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# Irreversibility, Thermodynamics and Finsler Spaces

R. M. Kiehn Physics Department, University of Houston

## Abstract

Recent activity in topological classifications of closed symplectic integrable Hamiltonian systems focuses attention on those properties of a Lagrangian formulation for which the fundamental 2-form is exact. The Lagrangian formulation, based on a Cartan-Hilbert Action which has n degrees of freedom, leads to an unconstrained symplectic system which is dissipative and of dimension 2n+2. Canonical momentum constraints lead to a contact submanifold of dimension 2n+1 with a unique extremal field. If the 2n+2 symplectic system is to exist, it is necessary that the momenta are not defined canonically, $\partial L/\partial v - p \neq 0$; and that there must exist anholonomic differential fluctuations $\Delta v = dv - Adt \neq 0$ in the velocity and/or in position, $\Delta x = dx - Vdt \neq 0$. The implication is that (non-extremal) evolution on the 2n+2 symplectic domain can be dissipative but the process is not described kinematically in terms of a single parameter group. The fluctuations in velocity lead to non-zero temperature gradients and the fluctuations in position lead to non-zero pressure gradients. Both types of fluctuations lead to distinct contributions to a zero point energy. These 2n+2 domains can act as a source of magnetic dynamo action in a plasma, where velocity fluctuations associated with temperature produce a charge acceleration mechanism in regions where $E \cdot B \neq 0$. Anholonomic differential fluctuations in position lead to the dissipative terms in the Navier-Stokes equations. Using the fact that Cartan's Lie derivative of the Action with respect to a vector field V is a cohomological equivalent to the First Law of Thermodynamics, it is possible to decide if a given process V is irreversible or not. On the 2n+2 symplectic domain, defined as Thermodynamic Space, two distinct evolutionary processes may be defined in terms of the Adiabatic Vector and the Torsion Current. The first process is a symplectomorphism, and therefore is reversible; the second process is not a symplectomorphism, and is irreversible in a thermodynamic sense.

## A. Some Results of Thermodynamic Experience.

Although the readers of this journal usually expect to encounter experimental results, the concept of irreversibility and aging is so ubiquitous in almost every branch of science, and yet is so poorly understood, that it seemed appropriate to use this journal to give a broad expose to a new theoretical approach to the topic. In short, using the methods of Cartan's exterior calculus, the concept of topological evolution may be used determine in a non-statistical manner whether or not a speci¯ed process acting on a physical system is irreversible in a thermodynamic sense. Before explaining the meaning of Cartan's methods, let us gather together certain thermodynamical perspectives.

1. The First Law of Thermodynamics is a statement of cohomology. The di®erence between the inexact 1-form of heat, Q, and the 1-form of work, W, is equal to a perfect di®erential of the internal energy, U:

$$Q - W = dU$$

2. The key to practical machines is due to the fact that the cyclic integral of the work 1-form is not zero

$$: \oint W \neq 0 \quad \text{which implies that} \quad \oint Q \neq 0:$$

The fact that the cyclic integral is not zero, implies that the 1-form integrands cannot be perfect di®erentials.

3. There are certain cyclic processes which are reversible (Carnot cycle), and certain cyclic processes which are irreversible.

4. Irreversibility implies that the heat 1-form does not admit an integrating factor (usually denoted as the reciprocal temperature function). Hence irreversibility is related to the Frobeneous theorem, such that

$$Q \wedge dQ = 0 \text{ implies reversibility;} \quad Q \wedge dQ \neq 0 \text{ implies irreversibility:}$$

(One objective will be to demonstrate that su±cient conditions for irreversibility - and the aging process - are arartifacts of a symplectic domain of four dimensions, or more.)

5. In equilibrium thermodynamics, two species of "conjugate" variables are recognized. They are the intensive (Pressure, Temperature...) variables and the extensive (Volume, Entropy...) variables. The classic thermodynamic functions are homogeneous of degree 1 in the extensive variables. This leads to the idea that at least equilibrium thermodynamics must be related to projectivized tangent bundles ( and therefore to Finsler spaces which are non-Riemannian).

