>

CUT AND CONNECT

R.M. Kiehn
Physics Department
University of Houston
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jjkrmk@swbell.net

connect.mws

The purpose of this article is to study topological features and the topological evolution of certain electromagnetic fields in the neighborhood of the origin. The examples are designed to develop insight into the problem of reconnection in plasmas. In order to overcome the algebraic complexities, a Maple symbolic mathematics program is devised to compute the fields and functions needed.

This Maple program starts from a given input set of functions describing the 4-vector of potentials (the vector and the scalar potentials over space-time).
The program constructs the 1-form of Action based in these potentials, and all elements of the Pfaff sequence. A, F=dA, A^F, dA^dA=F^F.
The non-zero elements of the 2-form F =dA are the components of the E and B fields.
The non-zero elements of the 3-form A^F define the Topological Torsion 3-form.
The non-zero element of the 4 form F^F generates the Second Poincare invariant (E.B)

Then an assumption is made for a constitutive set of equations, permitting the computation of the elements of G, or the components of the field excitations, D and H.
The program then computes the charge-current density, J, rho
as the exterior derivative of J=dG
The 3-form of Spin is defined as A^G
The 4 form d(A^G) = F^H-A^J defines the first Poincare invariant.

The 3-form fields of Spin and Torsion are shown to exist, and their utility is made evident.

The output is also expressed in Engineering Format, but has its topological foundations in the differential form description. The Spin and Torsion field are not just arbitrary constructs of vector fields, they have important topological significance.

There are eight basic cases of symmetry (in two categories) to study in the vicinity of the origin. For a given category, the Magnetic Field in the z = 0 plane, at some time in its evolution exhibits the origin as a saddle, a source, a rotation center, or a sink of vector field lines.

The saddle situation (as mentioned by Hornig) is the most interesting to the reconnection problem

Category 1: B field evolution induces a rotation (plus or minus), or a saddle.

```
> restart:with(plots):with(DEtools):with(linalg):with(difforms):
  with(liesymm):with(plots):setup(x,y,z,t,s,r):defform(x=0,y=0,z=0,r=0,t=0,s=0,a=c
  onst,b=const,c=const,k=const,mu=const,omega=const,m=const,n=const,
  A=const,B=const,eta=const):
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for `&^`
Warning, new definition for close
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for translate
Warning, new definition for wdegree
> T:=1/2*(1+tanh(t));F:=B*tanh(z);G:=(tanh(t)^2-1)*C/(x^2+y^2+z^2)^(2/2):
```

$$T := \frac{1}{2} + \frac{1}{2}\tanh(t)$$

$$F := B\tanh(z)$$

Choose you own functions for the 4 vector of potentials here. The time history is given by the kink function T. The constants m and n are presumed to take on values plus or minus 1. If m = +n, the origin becomes a saddle. If m = -n the origin becomes a rotation. The rotation can be left handed or right handed..

```
> A1:=m*(x)*(1-T)*(F)+0*G;A2:=-n*T*(y)*F+0*G;A3:=0*G;A4:=+0;Action:=A1*d(x)+A2*d(y
  )+A3*d(z)-A4*d(t);
```

$$A1 := m\,x\left(\frac{1}{2} - \frac{1}{2}\tanh(t)\right)B\tanh(z)$$

$$A2 := -n\left(\frac{1}{2} + \frac{1}{2}\tanh(t)\right)y\,B\tanh(z)$$

$$A3 := 0$$

$$A4 := 0$$

$$Action := m\,x\left(\frac{1}{2} - \frac{1}{2}\tanh(t)\right)B\tanh(z)\,\mathrm{d}(x) - n\left(\frac{1}{2} + \frac{1}{2}\tanh(t)\right)y\,B\tanh(z)\,\mathrm{d}(y)$$

Next compute the Pfaff sequence, the EB field 2-form, the Torsion 3-form, and its divergence. F^F.

```
> EBFields:=(d(Action));
  TORS:=simplify(Action&^EBFields);Poincare2:=simplify(EBFields&^EBFields);
```

$$EBFields := -\frac{1}{2}m\,x\,(\tanh(t)^2 - 1)\,B\tanh(z)\,((\mathrm{d}(x))\ \&\hat{}\ (\mathrm{d}(t))) - \frac{1}{2}m\,x\,(-1 + \tanh(t))\,B\,(-1 + \tanh(z)^2)\,((\mathrm{d}(x))\ \&\hat{}\ (\mathrm{d}(z))) - \frac{1}{2}n\,(\tanh(t)^2 - 1)\,y\,B\tanh(z)\,((\mathrm{d}(y))\ \&\hat{}\ (\mathrm{d}(t))) - \frac{1}{2}n\,(1 + \tanh(t))\,y\,B\,(-1 + \tanh(z)^2)\,((\mathrm{d}(y))\ \&\hat{}\ (\mathrm{d}(z)))$$

$$TORS := \frac{1}{2}\frac{n\,y\,B^2\,m\,x\ \&\hat{}(\mathrm{d}(x), \mathrm{d}(y), \mathrm{d}(t))\,(\cosh(z)^2 - 1)}{\cosh(t)^2\cosh(z)^2}$$

$$Poincare2 := \frac{n\, y\, B^2\, m\, x\, \sinh(z)\ \&\text{^}(\mathrm{d}(x), \mathrm{d}(y), \mathrm{d}(z), \mathrm{d}(t))}{\cosh(t)^2 \cosh(z)^3}$$

