\partial t + gradU(x)]. \tag{2.26}$$

In the example, this expression is equal to zero (no dissipation) for null fluctuations, as the "divergence" of the stress energy term vanishes. As there is only one parameter, the partial derivative is a total derivative, and the above expression leads to the classical Newtonian formula for the "equations of motion" of a conservative system, $d\mathbf{p}/dt = -gradU(x)$. Note that the Newtonian force, $d\mathbf{p}/dt$, is equal to the Lagrangian force, $f_\mu^{Lagrange}$, only when the gradient of the potential vanishes.

When kinematic fluctuations are admitted, then the fluctuations contribute to the work 1-form through the expression

$$W = (\partial L/\partial \mathbf{\Psi}^\mu)\omega^\mu \approx -gradU(x) \circ (d\mathbf{x} - \mathbf{v}dt). \tag{2.27}$$

Note that this fluctuation term can vanish, even in the presence of kinematic fluctuations, if the fluctuations are orthogonal (transverse) to the gradient of the potential. The result implies that when fluctuations (if they exist) reside on the surface of constant potential energy, they contribute nothing to the dissipative power. Only the fluctuations normal to the surface of constant potential energy will contribute to the 1-form of virtual work. In thermodynamics, the assumption of a closed system implies that no massive particles pass through the system boundary. This idea is analogous to the statement that the normal component of the fluctuations must vanish on the surface of constant potential. The implication is that the surface of constant potential energy defines the "boundary" of a physical system, a well known result in electrostatics.

All of the above ideas are restricted to those physical systems that can be described by a single evolutionary parameter, t, and a Lagrange function which is explicitly independent from t.

2.1.2. Example 2. The two dimensional dynamical string; Laplace's equation.

Let

$$\Phi : \{s, ct\} \Rightarrow \{\phi(s,t), \phi_s, \phi_{ct}\}; \tag{2.28}$$

where in the notation above

$$\mathbf{\Psi}^\mu \approx \phi, and \mathbf{\Psi}_s^\mu \approx \phi_s. \mathbf{\Psi}_{ct}^\mu \approx \phi_{ct}. \tag{2.29}$$

$$\omega^{\mu} \approx d\phi - \phi_s ds - \phi_{ct} d(ct) \tag{2.30}$$

Note that the closure condition, $d\omega^{\mu} = 0$ implies $d\phi_s \hat{} ds + \phi_{ct} \hat{} d(ct) = 0$ and therefore

$$\partial\phi_s/\partial ct = \partial\phi_{ct}/\partial s. \tag{2.31}$$

Suppose the LaGrange function is

$$L(\phi(s,t), \phi_s, \phi_{ct}) = (\phi_s^2 + \phi_{ct}^2)/2 \tag{2.32}$$

Then

$$\partial L/\partial \mathbf{\Psi}^{\mu} \approx \partial L/\partial\phi = 0; \tag{2.33}$$

$$\partial L/\partial \mathbf{\Psi}_k^{\mu} \approx \partial L/\partial\phi_s or, \partial L/\partial\phi_{ct}. \tag{2.34}$$

The Stress Energy tensor becomes a two dimensional matrix.

$$[W_j^k] = [(\partial L/\partial \mathbf{\Psi}_k^{\mu})\mathbf{\Psi}_j^{\mu} - \overline{L}\delta_j^k] \approx \begin{bmatrix} (\phi_s^2 + \phi_{ct}^2)/2 & \phi_s\phi_{ct} \\ \phi_s\phi_{ct} & (\phi_s^2 + \phi_{ct}^2)/2 \end{bmatrix} \tag{2.35}$$

The Lagrange Euler force on N is of one component,

$$f_{\mu}^{Lagrange} = \{\partial(\partial L/\partial \mathbf{\Psi}_k^{\mu})/\partial x^k - \partial L/\partial \mathbf{\Psi}^{\mu}\} \approx \partial\phi_s/\partial s + \partial\phi_{ct}/\partial ct, \tag{2.36}$$

which if the fluctuations go to zero becomes equal to the Laplacian of the function ϕ. In other words the generalized force is completely determined from the functional forms assumed for $\phi_s(s,ct)$ and $\phi_{ct}(s,ct)$.

The work one form on M however consists of two components,

$$W = [\partial W_j^k/\partial x^k]dx^j = (\partial\phi_s/\partial s + \partial\phi_{ct}/\partial ct)\{\phi_s ds + \phi_{ct} d(ct)\} \tag{2.37}$$

. Note for later discussion that the stress energy tensor, $[W_j^k]$, is symmetric.

