TOPOLOGY AND TOPOLOGICAL EVOLUTION OF CLASSICAL ELECTROMAGNETIC FIELDS AND CURRENTS.

R. M. Kiehn

Emeritus, Physics Dept., Univ. Houston
http://www.cartan.pair.com
first draft 06/01/91, Updated 06/19/2003

Abstract

The theory of classical electromagnetism can be put into correspondence with two topological constraints placed on the variety of independent variables $\{x, y, z, t\}$. The topological constraints are formulated in terms of the exterior differential systems, $F - dA = 0$, and $J - dG = 0$. These topological constraints imply that the domains of support for finite non-zero electromagnetic field intensities, and finite non-zero electromagnetic currents, in general, cannot be compact without boundary. The method emphasizes the physical importance of the potentials, for the two fundamental constraints lead to the independent concepts of topological torsion, $A\hat{}F$, and topological spin, $A\hat{}G$, with topological features that are explicitly dependent on the potentials, A. The exterior derivatives of these two 3-forms create the familiar Poincare invariants as 4-forms, whose closed integrals are evolutionary deformation invariants. The zero sets of each 3-form can be used to define the concepts of transverse magnetic and transverse electric waves on topological grounds. The direction fields of the 3-forms $A\hat{}F$ and $A\hat{}G$ can exhibit linking and separation into component domains. The possible evolution of these topological properties is studied with respect to classes of processes that can be defined in terms of singly parameterized vector fields. Non-zero values of the Poincare 4-forms are the source of topological change and non-equilibrium thermodynamics.

1. Introduction

In the language of exterior differential systems [1] it becomes evident that classical electromagnetism is equivalent to a set of topological constraints on a variety of independent variables. Certain integral properties of such an electromagnetic system are deformation invariants with respect to all continuous evolutionary processes that can be described by a singly parameterized vector field. These deformation (topological) invariants lead to the fundamental topological conservation laws described in the physical literature as the conservation of charge-current and the conservation of flux. Recall the definitions:

> **Definition 1:** A continuous process is a map from an initial state with a topology $T_{initial}$ into a final state with perhaps a different topology T_{final} such that the limit points of the initial state are permuted among the limit points of the final state. [2]
>
> **Definition 2:** A deformation invariant is an integral over a closed manifold, $\int \cdots \int_{closed} \omega$ such that the Lie derivative of the closed integral with respect to a singly parameterized vector field, βV^k, vanishes, for any choice of parametrization, β.

It is also important to recall that a given variety of independent variables can support more than one topology. In classical electromagnetism, experience indicates that there are topological concepts related to the concept of Field Intensities, $(\mathbf{E}, \mathbf{B})$, and forces, which are distinct from the topological concepts related to Field Excitations $(\mathbf{D}, \mathbf{H})$, and sources. The Field Intensities have functional components which transform as a covariant tensor, while the Field Excitations have components that transform as a tensor density. These distinctions are often masked by the imposition of a metric structure, or a limitation to volume preserving (often non-dissipative) evolutionary processes.

The idea of a deformation invariant comes from the Cartan concept of a tube of trajectories as applied to Hamiltonian mechanics. Consider a tubular domain of trajectories (a fiber bundle in modern lingo) defined by the direction field of an arbitrary (singly parametrized) vector field, V. Construct a closed curve C1 connecting arbitrary points on the trajectories (fibers) of this vector field. It is common in physical applications, but not necessary, to assume that the curve C1 consists of isochronous points (defined as the set of points that had a common set of base points, or initial conditions). Consider an arbitrary deformation of the points that make up this curve C1 (integration chain) onto a new set of points, C2, with the only restriction being that a point on a given trajectory stays on the same trajectory. The deformation of the curve C1 in to the curve C2 can be accomplished naturally, in the sense of a hydrodynamic convection of points down the flow lines, or abstractly, by reparametrizing the original vector field. Note that deformation by reparametrization does not change the direction field, but only how "fast" the points flow along specific flow lines. Also note that the tube of trajectories need not be solid, and can contain interior domains for which the flow is null, (or for which there is no trajectory from the base points of intial conditions). Next evaluate the line integral of some Action 1-form over C1 and also over C2. If the values of the two integrals are the same, the integral is defined as a deformable evolutionary invariant. The question is, given a specific 1-form, A, what are the direction fields that leave the closed integrals as deformation invariants.

Cartan studied this problem on the state space of variables, $\{p, q, t\}$, for a 1-form of Action defined as, $A(p, q, t) = pdq - H(p, q, t)dt$. He showed that closed integral of this 1-form of Action, $\oint_C A$, is a deformation (topological) invariant with respect to a uniquely defined direction field. This unique vector field is defined as an extremal field, and generates the classic Hamiltonian flow that conserves energy. [3] Cartan's proof is not restricted to state space, but instead applies to any 1-form of Action whose Pfaff dimension, or class, is odd. Such Action 1-forms always admit a Hamiltonian representation for the evolutionary vector field. The odd Pfaff dimension produces a contact manifold.

However, for arbitrary physical systems that can be defined by a 1-form of Action, A:

Theorem 1. The closed integral of the derived 2-form $F = dA$ is a evolutionary deformation invariant with respect to *all* continuous processes that can be defined by a singly parameterized vector field.

Proof: Using Cartan's Magic formula [4]

$$L_{(\beta\mathbf{V})} \iint_{closed} F = \iint_{closed} i(\beta\mathbf{V})dF + \iint_{closed} di(\beta\mathbf{V})F = 0 + 0 \qquad (1.1)$$

The integration domain is, in this case, a two dimensional closed two surface (which need not be a boundary). This concept is at the basis of the Helmholtz theorems in hydrodynamics, and the conservation of flux in classical electromagnetism. A certain subclass of all vector fields (defined as symplectomorphisms) can be determined which will leave the flux integrals $\iint F$ invariant even if the integration domain is an open 2-surface.

The necessary condition that a 2-form be an evolutionary deformation invariant for all continuous processes is that the 2-form be closed, $dF = 0$. This requirement is satisfied by the constraint of the exterior differential system, $F - dA = 0$, and c2 differentiability. In the case of electromagnetism, the 2-form F is said to be exact. Exact 2-forms, in general, do not have domains of support that are compact without boundary. The domains of support for an exact 2-form are either open (and extend to infinity) or are compact with boundary. Other physical investigations, and many mathematical developments, are based upon the assumption that there exists a closed but not exact 2-form, for which the concept of a compact surface without boundary is without paradox. A classic example is the compact Riemann surface. Almost all compact domains of support for the Field Intensities, F, are excluded by the exterior differential system, $F - dA = 0$. On the otherhand, the domain of support for the Field Excitations, G,can be compact without boundary.

1.1. The Postulate of Potentials

Herein, the assumption that classical electromagnetic systems are defined by the topological constraint that the 2-form F is exact will be called the *Postulate of Potentials*. It is an essential point of departure form other theoretical developments, for physical meaning is associated with topological equivalence classes of potentials. When written as the equation, $F - dA = 0$, the postulate of potentials is to be recognized as an exterior differential system constraining the topology of the independent variables. What is remarkable about the choice that F is exact, is that the Poincare lemma, $ddA = dF = 0$, implies that the Maxwell-Faraday equations for the first four variables form a nested set of partial differential equations that are the exactly the same no matter what the dimension ($\geqq 4$) of the independent variables. The Maxwell-Faraday equations are universal and applicable for all physical systems that support a 1 form of Action.

The proof that the (2 dimensional) domain of finite support for F (and hence the **E** and **B** fields) can not, in general, be compact without boundary follows from Stokes theorem, $\oint A = \iint F$. Suppose the compact two surface consists of isochronous points. On the RHS, the domain of support requires that the coefficients of the two dimensional integrand can not vanish. On the LHS the integral over a boundary (which is empty for a compact surface without boundary) vanishes. Hence there is a contradiction unless the components of the 2-form (in this case the $\mathbf{B}$ field) are tangential to the surface, and are without singularities. For such surfaces, the Euler characteristic must vanish, and it is known that the only two exceptions are the Torus and the Klein-Bottle. However, these situations require the additional topological constraint that $F\hat{}F = 0$. For an electromagnetic action, the exceptional compact without boundary cases can only exist if $\mathbf{E} \circ \mathbf{B} = 0$. The resulting statement is that there do not exist compact domains of support without boundary when $\mathbf{E} \circ \mathbf{B} \neq 0,$ a statement that will be of interest to thermodynamics of irreversible systems.

The concepts developed in this article subsume that the potentials, A, have physical meaning in a topological sense of equivalence classes. In fact, topological evolution will be observed most often when the potentials evolve from one equivalence class to another. The theory is not strictly gauge invariant, for there are many A that satisfy the equation $F - dA = 0$. It is the closed components of A that are not exact that determine many of the topological, multiply connected, features of the electromagnetic system. Strict adherence to gauge invariance would not permit topological evolution, and the study of irreversible processes.

1.2. The Postulate of Conserved Currents

Herein, it is stipulated that the classic electromagnetic system requires a second topological constraint to be imposed upon the domain of independent variables. This postulate will be called the *Postulate of Conserved Currents.* The electromagnetic domain not only supports the 1-form A, but also supports an N-1=3 form, J, which is exact. The equivalent differential system, $J - dG = 0,$ requires that the (N-1 dimensional) domain of support for J cannot be compact without boundary. However,

Theorem 2: The closed integrals of J are deformation invariants for *any* continuous evolutionary process that can be defined in terms of a singly parameterized vector field.

Proof: Using Cartan's Magic formula:

$$L_{(\beta\mathbf{V})} \iint_{closed} J = \iint_{closed} i(\beta\mathbf{V})dJ + \iint_{closed} di(\beta\mathbf{V})F = 0 + 0 \qquad (1.2)$$

For the 3-forms of charge current, a similar argument indicates that the compact domains of support are limited to those of zero Euler characteristic. The classic example is the three sphere, S^3. The three sphere (that will support currents with out zeros) has a famous map to compact two sphere. Hence, there can exist domains of field excitations on compact two spheres, such that the induced current, $J = dG$, resides on the three sphere. The image is the Hopf map, which can have torsion. Such currents are in the direction of the torsion vector, $A\hat{}dA = A\hat{}F$, and have extraordinary properties, as will be shown below.

In section 2, the classical Maxwell system will be displayed in terms of the vector formalism of Sommerfeld and Stratton. The key feature is to note that the fields of intensities (**E** and **B**) are considered as separate and distinct from the fields of excitation (**D** and **H**), a historical distinction that is often masked in modern exposes of electromagnetic theory.

In section 3, it will be demonstrated explicitly that the classic formalism of electromagnetism in section 2 is a consequence of a system of two fundamental topological constraints

$$F - dA = 0, \qquad J - dG = 0. \qquad (1.3)$$

defined on a domain of four independent variables. The theory requires the existence of four fundamental exterior differential forms, $\{A, F, G, J\}$, which can be used to construct the complete Pfaff sequence [5] of forms by the processes of exterior differentiation and exterior multiplication. On a domain of four independent variables, the complete Pfaff sequence contains three 3-forms: the classic 3-form of charge current density, J, and the (apparently novel to many researchers) 3-forms of Spin Current density, $A\hat{}G$,[6] and Topological Torsion-Helicity, $A\hat{}F$ [7]. For an electromagnetic system, the Action 1-form has the physical dimensions of the flux quantum, h/e, the 2-form, G, has the physical dimensions of charge, e, the 3-form, $A\hat{}G$, has the physical dimensions of spin, h, and the 3-form $A\hat{}F$,has the physical dimensions of spin multiplied by the Hall impedance, $(h/e)^2 = h(h/e^2) = hZ_{hall}$. These last two 3-forms are explicitly dependent upon postulate of potentials, and demonstrate the physical significance of the vector and scalar potentials. These physical 3-form objects are not independent of "gauge".

As the charge current 3-form, J, is a deformation invariant by construction, it is of interest to determine topological refinements or constraints for which the 3-forms of Spin Current and Topological Torsion will define physical topological conservation laws in the form of deformation invariants. The additional constraints are equivalent to the topological statement that the closure (exterior derivative) of each of the three forms is empty (zero). It will be demonstrated in section 4 that these closure conditions define the two classic Poincare invariants (4-forms) as deformation invariants, and when each of these invariants vanish the corresponding 3-form generates a topological quantity (Spin or Torsion respectively) which is also a deformation invariant. The possible values of the topological quantities, as deRham period integrals [8], form rational ratios.

