The Chiral Vacuum

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Abstract

The consequences of modifying the constitutive equations that describe the classical Lorentz Vacuum to include a chiral term in the format, $\mathbf{D} = \varepsilon_0\mathbf{E} + [\gamma] \circ \mathbf{B}$ and $\mathbf{H} = \mathbf{B}/\mu_0 - [\gamma^\ddagger] \circ \mathbf{E}$ are studied. Wave solutions to the Maxwell Faraday and the Maxwell Ampere equations can be found which are free from real charge densities and current densities, and therefor appear to define a Chiral Vacuum state. The assumption of a simple complex scalar form for chiral constitutive matrix, $[\gamma] = (g + i\gamma)$, leads to cases where the only detectable difference between the Chiral vacuum and the Lorentz vacuum is to be found in the value for radiation impedance, Z, a value which depends on the chiral coefficients g and γ, as well as the ratio $\sqrt{\mu_0/\varepsilon_0}$, through the determinant of the constitutive matrix.

1. Introduction

From the disciplines of Astronomy, General Relativity, and Quantum Mechanics comes an increased interest in possible chiral phenomena that could be associated with the vacuum state. Yet the classic literature of electromagnetism does not seem to address such a chiral effect. The conventional Lorentz Vacuum state for classical electromagnetism is defined in terms of solutions to the Maxwell Faraday equations for the intensities, $\mathbf{E}$ and $\mathbf{B}$, and the Maxwell Ampere equations for the excitations, $\mathbf{D}$ and $\mathbf{H}$, which produce no charge densities or current densities, and satisfy the constitutive equations of constraint, $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu_0$.

Such solutions for the field intensities satisfy not only both Maxwell equations, but also the vector wave equation with a propagation speed of $c = 1/\sqrt{\varepsilon_0\mu_0}$. The permittivity, ε_0, and the permeability, μ_0, of the Lorentz Vacuum domain are presumed to be isotropic and homogeneous constants.

It is remarkable that a chiral constitutive relation of the form $\mathbf{D} = \boldsymbol{\varepsilon}_0\mathbf{E} \ +[\gamma]\circ \mathbf{B}$ and $\mathbf{H} = \mathbf{B}/\mu_0 - [\gamma^{\ddagger}] \circ \mathbf{E}$ will also satisfy both Maxwell equations, with out generating real charge densities and real current densities. The assumption of a simple complex scalar form for chiral constitutive matrix, $[\gamma] = (g + i\gamma)$, leads to two general cases, described below in detail. In one case, the only detectable difference between the Chiral vacuum and the Lorentz vacuum is to be found in the value for radiation impedance, Z, a value which depends on the chiral coefficients g and γ, as well as the ratio $\sqrt{\mu_0/\boldsymbol{\varepsilon}_0}$, through the determinant of the constitutive matrix. In the other case, the propagation phase velocities of left handed and right handed helical waves can be slightly different leading to a reactive impedance contribution to the classic radiation impedance of the Lorentz vacuum.

2. Bateman's development

In 1914, in a small monograph entitled Electrical and Optical Wave Motion, H. Bateman, introduced a number of interesting solutions to Maxwell's equations that emulate propagating singular strings (not plane waves). Bateman is perhaps more famous for his work on the equations that describe the decay chains of radioactive species. However, as pointed out by Whittaker [2], it was Bateman who determined in 1910 that the Maxwell equations were invariant with respect to the conformal group, a much wider group than the Lorentz transformations. Bateman in 1910 also recognized the relationship of his work to the tensor calculus of Ricci and Levi-Civita, several years before the Einstein development of general relativity x.

Maxwell's equations are defined as the Maxwell Faraday equations for the field intensities,

$$curl\,\mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0 \tag{2.1}$$

and the Maxwell Ampere equations for the field excitations, with source.