The extremal case of zero virtual work implies that either the "string displacement" is constrained to the surface $\{\phi_s ds + \phi_{ct} d(ct)\} = 0$,or $(\partial\phi_s/\partial s + \partial\phi_{ct}/\partial ct) = 0$. If the fluctuations are zero, then the last equation implies that the virtual work is zero only if the function ϕ is harmonic, $\nabla^2\phi = 0$. In this case the function ϕ satisfies Laplace's equation.

2.1.3. Example 3. Another two component Lagrangian; the Wave Equation.

Let the map ϕ be defined as:

$$\phi : \{s, ct\} \Rightarrow \{\mathbf{\Psi}^{\mu}, \mathbf{\Psi}_{s}^{\mu}, \mathbf{\Psi}_{ct}^{\mu}\} = \{\phi, \phi_{s}, \phi_{ct}\} \ . \tag{2.38}$$

The target space consists of a single field amplitude (or wave function, ϕ) that corresponds to a "coordinate", but has several (two) components that correspond to a "velocity".

Note that $\omega^{\mu} \Leftrightarrow d\phi - \phi_{ct}d(ct) - \phi_{s}ds$. Now suppose that the Lagrange function for this system is given in the form

$$L(\phi,\phi_{s}, \phi_{ct}) = \{\phi_{s}^{2} - \phi_{ct}^{2}\}/2. \tag{2.39}$$

This differs by a sign from Example 2. Then the field momentum is the two component contravariant vector

$$\Pi_{\mu}^{k} = \partial L/\partial \mathbf{\Psi}_{k}^{\mu} \Rightarrow \pi^{k} = \begin{bmatrix} \phi_{s} \\ -\phi_{ct} \end{bmatrix}. \tag{2.40}$$

The stress energy tensor becomes a two by two matrix

$$W_{j}^{k} = [\partial L/\partial \mathbf{\Psi}_{k}^{\mu})\mathbf{\Psi}_{j}^{\mu}\} - \underline{L}\delta_{j}^{k}] = \begin{bmatrix} \{\phi_{s}^{2} + \phi_{ct}^{2}\}/2 & \{\phi_{s}\phi_{ct}\} \\ -\{\phi_{s}\phi_{ct}\} & \{\phi_{s}^{2} + \phi_{ct}^{2}\}/2 \end{bmatrix} \tag{2.41}$$

Note that the stress energy tensor is NOT symmetric (therefore the system will have a spin component of angular momentum as well as a classical component of angular momentum – see below).

The Lagrange force becomes a "scalar" 1 component object

$$f_{\mu}^{Lagrange} = \{\partial(\partial L/\partial \mathbf{\Psi}_{k}^{\mu})/\partial x^{k} - \partial L/\partial \mathbf{\Psi}^{\mu}\} = \partial(\phi_{s})/\partial s - \partial(\phi_{ct})/\partial(ct) - 0 \tag{2.42}$$

Suppose that the fluctuations are zero, and that $\phi_{s} \Rightarrow \partial\phi/\partial s$ and $\phi_{ct} \Rightarrow \partial\phi/\partial(ct)$. Then if this "generalized" force is to vanish, the amplitude function, ϕ, satisfies the wave equation in the two variables, s and ct. That is, the Field Equations are the one dimensional wave equation:

$$\partial^{2}\phi/\partial s^{2} - \partial^{2}\phi/\partial(ct)^{2} = 0. \tag{2.43}$$

Problem 1. Consider the complex (or two dimensional) oscillator. Map t into $z = x + iy$, $\overline{z} = x - iy$, $\phi(z, \overline{z}), \chi(z, \overline{z})$. Note that N is 4 dimensional. Set up a Lagrangian equivalent to the Harmonic oscillator. Compute the stress energy tensor, etc.

3. Comments about the stress energy tensor

3.1. Reversible processes

From the discussion above it is evident that if there are no fluctuations, and if the virtual work 1-form vanishes, then the power theorem yields the classic Lagrange-Euler formulas for the equations of motion. In general, the map and the functional format of the Lagrangian will permit the computation of the generalized (or external) force, and the virtual work 1-form, W. It is often difficult to find the functional form of L that will match a specific generalized force. However, classes of such "boundary" conditions can be studied for there topological properties.