Repeat the computation in Engineering format.

```
> APOT2:=evalm([A1,A2,A3]):
> Bf2:=evalm(curl(APOT2,[x,y,z])):B1:=simplify(Bf2[1]);B2:=simplify(Bf2[2]);B3:=fa
  ctor(Bf2[3]);Helicity2:=factor(innerprod(APOT2,Bf2));Ef2:=evalm([diff(-A1,t)-dif
  f(A4,x),diff(-A2,t)-diff(A4,y),diff(-A3,t)-diff(A4,z)]):E1:=simplify(Ef2[1]);E2:
  =simplify(Ef2[2]);E3:=simplify(Ef2[3]);EotB:=simplify(innerprod(Ef2,Bf2));
```

$$B1 := \frac{1}{2} \frac{n\, y\, B\, (\cosh(t) + \sinh(t))}{\cosh(t) \cosh(z)^2}$$

$$B2 := \frac{1}{2} \frac{m\, x\, B\, (\cosh(t) - \sinh(t))}{\cosh(t) \cosh(z)^2}$$

$$B3 := 0$$

$$Helicity2 := 0$$

$$E1 := \frac{1}{2} \frac{m\, x\, B \sinh(z)}{\cosh(t)^2 \cosh(z)}$$

$$E2 := \frac{1}{2} \frac{n\, y\, B \sinh(z)}{\cosh(t)^2 \cosh(z)}$$

$$E3 := 0$$

$$EotB := \frac{1}{2} \frac{m\, x\, B^2 \sinh(z)\, n\, y}{\cosh(t)^2 \cosh(z)^3}$$

The B field in the z= 0 plane starts off at t = minus infinity with only a y component proportional to -mx. As t goes to plus infinity the B field has only a x component proportional to -ny. At t = 0 the B field lines are such that there is a saddle point at the origin for m=n and a rotational center for m= -n.

In the next three figures, the direction fields have the topology of a distorted saddle (in the z = 0 plane). Starting with two initial conditions, [x=1,y=1] and [x=-1,y=-1], two trajectories are actually computed for the early time t= - 1.5, t= 0 and the late time t = +1.5. The "field lines" (corresponding to the "stream lines" of the B field, undergo an apparent "CUT and CONNECT" process.

The Helicity AdotB is zero for the example field.
Note the time dependent "burst" of the E field as a pulse, leading to a pulse of E dot B

Use the Vacuum consitutive equations B = mu H D = epsilon E to construct the Amperian currents
The TIME DEPENDENT amperian current in the z direction is not zero, and a burst of charge density is created if m = n (the saddle case)

```
> Jf2:=curl([B1,B2,B3]/mu,[x,y,z]):J1:=factor(Jf2[1]);J2:=factor(Jf2[2]);J3:=facto
  r(Jf2[3]);
```

$$J1 := -\frac{m\, x\, B\, (-\cosh(t) + \sinh(t)) \sinh(z)}{\mu \cosh(t) \cosh(z)^3}$$

$$J2 := -\frac{n\, y\, B\, (\cosh(t) + \sinh(t)) \sinh(z)}{\mu \cosh(t) \cosh(z)^3}$$

$$J3 := -\frac{1}{2}\frac{B\,(-m\cosh(t)+m\sinh(t)+n\cosh(t)+n\sinh(t))}{\mu\cosh(t)\cosh(z)^2}$$

Construct the displacement currents. Note that the Amperian current in the z direction is not zero

```
> rho:=simplify(epsilon*diverge(Ef2,[x,y,z]));JDf2:=evalm([diff(Ef2[1],t),diff(Ef2[2],t),diff(Ef2[3],t)]):
```

$$\rho := \frac{1}{2}\frac{\varepsilon\,B\sinh(z)\,(m+n)}{\cosh(t)^2\cosh(z)}$$

If m=n then there is a burst of charge density (the saddle case) If m=-n (the rotational case) the charge density is always zero, typical of a plasma.

Add the displacement currents to the Amperian currents to obtain the total current.

```
> Jtot:=evalm(Jf2-epsilon*JDf2):Jtot1:=simplify(Jtot[1]);Jtot2:=simplify(Jtot[2]);Jtot3=factor(Jtot[3]);
```

$$Jtot1 := -\frac{m\,x\,B\sinh(z)\,(-\cosh(t)^3+\cosh(t)^2\sinh(t)-\varepsilon\sinh(t)\,\mu\cosh(z)^2)}{\mu\cosh(t)^3\cosh(z)^3}$$

$$Jtot2 := -\frac{n\,y\,B\sinh(z)\,(\cosh(t)^3+\cosh(t)^2\sinh(t)-\varepsilon\sinh(t)\,\mu\cosh(z)^2)}{\mu\cosh(t)^3\cosh(z)^3}$$

$$Jtot3 = -\frac{1}{2}\frac{B\,(-m\cosh(t)+m\sinh(t)+n\cosh(t)+n\sinh(t))}{\mu\cosh(t)\cosh(z)^2}$$

```
> JxB:=crossprod(Jtot,Bf2):
```

```
> JxBX:=factor(JxB[1]);JxBY:=factor(JxB[2]);JxBZ:=simplify(JxB[3]);
```

$$JxBX := \frac{1}{4}\frac{B^2\,(-m\cosh(t)+m\sinh(t)+n\cosh(t)+n\sinh(t))\,m\,x\,(-1+\tanh(t))\,(\tanh(z)-1)\,(\tanh(z)+1)}{\mu\cosh(t)\cosh(z)^2}$$

$$JxBY := \frac{1}{4}\frac{B^2\,(-m\cosh(t)+m\sinh(t)+n\cosh(t)+n\sinh(t))\,n\,(1+\tanh(t))\,y\,(\tanh(z)-1)\,(\tanh(z)+1)}{\mu\cosh(t)\cosh(z)^2}$$

$$JxBZ := -\frac{1}{2}B^2\sinh(z)\,(-m^2x^2\varepsilon\,\mu\cosh(z)^2-2\,m^2x^2\cosh(t)^4+m^2x^2\cosh(t)^2-m^2x^2\varepsilon\,\mu\cosh(z)^2\cosh(t)\sinh(t)$$
$$+m^2x^2\varepsilon\,\mu\cosh(z)^2\cosh(t)^2-2\,n^2y^2\cosh(t)^3\sinh(t)-n^2y^2\varepsilon\,\mu\cosh(z)^2+n^2y^2\varepsilon\,\mu\cosh(z)^2\cosh(t)\sinh(t)$$
$$+n^2y^2\varepsilon\,\mu\cosh(z)^2\cosh(t)^2-2\,n^2y^2\cosh(t)^4+n^2y^2\cosh(t)^2+2\,m^2x^2\cosh(t)^3\sinh(t))\Big/(\mu\cosh(t)^4\cosh(z)^5)$$