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \qquad div\mathbf{D} = \rho \tag{2.2}$$

The vacuum is defined by the constraints of a source free domain, such that the current density vanishes, $\mathbf{J} = 0$, and the charge density vanishes, $\rho = 0$. In addition, the vacuum is defined by a set of generalized constitutive constraints between the field intensities and the field excitations, which take the form of the 6x6 matrix equation,

$$\left| \begin{array}{c} \mathbf{D} \\ \mathbf{H} \end{array} \right\rangle = \left[\begin{array}{cccccc} -\varepsilon_0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & -\varepsilon_0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & -\varepsilon_0 & 0 & 0 & \gamma \\ \gamma^\dagger & 0 & 0 & 1/\mu_0 & 0 & 0 \\ 0 & \gamma^\dagger & 0 & 0 & 1/\mu_0 & 0 \\ 0 & 0 & \gamma^\dagger & 0 & 0 & 1/\mu_0 \end{array} \right] \circ \left| \begin{array}{c} -\mathbf{E} \\ \mathbf{B} \end{array} \right\rangle . \qquad (2.3)$$

The sign conventions are those established by Post [3].

Bateman presumed, as have most authors, that the vacuum state requires that $\gamma = 0$. Indeed, the Lorentz vacuum will be defined as the case where $\gamma = 0, \gamma^\dagger = 0$, and the Chiral vacuum will be defined as the case when $\gamma \neq 0, \gamma^\dagger \neq 0$.

Substitution of the Lorentz Vacuum constraints

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \qquad \mathbf{H} = \mathbf{B}/\mu_0 \ . \qquad (2.4)$$

into the Maxwell-Ampere equation yields

$$\mathbf{J} = curl\, \mathbf{H} - \partial \mathbf{D}/\partial t = \{curl \mathbf{B} - \varepsilon\mu \partial \mathbf{E}/\partial t\}/\mu \Rightarrow 0 \qquad (2.5)$$

$$\rho = div \mathbf{D} = div \mathbf{E}/\varepsilon \Rightarrow 0 \qquad (2.6)$$

Each term must vanish for the "vacuum" condition of no charge density and no current density. Suppose the conditions are true. Then differentiating the first expression with respect to time, and taking the curl of the Maxwell-Faraday expression, combine to yield

$$grad\, div \mathbf{E} - curl\, curl\, \mathbf{E} - \varepsilon\mu \partial^2 \mathbf{E}/\partial t^2 \qquad (2.7)$$

In other words a necessary condition for the Lorentz vacuum is that the fields satisfy the Vector Wave Equation (with $div \mathbf{E} = 0$).

Following Bateman, form the inner 3D product of the Maxwell Faraday equation with $\mathbf{H} = \mathbf{B}/\mu$, and the inner product of the source free Maxwell Ampere equation with $\mathbf{E}$. Use the constitutive definitions for the Lorentz vacuum where

$\mathbf{H} = \mathbf{B}/\mu$ and $\mathbf{D} = \varepsilon\mathbf{E}$. Subtract the second resultant from the first, (assuming $\gamma = 0$), to produce the famous Poynting equation,

$$div(\mathbf{E} \times \mathbf{H}) + \mathbf{H} \circ \partial\mathbf{B}/\partial t + \mathbf{E} \circ \partial\mathbf{D}/\partial t \Rightarrow \qquad (2.8)$$
$$div(\mathbf{E} \times \mathbf{H}) + \partial(1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2)/\partial t = 0. \qquad (2.9)$$

The result is an equation of continuity in terms of the field variables. By comparison to a "fluid", this "equation of continuity" yields a field energy density, ρ_e, and an energy current density, $\rho_e\mathbf{v}$, given by the expressions:

$$\rho_e c^2\mathbf{v} = (\mathbf{E} \times \mathbf{H}) = (\mathbf{D} \times \mathbf{B})c^2 \quad and \quad \rho_e c^2 = (1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2). \qquad (2.10)$$

It is important to note that the energy flux, $(\mathbf{E} \times \mathbf{H})$, and the momentum flux, $(\mathbf{D} \times \mathbf{B})$, are in the same direction and propagate with the same speed.