First consider the case where the fluctuations vanish. Even then it is extraordinary that the virtual work 1-form should vanish for arbitrary direction fields. This extremal constraint, for arbitrary direction fields, requires that the stress energy tensor should have a zero divergence. There may exist specific (not arbitrary) direction fields which cause the work 1-form to vanish. In these directions the power (force times velocity) is zero. The Lagrange generalized forces then are orthogonal to these special direction fields in the sense of Bernoulli.

A less restrictive condition would be to require that the cyclic integral of the Work 1-form should vanish. In such cases, it must be true that the exterior derivative of the Work 1-form should vanish. In such circumstances, the Work 1-form must reduce to a perfect differential of some other function. In such circumstances, the new function has the appearance of a potential U in the equations of motion. The generalized force is not zero, but is a gradient field which does not contribute to cyclic integrals. The functional form of L dictates whether or not such an option is possible. In certain instances the perfect differential representing the work 1-form is such that the equations of motion are locally adiabatic.

A even less restrictive condition is the constraint that the 1-form of virtual work should be closed, but not identically zero ($dW = 0$, but $W \neq 0$). In this case the "curl" of the Lagrange force should vanish, but the cyclic work need not vanish. From the first law of thermodynamics, $Q\hat{}dQ = (W + dU)\hat{}dW$. Hence,

$dW = 0$, implies that $Q\hat{}dQ = 0$, and therefore Q admits an integrating factor. Such processes are reversible [G]. In physical examples, the closure constraint is equivalent to the Helmholtz theorems in fluid dynamics, and the Master equation constraint in plasma physics.

The stress energy tensor density, $[W_j^k] = [(\partial L/\partial \mathbf{\Psi}_k^\mu)\mathbf{\Psi}_j^\mu - \overline{L}\delta_j^k]$, is represented on M by a m by m matrix of functions, and is not necessarily a symmetric matrix. For a 1-dimensional space M of a single parameter (usually denoted as time, t, in engineering and physics), the stress energy tensor is a single component function and typically represents the total energy of a physical system. The symmetry question of the single component matrix is mute . For a two parameter M space, the stress energy tensor is represented by a two by two matrix of functions. Depending on the format of the Lagrange function, the matrix may be symmetric or not. (See the examples above). For a four dimensional parameter space equivalent to space-time, {x,y,z,t}, the stress energy tensor is a 4 by 4 matrix.

When the stress energy tensor is symmetric, it will be shown below that there exists a conserved third rank tensor field, symbolized by **L**, which can be interpreted as the field Orbital Angular Momentum. When the stress energy tensor is not symmetric, and **L** is not an evolutionary invariant, it is sometimes possible to find a conserved third rank tensor field, **J**, which is conserved. The classic technique is to define a Spin component of angular momentum, **S**, to be added to the conventional and classic Orbital angular momentum, **L**. The result is that $\mathbf{J} = \mathbf{L} + \mathbf{S}$ can be a conserved quantity (an invariant of the evolution) but the individual components **L** or **S** are not. This concept was used by Dirac to justify the spin properties of the electron in a Quantum Mechanical setting. Note that the concept of Spin is a not solely a quantum mechanical effect, but also exists in non-quantized classical fields when the stress energy tensor is not symmetric.

None of these results above required the explicit use of the calculus of variations. When the system is without fluctuations (in the sense that the fluctuation 1 forms are both closed and zero) then the functions $\Psi_k^\mu \Rightarrow \partial\Psi^\mu/\partial x^k$ are indeed the Jacobian coefficients of the mapping. It follows that the "generalized force" $f_\mu^{LaGrange}\mathbf{\Psi}_j^\mu$ on M is the covariant pullback of the Lagrange force, $f_\mu^{LaGrange}$, on N.

Classical Hamiltonian mechanics on a state space of odd (2n+1) dimensions is a study of those special extremal direction fields such that the Virtual work 1-form vanishes. As will be shown below, such a process is unique to a maximal rank manifold of odd dimensions. Such extremal processes do not exist on maximal rank manifolds of even dimension.