Magnetic Energy, Electric Enery and the Interaction Energy

The system is dominated by magnetic energy density at large plus and minus values of time.

```
> MagE2:=simplify(innerprod(Bf2,Bf2)/mu);
```

$$MagE2 :=$$
$$-\frac{1}{4}\frac{B^2\,(2\,m^2x^2\sinh(t)\cosh(t)-2\,m^2x^2\cosh(t)^2-2\,n^2y^2\cosh(t)^2-2\,n^2y^2\sinh(t)\cosh(t)+n^2y^2+m^2x^2)}{\cosh(t)^2\cosh(z)^4\,\mu}$$

The electric energy undergoes a burst at t = 0

```
> ElecE1:=factor(epsilon*innerprod(Ef2,Ef2));
```

$$ElecE1 := \frac{1}{4}\varepsilon B^2 \tanh(z)^2 (-1+\tanh(t))^2 (1+\tanh(t))^2 (n^2 y^2 + m^2 x^2)$$

The Intereaction energy is AdotJ - rho phi

```
> IntE1:=(factor(innerprod(APOT2,Jtot)));
```

$$IntE1 := \frac{1}{2} B^2 \tanh(z) (-m^2 x^2 \tanh(t) \sinh(z) \cosh(t) + m^2 x^2 \tanh(t) \sinh(z) \sinh(t) - m^2 x^2 \varepsilon \tanh(t)^2 \tanh(z) \mu \cosh(t) \cosh(z)^3 + m^2 x^2 \varepsilon \tanh(t)^4 \tanh(z) \mu \cosh(t) \cosh(z)^3 + m^2 x^2 \sinh(z) \cosh(t) - m^2 x^2 \sinh(z) \sinh(t) + m^2 x^2 \varepsilon \tanh(t) \tanh(z) \mu \cosh(t) \cosh(z)^3 - m^2 x^2 \varepsilon \tanh(t)^3 \tanh(z) \mu \cosh(t) \cosh(z)^3 + n^2 y^2 \tanh(t) \sinh(z) \cosh(t) + n^2 y^2 \tanh(t) \sinh(z) \sinh(t) - n^2 y^2 \varepsilon \tanh(t)^2 \tanh(z) \mu \cosh(t) \cosh(z)^3 + n^2 y^2 \varepsilon \tanh(t)^4 \tanh(z) \mu \cosh(t) \cosh(z)^3 + n^2 y^2 \sinh(z) \cosh(t) + n^2 y^2 \sinh(z) \sinh(t) - n^2 y^2 \varepsilon \tanh(t) \tanh(z) \mu \cosh(t) \cosh(z)^3 + n^2 y^2 \varepsilon \tanh(t)^3 \tanh(z) \mu \cosh(t) \cosh(z)^3) \Big/ (\mu \cosh(t) \cosh(z)^3)$$

\>

The topological TORSION 4-vector:

```
> Tors:=evalm(crossprod(Ef2,APOT2)+A4*Bf2):TORSION:=([factor(Tors[1]),factor(Tors[
2]),simplify(Tors[3]),Helicity2]);
```

$$TORSION := \left[0, 0, -\frac{1}{2}\frac{m\, x\, B^2\, n\, y\, (\cosh(z)^2 - 1)}{\cosh(t)^2 \cosh(z)^2}, 0\right]$$

\>

The spatial part of the Torsion vector is NOT Zero, even though the Helicity density (4th component of the Torsion 3-form) is zero. The third component of the torson has a time-dependent burst like quality.

The topological Spin 4-vector

```
>
> Spin:=crossprod(APOT2,Bf2):sp1:=factor(Spin[1]);sp2:=factor(Spin[2]);sp3:=simpli
fy(subs((Spin[3])));Poincare1:=simplify(diverge(Spin,[x,y,z]));
```

$$sp1 := 0$$

$$sp2 := 0$$

$$sp3 := -\frac{1}{4}\frac{B^2 \sinh(z) (2 m^2 x^2 \sinh(t) \cosh(t) - 2 m^2 x^2 \cosh(t)^2 - 2 n^2 y^2 \cosh(t)^2 - 2 n^2 y^2 \sinh(t) \cosh(t) + n^2 y^2 + m^2 x^2)}{\cosh(t)^2 \cosh(z)^3}$$

$$Poincare1 := \frac{1}{4} B^2 (-3 m^2 x^2 + 4 \cosh(z)^2 m^2 x^2 \cosh(t) \sinh(t) - 4 \cosh(z)^2 m^2 x^2 \cosh(t)^2 + 2 \cosh(z)^2 m^2 x^2 - 4 \cosh(z)^2 n^2 y^2 \cosh(t)^2 + 2 \cosh(z)^2 n^2 y^2 - 4 \cosh(z)^2 n^2 y^2 \cosh(t) \sinh(t) - 6 m^2 x^2 \sinh(t) \cosh(t) + 6 m^2 x^2 \cosh(t)^2 + 6 n^2 y^2 \cosh(t)^2 + 6 n^2 y^2 \sinh(t) \cosh(t) - 3 n^2 y^2) \Big/ (\cosh(t)^2 \cosh(z)^4)$$

The spin is only along the z axis, and exists as a pulse that starts with a value of almost zero and ends up with a value of almost zero.