The concepts of Spin Current and the Torsion vector have had almost no utilization in applications of classical electromagnetic theory. Just as the vanishing of the 3-form of charge current, $J = 0$, defines the topological domain called the vacuum, the vanishing of the two other 3-forms will refine the fundamental topology of the Maxwell system. Such constraints permit a definition of transversality to be made on topological (rather than geometrical) grounds. If both $A\hat{}G$ and $A\hat{}F$ vanish, the vacuum state supports topologically transverse modes only (TTEM). Examples lead to the conjecture that TTEM modes do not transmit power, a conjecture that has been verified when the concept of geometric transversality (TEM) and topological transversality (TTEM) coincide. A topologically transverse magnetic (TTM) mode corresponds to the topological constraint that $A\hat{}F = 0$. A topologically transverse electric mode (TTE) corresponds to the topological constraint that $A\hat{}G = 0$. Examples, both novel and well-known, of vacuum solutions to the electromagnetic system which satisfy (and which do not satisfy) these topological constraints are given in section 4. The ideas should be of interest to those working in the field of Fiber Optics and wave guides with open boundary conditions. Recall that classic solutions which are geometrically and topologically transverse (TEM$\equiv$TTEM) do not transmit power [9]. However, in section 4 an example vacuum wave solution is given which is geometrically transverse (the fields are orthogonal to the field momentum and the wave vector), and yet the geometrically transverse wave transmits power at a constant rate: the example wave is not topologically transverse as $A\hat{}F \neq 0$.

In section 5, an additional topological constraint will be used to define the plasma process as a restriction on all processes which can be described in terms of a singly parameterized vector field. The plasma process (which is to be distinguished from a Hamiltonian process) will be restricted to those vector fields which leave the closed integrals of G a deformation invariant. (Compare to the Cartan definition that a Hamiltonian process is a restriction on arbitrary processes such that the closed integrals of A are deformation invariants with respect to Hamiltonian processes). A plasma process need not conserve energy. A *perfect* plasma process is a plasma process which is also a Hamiltonian process. Again, the three forms, J, $A\hat{}G$ and $A\hat{}F$ are of particular interested for their tangent manifolds define "lines" in the 4-dimensional variety of space and time. Relative to plasma processes, the topological evolution associated with such lines, and their entanglements, is of utility in understanding solar corona and plasma instability. [10]

2. The Domain of Classical Electromagnetism

2.1. The classical Maxwell-Faraday and the Maxwell-Ampere equations.

Using the notation and the language of Sommerfeld and Stratton [11], the classic definition of an electromagnetic system is a domain of space-time $\{x, y, z, t\}$ which supports both the Maxwell-Faraday equations,

$$curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\ \mathbf{B} = 0, \tag{2.1}$$

and the Maxwell-Ampere equations,

$$curl\ \mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}, \qquad div\ \mathbf{D} = \rho. \tag{2.2}$$

2.2. The conservation of charge current

In every case, the charge current density for the Maxwell system satisfies the conservation law,

$$div\,\mathbf{J} + \partial\rho/\partial t = 0. \tag{2.3}$$

The charge-current densities are subsumed to be zero $[\mathbf{J},\ \rho] = 0$ for the vacuum state.

For the Lorentz vacuum state, the field excitations, $\mathbf{D}$ and $\mathbf{H}$, are linearly connected to the field intensities, $\mathbf{E}$ and $\mathbf{B}$, by means of the Lorentz (homogeneous and isotropic) constitutive relations:

$$\mathbf{D} = \varepsilon\mathbf{E}\,, \quad \mathbf{B} = \mu\mathbf{H}. \tag{2.4}$$

The two vacuum constraints imply that the solutions to the homogeneous Maxwell equations also satisfy the vector wave equation, typically of the form

$$grad\ div\ \mathbf{B} - curl\, curl\, \mathbf{B} - \varepsilon\mu\partial^2\mathbf{B}/\partial t^2 = 0. \tag{2.5}$$

The constant wave phase velocity, v_p, is taken to be

$$v_p^2 = 1/\varepsilon\mu \equiv c^2 \tag{2.6}$$

Similar results can be obtained for the solid state where the constitutive constraints can be more complex [12], and for the plasma state where the charge-current densities are not zero.

2.3. The existence of potentials

It is further subsumed that the classic Maxwell electromagnetic system is constrained by the statement that the field intensities are deducible from a system of twice differentiable potentials, $[\mathbf{A}, \phi]$:

$$\mathbf{B} = curl\ \mathbf{A}, \qquad \mathbf{E} = -grad\ \phi - \partial\mathbf{A}/\partial t. \tag{2.7}$$

This constraint topologically implies that domains that support non-zero values for the covariant field intensities, $\mathbf{E}$ and $\mathbf{B}$, can *not* be compact domains without a boundary. It is this constraint that distinguishes classical electromagnetism from Yang Mills theories. Two other classical 3-vector fields are of interest, the Poynting vector $\mathbf{E} \times \mathbf{H}$ representing the flux of electromagnetic radiative energy, and the field momentum flux, $\mathbf{D} \times \mathbf{B}$.

3. The Fundamental Exterior Differential Systems.

The formulation of Maxwell theory in section 2 is relative to a choice of independent variables $\{x, y, z, t\}$ using classical vector analysis developed in euclidean 3-space. The topological features of the formalism are not immediately evident. However, electromagnetism has a formulation in terms of Cartan's exterior differential forms [13]. Exterior differential forms do not depend upon a choice of coordinates, do not depend upon the a choice of metric, and are independent of the constraints imposed by gauge groups and connections. In such a formulation the equations of an electromagnetic system become recognized as consequences of topological constraints on a domain of independent variables.

The use of differential forms should not be viewed as just another formalism of fancy. The technique goes beyond the methods of tensor calculus, and admits the study of topological evolution. Recall that if an exterior differential system is valid on a final variety of independent variables {x,y,z,t}, then it is also true on any initial variety of independent variables that can be mapped onto {x,y,z,t}. The map need only be differentiable, such that the Jacobian matrix elements are well defined *functions*. The Jacobian matrix does not have to have an inverse, so that the exterior differential system is not restricted to the equivalence class of diffeomorphisms. The field intensities on the initial variety are functionally well defined by the pullback mechanism, which involves algebraic composition with components of the Jacobian matrix transpose, and the process of functional substitution. This independence from a choice of independent variables (or coordinates) for Maxwell's equations was first reported by Van Dantzig [14]. It follows that the Maxwell differential system is well defined in a covariant manner for both Galilean transformations as well as Lorentz transformations, or any other diffeomorphism. (The singular solution sets to the equations do not enjoy this universal property). In addition, it should be noted that the ideas of the exterior differential system imply that the closure equations of the Maxwell-Faraday type form a nested set, with exactly the same format, independent of the choice of the *number* of independent variables. Every physical system (such as fluid) that supports a 1-form of Action, also has its version of the Maxwell-Faraday equations.

3.1. The Maxwell-Faraday exterior differential system.

The Maxwell-Faraday equations are a consequence of the exterior differential system

$$F - dA = 0, \tag{3.1}$$

where A is a 1-form of Action, with twice differentiable coefficients (potentials proportional to momenta) which induce a 2-form, F, of electromagnetic intensities ($\mathbf{E}$ and $\mathbf{B}$, related to forces). The exterior differential system is a topological constraint that in effect defines field intensities in terms of the potentials. On a four dimensional space-time of independent variables, (x, y, z, t) the 1-form of Action (representing the postulate of potentials) can be written in the form

$$A = \Sigma_{k=1}^{3} A_k(x, y, z, t)dx^k - \phi(x, y, z, t)dt = \mathbf{A}\circ d\mathbf{r} - \phi dt. \tag{3.2}$$

Subject to the constraint of the exterior differential system, the 2-form of field intensities, F, becomes:

$$\begin{aligned} F &= dA = \{\partial A_k/\partial x^j - \partial A_j/\partial x^k\}dx^j\hat{}dx^k = F_{jk}dx^j\hat{}dx^k \\ &= +\mathbf{B}_z dx\hat{}dy... + \mathbf{E}_x dx\hat{}dt... \end{aligned} \tag{3.3}$$

where in usual engineering notation,

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\phi, \qquad \mathbf{B} = curl\ \mathbf{A} \equiv \partial A_k/\partial x^j - \partial A_j/\partial x^k. \tag{3.4}$$

The closure of the exterior differential system, $dF = 0$,

$$dF = ddA = \{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t\}_x dy\hat{}dz\hat{}dt - .. + .. - div\,\mathbf{B}dx\hat{}dy\hat{}dz\} \Rightarrow 0, \tag{3.5}$$

generates the Maxwell-Faraday partial differential equations.:

$$\{curl\ \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \qquad div\,\mathbf{B} = 0\}. \tag{3.6}$$

The component functions ($\mathbf{E}$ and $\mathbf{B}$) of the 2-form, F, transform as covariant tensor of rank 2. The topological constraint that F is exact, implies that the domain of support for the field intensities cannot be compact without boundary, unless the Euler characteristic vanishes. These facts distinguish classical electromagnetism from Yang-Mills field theories. Moreover, the fact that F is subsumed to be exact and C1 differentiable excludes the concept of magnetic monopoles from classical electromagnetic theory on topological grounds. By theorem 1, the integral of the 2-form F over any closed 2-manifold is a deformation (topological) invariant of any evolutionary process that can be described by a singly parameterized vector field.

3.2. The Maxwell Ampere exterior differential system

The Maxwell Ampere equations are a consequence of second exterior differential system,

$$J - dG = 0, \tag{3.7}$$

where G is an N-2 form *density* of field excitations ($\mathbf{D}$ and $\mathbf{H}$, related to sources), and J is the N-1 form of charge-current densities. The partial differential equations equivalent to the exterior differential system are precisely the Maxwell-Ampere equations. This second postulate, on a four dimensional domain of independent variables, assumes the existence of a N-2 form density given by the expression,[1]

$$G = G^{34}(x,y,z,t)dx\hat{}dy... + G^{12}(x,y,z,t)dz\hat{}dt... = -\mathbf{D}^z dx\hat{}dy...\mathbf{H}^z dz\hat{}dt... \tag{3.8}$$

Exterior differentiation produces an N-1 form,

$$J = \mathbf{J}^z(x,y,z,t)dx\hat{}dy\hat{}dt... - \rho(x,y,z,t)dx\hat{}dy\hat{}dz. \tag{3.9}$$

Matching the coefficients of the exterior expression $dG = J$ leads to the Maxwell-Ampere equations,

$$curl\,\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J} \quad and \quad div\,\mathbf{D} = \rho. \tag{3.10}$$

The fact that J is exact leads to the charge conservation law, $dJ = ddG = 0$, or

$$\partial\mathbf{J}^x/\partial x + \partial\mathbf{J}^y/\partial y + \partial\mathbf{J}^z/\partial z + \partial\rho/\partial t = 0. \tag{3.11}$$

The exterior differential system is a topological constraint for by Stokes theorem the support for G can be compact without boundary only if the domain is without charge-currents. The closure of the exterior differential system, $dJ = 0$, generates the charge-current conservation law. By theorem 2, the integral of J over a closed 3 dimensional domain is a relative integral invariant (a deformation invariant) of any process that can be described in terms of a singly parametrized vector field. The Lie derivative of $\iiint_{closed} J$ is equal to zero for any 4-vector field V, when $dJ = 0$. The integral is then a deformation invariant, for the result is valid

[1] Minus sign on D june 19 2003

even if the 4-vector field is distorted by an arbitrary function, $\beta\{x, y, z, t\}$, such that $\mathbf{V} \Rightarrow \beta(x, y, z, t)\mathbf{V}$.

3.3. The Torsion and Spin 3-forms

As mentioned above, the method of exterior differential forms goes beyond the domain of classical tensor analysis, for it admits of maps from initial to final state that are without inverse. (Tensor analysis and coordinate transformations require that the Jacobian map from initial to final state has an inverse - the method of exterior differential forms does not.) Hence the theory of electromagnetism expressed in the language of exterior differential forms admits of topological evolution, at least with respect to continuous processes without Jacobian inverse. With respect to such non-invertible maps, both tensor fields and differential forms are not functionally well defined in a predictive sense [15]. Given the functional forms of a tensor field on an initial state, it is impossible to predict uniquely the functional form of the tensor field on the final state unless the map between initial and final state is invertible. However differential forms are functionally well defined in a retrodictive sense, by means of the pullback. Covariant anti-symmetric tensor fields pull back retrodictively with respect to the transpose of the Jacobian matrix (of functions) and functional substitution, and contravariant tensor densities pullback retrodictively with respect to the adjoint of the Jacobian matrix, and functional substitution. The transpose and the adjoint of the Jacobian exist, even if the Jacobian inverse does not.