4. The Method of Cartan Hilbert

4.1. Non-canonical evolution

The method of Lagrange can be modified to include constraints by using the so-called Lagrange multipliers. From the point of view of the calculus of variations, the variational integrand is prolonged from $L(\mathbf{q}, \mathbf{v}, t)dt$ (which is of Pfaff dimension 2, and therefor always integrable) to a 1-form of possibly larger Pfaff dimension. Consider those systems that can be described by a Lagrange function $L(\mathbf{q}, \mathbf{v}, t)$ and a 1-form of Action given by the expression:

$$\mathcal{A} = L(\mathbf{q}, \mathbf{v}, t)dt + \mathbf{p}\cdot(d\mathbf{q} - \mathbf{v}dt), \tag{4.1}$$

This formula defines the Cartan-Hilbert 1-form of Action.

At first glance it appears that the domain of definition is a 3n+1 dimensional variety of independent variables, $\{\mathbf{q}, \mathbf{v}, \mathbf{p}, t\}$. Where $\mathbf{p} \neq \partial L/\partial \mathbf{v}$ and $(d\mathbf{q} - \mathbf{v}dt) \neq 0$. The original Lagrange 1-form of Action, $L(\mathbf{q}, \mathbf{v}, t)dt$, has been prolonged by the addition of (possibly zero) fluctuations, $(d\mathbf{q} - \mathbf{v}dt)$, and appropriate Lagrange multipliers, $\mathbf{p}$. It will not be necessary to impose a condition on the fluctuations that will make them holonomic. In fact anholonomic constraints lead to interesting effects of torsion.

For the given (prolonged) Action, construct the Pfaff sequence $\{\mathcal{A}, d\mathcal{A}, \mathcal{A}\hat{}d\mathcal{A}, d\mathcal{A}\hat{}d\mathcal{A}...\}$. The Top (non-zero) Pfaffian of this sequence is given by the formula,

$$(d\mathcal{A})^{n+1} = \tag{4.2}$$

$$(n+1)!\{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k) \bullet dv^k\}\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt \tag{4.3}$$

which indicates that the Pfaff topological dimension is 2n+2, and not the geometrical dimension 3n+1. This result means that there exits a map from the 3n+1 dimensional space to a 2n+2 dimensional space that captures the topological features of interest. It follows that the exact two form $d\mathcal{A}$ satisfies the equations

$$(d\mathcal{A})^{n+1} \neq 0, \; but \quad \mathcal{A}\hat{}(d\mathcal{A})^{n+1} = 0. \tag{4.4}$$

The result is true for any closed addition to the 1-form $\mathcal{A}$; e.g., $\mathcal{A} + \gamma$ with $d\gamma = 0$. The Pfaff dimension is independent from any "gauge" addition to the prolonged Lagrangian 1-form. On the 2n+2 domain, the components of 2n+1 form $\mathcal{T} = \mathcal{A}\hat{}(d\mathcal{A})^n$ generate what is herein defined as the Torsion Current, whose

2n+2 components behave as a contravariant vector density, $\mathbf{T}^m$. The components of the "Torsion current" are orthogonal (transversal) to the 2n+2 components of the covariant vector, $\mathbf{A}_m$, that make up the coefficients of the Action 1-form. In other words,

$$\mathcal{A}\hat{}\mathcal{T} = \mathcal{A}\hat{}(\mathcal{A}\hat{}(d\mathcal{A})^n) = 0 \Rightarrow i(\mathbf{T})(\mathcal{A}) = \mathbf{T}^m\mathbf{A}_m = 0. \tag{4.5}$$

This topological result does not depend upon geometric ideas such as metric.

The 2n+2 domain will be defined as **Thermodynamic Space**. The 2n+2 domain so constructed can not be compact without boundary for it has a volume element which is exact. It is orientable if the 2n+2 volume element does not change sign.

For the 2n+2 domain to be **symplectic**, the top Pfaffian can never vanish, therefore:

$$(\partial L/\partial v^k - p_k) \bullet dv^k \neq 0 \tag{4.6}$$

If, however, the constraints of canonical momenta are subsumed, such that $\partial L/\partial v^k - p_k = 0$, then the 2-form $d\mathcal{A}$ is not symplectic on its maximal dimension 2n+2, but instead the top Pfaffian defines a contact manifold on a **State Space** of topological dimension 2n+1 with the formula

$$\mathcal{A}\hat{}(d\mathcal{A})^n = n!\{p_k v^k - L(t,q,v\}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt + ...\text{terms involving } \gamma. \tag{4.7}$$

The Torsion current reduces to a single component on the contact manifold. It is this 2n+1 dimensional contact manifold that serves as the arena for most of classical mechanics prior to 1965, especially for those theories which were built from the calculus of variations. The 2n+1 dimensional contact manifold, or State Space, admits a unique "extremal" evolutionary field, that satisfies "Hamilton's equations" $i(\mathbf{V})d\mathcal{A} = 0$. The coefficient of the state space volume is to be recognized as the Legendre transform of the physicist's Hamiltonian energy function.