SADDLE CASE m = n

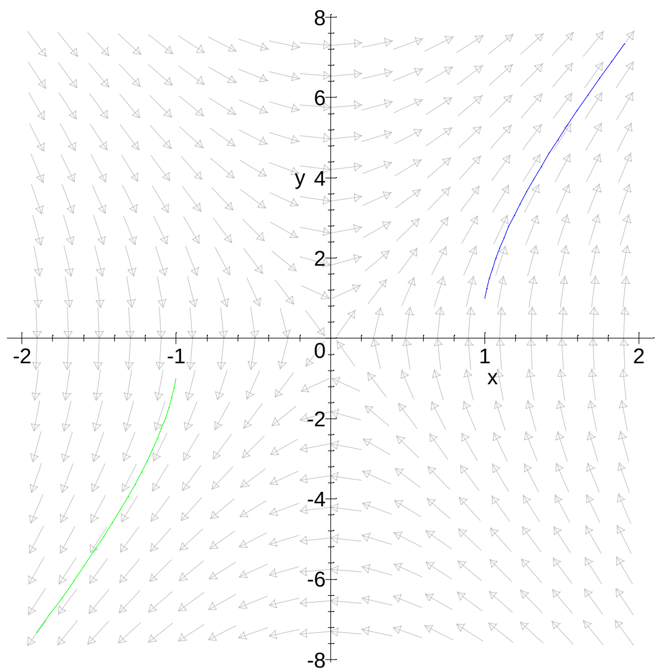
```
> BX:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-1.5,z=0.0,(B1)))):
> BY:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-1.5,z=0.0,(B2)))):
> BZ:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-1.5,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BX,diff(y(t),t)=BY], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat
```

```
  1 B field at t = -1.5, m=n`,colour=grey,linecolor = [blue,green]);
```

EARLY TIME

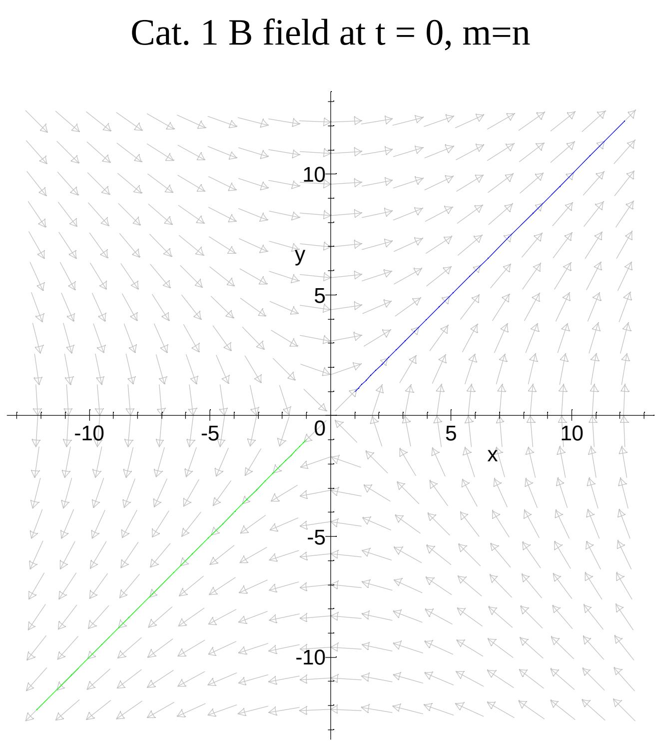
```
>
Warning, new definition for translate
```

Cat 1 B field at t = -1.5, m=n



Category 1: Plot is at z=0 for t = - 1.5, m = n. The plot is the value of the B field lines at constant t. These are streamlines not streaklines. The x,y scales are distorted. This is the evolutionary saddle at early times. Two flow lines are shown for initial conditions of [x=1,y=1] in blue and [x=-1,b=-1] in green

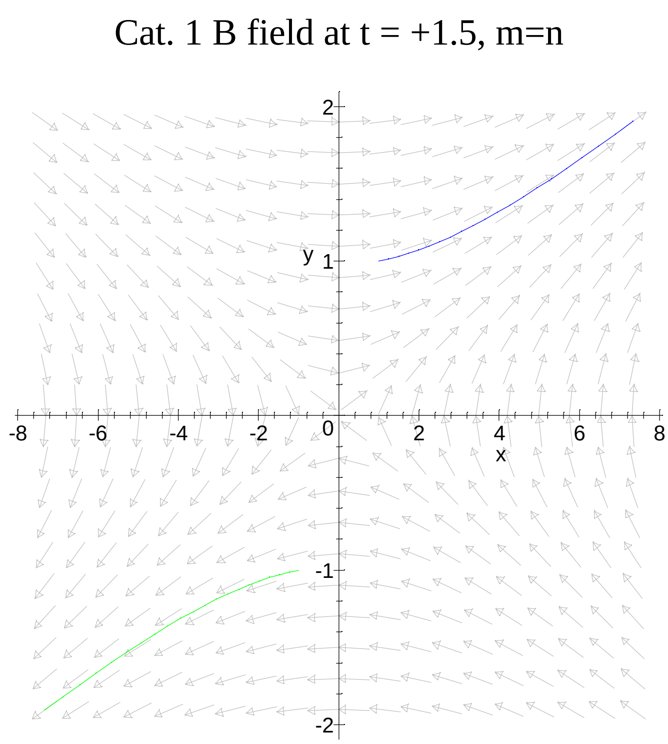
```
> BX:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-0,z=0.0,(B1)))):
> BY:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-0,z=0.0,(B2)))):
> BZ:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-0,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BX,diff(y(t),t)=BY], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  1 B field at t = 0, m=n`,colour=grey,linecolor = [blue,green]);
```



Category 1: Plot is at z= 0 for t = 0, m= n. The plot is the value of the B field lines at constant t. (stream lines, not streak lines) The x,y scales are not constrained. This is the evolutionary saddle mode for t=0. Two flow lines are shown for initial conditions of [x=1,y=1] in blue and [x=-1,b=-1] in green

```
>
> BX:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=1.5,z=0.0,(B1)))):
> BY:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=1.5,z=0.0,(B2)))):
> BZ:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=1.5,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BX,diff(y(t),t)=BY], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  1 B field at t = +1.5, m=n`,colour=grey,linecolor = [blue,green]);
```

LATE TIME



Category 1: Plot is at z= 0 for t = + 1.5, m= n. The plot is the value of the B field lines at constant t. (stream lines, not streak lines) The x,y scales are not constrained. This is the evolutionary saddle mode for late times > 0.