The exterior differential forms that make up the electromagnetic system consist of the primitive 1-form, A, and the primitive N-2 form density, G, their exterior derivatives, and their algebraic intersections defined by all possible exterior products. The complete Maxwell system of exterior differential forms (the Pfaff sequence for the Maxwell system) is given by the set:

$$\{A, F = dA, G, J = dG, A\hat{}F, A\hat{}G, A\hat{}J, F\hat{}F, G\hat{}G\}. \tag{3.12}$$

These forms and their unions may be used to form a topological base on the domain of independent variables. The Cartan topology constructed on this system of forms has the useful feature that the exterior derivative may be interpreted as a limit point, or closure, operator in the sense of Kuratowski [16]. The exterior differential systems that define the Maxwell-Ampere and the Maxwell-Faraday equations above are essentially topological constraints of closure. Note that the complete Maxwell system of differential forms (which assumes the existence of A)

also generates two other exterior differential systems:

$$d(A\hat{}G) - (F\hat{}G - A\hat{}J) = 0, \tag{3.13}$$

and

$$d(A\hat{}F) - F\hat{}F = 0. \tag{3.14}$$

The two objects, $A\hat{}G$ and $A\hat{}F$ are three forms, not usually found in discussions of classical electromagnetism. The closed components of the first 3-form (density) were called topological spin [17] and the closed components of the second 3-form were called topological torsion (or helicity) [18]. By direct evaluation of the exterior product, and on a domain of 4 independent variables, each 3-form will have 4 components that can be symbolized by the 4-vector arrays

$$Spin - Current : \mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}] \equiv [\mathbf{S}, \sigma], \tag{3.15}$$

and [2]

$$Torsion - vector : \mathbf{T}_4 = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] \equiv [\mathbf{T}, h], \tag{3.16}$$

which are to be compared with the charge current 4-vector density:

$$Ch\arg e - Current : \mathbf{J}_4 = [\mathbf{J}, \rho], \tag{3.17}$$

The 3-forms then can be defined by the equivalent contraction processes

$$\begin{aligned} & Topological\ \ Spin\ \ 3 - form \doteq\ A\hat{}G \\ = \ & i(\mathbf{S}_4)dx\hat{}dy\hat{}dz\hat{}dt = +\mathbf{S}^x dy\hat{}dz\hat{}dt..... - \sigma dx\hat{}dy\hat{}dz \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} & Topological\ Torsion - helicity\ \ 3 - form \doteq A\hat{}F \\ = \ & i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = \ +\mathbf{T}^x dy\hat{}dz\hat{}dt..... - hdx\hat{}dy\hat{}dz. \end{aligned} \tag{3.19}$$

The vanishing of the first 3-form is a topological constraint on the domain that defines topologically transverse electric (TTE) waves: the vector potential, $\mathbf{A}$, is

[2] Minus sign on T june 19,2003

orthogonal to $\mathbf{D}$, in the sense that $\mathbf{A} \circ \mathbf{D} = 0$. The vanishing of the second 3-form is a topological constraint on the domain that defines topologically transverse magnetic (TTM) waves: the vector potential, $\mathbf{A}$, is orthogonal to $\mathbf{B}$,in the sense that $\mathbf{A} \circ \mathbf{B} = 0$. When both 3-forms vanish, the topological constraint on the domain defines topologically transverse (TTEM) waves. For classic real fields this double constraint would require that vector potential, $\mathbf{A}$, is collinear with the field momentum, $\mathbf{D} \times \mathbf{B}$, and in the direction of the wave vector, $\mathbf{k}$.

The geometric notion of distinct transversality modes of electromagnetic waves is a well known concept experimentally, but the association of transversality to topological issues is novel herein. For certain examples that appear in section 4, it is apparent that the concept of geometric and topological transversality are the same. In the classic case of waveguides, with open boundary conditions as considered in fiber optic theory, it is known that certain TEM modes do not transmit power. In such cases the geometrical defintion and the topological definition coincide TEM≃TTEM. However, in section 4 a vacuum wave solution is given which satisfies the geometric concept of transversality (it is both a TM and a TE solution) but the mode radiates for it is not both a TTM and a TTE solution. The conjecture obtained from examples is that a TTEM solution does not radiate.

Note that if the 2-form F was not exact, such topological concepts of transversality would be without meaning, for the 3-forms of Topological Spin and Topological Torsion depend upon the existence of the 1-form of Action. The torsion vector $\mathbf{T}_4$ and the Spin vector $\mathbf{S}_4$ are associated vectors to the 1-form of Action in the sense that

$$i(\mathbf{T}_4)A = 0 \quad and \quad i(\mathbf{S}_4)A = 0 \tag{3.20}$$

3.4. The Poincare Deformation Invariants

The exterior derivatives of the 3-forms of Spin and Torsion produce two 4-forms, $F\hat{}G - A\hat{}J$ and $F\hat{}F$, whose closed integrals over closed 4 dimensional domains are deformation invariants for the Maxwell system. (The deformation invariance follows from Cartan's Magic formula and the fact that the 4-forms are exact). These topological objects are related to the conformal invariants of a Lorentz system as discovered by Poincare and Bateman. Note that their topological properties are valid even in the domain of dissipative charge currents and radiation.

In the format of independent variables $\{x, y, z, t\}$, the exterior derivative corresponds to the 4-divergence of the 4-component Spin and Torsion vectors, $\mathbf{S}_4$ and $\mathbf{T}_4$. The functions so created define the Poincare conformal invariants of the Maxwell system [3]:

$$\begin{aligned} Poincare\ 1 &= d(A\hat{}G) = F\hat{}G - A\hat{}J && (3.21) \\ &= \{div_3(\mathbf{A}\times\mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A}\circ\mathbf{D})/\partial t\}\Omega_4 && (3.22) \\ &= \{(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) - (\mathbf{A}\circ\mathbf{J} - \rho\phi)\}\Omega_4 && (3.23) \end{aligned}$$

$$\begin{aligned} Poincare\ 2 &= d(A\hat{}F) = F\hat{}F && (3.24) \\ &= -\{div_3(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A}\circ\mathbf{B})/\partial t\}\Omega_4 && (3.25) \\ &= \{+2\mathbf{E}\circ\mathbf{B}\}\Omega_4 && (3.26) \end{aligned}$$

For the vacuum state, with $J = 0$, zero values of the Poincare invariants require that the magnetic energy density is equal to the electric energy density ($1/2\mathbf{B}\circ\mathbf{H} = 1/2\mathbf{D}\circ\mathbf{E}$), and, respectively, that the electric field is orthogonal to the magnetic field ($\mathbf{E}\circ\mathbf{B} = 0$). Note that these constraints often are used as elementary textbook definitions of what is meant by electromagnetic waves. When either Poincare invariant vanishes, the corresponding closed 3-dimensional integral becomes a topological quantity in the sense of a deRham period integral.

Definition 3: Spin is defined as the closed integral of the 3-form $A\hat{}G$

$$Spin = \iiint_{closed} A\hat{}G \qquad (3.27)$$

Theorem 3: If the First Poincare Invariant vanishes, the Spin is an evolutionary deformation invariant with values whose ratios are rational.

[3] signs on T4 and div as of june 19,2003

Proof: Using Cartan's magic formula and the constraint that $d(A\hat{}G)=0$

$$\begin{aligned} L_{\beta\mathbf{V}}\,(Spin) &= \iiint_{closed} \{i(\beta\mathbf{V})d(A\hat{}G)+d(i(\beta\mathbf{V})(A\hat{}G)\} \quad (3.28) \\ &= \iiint_{closed} \{0+d(i(\beta\mathbf{V})(A\hat{}G)\} = 0. \end{aligned}$$

The quantized (integer) ratios comes from the deRham cohomology theorems on closed integrals of closed p-forms.

Similarly, when the second Poincare invariant vanishes, the closed integral of the 3-form of Torsion-Helicity becomes a deformation invariant with quantized values:

Definition 4: Chirality is defined as the closed integral of the 3-form $A\hat{}F$

$$Chirality = \iiint_{closed} A\hat{}F \tag{3.29}$$

Theorem 4. If the second Poincare Invariant vanishes, the Torsion Helicity is an evolutionary deformation invariant with values whose ratios are rational.

Proof: Using Cartan's magic formula and the constraint that $d(A\hat{}F)=0$

$$\begin{aligned} L_{\beta\mathbf{V}}\,(Chirality) &= \iiint_{closed} \{i(\beta\mathbf{V})d(A\hat{}F)+d(i(\beta\mathbf{V})(A\hat{}F)\} \quad (3.30) \\ &= \iiint_{closed} \{0+d(i(\beta\mathbf{V})(A\hat{}F)\} = 0. \end{aligned}$$

The quantized (integer) ratios comes from the deRham cohomology theorems on closed integrals of closed p-forms.

It is important to realize that these topological conservation laws are valid in a plasma as well as in the vacuum, subject to the conditions of zero values for the Poincare invariants. On the other hand, topological transitions between "quantized" states of Spin or Chirality require that the respective Poincare invariants are not zero.

4. Electromagnetic Waves in the Vacuum with Spin and Torsion

4.1. Solutions Old and New

As the Spin 4-vector and the Torsion 4-vector formalism may be unfamiliar to many readers, it is useful to compare four classes of unusual vacuum wave solutions with the usual waveguide solutions. The "unusual waves" have their vector potential, **A**, orthogonal to the wave vector, **k**, describing the direction of the wave front. In each unusual example, the current density is in the direction of the vector potential and therefore also orthogonal to the wave vector. The usual wave solutions have their vector potential parallel to the wave vector. The four unusual cases belong to equivalence classes defined by the constraints

$$\begin{aligned}
(A\hat{}F &= 0, A\hat{}G \neq 0) \\
(A\hat{}F &\neq 0, A\hat{}G = 0) \\
(A\hat{}F &= 0, A\hat{}G = 0) \\
(A\hat{}F &\neq 0, A\hat{}G \neq 0).
\end{aligned}$$

Each component of the potentials satisfies the wave equation subject to the dispersion relation, $\omega/k \pm \sqrt{1/\xi\mu} = 0$. The examples do not generate any charge current distributions when the vacuum dispersion equation is satisfied (the phase velocity equals to the group velocity equals the speed of light as determined by the constitutive equations). The choice of dispersion equation solution determines the direction of wave propagation.

In each example given below, the 1-form of Action is specified and the field intensities are computed. Then the Spin Current and the Torsion vector are evaluated. The functions have been chosen to satisfy the Lorentz vacuum conditions of zero charge current densities, subject to a dispersion relation. The Poynting vector is computed, and the Poincare invariants are evaluated.

The four classes of these simple (but unusual) wave types correspond to:

4.1.1. Example 1. Real Linear Polarization:

Consider the Potentials

$$A = [\cos(kz-\omega t), \cos(kz-\omega t), 0, 0] \tag{4.1}$$

and their induced fields:

$$\mathbf{E} = [-\sin(kz-\omega t), -\sin(kz-\omega t), 0]\omega$$

$$\mathbf{B} = [+\sin(kz-\omega t), -\sin(kz-\omega t), 0]k$$

$$\mathbf{J}_4 = [\cos(kz-\omega t), \cos(kz-\omega t), 0, 0](k^2 - \varepsilon\mu\omega^2)/\mu$$

$$\mathbf{S}_4 = [0, 0, -k/\mu, -\varepsilon\omega]\, 2\cos(kz-\omega t)\sin(kz-\omega t).$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\mathbf{E}\times\mathbf{H} \;= [0, 0, 1](\omega k/\mu)(2\cos(kz-\omega t)^2 - 1)$$

$$\begin{aligned}(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) &= -2\{\cos(kz-\omega t)^2 - \sin(kz-\omega t)^2\}(k^2 - \varepsilon\mu\omega^2)/\mu \\ (\mathbf{E}\circ\mathbf{B}) &= 0\end{aligned}$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. The Torsion vector vanishes indentically, independent from the dispersion condition, but the Spin vector does not. The first Poincare invariant vanishes subject to the constraint of the dispersion relation. The second Poincare invariant vanishes identically. The solution corresponds to a linear state of polarization at 45° with respect to the x-axis, with the electric and the magnetic fields in phase. There is a non-zero Poynting vector along the z axis., which is orthogonal to the vector potential. Note that the radiated power has a time average which is zero. If the charge current density is not zero (due to a fluctuation in the dispersion relation) the charge current vector is orthogonal to the Spin current vector.