$$H(t,q,v,p) = \{p_k v^k - L(t,q,v)\} \tag{4.8}$$

The Hamiltonian function can be decomposed into two parts:

$$H(t,q,v,p) = H_0(t,q,p) + E(t,q,p,v), \tag{4.9}$$

where the excess function $E(t,q,p,v)$ vanishes if the momenta are canonically defined. In this case the Hamiltonian is related to the Lagrangian by means of a Legendre transformation. The constraint of zero excess function implies that the LaGrange multipliers, $\mathbf{p}$, satisfy the equations

$$\mathbf{p}-\partial L/\partial\mathbf{v}=0, \qquad \mathbf{v}-\partial H/\partial\mathbf{p}=0. \tag{4.10}$$

This constraint is equivalent to the statement that the "Hamiltonian" is to be expressed in terms of the variables $\{t,q,p\}$ only. The 2n+1 space maintains its contact structure as long as the "total Hamiltonian energy" is never zero, and the momenta are canonically defined. However, if the Lagrangian is homogeneous of degree 1 in the velocities, $\mathbf{v}$, and if the momenta are canonically defined such that $\partial L/\partial v^k = p_k$, then the top Pfaffian of the sequence, now doubly constrained, defines yet another non-compact symplectic manifold of Pfaff dimension 2n (a distinguished form of Phase Space) mod the closed contributions due to γ. These aforementioned constraints are precisely Chern's constraints used to define a Finsler space which admits non-Riemannian geometries (when the Lagrange function contains more than quadratic powers of $\mathbf{v}$) and spaces with torsion.[3] Note that the processes of topological reduction described above are not equivalent to forming an arbitrary section(s) in the form of holonomic constraints.

4.2. More on Fluctuations

Consider the extended format of the Cartan-Hilbert invariant integrand written in slightly different notation,

$$\mathcal{A}=L(\mathbf{x},\mathbf{v},t)dt+\mathbf{p}\cdot(d\mathbf{x}-\mathbf{v}dt), \tag{4.11}$$

Note that the original space-time, $\{\mathbf{x},t\}$ has been extended to a 10 dimension space of functions, $\{\mathbf{x},t;\mathbf{v},\mathbf{p}\ \}$. On this set, it is convenient to define a vector field, $\mathbf{V}$, with 10 components given by the functions, $\{\mathbf{v},1;\mathbf{a},\mathbf{f}\}$. Differential and functional constraints will be imposed on this 10 dimension space thereby defining a topology. As before, the Cartan-Hilbert action involves a classic Lagrange function, $L(\mathbf{x},\mathbf{v},t)$, and a linear combination of non-zero position "fluctuation or deformation" 1-forms, defined as:

$$\Delta\mathbf{x}=(d\mathbf{x}-\mathbf{v}dt)\neq 0. \tag{4.12}$$

The covariant array, $\mathbf{p}$, of coefficients of the fluctuation 1-forms may be described as set of Lagrange multipliers. It will be demonstrated below that this covariant field, $\mathbf{p}$, dual to the contravariant velocity field, $\mathbf{v}$, plays the role of the canonical momentum, when subjected to additional, but classic, constraints that are equivalent to the constraint of zero temperature.

The 1-forms, $\Delta\mathbf{x}$, are defined as fluctuation-deformation 1-forms, for they represent deviations from the pure kinematic point of view associated with a rigid body dynamics or the evolution of a point particle in terms of a single parameter group of transformations. Although not always true, these deviations are often small corrections to the kinematic constraints, $\Delta\mathbf{x} = 0$, and have the appearance of "fluctuations" about the "kinematic" lines that act as guiding centers for the evolution.