AN APPARENT CUT AND CONNECT OF THE B FIELD LINES HAS TAKEN PLACE AS t EVOLVES FROM NEGATIVE TO POSITIVE VALUES.

For m = n , there is a burst of charge density, a burst of EdotB, and a burst of torsion as the apparent cut and connect process takes place.

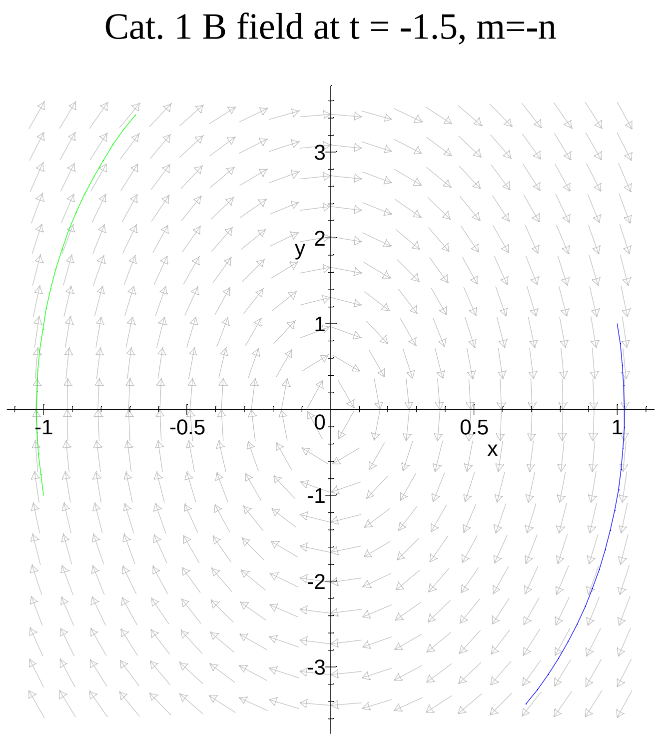
At t=0, the saddle features are still preserved, with out x-y scale distortions.
The initial point [x=1,y=1], at t = 0, lies on the separatrix.

```
>
```

ROTATIONAL CASE m = -n

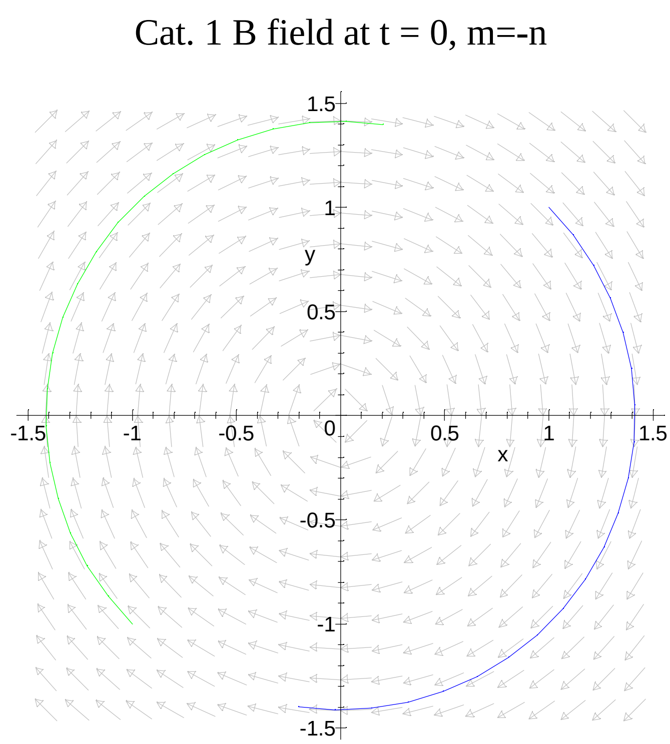
```
> BX:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-1.5,z=0.0,(B1)))):
> BY:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-1.5,z=0.0,(B2)))):
> BZ:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-1.5,z=0.0,(B3)))):
>
> with(DEtools):
  DEplot([diff(x(t),t)=BX,diff(y(t),t)=BY], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  1 B field at t = -1.5, m=-n`,colour=grey,linecolor = [blue,green]);
```

EARLY TIME



Category 1: Plot is at z= 0 for t = - 1.5, m= - n. The plot is the value of the B field lines at constant t. (stream lines, not streak lines) The x,y scales are not constrained. This is the evolutionary rotational mode for early times < 0.

```
>
>
> BX:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-0,z=0.0,(B1)))):
> BY:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-0,z=0.0,(B2)))):
> BZ:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-0,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BX,diff(y(t),t)=BY], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  1 B field at t = 0, m=-n`,colour=grey,linecolor = [blue,green]);
```