4.1.2. Example 2. Real Circular Polarization:

Consider the Potentials

$$A = [\cos(kz - \omega t), \sin(kz - \omega t), 0, 0] \tag{4.2}$$

and their induced fields:

$$\mathbf{E} = [-\sin(kz - \omega t), +\cos(kz - \omega t), 0]\omega$$

$$\mathbf{B} = [-\cos(kz - \omega t), -\sin(kz - \omega t), 0]k$$

$$\mathbf{J}_4 = [\cos(kz - \omega t), \sin(kz - \omega t), 0, 0](k^2 - \varepsilon\mu\omega^2)/\mu$$

$$\mathbf{S}_4 = [0, 0, 0, 0].$$

$$\mathbf{T}_4 = [0, 0, -\omega, -k].$$

$$\mathbf{E} \times \mathbf{H} = [0, 0, 1]\,\omega\, k/\mu$$

$$\begin{aligned} (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) &= 0 \\ (\mathbf{E} \circ \mathbf{B}) &= 0 \end{aligned}$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. The Spin vector vanishes identically, but the Torsion vector does not. In fact, the torsion vector is constant. The solution corresponds to a circular state of polarization with the constant magnetic and electric amplitudes rotating about the z axis. The Poynting vector is not zero and is a constant, time independent, vector. This wave solution is geometrically transverse (TEM), yet it produces power as it is not topologically transverse (TTEM). If the dispersion relation is not precisely satisfied, the current vector is orthogonal to the Torsion vector and parallel to the vector potential. Both Poincare invariants vanish identically. The soliton like solution should be compared to the wave guide solution of example 5 below, which is also TEM, but does not radiate.

4.1.3. Example 3. Complex Linear Polarization:

Consider the Potentials

$$A = [\cos(kz - \omega t), i\cos(kz - \omega t), 0, 0] \tag{4.3}$$

and their induced fields:

$$\mathbf{E} = [-\sin(kz - \omega t), -i\sin(kz - \omega t), 0]\omega$$

$$\mathbf{B} = [+i\sin(kz - \omega t), -\sin(kz - \omega t), 0]k$$

$$\mathbf{J}_4 = [\cos(kz - \omega t), i\cos(kz - \omega t), 0, 0](k^2 - \varepsilon\mu\omega^2)/\mu$$

$$\mathbf{S}_4 = [0, 0, 0, 0].$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\mathbf{E} \times \mathbf{H} = [0, 0, 0]$$

$$\begin{aligned} (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) &= 0 \\ (\mathbf{E} \circ \mathbf{B}) &= 0 \end{aligned}$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. The fields are said to be complex linearly polarized because the complex **B** field is a complex scalar multiple of the complex **E** field. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. Note that both the Torsion vector and the Spin vector vanish identically. The complex square of both the electric and the magnetic field vectors vanish. Both Poincare invariants vanish independent from the dispersion constraint. Although the fields are propagating, there is no momentum flux and the Poynting vector is zero. The **E** and **B** fields are (complex) collinear. This example is perhaps the simplest member of the class of Bateman-Whittaker complex solutions described in Example 10, below.

4.1.4. Example 4. Complex Circular Polarization:

Consider the Potentials

$$A = [\cos(kz - \omega t), i\sin(kz - \omega t), 0, 0] \tag{4.4}$$

and their induced fields:

$$\mathbf{E} = [-\sin(kz - \omega t), +i\cos(kz - \omega t), 0]\omega$$

$$\mathbf{B} = [-i\cos(kz - \omega t), -\sin(kz - \omega t), 0]k$$

$$\mathbf{J}_4 = [\cos(kz - \omega t), i\sin(kz - \omega t), 0, 0](k^2 - \varepsilon\mu\omega^2)/\mu$$

$$\mathbf{S}_4 = [0, 0, -k/\mu, -\varepsilon\omega]\, 2\cos(kz - \omega t)\sin(kz - \omega t).$$

$$\mathbf{T}_4 = i[0, 0, -\omega, -k].$$

$$\mathbf{E} \times \mathbf{H} \ = [0, 0, -1]\, (\omega\, k/\mu)(2\cos(kz - \omega t)^2 - 1)$$

$$\begin{aligned} (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) &= -2\{\cos(kz - \omega t)^2 - \sin(kz - \omega t)^2\}(k^2 - \varepsilon\mu\omega^2)/\mu \\ (\mathbf{E} \circ \mathbf{B}) &= 0 \end{aligned}$$

This class of potentials generates a set of complex field intensities and excitations, and a current density proportional to the vector potential. If the dispersion relation $(k^2 - \varepsilon\mu\omega^2) = 0$ is satisfied, then the solutions are acceptable vacuum solutions, with a vanishing charge current density. Both the Torsion vector (imaginary) and the Spin vector (real) do not vanish. The second Poincare invariant vanishes identically, and the first Poincare invariant vanishes subject to the dispersion constraint. The current vector, if non-zero due to fluctuations in the dispersion relation, is orthogonal to both the Torsion vector and the Spin vector.

Examples 1 through 4 above are geometrically transverse waves in the engineering sense that the propagation direction (along the z axis) is in the direction of the momentum flux, $\mathbf{D} \times \mathbf{B}$. However, the waves are not "topologically transverse" in that the sense that the $\mathbf{D}$ and $\mathbf{B}$ fields are not necessarily transverse to the components of the vector potential $\mathbf{A}$.

4.1.5. Example 5. Waveguide TEM modes

Consider the Potentials

$$A = [0, 0, \phi(x,y), (\omega/k)\phi(x,y)]\cos(kz - \omega t) \tag{4.5}$$

and their induced fields:

$$\mathbf{E} = [-(\omega/k)\partial\phi/\partial x, -(\omega/k)\partial\phi/\partial y, 0]\cos(kz - \omega t)$$

$$\mathbf{B} = [\partial\phi/\partial y, -\partial\phi/\partial x, 0]\cos(kz - \omega t)$$

$$\begin{aligned}\mathbf{J}_4 \quad = \quad & [\partial\phi/\partial x(\varepsilon\mu(\omega/k)^2 - 1)\sin(kz - \omega t), \\ & \partial\phi/\partial y(\varepsilon\mu(\omega/k)^2 - 1)\sin(kz - \omega t), \\ & \nabla^2\phi\cos(kz - \omega t), \\ & (\varepsilon\mu\omega/k)\nabla^2\phi\cos(kz - \omega t)]/\mu\end{aligned}$$

$$\begin{aligned}\mathbf{S}_4 \quad = \quad & [\phi\partial\phi/\partial x\cos(kz - \omega t)^2(1 - \varepsilon\mu(\omega/k)^2), \\ & \phi\partial\phi/\partial y\cos(kz - \omega t)^2(1 - \varepsilon\mu(\omega/k)^2), \\ & 0, \\ & 0]/\mu\end{aligned}$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\begin{aligned}\mathbf{E} \times \mathbf{H} \quad = \quad & [\phi(\partial\phi/\partial x)k\cos(kz - \omega t)\sin(kz - \omega t)(v_g - v_p), \\ & \phi(\partial\phi/\partial y)k\cos(kz - \omega t)\sin(kz - \omega t)(v_g - v_p), \\ & (v_g)\cos(kz - \omega t)^2(\nabla^2\phi)]/\mu\end{aligned}$$

$$\begin{aligned}(\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) \quad &\neq \quad 0 \\ (\mathbf{E} \circ \mathbf{B}) \quad &= \quad 0\end{aligned}$$

Note that the vector potential, $\mathbf{A}$, is parallel to both the wave vector, $\mathbf{k}$, and the field momentum, $\mathbf{D} \times \mathbf{B}$. The Torsion vector and the second Poincare invariant are indentically zero. The solution produces current and spin densities unless a dispersion relation, $\varepsilon\mu(\omega/k)c^2 = 1$, is satisfied. Subject to the dispersion constraints, this classic solution has both a zero Torsion vector and a zero Spin vector. Both $\mathbf{A} \circ \mathbf{D} = 0$ and $\mathbf{A} \circ \mathbf{B} = 0$. The wave front is in the spatial direction of the potential, by construction. The candidate solution subject to the dispersion relation is both topologically transverse TTEM and geometrically transverse, TEM .

However, even if the dispersion relations are satisfied, the geometric TEM solution produces finite charge current densities, unless the function $\phi(x, y)$ is a solution of the two dimensional Laplace equation, $\nabla^2\phi = 0$. This further constrain implies that the TEM solution produces no radiated power in the charge free state, for $\mathbf{E} \times \mathbf{H} \Rightarrow 0$ as $\nabla^2\phi \Rightarrow 0$. In the next example the constraint that the system be TTEM is relaxed, and radiated power is achieved. in a TTM mode.

4.1.6. Example 6. Waveguide TM modes

Consider the Potentials

$$A = [0, 0, \phi(x, y)\cos(kz - \omega t), v_g\phi(x, y)\cos(kz - \omega t) \tag{4.6}$$

and their induced fields (note that example 6 differes from example 5 in that a "group" velocity v_g is used in the definition of the potentials, instead of the phase velocity, $v_p = \omega/k$):

$$\mathbf{E} = [-v_g\partial\phi/\partial x, -v_g\partial\phi/\partial y, \phi(x, y)\tan(kz - \omega t)(v_g k - \omega)]\cos(kz - \omega t)$$

$$\mathbf{B} = [\partial\phi(x, y)/\partial y\cos(kz - \omega t), -\partial\phi(x, y)/\partial x\sin(kz - \omega t), 0]$$

$$\begin{aligned}\mathbf{J}_4 \;=\; & [k\partial\phi/\partial x(\varepsilon\mu v_g v_p - 1)\sin((kz - \omega t), \\ & k\partial\phi/\partial y\sin((kz - \omega t)(\varepsilon\mu v_g v_p - 1), \\ & -(\nabla^2\phi + \alpha\phi)\cos(kz - \omega t), \\ & -v_g\varepsilon\mu(\nabla^2\phi + \beta\phi)\cos(kz - \omega t)]/\mu\end{aligned}$$

$$\alpha = k^2\varepsilon\mu v_p(v_p - v_g), \qquad \beta = k^2 v_g(v_p/v_g - 1)$$

$$\begin{aligned}\mathbf{S}_4 \;=\; & [-(v_g/v_p - 1)\phi\partial\phi/\partial x\cos(kz-\omega t)^2,\\ & -(v_g/v_p - 1)\phi\partial\phi/\partial x\cos(kz-\omega t)^2,\\ & -k(v_g/v_p - 1)\phi^2\sin(kz-\omega t),\\ & -\mu k(v_g - v_p)\phi^2\sin(kz-\omega t)]/\mu\end{aligned}$$

$$\mathbf{T}_4 = [0, 0, 0, 0].$$

$$\begin{aligned}\mathbf{E}\times\mathbf{H} \;=\; & [(v_p/v_g - 1)\phi\partial\phi/\partial x\sin(kz-\omega t),\\ & (v_p/v_g - 1)\phi\partial\phi/\partial y\sin(kz-\omega t),\\ & ((\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2)\cos(kz-\omega t)](\,v_g/\mu)\cos(kz-\omega t)\end{aligned}$$

$$\begin{aligned}(\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) &= -(\{\varepsilon\mu(\omega/k)^2 - 1\}/\mu)\cos(kz-\omega t)^2\{(\nabla\phi)^2 \;+\phi(\nabla^2\phi)\}\\ (\mathbf{E}\circ\mathbf{B}) &= 0\end{aligned}$$

Note that in this solution, the fourth component of the Action is scaled by the "group velocity", v_g, not the "speed of light", as determined by the constitutive properties: $c = \sqrt{1/\xi\mu}$. This class of potentials requires that the function $\phi(x, y)$ be a solution of the two dimensional Helmholtz equation, $\nabla^2\phi + \lambda^2\phi = 0$. The phase velocity, $v_p = \omega/k$, differs from the group velocity, v_g. Again, two constraint conditions (dispersion relations) are required for the solution to be a vacuum solution without charge currents. One of the constraint conditions demands that the product of the group and the phase velocity, $v_p = \omega/k$, to be equal to the square of the speed of light as determined from the constitutive properties:

$$v_p \cdot v_g = 1/\varepsilon\mu = c^2.$$

The second constraint required for the vacuum state $(\mathbf{J} = 0, \rho = 0)$ is determined by the Helmholtz parameter, λ,and is satisfied when

$$\lambda^2 = k^2(v_p/v_g - 1).$$

Such TM modes are also TTM modes; the Torsion vector is identically zero, but the Spin vector is not. Note that the solution becomes a TEM mode solution when the phase velocity equals the group velocity, and the function ϕ satisfies the Laplace equation, $\nabla^2\phi = 0$. Further note that the $\mathbf{E}$ field has a longitudinal component when the group velocity and the phase velocity are not the same. For the transverse magnetic mode, $\mathbf{A} \circ \mathbf{B} = 0$, but $\mathbf{A} \circ \mathbf{D} \neq 0$. The second Poincare invariant vanishes, $\mathbf{E} \circ \mathbf{B} = 0$,but for this solution, the first Poincare invariant does not vanish. Not only is the Spin vector not zero, but also its divergence is not zero. The energy flow is in the direction of the wave vector, $\mathbf{k}$, but not in the direction of the field momentum, $\mathbf{D} \times \mathbf{B}$, and the energy propagates with the group velocity v_g.