The fluctuation 1-forms are not necessarily zero for deformable media. If it is assumed that the "points", $\mathbf{x}$, evolve in terms of a map, f^k, from a set of initial conditions, $\mathbf{y}$, then the map

$$x^k = f^k(\mathbf{y}, t) \tag{4.13}$$

with the classic assumption that $\mathbf{v} = \partial f^k(\mathbf{y}, t)/\partial t$ leads to the differential expression,

$$\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) = \{\partial f^k(\mathbf{y}, t)/dy^j\}dy^j \tag{4.14}$$

First suppose that the fluctuations vanish, $\Delta\mathbf{x} = 0$. If the Jacobian determinant, $det[\partial f^k(\mathbf{y}, t)/dy^j]$, does not vanish, then the parameters, $\mathbf{y}$, must be constants, such that $dy^j = 0$; that is, $d\mathbf{y}$ is a zero vector. The trajectories are retraceable, for an inverse mapping exists. However, if the Jacobian determinant vanishes, then the differentials $d\mathbf{y}$ need not be the zero vector. For zero fluctuations, $d\mathbf{y}$ must be a null vector of the Jacobian.

Next suppose the fluctuations do not vanish. If the Jacobian determinant does vanish, then the differentials $d\mathbf{y}$ must not be a null vector of the Jacobian, for finite fluctuations. If the Jacobian determinant does not vanish, then the differentials $d\mathbf{y}$ must not be a zero vector for finite fluctuations; that is, the $\mathbf{y}$ are not constants. In otherwords, the "initial conditions" are not constants in a single parameter evolutionary system that has fluctuations.

The criteria that $\Delta\mathbf{x} = 0$ has solutions is established by the Frobenius condition:

$$d(\Delta x^k)\hat{}(\Delta x^1)\hat{}(\Delta x^2)\hat{}...(\Delta x^n) = 0. \tag{4.15}$$

But the Frobenius construction produces a n+2 form on an n+1 dimensional space. Hence it always vanishes; there always exist solutions with zero fluctuations for single parameter evolution of "points". The idea is that a set of points that do not interact (ideal gas) always has an "equilibrium" state of zero fluctuations.

However, suppose that the system has a two parameter evolutionary map, representing the evolution of a "string" rather than a "point". Consider the map

$$x^k = f^k(\mathbf{y}, t, \sigma) \quad \text{with} \quad \mathbf{v} = \partial f^k(\mathbf{y}, t)/\partial t, \mathbf{u} = \partial f^k(\mathbf{y}, t)/\partial\sigma \tag{4.16}$$

$$\text{such that} \quad \Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt - \mathbf{u}d\sigma) = \{\partial f^k(\mathbf{y}, t)/dy^j\}dy^j \tag{4.17}$$

Then the Frobenius condition is a 2n+2 form on a 2n+2 dimensional space which need not vanish. Hence, it is not always true that solutions with zero fluctuations exist. The 2 parameter concept is the simplest representation of a set of interacting points. The second parameter, σ, is a description of a constraint in the sense that the points make up a "string", with an arc length parameter, σ. If $d\sigma$ is zero the string does not stretch as it evolves with dt. But if $d\sigma$ is not zero as the evolution proceeds, it is not true that all such systems admit a state of zero fluctuations. (Is this the meaning of a "zero-point energy"?)

If the parameters, $\mathbf{y}$, which could be interpreted either as initial conditions, or as the coordinates of the origin, are not constants, then the system does not evolve according to the kinematic rules associated with a single parameter group. The basic idea is that the statement, $d\mathbf{x} - \mathbf{v}dt = 0$, must be interpreted as a topological constraint, just as the statement $xdx + ydy + zdz = 0$ is a topological constraint on Euclidean 3-space that produces the topology of a spherical surface.

In classical hydrodynamics, the non-zero fluctuations $d\mathbf{x} - \mathbf{v}dt$ are usually constrained by topological conditions such that their associated 3-form admits an integrating factor, . This topological constraint means that there must exist a function, $\rho(x, y, z, t)$, such that the non-zero 3-form,

$$\text{-} \ = (dx - v^x dt)\hat{}(dy - v^y dt)\hat{}(dz - v^z dt) \tag{4.18}$$

has a vanishing exterior derivative,

$$d\text{-} \ = \{div_3 \rho\mathbf{v} + \partial\rho/\partial t\}dx\hat{}dy\hat{}dz\hat{}dt = 0. \tag{4.19}$$

This topological constraint is usually called the "equation of continuity" for deformable media. Note that the 3-form is trivially zero if there are no fluctuations.

It may be shown that this topological constraint makes an absolute invariant of the evolution. If the flow lines are retraceable, implying that the Jacobian determinant of the assumed mapping is of rank 3, then the topological constraint may be interpreted as the "conservation of mass". However, it is not apparent that nature always insists on the assumed topological constraint among the fluctuation-deformation 1-forms. Such a constraint is a matter for test, especially in the case of a turbulent, irreversible, evolutionary process.