It should be remembered that these equations can be complex. The energy current density and the energy density can be formed from complex numbers. Bateman finds the extraordinary result, equivalent to the expression,

$$\rho_e^2(1/\mu\varepsilon - \mathbf{v} \circ \mathbf{v}) = \rho_e^2(c^2 - \mathbf{v} \circ \mathbf{v}) \qquad (2.11)$$
$$\equiv (1/c^2)\{[(1/2)(\mathbf{D} \circ \mathbf{E}) - (1/2)(\mathbf{B} \circ \mathbf{H})]^2 \qquad (2.12)$$
$$+(\mathbf{E} \circ \mathbf{B}/Z_{freespace})^2\}. \qquad (2.13)$$

under the assumption that $\varepsilon\mu c^2 = 1$. The factor (μ/ε) is the square of the radiation impedance of free space, $Z_{freespace} = \sqrt{\mu/\varepsilon}$. It is apparent that the first term on the right is the first Poincare (conformal) invariant equivalent to the Lagrange energy density of the field (the difference between the deformation and the kinetic energy densities). The second term is the second Poincare invariant of the field, and is to be associated with topological parity and thermodynamic irreversibility [4]. Bateman remarks that "the rate at which energy flows through the field is less that the velocity of light", unless the two Poincare invariants on the RHS vanish. The importance of the null Poincare invariants becomes obvious, as they furnish the requirement that the field energy propagates with the speed of light. It is important to remember that these equations can involve complex vector fields.

In general, for the Lorentz vacuum, the energy density of the field is defined as

$$Ham = (1/2)(\mathbf{D} \circ \mathbf{E}) + (1/2)(\mathbf{B} \circ \mathbf{H}) = 1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2 \tag{2.14}$$

while the field Lagrangian is defined classically as

$$Lag = (1/2)(\mathbf{D} \circ \mathbf{E}) - (1/2)(\mathbf{B} \circ \mathbf{H}) = 1/2\varepsilon\mathbf{E}^2 - 1/2\mathbf{B}^2/\mu. \tag{2.15}$$

3. Chirality

The development above describes classic results valid for a Lorentz Vacuum, but now the question arises as to how these results change for a Chiral Vacuum. A Chiral Vacuum will be defined as a vacuum for which the constitutive matrices represented by $[\gamma]$ are not zero, but for which there are no real charge densities or current densities. The objective of this article is to assume that $[\gamma]$ is a complex domain constant, not zero, and then to determine what are the consequences of such an assumption. Such an assumption, which if applicable to the vacuum, would imply that the Chiral vacuum, and therefor the universe itself, may not have a center of symmetry. The Chiral adjective is appropriate, for a pure imaginary $[\gamma]$ replicates certain features of media which are optically active. The classic example of an optically active media is a solution of right handed helical molecules, such as sugar, in water. The phenomena has practical use in the wine industry and has been used to permit the grower to determine the sugar content of his grapes. (This is the basis of the words $\circ brix$ often found on French wine labels).

Once a constitutive matrix is assumed it is possible to compute the characteristics of the combined Maxwell Faraday and Maxwell Ampere partial differential system. These surfaces, independent from any gauge assumptions, define point sets upon which the solutions to the partial differential system are not unique. The characteristic point sets, in general, form non-stationary Kummer-Fresnel quartic surfaces, of which the constitutive equations of the Chiral Vacuum generate a special case [5]. The theory for such surfaces has been worked out in detail, and the references below contain links to Maple programs that will generate such surfaces for arbitrary constitutive equations. There is an added importance to the recognition that the characteristic surfaces are Kummer surfaces, for then a connection between classical electromagnetism and Clifford algebras can be made, with the possibility that classical solutions to Maxwell's equations can involve spinors. Examples of such quaternionic solutions that indicate that the phase velocity of propagation in the inbound and outbound directions are not the same have been published [6].