4.1.7. Example 7. A vacuum solution for which $\mathbf{E} \circ \mathbf{B} \neq 0$

Consider the potentials

$$\mathbf{A} = [+y, -x, ct]/\lambda^4 \ , \quad \phi = cz/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{4.7}$$

and their induced fields:

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6$$

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), +(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6.$$

$$\mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}$$

$$\mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8.$$

$$\begin{aligned} (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) &= 0 \\ (\mathbf{E} \circ \mathbf{B}) &= -4c/\lambda^8 \end{aligned}$$

Both the Spin current and the Torsion vector are non-zero, which implies that this solution represents waves which are neither TTM nor TTE. They are not transverse waves in any sense. However, the first Poincare invariant vanishes, implying that the Spin integral is a deformation invariant, and is conserved. The second Poincare invariant is not zero, which implies that the Torsion-Helicity integral is not a topological invariant. These solutions are not simple transverse waves for both $\mathbf{A} \circ \mathbf{B} \neq 0$, and $\mathbf{A} \circ \mathbf{D} \neq 0$. Note that the physical units of the second Poincare invariant are that of an energy density multiplied by an impedance (ohms). As the second Poincare invariant is not zero, it is impossible to find a compact without boundary two surface that contains non-zero lines of magnetic field. That is, a closed 2-torus of magnetic field lines does not exist.

However, as the first Poincare invariant is zero it is possible to construct a deformation invariant in terms of the deRham period integral over a closed 3 dimensional submanifold:

$$Spin = \underset{closed}{\iiint} \{S_x dy\hat{}dz\hat{}dt - S_y dx\hat{}dz\hat{}dt + S_z dx\hat{}dy\hat{}dt - \sigma dx\hat{}dy\hat{}dz\}.$$

4.1.8. Example 8. Another vacuum solution for which $\mathbf{E} \circ \mathbf{B} \neq 0$,

4.1.9. complimentary to example 7.

Consider the potentials

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4 \ , \quad \phi = cx/\lambda^4, \quad where \ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2 \quad (4.8)$$

and their induced fields:

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6$$

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6.$$

As in example 7, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for example 7:

$$\mathbf{E} \circ \mathbf{B} = +4c/\lambda^8 \quad and \qquad \mathbf{A} \circ \mathbf{B} = +2ct/\lambda^8 \ .$$

4.1.10. Example 9 Superposition of the two complimentary examples 7 and 8.

When the potentials of examples 7 and 8 are combined by addition or subtraction, the resulting wave is topologically transverse magnetic, but not topological transverse electric. Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist topologically transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector does not vanish, hence the example superposition is a wave that is not topologically transverse electric. However, for the examples above and their superposition, the first Poincare invariant vanishes, which implies that the Spin remains a conserved topological quantity for the superposition. The spin current density for the combined examples is given by the formula:

$$\begin{aligned} \mathbf{S}_4 &= [-2cx(y+ct)^2, cy(y+ct)(x^2-y^2+z^2-2cty-c^2t^2), -2cz(y+ct)^2, \\ &\quad -(y+ct)(x^2+y^2+z^2+2cty+c^2t^2)]4c/\lambda^{10} \end{aligned} \tag{4.9}$$

while the Torsion current is a zero vector

$$\mathbf{T}_4 = [0,0,0,0].$$

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by γ, such that

$$\mathbf{E} \times \mathbf{H} = \gamma\ \mathbf{S}, \quad with\ \gamma = -(x^2+y^2+z^2+2cty+c^2t^2)/2c(y+ct)\lambda^2.$$

These results seem to give classical credence to the Planck assumption that vacuum state of Maxwell's electrodynamics supports quantized angular momentum, (the conserved spin integral) and that the energy flux must come in multiples of the spin quanta. In other words, these combined solutions of examples 7 and 8 have the appearance of the photon.

4.1.11. Example 10. Bateman-Whittaker solutions.

In the modern language of differential forms, Bateman [19] (and Whittaker) determined that if two *complex* functions $\alpha(x, y, z, t)$ and $\beta(x, y, z, t)$ are used to define the 1-form of Action,

$$A = \alpha d\beta - \beta d\alpha \Rightarrow \mathbf{A} = \alpha\nabla\beta - \beta\nabla\alpha, \quad \phi = -(\alpha\partial\beta/\partial t - \beta\partial\alpha/\partial t) \tag{4.10}$$

then the derived 2-form

$$F = 2d\alpha \hat{} d\beta$$

generates the complex field intensities,

$$\mathbf{E} = (\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \quad \textit{and} \quad \mathbf{B} = \nabla\alpha \times \nabla\beta,$$

which of course satisfy the Maxwell-Faraday equations. If in addition, the functions α and β satisfy the complex Bateman constraints:

$$\nabla\alpha \times \nabla\beta = \pm(i/c)[(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha],$$

then the complex field excitations computed from the Lorentz vacuum constitutive constraints will satisfy the Maxwell-Ampere equations for the vacuum, without charge currents. It is apparent immediately that the second Poincare invariant is identically zero for such solutions. It is also apparent immediately that the Torsion vector is identically zero. What is not immediately apparent is that the first Poincare invariant and the Spin vector vanish identically as well. In fact, the constrained complex solutions of the Bateman type are examples of topologically transverse (TTEM) waves. The Bateman solutions do not radiate!

As an explicit example, consider

$$\alpha = (x \pm iy)/(z - r), \quad \beta = (r - ct), \quad r = \sqrt{x^2 + y^2 + z^2)}.$$

These functions satisfy the Bateman conditions (and, it should be mentioned, the Eikonal equation subject to the dispersion relation $\varepsilon\mu c^2 = 1$). The $\mathbf{E}$ and the $\mathbf{B}$ fields are complex (and complicated algebraically)

$$\mathbf{B} = [yx + \sqrt{-1}(z^2+y^2-rz), -(z^2+x^2-rz) - \sqrt{-1}xy, (r^2+z^2-2rz)/(r-z)\,)(y-\sqrt{-1}x)]2/(r(z-r)$$

$$\mathbf{E} = [-\sqrt{-1}yx + (y^2+z^2-rz), \sqrt{-1}(x^2+z^2-rz) - xy, (z-r\,)(x+\sqrt{-1}y)]2c/(r(z-r)^2)$$

$$\mathbf{S}_4 = [0,0,0,0].$$

$$\mathbf{T}_4 = [0,0,0,0].$$

$$\mathbf{E}\times\mathbf{H} = [0,0,0], \quad \mathbf{D}\times\mathbf{B} = [0,0,0], \quad \mathbf{E}\circ\mathbf{E} = 0, \quad \mathbf{B}\circ\mathbf{B} = 0$$

$$\begin{aligned} (\mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E}) &= 0 \\ (\mathbf{E}\circ\mathbf{B}) &= 0 \end{aligned}$$

The functions α and β that satisfy the Bateman condition may be used to construct an arbitrary function, $F(\alpha,\beta)$, and remarkably enough, the arbitrary function $F(\alpha,\beta)$ satisfies the Eikonal equation,

$$(\nabla F)^2 - \varepsilon\mu(\partial F/\partial t)^2 = 0.$$

From experience with Eikonal solutions and wave equations, it might be thought that Eikonal solutions are sufficient. However, the Bateman conditions are necessary, for both the candidate solutions

$$\alpha = (x \pm iy)/(z-ct), \quad \beta = (r-ct), \quad r = \sqrt{x^2+y^2+z^2)}.$$

satisfy the Eikonal equation, but not the Bateman conditions. They do not generate TTEM modes in the vacuum. For arbitrary functions the algebra can become quite complex. A Maple symbolic mathematics program for computing the various terms is available (see references below)

4.1.12. Example 11. A Plasma Accretion disc from HedgeHog B field solutions.

An interesting static solution that has chiral symmetry breaking can be obtained from the potentials

$$\begin{aligned} \mathbf{A} &= \Gamma(x,y,z,t)[-y,x,0]/(x^2+y^2)\ , \qquad (4.11) \\ with\quad \Gamma &= -z\,m/\sqrt{(x^2+y^2+\epsilon z^2)} \\ and\quad \phi &= 0. \end{aligned}$$

These potentials induce the field intensities:

$$\mathbf{E} = [0,0,0]$$

$$\mathbf{B} = m\,[x,y,z]/(x^2+y^2+\epsilon\,z^2)^{3/2}.$$

The **B** field is the famous Dirac Hedgehog field often associated with "magnetic monopoles". However, the radial **B** field has zero divergence everywhere except at the origin, which herein is interpreted as a topological obstruction. The factor ϵ is to be interpreted as an oblateness factor associated with rotation of a plasma, and is a number between zero and 1. It is apparent that the helicity density and the second Poincare invariant are zero:

$$\mathbf{E}\circ\mathbf{B} = 0 \qquad and \qquad \mathbf{A}\circ\mathbf{B} = 0\,.$$

In fact, the 3-form of topological torsion vanshes identically (as $\phi = 0$),

$$\mathbf{T}_4 = [0,0,0,0].$$

In this example, there is a non-zero value for the Amperian current density, even though the potentials are static. The Current Density 3-form has components,

$$\mathbf{J}_4 = (3m/2\mu)\ (1-\epsilon)\,z\,[-y,x,0,0]/(x^2+y^2+\epsilon\,z^2)^{5/2}..$$

which do not vanish if the system is "oblate" ($0 < \epsilon < 1$). This current density has a sense of "circulation" about the z axis, and is proportional to the vector potential reminiscent of a London current, $\mathbf{J} = \lambda\mathbf{A}$. The "order" parameter is $(3/2\mu)\ (1-\epsilon)/(x^2+y^2+\epsilon\,z^2)^2$.

The Lorentz force can be computed as:

$$\mathbf{J}\times\mathbf{B} = (3m^2/4\mu)\ (1-\epsilon)[xz^2, yz^2, -z]/(x^2+y^2+\epsilon\, z^2)^2$$

The formula demonstrates that the Lorentz force on the plasma, for the given system of circulating currents, is directed radially away (centrifugally) from the rotational axis, and yet is such that the plasma is attracted to the $z=0, xy$ plane. The Lorentz force is divergent in the radial plane and convergent in the direction of the z axis, towards the z=0 plane. This electromagnetic field, therefor, would have the tendency to form an accretion disk of the plasma in the presence of a central gravitational field.

Although the 3-form of Topological Torsion vanishes identically, the 3-form of Spin is not zero. The spatial components of the Spin are opposite to the direction field of the Lorentz force (in the sense of a radiation reaction).

$$\mathbf{S}_4 = (m^2/4\mu)[xz^2, yz^2, -z, 0]/(x^2+y^2+\epsilon\, z^2)^2.$$

The components of the Spin 3-form are in fact proportional to the components of the virtual work 1-form. (See section 6) with the ratio $-3(1-\epsilon)$ depending on the oblateness factor.

It is also true that the divergence of the 3-form of spin is not zero, for the first Poincare invariant is

$$d(A\hat{\ }G) \Rightarrow P1 = (m^2/4\mu)(x^2+y^2+4(1-\epsilon)\, z^2)/(x^2+y^2+\epsilon\, z^2)^3$$

4.2. Self dual solutions

It is possible to construct a two-form G (without using the Lorentz vacuum constitutive definitions) in terms of two arbitrary functions, α and β, from the dual relations:

$$G = i(*d\alpha)\hat{\ }i(*d\beta)\Omega = i(*d\alpha)\hat{\ }i(*d\beta)\Omega_4. \qquad (4.12)$$

The functions α and β used in the dual construction are not required to be solutions of the Bateman condition. However, the resulting "self-dual" field excitations are **not** the same as those generated by the Bateman method, unless the functions also satisfy the Bateman conditions of complex collinearity. In the self dual formulas the * operator is the Hodge * operator with respect to the Lorentz metric modified by the impedance of free space. The resulting self-dual excitations constructed from the two arbitrary functions indeed satisfy the Maxwell-Ampere equations, in virtue of the Maxwell-Faraday equations and the dispersion relation. The construction yields:

$$\mathbf{H} = \sqrt{-1}/\mu c(\partial\alpha/\partial t)\nabla\beta - (\partial\beta/\partial t)\nabla\alpha \quad and \quad \mathbf{D} = -\sqrt{-1}\varepsilon/c\nabla\alpha\times\nabla\beta. \quad (4.13)$$

The self-dual construction, however, implies a chiral (non-Lorentz) constitutive relation of the type $\mathbf{D} = -[\gamma]\circ\mathbf{B}$ and $\mathbf{H} = [\gamma^{\dagger}]\circ\mathbf{E}$, and will not be considered further in this article.