6. The van der Waals gas is ubiquitous from a topological point of view. The Chemical Engineer's law of corresponding states indicates that all equilibrium substances exhibit the swallow tail singularity of the Gibbs free energy surface. This is

a natural result of 3 dimensional evolution, where the Cayley-Hamilton polynomial for the Jacobian of the evolutionary process is cubic. The universal features of the Gibbs surface are determined by the zero sets of the Mean and Gauss curvatures. The spinodal line of phase stability is where the Gauss curvature of the Gibbs surface vanishes, and the critical point is where both the Mean curvature and the Gauss curvature vanish.

7. Cartan's "magic formula" (quoting Marsden) of topological evolution is equivalent to the first law of thermodynamics.

$$L_{(v)}A = i(V)dA + d(i(V)A) = W + dU = Q.$$

In this formula, A is the "Action" 1-form that describes the physical system. V is the vector field that defines the evolutionary process. W is the 1-form of (virtual) work 1-form. Q is the 1-form of heat. When Q = 0, the equation $L_{(v)}A = 0$ is interpreted to mean that the Action 1-form, A, is an invariant of the evolutionary process generated by the vector field, V. Hence the Cartan topology induced by the Pfaff sequence, $\{A, dA, A\wedge dA, dA\wedge dA, \ldots\}$ is an evolutionary invariant of such processes; they are called adiabatic processes. The adiabatic processes are homeomophisms relative to the Cartan topology. That is, Adiabatic process do not cause topological change.

8. There are certain processes (which are not adiabatic) for which $\oint A$ is not an evolutionary invariant, but indeed changes in rational amounts (the photon).

$$L_{(v)} \int A = \int Q = \text{integer}\,(hf)$$

This equation of constrain implies that in these circumstances Q is closed but not exact; i.e., $dQ = 0$, but $\oint_{cycle} Q \neq 0$. Note that if $dQ = 0$, then $L_{(v)}dA = dQ = 0$, which is the equivalent of Helmholtz theorem for the conservation of "vorticity". The statement is equivalent to the constraint that the virtual work is closed.

$$\text{Helmholtz (symplectic) processes} \supset dW = 0$$

Now modern "Hamiltonian" processes are defined as those processes for which the virtual work 1-form is both closed and exact. Extremal processes (the classic Hamiltonian process) are those for which the 1-form of virtual work vanishes - a special case.

$$\text{Hamiltonian processes} \supset W = d\Theta$$

$$\text{Extremal processes} \supset W = 0$$

Neither class of Hamiltonian processes will produce $\oint_{cycle} Q \neq 0$. The implication is that Hamiltonian processes do not radiate. It will be shown that all such Hamiltonian processes are reversible. However, there exist Helmholtz (or Symplectic) processes for which the virtual work is closed but not exact. It is these processes (symplectic, closed but not exact) that must be studied in order to understand the details of quantum transitions. Pure Hamiltonian analysis can describe only the "stationary states", and not the details of the atomic transitions.

9. Both Helmholtz, and its subclass of Hamiltonian processes, are reversible in a thermodynamic sense, for the closed but not exact condition implies that $Q \wedge dQ = 0$. The heat 1-form admits an integrating factor, hence the process is reversible in a thermodynamic sense.

However, on an even dimensional symplectic space consider that vector, T, such that $i(T)\Omega = A \wedge dA$.... Then homogeneous processes of the type

$$L_{(T)}A = \Gamma A = Q.$$

Then

$$Q \wedge dQ = (\Gamma^2 A \wedge dA) \neq 0$$

is a necessary condition for irreversibility of such processes, and $\Gamma = \mathrm{dual}(dA \wedge dA) \neq 0$ is a sufficient condition. This last sufficient condition implies that irreversibility is an artifact of dimension 4 or more.

## B. Extensions of the Cartan-Hilbert Action 1-form and Finsler Spaces

Consider those systems that can be described by a Lagrange function $L(q, v, t)$ and a 1-form of Action given by the expression:

$$A = L(q, v, t)dt + p \cdot (dq - v dt), \tag{1}$$

At first glance it appears that the domain of definition is a 3n+1 dimensional variety of independent variables, $\{q, v, p, t\}$. Where $p \neq \partial L/\partial v$ and $(dq - v dt) \neq 0$.