The Cartan 1-form will be used not only to generate the Cartan topology, but also to generate, by means of a procedure equivalent to a variational principle, a set of partial differential equations of evolution with solution vector fields, $\mathbf{V}$. The continuous smooth curves tangent to the vector fields, $\mathbf{V}$, in the higher dimensional geometry of the partial differential system may pullback, or intersect, or project, discontinuously in the lower dimensional geometry. The ordinary kinematic differential equations based on $\mathbf{v}$, without fluctuations, yield solution curves that act as "guiding centers" for the fluctuation fields, $\mathbf{V}$, in the limit that the fluctuations are small. The projections of the continuous curves in the geometry of the higher dimensional space may have gaps and tangential discontinuities on space-time. The discontinuities would be interpreted as defects or fluctuations in an otherwise homogeneous and continuous system. These ideas may be compared to the concept of Poincare sections in the theory of non-linear dynamics. The Cartan method permits the concepts of discontinuous fluctuations to be put on a continuous basis in a space of higher dimension. This topological idea of removing apparent discontinuities by embedding in a space of higher dimensions is similar to the geometric idea where a curved space may be embedded in a higher dimensional euclidean flat space.

Physicists often recognize the Cartan Action in the format,

$$\mathcal{A} = L(\mathbf{x}, \mathbf{v}, t)dt + \mathbf{p}\circ(d\mathbf{x} - \mathbf{v}dt) = \mathbf{p}\circ d\mathbf{x} - H(t, \mathbf{x}, \mathbf{v}, \mathbf{p})dt, \tag{4.20}$$

but do not seem to appreciate that this composition may be interpreted in terms of a fluctuation geometry on a space of 10 dimensions. The additional assumption that the momenta are canonically defined corresponds to a functional relationship or constraint between the variables such that the excess function vanishes. If the relationship is linear, then there would exist a constitutive or metrical relationship between the dual fields, $\mathbf{v}$ and $\mathbf{p}$. Such assumptions are NOT made a priori in this article.

Consider first a Cartan 1-form of action where the fluctuations vanish over a domain. Then the 2-form of limit points is $F = dA = dL\hat{}dt$. It follows that $H = A\hat{}dA = 0$, and $K = dA\hat{}dA = 0$. The Pfaff dimension of such systems is 2 at most. Such systems can have vorticity but are without helicity, or Topological

Torsion. However, in this article, systems of non-zero H and non-zero K are of interest. Examples of systems that do support Topological Torsion are presented in reference [Kiehn 1991a]. From this point of view, both Topological Torsion and Topological Parity are to be associated with the concept of non-null kinematic fluctuations which are not transversal to the system momentum, p x 0.

When fluctuations are permitted, then the exterior derivative of the Cartan action on the 10 dimensional space becomes explicitly,

$$dA = (\partial L/\partial \mathbf{v} - \mathbf{p})d\mathbf{v}\hat{}dt + d\mathbf{p}\circ(d\mathbf{x} - \mathbf{v}dt) + \partial L/\partial \mathbf{x}(d\mathbf{x}\hat{}dt) \tag{4.21}$$

which can be rewritten in the equivalent suggestive format

$$dA = (\partial L/\partial \mathbf{v} - \mathbf{p})\Delta\mathbf{v}\hat{}dt + \Delta\mathbf{p}\circ\Delta\mathbf{x}. \tag{4.22}$$

The term $\Delta\mathbf{v} = d\mathbf{v} - \mathbf{a}dt$ represents the fluctuations in velocity (temperature?) in the same sense that $\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt)$ represents fluctuations in position (pressure?). The term $\Delta\mathbf{p} = (d\mathbf{p} - \partial L/\partial\mathbf{x})dt$ has the appearance of a fluctuation in the Newtonian force. The functions, **a**, are defined to be to the contravariant acceleration vector field (with velocity fluctuations) in the same extremal sense that **v** is defined as the contravariant velocity vector field (with position fluctuations).

The Heisenberg like notation, $\Delta\mathbf{p}\circ\Delta\mathbf{x}$, stands for the sum of 2-forms,

$$\Delta\mathbf{p}\circ\Delta\mathbf{x} = (d\mathbf{p} - \partial L/\partial\mathbf{x})dt\hat{}(d\mathbf{x} - \mathbf{v}dt). \tag{4.23}$$

which is similar to the dot product of two vectors, but here the combinatorial action is through the exterior product, ^. Although closely related to an expectation value generated by an inner product, or to the integrand of a cross-correlation integral, no statistical or ensemble averaging is assumed in this article. The beauty of the Cartan analysis is that it is retrodictively deterministic and well defined in a pullback sense, even when unique, deterministic prediction is impossible [Kiehn,1976b].