5. 5. Deformation Invariants and the Plasma State.

In section six, below, thermodynamic arguments will be used to form equivalence classes of evolutionary processes based upon the topological properties of the 1-form of work, and the 1-form of heat. In this section, categories of evolutionary processes will be described based upon certain constraints that correspond to the conservation of charge quanta and flux quanta. These constraints will be related to various approximations used to define the plasma state and MHD.

5.1. Special evolutionary processes. The plasma process

As described above, the fundamental equation of topological evolution is given by Cartan's magic formula [4], acting as a propagator on the forms that make up the exterior differential system. As stated in the first paragraph, an evolutionary process is defined herein as a map that can be described by a singly parameterized vector field. If the Action of the Lie derivative on the complete system of Maxwell exterior differential forms vanishes for a particular choice of process, then that process leaves the entire Maxwell system absolutely invariant. As a topology can be constructed in terms of an exterior differential system, and if a special process leaves that system of forms invariant, then the topology induced by the system of forms is invariant, and the process must be a homeomorphism.

However, for a given Maxwell system, it is more likely that only some of the exterior differential forms that make up the Maxwell system are invariant relative to an arbitrary process; others are not. Of particular interest are those forms which are relative integral invariants of continuous deformations. The closed integral of the form is not only invariant with respect to a process represented by particular vector field, but also with respect to longitudinal deformations of that process obtained by multiplying the particular vector field by an arbitrary function. For vector fields which are singly parameterized, this concept of longitudinal deformation is equivalent to a reparameterization of the vector field.

The development that follows is guided by Cartan's pioneering work, in which he examined those specialized processes for mechanical systems that leave the 1-form of Action, A, on state space, a deformation invariant. Cartan proved that such processes on domains of odd Pfaff dimension always have a Hamiltonian representation. An electromagnetic system differs in that it is constructed not only the primitive 1-form, A, but also the N-2 form, G. Both objects can undergo evolutionary processes, and moreover the domains of interest are not necessarily of odd Pfaff dimension.

For electromagnetic systems, a particular interesting choice of specialized processes are those that leave the closed integrals of the N-2 form, G, of field excitations a deformation (relative) integral invariant. Such processes preserve the net *number* of charges, globally.

Definition 5: A plasma process is a process for which $V^4\mathbf{J} = \rho\mathbf{V}+\lambda\mathbf{B}$.

Lemma 1: (Gauss law) The net number of charges in an interior domain of a physical system is given by the formula $Q = \iint_{closed} G$.

Theorem 5: The closed integral of G is a evolutionary deformation invariant with respect to plasma processes. Hence, the net number of charges in an interior domain is preserved by a plasma process.

Proof: Using Cartan's Magic formula and the definition of plasma process:

$$L_{\beta V}(\iint_{closed} G) = \iint_{closed} i(\beta V)dG = \iint_{closed} i(\beta V)J \tag{5.1}$$

$$= \iint_{closed} \beta\{(V^4\mathbf{J} - \rho\mathbf{V})^x dy\hat{}dz - ... + (\mathbf{J}\times\mathbf{V})^x dx\hat{}dt... \Rightarrow 0 \tag{5.2}$$

From the Lemma:

$$L_{\beta V}(\iint_{closed} G) = L_{\beta V} Q = 0. \tag{5.3}$$

As the closed integrals of G are, by Gauss law, the counters of net charge within the closed domain, the classical plasma equation is to be recognized as the statement that in the closed domain the net number of charges is a deformation invariant. If the various terms in the integrand vanish, such that $V^4\mathbf{J} = \rho\mathbf{V}$, then the integral is satisfied. However, this constraint is too stringent, for all that is required is that the terms in the integrand be exact. The only available 2-form in the Maxwell system that is exact is $F = dA$. Hence the requirement for preserving the number of charges is given by the equations $i(\beta V)J = \lambda dA$,or

$$\beta\{(V^4\mathbf{J} - \rho\mathbf{V}) = \lambda\mathbf{B} \tag{5.4}$$

$$\beta(\mathbf{J} \times \mathbf{V})^x = \lambda\mathbf{E}. \tag{5.5}$$

Note that the requirement for global charge conservation is that $\mathbf{E} \circ \mathbf{B} = 0$; the Pfaff dimension of the 1-form of Action must be 3 or less.

If the integration domain is a boundary, rather than a closed cycle, then slight different conditions will apply. In the boundary case, it follows that

$$i(\beta V)J = \lambda dA + \lambda_k G_0^k \tag{5.6}$$

where the G_0^k are the contributions to the 2-form G which are closed, but not exact. In all cases, charges must be produced only in equal and opposite pairs by a "plasma process", or not at all, in order to preserve the net number of charges. A plasma process does not involve net charge production. This constraint, $V^4\mathbf{J} = \rho\mathbf{V} + (\lambda/\beta)\mathbf{B}$, is used to define the "Plasma state" in this article.

This invariance principle is to be compared to the Helmholtz theorem which checks on the validity of the deformation integral invariance of the 2-form F.

$$L_{\beta\mathbf{v}}(\iint_{closed} F) = \iint_{closed} i(\beta V)dF = 0 \tag{5.7}$$

The closed integral of Helmholtz is an intrinsic topological (deformation) invariant of an electromagnetic system, for the 2-form F is exact by construction (the postulate of potentials). The Helmholtz integral is a deformation invariant for all evolutionary processes that can be described by a singly parameterized vector field. (This statement is not true for Yang Mills fields). Hence in a plasma, for which the evolutionary processes are constrained such that $V^4\mathbf{J} = \rho\mathbf{V} + (\lambda/\beta)\mathbf{B}$, both the closed integrals of F and G are deformation invariants. In the sense, the plasma is a topological refinement of the complete Maxwell system.

In the subsections that follow, various topological sub-categories of plasma processes will be examined. The ideal and semi-ideal plasma processes will obey the plasma master equation, and the non-ideal plasma processes will not. The electromagnetic flux is a local (absolute) invariant of all semi-ideal plasma processes. This statement is similar to the classification of hydrodynamic flows. Ideal and semi-ideal hydrodynamic flows satisfy the Helmholtz theorem, and the "local" conservation of vorticity.

5.2. The ideal plasma process is a plasma process which is also a Hamiltonian process.

Next consider the evolutionary properties of the 1-form of Action in the plasma state by evaluating the possible deformation invariance of the 1-form of Action, A, with respect to motions that preserve the plasma state:

> **Definition 6:** A ideal plasma process, or equivalently a force-free plasma process, is a plasma process for which the Lorentz force $\mathbf{E} + \mathbf{V} \times \mathbf{B}$ vanishes.
>
> **Theorem 6:** An ideal plasma process is a Conservative Hamiltonian process that preserves the net charge number. The second Poincare invariant vanishes, implying that the closed integrals of Torsion-Helicity are deformation invariants with respect to force-free plasma processes.
>
> Proof: Using Cartan's magic formula and the constraint that $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = 0$,

$$L_{\rho\mathbf{V}}(\oint A) = \oint i(\rho\mathbf{V})dA = \oint W = \oint\{(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})_k dx^k + (\mathbf{J} \circ \mathbf{E})dt\} \Rightarrow 0. \quad (5.8)$$

From Cartan, when $L_{\rho\mathbf{V}}(\oint A) = 0$, the evolutionary process admits a Hamiltonian representation. When $i(\rho\mathbf{V})dA = 0$, the Pfaff dimension (of A) must be odd, hence $F\hat{}F = 0$. The deformation invariance of the Torsion-Helicity follows from Theorem 4.

Such a set of constraints, $W = i(\rho V)dA = 0$, topologically defines the "ideal" or force-free plasma state as a plasma process for which both the 1-form of virtual Work vanishes, and there is no net charge production, and Torsion Helicity is conserved.

5.3. The Bernoulli-Casimir plasma process is a semi-ideal plasma process.

The topological constraint that the 1-form of virtual work vanishes is sufficient, but not necessary, for a plasma process to preserve the closed integrals of the Action 1-form. The 1-form of virtual Work, $W,$ need not be zero, but only closed: $dW \Rightarrow 0$. For, by using Stokes theorem:

$$L_{\rho\mathbf{V}}(\oint A) = \oint i(\rho\mathbf{V})dA = \oint W = \iint dW = 0. \tag{5.9}$$

Consider the anti-symmetric matrix of functions constructed from the 2-form, $F = dA$. If the Pfaff dimension is odd, say 2n+1, then the rank of the antisymmetric matrix is 2n. Hence there exists one (unique) eigenvector of the matrix, such that $W = i(V)dA = 0$. Such is the domain of Cartan's theory of Hamiltonian systems on state space. Next consider the case where the Pfaff dimension of the 1-form is even. The anti-symmetric matrix of functions constructed from the 2-form, F, is of maximal rank 2n+2. There do not exist *any* null eigenvectors. Hence in such a system of even Pfaff dimension, the virtual work is never zero, $W = i(V)dA \neq 0$.

In the electromagnetic case, the case of Pfaff dimension of interest is 4. In such a case, the second Poincare invariant does not vanish: $F\hat{}F \neq 0$, (implying that $\mathbf{E} \circ \mathbf{B} \neq 0$). There are two equivalence classes of evolutionary processes on the domains of even Pfaff dimension, that leave the closed integrals of Action invariant. The first class will be defined by those processes where the virtual work 1-form, W is exact:

Definition 7: A Bernoulli-Casimir function, Θ, is a function whose gradient is proportional to the virtual work created by any process:

$$W = i(\rho\mathbf{V})dA = d\Theta. \tag{5.10}$$

On a domain of even Pfaff dimension, the function $\Theta(x, y, z, t)$ cannot be constant. However, specification of a function, $\Theta(x, y, z, t)$, permits a unique construction of the process, V, that generates the selected function. The antisymmetric matrix of coefficients of dA is invertible on domains of even Pfaff dimension. However,

> **Theorem 7:** A Bernoulli-Casimir function is an evolutionary invariant, but not a domain constant.
>
> Proof: Using Cartan's magic formula,
>
> $$L_{\rho\mathbf{V}}(\Theta) = i(\rho\mathbf{V})d\Theta = i(\rho\mathbf{V})i(\rho\mathbf{V})A = 0. \tag{5.11}$$
>
> If $\Theta(x, y, z, t) = \text{constant}$, then $d\Theta = 0$, which implies that $W = 0$. However, $W = i(\rho\mathbf{V})dA \neq 0$ on a domain of even Pfaff dimension as there are no null eigenvectors of a maximal rank antisymetric matrix on the domain of even Pfaff dimension - a contradiction.

The Bernouilli-Casimir function, $\Theta(x, y, z, t)$, is not the same as the Hamiltonian energy function on state space, $H(p, q, t)$, but is more closely related to a thermodynamic concept such as the Helmholtz free energy. Theorem 7 states that the Bernouilli-Casimir function, Θ, behaves like a constant on any particular trajectory, but does not have the same value on all trajectories.

> **Definition 8:** A semi-ideal plasma process, or equivalently a symplectic plasma process. is a plasma process for which the Lorentz force $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$ is exact, hence has zero curl.
>
> **Theorem 8:** A semi-ideal plasma process is a Symplectic process that preserves the net charge number, and the closed integrals of Action. The second Poincare invariant does not vanish, but the closed integrals of Torsion-Helicity are deformation invariants with respect to symplectic processes.
>
> Proof: Using Cartan's magic formula and the constraint that $W = d\Theta$,

$$L_{\rho\mathbf{V}}(\oint A) = \oint i(\rho\mathbf{V})dA + dU = \oint W + dU = \oint d(\Theta + U) \Rightarrow 0 \quad (5.12)$$

$$L_{\rho\mathbf{V}}(\iiint_{closed} A\hat{\ }F) = (\iiint_{closed} d\{(\Theta + U)\hat{\ }F\}) = 0 \quad (5.13)$$

For an electromagnetic action constructed in terms of the potentials, and for plasma processes, the Lorentz force is represented by a spatial gradient, $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \nabla\Theta$, and the dissipative power becomes, $-\mathbf{J} \circ \mathbf{E} = \partial\Theta/\partial t$. For such symplectic plasma processes, the gradient of the Bernoulli-Casimir function is transverse to the **B** field only when the second Poincare invariant vanishes.

$$\rho\,\mathbf{E} \circ \mathbf{B} = \nabla\Theta \circ \mathbf{B}. \quad (5.14)$$

A similar expression was studied in conjunction with topological conservation of Helicity in MHD by Hornig and Schindler [20].