For the given Action, construct the Pfaff sequence $\{A, dA, A \wedge dA, dA \wedge dA ...\}$ The Top (non-zero) Pfaffian of this sequence is given by the formula,

$$(dA)^{n+1} = (n + 1)! \{\Sigma_{k=1}^{n}(\partial L/\partial v^k - p_k) \bullet dv^k\} \wedge dp_1 \wedge ... dp_n \wedge dq^1 \wedge .. dq^n \wedge dt, \tag{2}$$

which indicates that the Pfaff topological dimension is 2n+2, and not the geometrical dimension 3n+1. It follows that the exact two form dA satisfies the equations

$$(dA)^{n+1} \neq 0, \text{ but } \quad A \wedge (dA)^{n+1} = 0. \tag{3}$$

The result is true for any closed addition to the 1-form A; e.g., $A + \gamma$ with $d\gamma = 0$. On the 2n+2 domain, the components of 2n+1 form $T = A \wedge (dA)^n$ generate what is herein defined as the Torsion Current, a contravariant vector density, $T^m$. The components of the "Torsion current" are orthogonal (transversal) to the 2n+2 components of the covector, $A_m$, that make up the coefficients of the Action 1-form. In other words,

$$A \wedge T = A \wedge (A \wedge (dA)^n) = 0 \Rightarrow i(T)(A) = T^m A_m = 0. \tag{4}$$

This topological result does not depend upon geometric ideas such as metric.

The 2n+2 domain will be defined as Thermodynamic Space. The 2n+2 so constructed can not be compact without boundary for it has a volume element which is exact. It is orientable.

For the 2n+2 domain to be symplectic, the top Pfaffian can never vanish, therefore:

$$(\partial L/\partial v^k - p_k) \bullet dv^k \neq 0 \tag{5}$$

If, however, the constraints of canonical momenta are subsumed, such that $\partial L/\partial v^k - p_k = 0$, then the 2-form $dA$ is not symplectic on its maximal dimension 2n+2, but instead the top Pfaffian defines a contact manifold on a State Space of topological dimension 2n+1 with the formula

$$A \wedge (dA)^n = n!\{p_k v^k - L(t,q,v\}dp_1 \wedge ... dp_n \wedge dq^1 \wedge ..dq^n \wedge dt + ...\text{terms involving } \gamma. \tag{6}$$

The Torsion current reduces to a single component on the contact manifold. It is this 2n+1 dimensional contact manifold that serves as the arena for most of classical mechanics prior to 1965, especially for those theories which were built from the calculus of variations. The 2n+1 dimensional contact manifold, or State Space, admits a unique "extremal" evolutionary field, that satisfies "Hamilton's equations" $i(V)dA = 0$. The coefficient of the state space volume is to be recognized as the Legendre transform of the physicist's Hamiltonian energy function.

$$H(t,q,v,p) = \{p_k v^k - L(t,q,v)\} \tag{7}$$

When the constraints of canonical momenta are valid, it follows that $\partial H(t,q,v,p)/\partial v = 0$; This result is interpreted by the statement that the "Hamiltonian" is to be expressed in terms of the variables $\{t,q,p\}$ only. The 2n+1 space maintains its contact structure as long as the "total Hamiltonian energy" is never zero, and the momenta are canonically defined. However, if the Lagrangian is homogeneous of degree 1 in the velocities, $v$, and if the momenta are canonically defined such that $\partial L/\partial v^k = p_k$, then the top Pfaffian of the sequence, now doubly constrained, defines yet another non-compact symplectic manifold of Pfaff dimension 2n (a distinguished form of Phase Space) mod the closed contributions due to $\gamma$. These aforementioned constraints are precisely Chern's constraints used to define a Finsler space which admits non-Riemannian geometries (when the Lagrange function contains more than quadratic powers of $v$) and spaces with torsion.[3] Note that the processes of topological reduction described above are not equivalent to forming an arbitrary section(s) in the form of holonomic constraints.