The bracket factor, $(\partial L/\partial\mathbf{v} - \mathbf{p}) = -\partial H/\partial\mathbf{v}$ will be defined as the scaled covariant vector field, $\mathbf{k}/S$. The topological constraint $\mathbf{k} = 0$ permits the Lagrange multipliers to be uniquely determined as the canonical momenta of classical mechanics, $\mathbf{p} = \partial L/\partial\mathbf{v}$.

A direct computation of the Topological Torsion, H, on the 10 dimensional space yields,

$$H = A^\wedge dA = Ldt^\wedge(\Delta\mathbf{p}^\wedge\Delta\mathbf{x}) + (\mathbf{k}/S \circ \mathbf{v})^\wedge(\mathbf{p} \circ \Delta\mathbf{x})^\wedge dt. \tag{4.24}$$

which may be evaluated in principle on 4 dimensional space time by functional substitution. A similar direct computation in the higher dimensional geometry of variables $\{\mathbf{x}, t; \mathbf{v}, \mathbf{p}\}$ of the exterior derivative, $K = dH$, produces a 4-form that also can be pulled back to $\{x, y, z, t\}$ by functional substitution. The Topological Parity 4-form becomes,

$$K = dA^\wedge dA = 2\{(\partial L/\partial\mathbf{v} - \mathbf{p}) \circ \Delta\mathbf{v}\}^\wedge(\Delta\mathbf{p}^\wedge\Delta\mathbf{x})^\wedge dt + 2(\Delta\mathbf{p}^\wedge\Delta\mathbf{x})^\wedge(\Delta\mathbf{p}^\wedge\Delta\mathbf{x}). \tag{4.25}$$

On a space-time variety, this 4-form becomes Chern's top Pfaffian whose integral gives information concerning the Euler characteristic of space-time. It is apparent that K depends on the exterior product of the fluctuations of position, Lagrange multipliers, and velocity, as well as the bracket factor, $(\partial L/\partial\mathbf{v} - \mathbf{p}) = -\partial H/\partial\mathbf{v}$, and dt.

The classic first variation of the Action integral, $\int A$, is an extremal principle which, for the action specified above, will generate a Finsler geometry. According to Chern, the Finsler variation is equivalent to setting $dA = 0$, mod $\Delta\mathbf{x}$. In addition, for Finsler geometries, the Lagrange function is presumed to be homogeneous of degree 1 in the $\mathbf{v}$. This constraint is used by Chern to construct or define a "projectivized" tangent bundle. The homogeneity condition implies that the variable, t, can be reparametrized, and the vector $\mathbf{v}$ forms the elements of a projective geometry.

In classical field theory, the Finsler constraint is often imposed arbitrarily:

$$\mathbf{k}/S = (\partial L/\partial\mathbf{v} - \mathbf{p}) = -\partial H/\partial\mathbf{v} = 0. \tag{4.26}$$

As mentioned above, such a constraint uniquely defines the Lagrange multipliers, $\mathbf{p}$, as the components of the canonical momentum. The Topological Parity 4-form is then dependent on the exterior product of "fluctuations" in position and momentum only, and has the same physical dimensions as the square of Planck's constant. From a qualitative point of view, fluctuations in velocity correspond to the property of temperature. Hence, the Finsler constraint can be interpreted as a constraint of constant (or zero) temperature

Now examine the closed loop integrals on the space M of the components that make up the power theorem. The loop integral of the work 1-form represents the net heat flow around the cycle, or equivalently the cyclic work in a thermodynamic sense. If the divergence of the stress energy tensor vanishes, then non-zero cyclic work must be caused by fluctuations. Note that even if there are non-zero fluctuations, $\Delta \mathbf{x} \neq 0$, it is possible that there is no cyclic work subject to the condition that the "gradient" terms $\partial L/\partial \mathbf{x}$ form a null (or adjoint) vector to all of the fluctuations. Such states should have long lifetimes, and be with minimal dissipation. (Wakes, superconductivity??)

5. References

[G] I.I. Gol'denskovi, "The Thermodynamics of Deformable Media" Riedel (1962) p123,