Note that it must be true for the case under consideration, that

$$\rho\,\mathbf{E} \circ \mathbf{V} = \nabla\Theta \circ \mathbf{V}. \quad (5.15)$$

Such a constraint implies that if the Ohmic dissipation vanishes, $\partial\Theta/\partial t = 0$, and the spatial gradient of the time independent Bernoulli function must be transverse to the current, or to the direction field that represents the evolutionary process. Suppose the contrary, where the Bernoulli function is not indepedent from time. If the Ohmic assumption is made for the plasma process, $\mathbf{J} = \rho\mathbf{V} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B})$, then the symplectic condition leads to a thermopower format of the type

$$\mathbf{J} = (1/\rho\sigma)grad(kT) \quad (5.16)$$

when it is subsumed that the Bernouilli-Casimir function is related to temperature. It would appear that for plasma motion along the **B** field lines, there can exist a dynamo action to produce an **E** field collinear with the magnetic field, and related to a thermodynamic gradient in the direction of the current.

5.4. The Helmholtz plasma process is a semi-ideal process that obeys the Master equation.

The constraint that the virtual work 1-form, W, generated by a plasma process, $W = i(\rho V)dA$, be closed, does not require that it be exact.

> **Definition 9:** A Helmholtz (or Stokes) process is a process for which the virtual work is closed but not necessarily exact.

The constraint of closure yields two vector conditions that must be applied to the plasma process:

$$\begin{aligned} dW &= 0 \Rightarrow curl(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) = 0 & (5.17) \\ and \quad \nabla(\mathbf{J} \circ \mathbf{E}) &= \partial(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})/\partial t. & (5.18) \end{aligned}$$

The first vector condition implies that

$$\nabla\rho \times (\mathbf{E} + \mathbf{V} \times \mathbf{B}) + \rho\, curl(\mathbf{E}) + curl(\mathbf{V} \times \mathbf{B}) = 0. \tag{5.19}$$

By using the Maxwell-Faraday equation, this topological constraint becomes the (modified) plasma master equation:

$$-\partial\mathbf{B}/\partial t + curl(\mathbf{V} \times \mathbf{B}) = -\nabla \ln \rho \times (\mathbf{E} + \mathbf{V} \times \mathbf{B}). \tag{5.20}$$

When the gradient of the charge density is collinear with the Lorentz force, then the RHS of the above equation vanishes, and the ubiquitous master equation is retrieved.

All of the ideal and semi ideal plasma processes (Hamiltonian flows), as well as the Helmholtz plasma processes, enjoy the property that the electromagnetic flux is conserved locally. That is

$$L_{\rho\mathbf{V}}(dA) = L_{\rho\mathbf{V}}F = d(i(\rho V)F) = dW = 0. \tag{5.21}$$

It will be demonstrated below that all such processes are thermodynamically reversible.

5.4.1. Frozen-in lines.

It is of some interest to examine the evolution of the differential forms that make up an electromagnetic system relative to Plasma processes. The method is to construct the Lie derivative with respect a plasma process, $\mathbf{J} = \rho\mathbf{V}$, of all forms that make up the electromagnetic Pfaff sequence.

For an arbitrary vector field $\mathbf{Z}$ whose tangents define a line in space time, the N-1 form

$$Z = i(\gamma\mathbf{Z})\Omega_4 \tag{5.22}$$

can be tested for evolutionary invariance relative to any other vector V. Suppose the effect of the evolutionary process is conformal:

$$L_{(V)}Z = i(V)d\,Z + d(i(V)Z) = \Gamma(x,y,z,t)Z \tag{5.23}$$

This statement implies that the points that make up the tangent line of the vector field W remain on the tangent line. The points may be permuted but they do not leave the line. Such is the concept of a frozen in field. The points on a line evolve into points on the same line. If for a given $\mathbf{V}$ the evolution of the lines of $\mathbf{Z}$ is conformal, then there exists a parametrization of $\mathbf{V}$ such that the evolution is uniform and invariant. A parametrization function $\beta(x,y,z,t)$ can be found such that

$$L_{(\beta\mathbf{V})}Z = \beta L_{(\mathbf{V})}Z + L_{(\mathbf{V})}\beta\hat{}Z = (\beta \cdot \Gamma + i(\mathbf{V})d\beta)Z \Rightarrow 0. \tag{5.24}$$

For the electromagnetic system there are three N-1 forms, which may or may not be frozen into the evolutionary process. Consider the 3-form of charge-current, J.

$$L_{(\mathbf{V})}J = i(\mathbf{V})d\,J + d(i(\mathbf{V})J) \tag{5.25}$$

As $dJ = 0$,

$$L_{(\mathbf{V})}J = d\{i(\mathbf{V})i(\mathbf{J})\Omega_4\} \tag{5.26}$$

It follows that if $i(\mathbf{V})i(\mathbf{J})dx\hat{}dy\hat{}dz\hat{}dt\} = 0$, the field lines of J are frozen-in (with $\Gamma = 0$). So the plasma evolutionary process evolutionary, with $\mathbf{J} = \rho\mathbf{V}$, is an example of a process that "freezes-in" the lines of charge-current. However, there are many other evolutionary processes for which the J lines are frozen in.

The formulas created by 5.16 are valid on any set of independent variables, but expressions on 4 dimensions of space time for "frozen-in" lines are not quite the same as those that appear in the engineering literature based on euclidean 3-space [21]. Either the time-like component of the 4-vector $\mathbf{W}$ must vanish, or the process $\mathbf{V}$ must be explicitly time-independent for the general formulas to be in precise agreement with the engineering expressions. [22]

5.5. Evolution of the lines of topological torsion with respect to plasma currents.

Consider the evolution of the lines of topological torsion

$$
\begin{aligned}
L_{(\rho \mathbf{V})} A\hat{\ } F &= i(\rho \mathbf{V})d(A\hat{\ } F) + d(i(\rho \mathbf{V})A\hat{\ } F) && (5.27) \\
&= i(\rho \mathbf{V})d(A\hat{\ } F) + d\{(i(\rho \mathbf{V})A)\hat{\ } F - A\hat{\ } i(\rho \mathbf{V})F\} && (5.28)
\end{aligned}
$$

First consider those systems where the second Poincare invariant vanishes, $F\hat{\ } F = 0$. The lines in space time which are tangent to the 3-form $A\hat{\ } F$ then have zero divergence. The lines can only start and stop on boundary points, or they are closed on themselves. The Torsion lines can be either parallel to the plasma current or they can be orthogonal to the plasma current. As the electromagnetic current is exact, any three dimensional domain of support for a finite plasma current cannot be compact without a boundary. If the lines of plasma current start and stop on boundary points, then the lines of torsion can form closed loops that link these current lines. It is the concept of linkages that is of interest to the theory of magnetic knots.

Consider that plasma process such that the evolution is in the direction of the Torsion lines. As in this situation,

$$
\begin{aligned}
(i(J)A\hat{\ } F) &= (i(\rho \mathbf{V})A\hat{\ } F) \Rightarrow (i(\gamma \mathbf{T}_4)A\hat{\ } F) && (5.29) \\
&= \gamma(i(\mathbf{T}_4)(i(\mathbf{T}_4)\Omega_4 = 0, && (5.30)
\end{aligned}
$$

the 3-form of Torsion is a local invariant whenever the second Poincare invariant vanishes; $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$. In otherwords, $F\hat{\ } F \neq 0$ is a local necessary condition for topological change. It is also a remarkable fact that any evolution in the direction of the Torsion vector leaves the Action 1-form conformally invariant, in the sense that:

$$L_{(\gamma \mathbf{T}_4)}A = i(\gamma \mathbf{T}_4)dA + di(\gamma \mathbf{T}_4)A = \gamma(\mathbf{E} \circ \mathbf{B})A + 0. \tag{5.31}$$

The torsion vector on a domain of 4 variables is transverse to the 1-form of Action, as $A\hat{}(A\hat{}F) = 0$. Evolution in the direction of the Torsion vector can not be extremal (a Hamiltonian flow), unless the second Poincare invariant vanishes. In section 6 below this idea will be related to thermodynamic irreversibility.

5.6. Evolution of the lines of Spin Current with respect to plasma currents.

Consider the evolution of the lines of Spin current

$$L_{(\rho \mathbf{V})}A\hat{}G = i(\rho \mathbf{V})d(A\hat{}G) + d(i(\rho \mathbf{V})A\hat{}G) \tag{5.32}$$

$$= i(\rho \mathbf{V})d(A\hat{}G) + d\{(i(\rho \mathbf{V})A)\hat{}G - A\hat{}i(\rho \mathbf{V})G\} \tag{5.33}$$

First consider those systems where the first Poincare invariant vanishes, $F\hat{}G - A\hat{}J = 0$. The lines in space time which are tangent to the 3-form $A\hat{}G$ then have zero divergence. The lines can only start and stop on boundary points, or they are closed on themselves. The Spin lines are either parallel to the plasma current or they are orthogonal to the plasma current. As the electromagnetic current is exact, any three dimensional domain of support for a finite plasma current cannot be compact without a boundary. If the lines of plasma current do not stop or start on boundary points (current loops), then the Spin lines which terminate on boundary points can be linked by the current loops.

The concept of the spin vector depends on the existence of G, but not on the concept of $J = dG$. That is, the Spin vector can be associated with separated domains of charges, which can be compact domains without boundary. Such domains are compliments of those domains of finite charge current densities, which are domains that can not be compact without boundary.

6. Thermodynamics

6.1. Topological Thermodynamics and Irreversibility

The basic tool for studying topological evolution is Cartan's magic formula [4], in which it is presumed that a physical (electrodynamic or hydrodynamic) system can be described adequately by a 1-form of Action, A, and that a physical process can be represented by the direction field of a contravariant vector field, $\mathbf{V}$. The application of Cartan's magic formula yields

$$L_{(\mathbf{V})} \int A = \int L_{(\mathbf{V})} A = \int \{ i(\mathbf{V}) dA + d(i(\mathbf{V})A) \} \tag{6.1}$$
$$= \int \{ W + d(U) \} = \int Q. \tag{6.2}$$

The basic idea behind this formalism (which is at the foundation of the Cartan-Hilbert variational principle) is that postulate of potentials is valid: $F - dA = 0$. The base manifold will be the 4-dimensional variety $\{x, y, z, t\}$ of engineering practice, but no metrical features are presumed a priori. If relative to the process, V, the RHS of equation 6 is zero, $\int Q. \Rightarrow 0$, then $\int A$ is said to be an integral invariant of the evolution generated by $\mathbf{V}$. In thermodynamics such processes are said to be adiabatic.

From the point of view of differential topology, the key idea is that the Pfaff dimension, or class [23], of the 1-form of Action specifies topological properties of the system. Given the Action 1-form, A, the Pfaff sequence, $\{A, dA, A\hat{}dA, dA\hat{}dA, ...\}$ will terminate at an integer number of terms $\leq$ the number of dimensions of the domain of definition. On a 2n+2=4 dimensional domain, the top Pfaffian, $dA\hat{}dA$, will define a volume element with a density function whose singular zero set (if it exists) reduces the symplectic domain to a contact manifold of dimension 2n+1=3. This (defect) contact manifold supports a unique extremal field that leaves the Action integral "stationary", and leads to the Hamiltonian conservative representation for the Euler flow in hydrodynamics. The irreversible regime will be on an irreducible symplectic manifold of Pfaff dimension 4, where $dA\hat{}dA \neq 0$. Topological defects (or coherent structures) appear as singularities of lesser Pfaff (topological) dimension, $dA\hat{}dA = 0$.