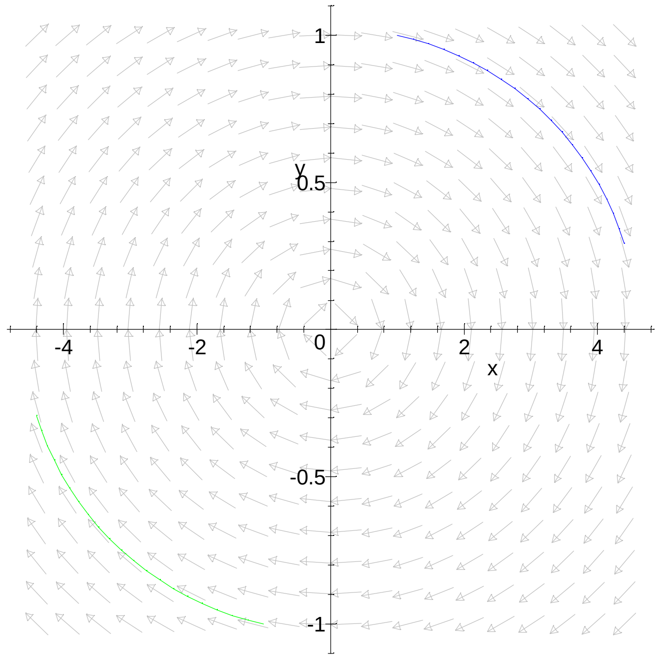
Category 1: The plot is a z=0 for t=0 and m= - n. The plot is of the B field streamlines at a fixed t. This is the rotational mode at t = 0

```
>
> BX:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=1.5,z=0.0,(B1)))):
> BY:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=1.5,z=0.0,(B2)))):
> BZ:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=1.5,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BX,diff(y(t),t)=BY], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  1 B field at t = +1.5, m=-n`,colour=grey,linecolor = [blue,green]);
```

LATE TIME

```
>
```

Cat. 1 B field at t = +1.5, m=-n



Category 1: Plot is at z= 0 for t =1.5, m= - n. The plot is the value of the B field lines at constant t. (stream lines, not streak lines) The x,y scales are not constrained. This is the evolutionary rotational mode for late times t > 0.

Category 2: B field evolution induces a source or a sink, or a saddle.

```
> restart:with(plots):with(DEtools):with(linalg):with(difforms):
  with(liesymm):with(plots):setup(x,y,z,t,s,r):defform(x=0,y=0,z=0,r=0,t=0,s=0,a=c
  onst,b=const,c=const,k=const,mu=const,omega=const,m=const,
  n=const,A=const,B=const,eta=const):
Warning, new definition for norm
Warning, new definition for trace
Warning, new definition for `&^`
Warning, new definition for close
Warning, new definition for d
Warning, new definition for mixpar
Warning, new definition for translate
Warning, new definition for wdegree
> T:=1/2*(1+tanh(t));F:=(B*tanh(z));G:=0*(tanh(t)^2-1)*C/(x^2+y^2+z^2)^(2/2):
```

$$T := \frac{1}{2} + \frac{1}{2}\tanh(t)$$

$$F := B\tanh(z)$$

```
>
> A1:=-m*y*(1-T)*(F)+G;A2:=-n*T*x*F+G;A3:=G;A4:=+0;Action:=A1*d(x)+A2*d(y)+A3*d(z)
-A4*d(t);
```

$$A1 := -m\,y\left(\frac{1}{2}-\frac{1}{2}\tanh(t)\right)B\tanh(z)$$

$$A2 := -n\left(\frac{1}{2}+\frac{1}{2}\tanh(t)\right)x\,B\tanh(z)$$

$$A3 := 0$$

$$A4 := 0$$

$$Action := -m\,y\left(\frac{1}{2}-\frac{1}{2}\tanh(t)\right)B\tanh(z)\,\mathrm{d}(x)-n\left(\frac{1}{2}+\frac{1}{2}\tanh(t)\right)x\,B\tanh(z)\,\mathrm{d}(y)$$

```
> EBFields:=(d(Action));
TORS:=simplify(Action&^EBFields);Poincare2:=simplify(EBFields&^EBFields);
```

$$EBFields := -\frac{1}{2}m\,(-1+\tanh(t))\,B\tanh(z)\,((\mathrm{d}(x))\ \&\wedge\ (\mathrm{d}(y)))$$
$$-\frac{1}{2}m\,y\,(\tanh(t)^2-1)\,B\tanh(z)\,((\mathrm{d}(t))\ \&\wedge\ (\mathrm{d}(x)))$$
$$-\frac{1}{2}m\,y\,(-1+\tanh(t))\,B\,(-1+\tanh(z)^2)\,((\mathrm{d}(z))\ \&\wedge\ (\mathrm{d}(x)))$$
$$+\frac{1}{2}n\,(\tanh(t)^2-1)\,x\,B\tanh(z)\,((\mathrm{d}(t))\ \&\wedge\ (\mathrm{d}(y)))-\frac{1}{2}n\,(1+\tanh(t))\,B\tanh(z)\,((\mathrm{d}(x))\ \&\wedge\ (\mathrm{d}(y)))$$
$$+\frac{1}{2}n\,(1+\tanh(t))\,x\,B\,(-1+\tanh(z)^2)\,((\mathrm{d}(z))\ \&\wedge\ (\mathrm{d}(y)))$$

$$TORS := -\frac{1}{2}\frac{n\,x\,B^2\,m\,y\ \&\wedge(\mathrm{d}(t),\mathrm{d}(x),\mathrm{d}(y))\,(\cosh(z)^2-1)}{\cosh(t)^2\cosh(z)^2}$$

$$Poincare2 := -\frac{m\,y\,B^2\sinh(z)\,n\,x\ \&\wedge(\mathrm{d}(z),\mathrm{d}(t),\mathrm{d}(x),\mathrm{d}(y))}{\cosh(t)^2\cosh(z)^3}$$