Along these lines, it is of interest to note that, in 1914, Bateman realized that a complex 3-dimensional vector, $\mathbf{M} = \mathbf{B} \pm i\sqrt{\varepsilon\mu}\mathbf{E}$ could be used to express both the Maxwell Faraday and the Maxwell Ampere equations for the Lorentz vacuum as one combined set of complex vector equations. Bateman determined that it is possible to find a conjugate pair of solutions $\mathbf{M}$ and $\mathbf{M}'$ that satisfy the complex equation

$$\mathbf{M} \circ \mathbf{M}'{=}0. \tag{3.1}$$

Each solution satisfies the equation

$$\mathbf{M} \circ \mathbf{M} = (\mathbf{B}^2 - \varepsilon\mu\mathbf{E}^2) \pm 2i(\sqrt{\varepsilon\mu}\mathbf{E} \circ \mathbf{B}) = \mathbf{I}_1 \pm 2i\,\mathbf{I}_2, \tag{3.2}$$

where $\mathbf{I}_1$ and $\mathbf{I}_2$ are the Poincare conformal invariants of the field, $\mathbf{M}$.

If the complex solution vector satisfies the complex equation of constraint,

$$\mathbf{M} \circ \mathbf{M} = (\mathbf{B}^2 - \varepsilon\mu\mathbf{E}^2) + 2i(\sqrt{\varepsilon\mu}\mathbf{E} \circ \mathbf{B}) = 0, \tag{3.3}$$

then such a vector not only satisfies both the Maxwell Faraday and the Maxwell Ampere (source free) equations for a Lorentz vacuum, but also - according to the derived result in equation ref6 - propagates the field energy with the speed of light. Such solutions were defined by Bateman as self conjugate solutions. (Translate to self dual solutions in modern day language.) The self dual equation of constraint also leads to the Clifford algebras, and therefor indicates that the Bateman solutions can have spinor representations, as well as complex number representations.

The Bateman self conjugate condition requires that the (complex) magnetic energy density be the same as the (complex) electric energy density, and the (complex) Electric field be orthogonal to the (complex) Magnetic field, $\mathbf{E} \circ \mathbf{B} = 0$. Both of these Poincare conformal invariants must be zero to satisfy the Bateman self duality condition. It is the self dual solutions, these self conjugate solutions, that satisfy the Eikonal expression, and therefore, as Bateman points out, can represent propagating electromagnetic discontinuities [7]. The Poincare invariants are additive, such that it is conceivable to construct a self-conjugate solution from two or more non-self conjugate solutions, each of which has different Poincare invariants, but which are equal to zero under addition.

Bateman apparently did not notice that the complex constraint equation of self duality on $\mathbf{M}$ is precisely the conditions that the complex position vector generated by $\mathbf{M}$ defines a minimal surface [8]. Moreover, Bateman did not notice that most of his results are to be obtained also for a Chiral vacuum. The details of the Chiral Vacuum condition are explored in the the next section. The application to minimal surfaces will appear elsewhere.

4. THE CHIRAL VACUUM

Use the (complex) Chiral Vacuum constitutive equations in the format of Post,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + [\gamma] \circ \mathbf{B} \qquad \mathbf{H} = -[\gamma^\dagger] \circ \mathbf{E} + \mathbf{B}/\mu_0 \ , \tag{4.1}$$

along with the Maxwell Faraday equations and the Maxwell Ampere equations, and replicate the steps of the preceding section. For simplicity, assume that the matrix

$$[\gamma] = (\, g + \sqrt{-1}\gamma)\sqrt{\mu/\varepsilon}\,[1] \tag{4.2}$$

and

$$[\gamma^\dagger] = (\alpha \cdot g - \sqrt{-1}\beta \cdot \gamma)\sqrt{\mu/\varepsilon}\,[1] \tag{4.3}$$

where $\alpha, \beta = \pm 1$. Note that if $\alpha = +1, \beta = +1$, then $[\gamma^\dagger]$ is the Hermitean conjugate of $[\gamma]$. If $\alpha = 1, \beta = -1$, then the imaginary part of $[\gamma]$ is anti-Hermitean. The Fresnel-Kummer wave surface equation for the characteristic of the Maxwell equations may be written as the polynomial,

$$\{R^4 + 1 - [2 - g^2(1-\alpha)^2 + \gamma^2(1+\beta)^2]R^2\} - i2\{g\gamma(1-\alpha)(1+\beta)\} = 0, \tag{4.4}$$

where $R^2 = {n_x}^2 + {n_y}^2 + {n_z}^2 = \mathbf{n} \circ \mathbf{n}$ represents the norm of the projectivized wave vector (index of refraction vector), $\mathbf{n} = \mathbf{k}/\omega$. Solutions of the characteristic polynomial yield the phase velocities of propagation in terms of the magnitude of the reciprocal index of refraction vector, $\mathbf{n}$. The phase velocity solutions are isotropic and homogeneous constants, determined by the root of the characteristic polynomial. The phase velocity is complex unless $\alpha = +1$,or the numeric factors are zero, e.g., $g = 0$ or $\gamma = 0$. For this reason, the case of $\alpha = -1$ is ignored in this article.