6.2. Reversible Processes

Classical hydrodynamic and electromagnetic processes can be represented by certain nested categories of vector fields, $\mathbf{V}$. Recall that in order to be Extremal, the process, $\mathbf{V}$, must satisfy the equation

$$Extremal - (unique\ Hamiltonian\, generator) \qquad i(\mathbf{V})dA = 0; \tag{6.3}$$

in order to be "Hamiltonian" the process must satisfy the equation

$$Bernoulli - Casimir - (\Theta\, is\, the\, generator): \qquad i(\mathbf{V})dA = d\Theta; \tag{6.4}$$

in order to be Symplectic, the process must satisfy the equation

$$Helmholtz - Stokes - -Symplectic: \qquad di(\mathbf{V})dA = 0. \tag{6.5}$$

The vector fields defined by 6.2a and 6.2b have generators that create a Hamiltonian flow. This Hamiltonian flow is uniquely defined, in the extremal case, in terms of the functions that define the physical system; i.e., in terms of the functions that define the 1-form of Action. In the $Bernoulli - Casimir$ case, the evolutionary process is not determined by the physical system alone. The possible evolutionary processes are not extremal. In fact, extremal processes cannot exist on the non-singular symplectic domain (which must be of even dimension), because a non-degenerate anti-symmetric matrix (the coefficients of the 2-form dA) does not have null eigenvectors on space of even dimensions.

Although unique extremal stationary states do not exist on the domain of Pfaff dimension 3, there can exist evolutionary invariant Bernoulli-Casimir functions, Θ, that generate non-extremal, "stationary" states. Such Bernoulli processes can correspond to energy dissipative symplectic processes, but they, as well as all symplectic processes, are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ, are evolutionary invariant(s), and may be used to describe non-unique stationary state(s).

The equations, above, which define several familiar categories of processes, are in effect constraints on the topological evolution of any physical system represented by an Action 1-form, A. The Pfaff dimension of the 1-form of virtual work, $W = i(\mathbf{V})dA$, is 2 or less for each of the three categories. The extremal constraint of equation 6.2a can be used to generate the Euler equations of hydrodynamics for a incompressible fluid. The Bernoulli-Casimir constraint of equation 6.2b can be used to generate the equations for a barotropic compressible fluid. The Helmholtz constraint of equation 6.2c can be used to generate the equations for a Stokes flow. However as will be shown below, all such processes are thermodynamically reversible.

An important idea is that it takes domains of Pfaff dimension 3, or more, with attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles, to yield the concepts of Spin and Torsion-Helicity. It takes systems of Pfaff dimension 4 to accommodate processes which are thermodynamically irreversible.

6.3. Irreversible Processes

Although there does not exist a unique extremal process on a symplectic manifold of Pfaff dimension 4, remarkably there does exist a unique (conformal) vector field whose direction field depends only upon the functional form of the 1-form, A, that is used to define the physical system. The direction field on the four dimensional domain is defined by the 3-form of topological torsion, $A\hat{}dA$, as discussed in Section 3. This unique (to within a factor) vector field is defined in component form as the Torsion Current, $\mathbf{T}_4$, and satisfies (on the 2n+2=4 dimensional manifold) the equation,

$$i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = i(\mathbf{T}_4)\Omega_4 = A\hat{}dA \tag{6.6}$$

This (four component) vector field, $\mathbf{T}_4$, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form $dA\hat{}dA$ vanishes, and the domain is no longer a symplectic manifold! The Torsion vector, $\mathbf{T}_4$, can be used to generate a dynamical system that will decay to the stationary states ($div_4(\mathbf{T}_4) \Rightarrow 0$) starting from arbitrary initial conditions. These processes are irreversible in the thermodynamic sense. It is remarkable that this unique evolutionary vector field, $\mathbf{T}_4$, is completely determined (to within a factor) by the physical system itself; e.g., the components of the 1-form, A, determine the components of the Torsion vector.

To understand what is meant by thermodynamic irreversibility, realize that Cartan's magic formula of topological evolution is equivalent to the first law of thermodynamics.

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q. \tag{6.7}$$

A is the "Action" 1-form that describes the hydrodynamic or electromagnetic system. $\mathbf{V}$ is the vector field that defines the evolutionary process. W is the 1-form of (virtual) work. Q is the 1-form of heat. From classical thermodynamics, a process is irreversible when the heat 1-form Q does not admit an integrating factor.

Definition 10: An irreversible (non-equilibrium) process is one for which the Heat 1-form Q does not admit an integrating factor.

From the Frobenius theorem, the lack of an integrating factor implies that $Q\hat{}dQ \neq 0$. Hence a simple test may be made for any process, $\mathbf{V}$, relative to a physical system described by an Action 1-form, A:

Theorem 10:

$$If \quad L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA \neq 0 \; then\ the\ process\ is\ irreversible. \tag{6.8}$$

Proof: Using Cartan's magic formula yields $L_{(\mathbf{V})}A = Q$ and $L_{(\mathbf{V})}dA = dQ$. Hence the requirement that an integrating factor does not exist is $Q\hat{}dQ = L_{(\mathbf{V})}A\hat{}L_{(\mathbf{V})}dA \neq 0$.

This topological definition implies that the three categories (above) of Symplectic $\supset$ Hamiltonian $\supset$ Extremal processes, $\mathbf{V} \subset \mathbf{S}$, are reversible in a thermodynamic sense (as $L_{(\mathbf{S})}dA{=}dQ = 0$).

However, for evolution in the direction of the Torsion vector, $\mathbf{T}_4$, direct computation demonstrates that the fundamental equations lead to a conformal evolutionary process:

$$L_{(\mathbf{T}_4)}A = \sigma A \quad and \quad i(\mathbf{T}_4)A = 0 \tag{6.9}$$

$$with \quad \sigma \sim div_4(\mathbf{T}_4) \sim d(A\hat{}dA) \tag{6.10}$$

Theorem 11: Evolution in the direction of the Torsion Vector is irreversible

Proof: The direction field associated with $\mathbf{T}_4$ is uniquely determined by the functional form of the 1-form of Action that defines the physical system on the four dimensional variety. By direct evaluation, using 6.6,

$$L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = Q\hat{}dQ = \sigma A\hat{}(\sigma dA + d\sigma\hat{}A) \qquad (6.11)$$
$$= \sigma^2 A\hat{}dA = \{div_4(\mathbf{T}_4)\}^2 A\hat{}dA. \qquad (6.12)$$

As the domain is of Pfaff dimension 4, it follows that $A\hat{}dA$ is not zero, and $dA\hat{}dA \sim div_4(\mathbf{T}_4)$ is not zero. Therefore, the RHS side of 6.7 is not zero, and the irreversibility result follows from theorem 10.

Explicit evaluations are carried out in the next section for electromagnetic systems of Pfaff dimension 4.

6.4. Applications to Electromagnetism

All of the development of section 6.1 will carry over to the electromagnetic system, which also subsumes the postulate of potentials. The topological torsion 3-form, $A\hat{}dA$, induces the torsion current

$$\mathbf{T}_4 = -\{(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi); \mathbf{A}\circ\mathbf{B}\} \equiv \{\mathbf{S}, h\}. \qquad (6.13)$$

If $div_4\mathbf{T} = 2\ \mathbf{E}\circ\mathbf{B} \neq 0$, the electromagnetic 1-form, A, defines a domain of Pfaff dimension 4. Such domains cannot support topologically transverse magnetic waves ($as\ \ A\hat{}F \neq 0$). Evolutionary processes (including plasma currents) that are proportional to the Torsion current are thermodynamically irreversible, if $\sigma = \mathbf{E}\circ\mathbf{B} \neq 0$. However, the conformal properties of evolution in the direction of the Torsion current lead to extraordinary properties when the plasma current is in the direction of the Torsion vector. From the thermodynamic arguments in section 6.1 based on the postulate of potentials, but using the notation of an electromagnetic system

$$L_{(\mathbf{T}_4)}A = \sigma A = (\mathbf{E}\circ\mathbf{B})A \qquad (6.14)$$

and

$$L_{(\mathbf{T}_4)}(A\hat{}F) = 2\sigma A = 2(\mathbf{E}\circ\mathbf{B})A\hat{}F. \qquad (6.15)$$

Hence, it follows that motion along the direction of the torsion vector freezes-in the lines of the torsion vector in space time, but the process is irreversible unless the second Poincare invariant is zero.

Recall that the definition of a plasma current, J, is equivalent to an evolutionary process such that

$$\text{Definition of a plasma Current } J: \qquad L_{(J)}G = 0. \tag{6.16}$$

Hence consider a plasma current which is also in the direction of the Torsion vector. Then

$$L_{(J)}A\hat{}G = (L_{(J)}A)\hat{}G + A\hat{}L_{(J)}G = (L_{(\gamma\mathbf{T}_4)}A)\hat{}G + A\hat{}L_{(J)}G \tag{6.17}$$
$$= \gamma \cdot (\mathbf{E} \circ \mathbf{B}) \;\; A\hat{}G + 0 \tag{6.18}$$

Hence for plasma motions in the direction of the (possibly dissipative) torsion vector, both the "lines" of the Spin vector are "frozen in" and the lines of the Torsion vector are "frozen in". Such "frozen in" objects can be used to give a topological definition of deformable coherent structures in a plasma. Moreover, as the evolutionary process causes the frozen in structures to deform and decay, it is conceivable that evolution could proceed to form stationary (not stagnant) states (where $\mathbf{E} \circ \mathbf{B} \Rightarrow 0$), such that the frozen in field line structures become local deformation invariants, or topological defects. Electromagnetic coherent structures are evolutionary deformable (and perhaps decaying) domains of Pfaff dimension 4, which form stationary states of topological defects (including the null state) in regions of Pfaff dimension 3, where $\mathbf{E} \circ \mathbf{B} = 0$.

Note that all semi-ideal (see section 5) plasma current processes are reversible in a thermodynamic sense.

References

[1] Bryant, R.L.,Chern, S.S., Gardner, R.B.,Goldschmidt, H.L., and Griffiths, P. A. (1991), Exterior Differential Systems, Springer Verlag

[2] K. Kuratowski, *Topology* (Warsaw, 1948), Vol. I.

Lipschutz, S. (1965) *General Topology* (Schaum, New York), 88.

References

[1] Cartan, E., (1958) *Lecons sur les invariants integrauxs*, Hermann, Paris .

[2] Marsden, J.E. and Riatu, T. S. *(1994) Introduction to Mechanics and Symmetry,* Springer-Verlag, p.122

[3] Schouten, J. A. and Van der Kulk, W., (1949) *Pfaff's Problem and its Generalizations*, Oxford Clarendon Press

[4] Kiehn, R. M. (1977) "Periods on manfolds, quantization and gauge", J. of Math Phys **18**, no. 4, p. 614

[5] Kiehn, R. M., (1990) "Topological Torsion, Pfaff Dimension and Coherent Structures", in: H. K. Moffatt and T. S. Tsinober eds, *Topological Fluid Mechanics*, Cambridge University Press, 449-458 .

[6] deRham,G. (1960) *Varietes Differentiables*, Hermann, Paris

[7] Fiber optics

[8] Hornig, G. and Schindler, K. (1996) "Magnetic topology and the problem of its invaraint definition" Physics of Plasmas, **3**, p.646

[9] Sommerfeld, A., (1952) *Electrodynamics,* Academic Press, New York. Also see Stratton, J.A., (1941) *Electromagnetic Theory* McGraw Hill N.Y. Sommerfeld carefully distingushes between intensities and excitations on thermodynamic grounds.

[10] Kiehn, R. M., Kiehn, G. P., and Roberds, R. B. (1991) "Parity and Time-reversal Symmetry Breaking, Singular Solutions", Phys Rev A, **43**, p. 5665

[11] Kiehn, R. M. (1991) "Are there three kinds of superconductivity" Int. J. Mod. Phys B Vol. 5 p.1779-1790. Also see Kiehn, R. M. and Pierce, J. F. (1969) "An Intrinsic Transport Theorem" Phys. Fluids **12**, p. 1971

[12] Van Dantzig. D., Proc. Cambridge Philos. Soc. **30**, 421 (1934). Also see: von Dantzig, D., 1934, "Electromagnetism Independent of metrical geometry", Proc. Kon. Ned. Akad. v. Wet. 37,

[13] Kiehn, R. M. (1976) "Retrodictive Determinism", Int. J. of Eng. Sci. **14**, p. 749

[14] Lipschutz, S. (1965) *General Topology* (Schaum, New York), 88.

[15] Bateman, H. (1914, 1955) *Electrical and Optical Wave Motion*, Dover p.12

[16] Kochin, N.E., Kibel, I.A., Roze, N. N., (1964) *Theoretical Hydrodynamics*, Interscience, NY p.157

[17] Kiehn, R. M. (1975) "Intrinsic hydrodynamics with applications to space time fluids", Int. J. Engng Sci **13**, p. 941-949