Repeat computation in Engineering format.

```
> APOT2:=evalm([A1,A2,A3]):
> Bf2:=evalm(curl(APOT2,[x,y,z])):B1:=simplify(Bf2[1]);B2:=simplify(Bf2[2]);B3:=fa
ctor(Bf2[3]);Helicity2:=factor(innerprod(APOT2,Bf2));Ef2:=evalm([diff(-A1,t)-dif
f(A4,x),diff(-A2,t)-diff(A4,y),diff(-A3,t)-diff(A4,z)]):E1:=simplify(Ef2[1]);E2:
=simplify(Ef2[2]);E3:=factor(Ef2[3]);EotB:=simplify(innerprod(Ef2,Bf2));
```

$$B1 := \frac{1}{2}\frac{n\,x\,B\,(\cosh(t)+\sinh(t))}{\cosh(t)\cosh(z)^2}$$

$$B2 := -\frac{1}{2}\frac{m\,y\,B\,(\cosh(t)-\sinh(t))}{\cosh(t)\cosh(z)^2}$$

$$B3 := -\frac{1}{2}\tanh(z)\,B\,(n+n\tanh(t)-m+m\tanh(t))$$

$$Helicity2 := 0$$

$$E1 := -\frac{1}{2}\frac{m\,y\,B\sinh(z)}{\cosh(t)^2\cosh(z)}$$

$$E2 := \frac{1}{2}\frac{n\,x\,B\sinh(z)}{\cosh(t)^2\cosh(z)}$$

$$E3 := 0$$

$$EotB := -\frac{1}{2}\frac{m\,y\,B^2\sinh(z)\,n\,x}{\cosh(z)^3\cosh(t)^2}$$

The Helicity AdotB is zero for the example field.
Note the burst of the E field as a pulse, as well as E dot B .
B3 is not zero.
When m = n the B field lines in the z=0 plane have a saddle singularity at t=0.. When m = - n the B field lines have a source or a sink singularity at the origin at t = 0.

Use the Vacuum consitutive equations B = mu H D = epsilon E to construct the Amperian currents
The amperian current in the z direction is constant in time.

```
> Jf2:=curl([B1,B2,B3]/mu,[x,y,z]):J1:=(Jf2[1]);J2:=(Jf2[2]);J3:=simplify(Jf2[3]);
```

$$J1 := -\frac{m\,y\,B\,(\cosh(t)-\sinh(t))\sinh(z)}{\mu\cosh(t)\cosh(z)^3}$$

$$J2 := -\frac{n\,x\,B\,(\cosh(t)+\sinh(t))\sinh(z)}{\mu\cosh(t)\cosh(z)^3}$$

$$J3 := 0$$

Construct the displacement currents.

```
> rho:=simplify(diverge(Ef2,[x,y,z]));JDf2:=evalm([diff(Ef2[1],t),diff(Ef2[2],t),d
  iff(Ef2[3],t)]):
```

$$\rho := 0$$

The charge density remains zero in every case. Add the displacement currents to the Amperian currents to obtain the total current.

```
> Jtot:=evalm(Jf2-epsilon*JDf2):Jtot1:=(Jtot[1]);Jtot2:=Jtot[2];Jtot3=factor(Jtot[
  3]);
```

$$Jtot1 := -\frac{m\,y\,B\,(\cosh(t)-\sinh(t))\sinh(z)}{\mu\cosh(t)\cosh(z)^3} - \varepsilon\,m\,y\tanh(t)\,(1-\tanh(t)^2)\,B\tanh(z)$$

$$Jtot2 := -\frac{n\,x\,B\,(\cosh(t)+\sinh(t))\sinh(z)}{\mu\cosh(t)\cosh(z)^3} + \varepsilon\,n\tanh(t)\,(1-\tanh(t)^2)\,x\,B\tanh(z)$$

$$Jtot3 = 0$$

Note that the z component of current is zero.
Compare to Category 1 examples given above.

```
> JxB:=crossprod(Jtot,Bf2):
```

```
> JxBX:=factor(JxB[1]);JxBY:=factor(JxB[2]);JxBZ:=factor(JxB[3]);
```

$$JxBX := \frac{1}{2}n\,x\,B^2 \left(\sinh(z)\cosh(t)+\sinh(z)\sinh(t)-\varepsilon\tanh(t)\tanh(z)\,\mu\cosh(t)\cosh(z)^3+\varepsilon\tanh(t)^3\tanh(z)\,\mu\cosh(t)\cosh(z)^3\right) \tanh(z)\,(n+n\tanh(t)-m+m\tanh(t)) \Big/ \left(\mu\cosh(t)\cosh(z)^3\right)$$

$$JxBY := \frac{1}{2} m\, y\, B^2 \big(-\sinh(z)\cosh(t) + \sinh(z)\sinh(t) - \varepsilon \tanh(t)\tanh(z)\,\mu \cosh(t)\cosh(z)^3 + \varepsilon \tanh(t)^3 \tanh(z)\,\mu\cosh(t)\cosh(z)^3\big) \tanh(z)\,(n + n\tanh(t) - m + m\tanh(t)) \,/\, (\mu \cosh(t)\cosh(z)^3)$$

$$JxBZ := -\frac{1}{2} B^2 (\tanh(z)-1)(\tanh(z)+1)\big(m^2 y^2 \sinh(z)\cosh(t) - m^2 y^2 \sinh(z)\cosh(t)\tanh(t) - m^2 y^2 \sinh(z)\sinh(t) + m^2 y^2 \sinh(z)\sinh(t)\tanh(t) + m^2 y^2 \varepsilon \tanh(t)\tanh(z)\,\mu\cosh(t)\cosh(z)^3 - m^2 y^2 \varepsilon\tanh(t)^2\tanh(z)\,\mu\cosh(t)\cosh(z)^3 - m^2 y^2 \varepsilon \tanh(t)^3\tanh(z)\,\mu\cosh(t)\cosh(z)^3 + m^2 y^2 \varepsilon\tanh(t)^4\tanh(z)\,\mu\cosh(t)\cosh(z)^3 + n^2 x^2 \sinh(z)\cosh(t) + n^2 x^2\sinh(z)\cosh(t)\tanh(t) + n^2 x^2 \sinh(z)\sinh(t) + n^2 x^2 \sinh(z)\sinh(t)\tanh(t) - n^2 x^2 \varepsilon\tanh(t)\tanh(z)\,\mu\cosh(t)\cosh(z)^3 - n^2 x^2 \varepsilon \tanh(t)^2\tanh(z)\,\mu\cosh(t)\cosh(z)^3 + n^2 x^2\varepsilon\tanh(t)^3\tanh(z)\,\mu\cosh(t)\cosh(z)^3 + n^2 x^2 \varepsilon\tanh(t)^4\tanh(z)\,\mu\cosh(t)\cosh(z)^3\big) \,/\, (\mu\cosh(t)\cosh(z)^3)$$