If the Hermitean conjugate constraints are used, $\alpha = 1$, and $\beta = 1$, then the phase velocity is determined from the formula for the (homogeneous, isotropic) index of refraction,

$$n = \pm\gamma \pm \sqrt{\gamma^2 + 1}. \tag{4.5}$$

For finite γ any g, there is a time-like dispersion of two helical waves. These chiral waves have phase velocities a bit greater and a bit less that the velocity of light

$c = \sqrt{1/\varepsilon\mu}$, as determined by the chiral factor γ, and these phase velocities are independent of the chiral factor g.

If the constraints $\alpha = 1$,and $\beta = -1$ are used, then the phase velocities are those of the Lorentz Vacuum, ($n = 1$), for any value of chiral factors, g and/or γ. The fundamental result is that the Chiral Vacuum and the Lorentz Vacuum are almost indistinguishable.

For the case $\alpha = 1$,and $\beta = 1$, the determinant of the constitutive matrix is real and equal to

$$\det[Constitutive] = -(\varepsilon/\mu + g^2 + \gamma^2)^3, \tag{4.6}$$

a value which is proportional to the reciprocal of the free space impedance cubed. For $\gamma = 0$, the only difference between the Chiral Vacuum and the Lorentz Vacuum would be in the value of the free space impedance, $Z = \sqrt{1/(\varepsilon/u + g^2)}$. If $\gamma \neq 0$, then there could exist a slight dispersion (in time) between left handed and right handed polarization states.

For the case $\alpha = 1$,and $\beta = -1$, the determinant of the constitutive tensor is more complicated. The determinant has complex values (implying dissipation) unless either $\gamma = 0$, or $g = 0$. In each non-dissipative case,

$$Z = \sqrt{1/(\varepsilon/u + g^2)} \;\; for\ \gamma = 0, \quad Z = \sqrt{1/(\varepsilon/u - \gamma^2)} \;\; for\ g = 0, \qquad n = 1. \tag{4.7}$$

Reality constraints imply that all cases of interest to this article are such that $\alpha = 1$. Substitution of the constitutive equations into the Maxwell Ampere equation yields

$$\begin{aligned}
\mathbf{J} &= curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \{curl\mathbf{B} - \varepsilon\mu\partial\mathbf{E}/\partial t\}/\mu & (4.8)\\
&\quad +g(-curl\,E - \partial B/\partial t) + \sqrt{-1}\gamma(\beta \cdot curl\,E - \partial B/\partial t) & (4.9)\\
\rho &= div\mathbf{D} = \varepsilon div\mathbf{E} + (g + \sqrt{-1}\gamma)(div\,\mathbf{B}) & (4.10)
\end{aligned}$$

The point of this exercise is to note that in virtue of the Maxwell Faraday equation, the Chiral Vacuum constitutive relations produce no real charge currents or charge densities if $\beta = -1$, independent of the choice of chiral coefficients. The field intensities satisfy the vector wave equation with phase velocities that are those of the Lorentz Vacuum.

If $\beta = +1$ then only an imaginary current density is created for non-zero γ. It is then possible to compute the reactive power, $\mathbf{J} \circ \mathbf{E}$, and therefor a reactive impedance that depends upon γ. (It is tempting to identify the chiral coefficient with the reciprocal Hall impedance, $\gamma = e^2/h$). The field intensities then satisfy a wave equation with a phase velocity that depends upon γ.

In no case do the Chiral Vacuum constitutive equations yield a free charge density, if $div\mathbf{E} = 0$ and $div\mathbf{B} = 0$ - a result which is valid if the field intensities are derived from a set of potentials. A second point is that the chiral factors of the type, g, do not have any effect on the Lorentz Vacuum except to modify the Radiation Impedance, Z.

Similar substitutions of the Chiral constitutive equations lead to the Poynting equation in the form:

$$\begin{aligned} div(\mathbf{E}\times\mathbf{H}) + \mathbf{H}\circ\partial\mathbf{B}/\partial t + \mathbf{E}\circ\partial\mathbf{D}/\partial t &= \\ div(\mathbf{E}\times\mathbf{H}) + \partial(1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2)/\partial t &= \{(\alpha-1)g - \sqrt{-1}(\beta+1)\gamma\}\mathbf{E}\circ\partial\mathbf{B}/\partial t. \end{aligned} \tag{4.11}$$

If the RHS of the equation above vanishes, then the Poynting theorem of equation 2.8 is retrieved without change in form. For the choice $\alpha = +1, \beta = -1$, again there are no differences between the Chiral Vacuum and the Lorentz Vacuum, for any value of the chiral factors. For the choice $\alpha = +1, \beta = +1$, the equation implies a chiral (imaginary or reactive) component to the Poynting equation, related to the time-like dispersion of the left handed and right handed helical waves. This term vanishes for $\gamma = 0$, and is independent from g.

The next step is to evaluate the expressions for the total field Hamiltonian energy density and the Lagrange density of the Chiral Vacuum. The expression for the Hamiltonian energy density becomes

$$\begin{aligned} Ham &= (1/2)(\mathbf{D}\circ\mathbf{E}) + (1/2)(\mathbf{B}\circ\mathbf{H}) = \\ &\quad 1/2\mathbf{B}^2/\mu + 1/2\varepsilon\mathbf{E}^2 + \{(\alpha-1)g + \sqrt{-1}(\beta+1)\gamma\}\mathbf{E}\circ\mathbf{B}/2 \end{aligned} \tag{4.12}$$

while the field Lagrangian is becomes:

$$\begin{aligned} Lag &= (1/2)(\mathbf{D}\circ\mathbf{E}) - (1/2)(\mathbf{B}\circ\mathbf{H}) = \\ &\quad 1/2\varepsilon\mathbf{E}^2 - 1/2\mathbf{B}^2/\mu + \{(\alpha+1)g + \sqrt{-1}(1-\beta)\gamma\}\mathbf{E}\circ\mathbf{B}/2 \end{aligned} \tag{4.13}$$

These results indicate that there are slight modifications to the energy density formulas, modifications that are dependent upon the second Poincare invariant.

However, for systems where the field intensities are deducible from a 1-form of potentials, and the 1-form is of Pfaff dimension 3 or less, then $\mathbf{E} \circ \mathbf{B}$ vanishes, and all computations of Hamiltonian or Lagrangian energy densities are identical for the Lorentz Vacuum, or for the Chiral vacuum. It is only for cases where the 1-form of potentials is of Pfaff dimension 4, such that $\mathbf{E} \circ \mathbf{B} \neq 0$, that the Chiral factors can make a difference in the expressions for Hamiltonian or Lagrangian energy density.

Again study the case $\alpha = 1$. Then the choice $\beta = -1$, implies that the Hamiltonian energy density is the same as the Lorentz Vacuum, but the Lagrangian depends upon the chiral factors. The choice $\beta = +1$, implies that the Lagrangian depends upon the chiral factor g and the Hamiltonian depends upon the chiral factor γ. All chiral effects on the energy densities disappear if $F\hat{}F = -2(\mathbf{E} \circ \mathbf{B})dx\hat{}dy\hat{}dz\hat{}dt = 0$.

> **These are a rather startling results for they demonstrate that the Lorentz vacuum and the Chiral vacuum can be formally indistinguishable, except for the impedance of free space (which is related to the determinant of the constitutive tensor and therefor to the chiral coefficients).**

5. SUMMARY

A set of constitutive equations that describe a Chiral Vacuum can be chosen to replicate most of the features of the Lorentz Vacuum, except for the Radiation Impedance, which depends upon the real and the imaginary coefficients of chirality.

6. REFERENCES

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