Magnetic Energy, Electric Enery and the Interaction Energy

```
> MagE2:=factor(innerprod(Bf2,Bf2)/mu);
```

$$MagE2 := \frac{1}{4} B^2 \big(2 n^2 \tanh(z)^2\tanh(t) + n^2 x^2 + m^2 y^2 + n^2 \tanh(z)^2\tanh(t)^2 - 2 n \tanh(z)^2 m + m^2 \tanh(z)^2 \tanh(t)^2 - 2 m^2 \tanh(z)^2\tanh(t) + m^2 y^2 \tanh(z)^4 - 2 m^2 y^2 \tanh(z)^2 + n^2 x^2 \tanh(z)^4 - 2 n^2 x^2 \tanh(z)^2 + m^2 y^2 \tanh(t)^2 - 2 m^2 y^2 \tanh(t) + 2 n^2 x^2 \tanh(t) + n^2 x^2 \tanh(t)^2 + 2 n \tanh(z)^2\tanh(t)^2 m - 4 n^2 x^2 \tanh(t)\tanh(z)^2 + 2 n^2 x^2 \tanh(t)\tanh(z)^4 - 2 n^2 x^2 \tanh(t)^2\tanh(z)^2 + n^2 x^2 \tanh(t)^2 \tanh(z)^4 + 4 m^2 y^2 \tanh(t)\tanh(z)^2 - 2 m^2 y^2 \tanh(t)\tanh(z)^4 - 2 m^2 y^2 \tanh(t)^2\tanh(z)^2 + m^2 y^2 \tanh(t)^2 \tanh(z)^4 + m^2 \tanh(z)^2 + n^2 \tanh(z)^2\big) / \mu$$

```
> ElecE1:=factor(epsilon*innerprod(Ef2,Ef2));
```

$$ElecE1 := \frac{1}{4} \varepsilon B^2 \tanh(z)^2 (-1+\tanh(t))^2 (1+\tanh(t))^2 (m^2 y^2 + n^2 x^2)$$

```
> IntE1:=factor(subs(innerprod(APOT2,Jtot)));
```

$$IntE1 := \frac{1}{2} B^2 \tanh(z)\big(m^2 y^2 \sinh(z)\cosh(t) - m^2 y^2\sinh(z)\cosh(t)\tanh(t) - m^2 y^2 \sinh(z)\sinh(t) + m^2 y^2 \sinh(z)\sinh(t)\tanh(t) + m^2 y^2 \varepsilon\tanh(t)\tanh(z)\,\mu\cosh(t)\cosh(z)^3 - m^2 y^2\varepsilon\tanh(t)^2\tanh(z)\,\mu\cosh(t)\cosh(z)^3 - m^2 y^2\varepsilon\tanh(t)^3\tanh(z)\,\mu\cosh(t)\cosh(z)^3 + m^2 y^2 \varepsilon\tanh(t)^4\tanh(z)\,\mu\cosh(t)\cosh(z)^3 + n^2 x^2 \sinh(z)\cosh(t) + n^2 x^2 \sinh(z)\cosh(t)\tanh(t) + n^2 x^2 \sinh(z)\sinh(t) + n^2 x^2\sinh(z)\sinh(t)\tanh(t) - n^2 x^2\varepsilon\tanh(t)\tanh(z)\,\mu\cosh(t)\cosh(z)^3 - n^2 x^2 \varepsilon\tanh(t)^2\tanh(z)\,\mu\cosh(t)\cosh(z)^3 + n^2 x^2\varepsilon\tanh(t)^3\tanh(z)\,\mu\cosh(t)\cosh(z)^3 + n^2 x^2\varepsilon\tanh(t)^4\tanh(z)\,\mu\cosh(t)\cosh(z)^3\big) \,/\, (\mu\cosh(t)\cosh(z)^3)$$

```
>
```

The topological TORSION 4-vector:

```
> Tors:=evalm(crossprod(Ef2,APOT2)+A4*Bf2):TORSION:=([factor(Tors[1]),factor(Tors[2]),factor(Tors[3]),Helicity2]);
```

$$TORSION := \left[0,\ 0,\ -\frac{1}{2} m\, y\, B^2 \tanh(z)^2\, n\, x\,(-1+\tanh(t))(1+\tanh(t)),\ 0\right]$$

```
>
```

The spatial part of the Torsion vector is NOT Zero, even though the Helicity density (4th component of the Torsion 3-form) is zero.

The topological Spin 4-vector

```
>
> Spin:=crossprod(APOT2,Bf2):sp1:=factor(Spin[1]);sp2:=factor(Spin[2]);sp3:=(Spin[
  3]);Poincare1:=simplify(diverge(Spin,[x,y,z]));
```

$$sp1 := \frac{1}{4} n\,(1+\tanh(t))\,x\,B^2 \tanh(z)^2\,(n+n\tanh(t)-m+m\tanh(t))$$

$$sp2 := \frac{1}{4} m\,y\,(-1+\tanh(t))\,B^2 \tanh(z)^2\,(n+n\tanh(t)-m+m\tanh(t))$$

$$sp3 := m^2 y^2 \left(\frac{1}{2}-\frac{1}{2}\tanh(t)\right)^2 B^2 \tanh(z)\,(1-\tanh(z)^2) + n^2 \left(\frac{1}{2}+\frac{1}{2}\tanh(t)\right)^2 x^2 B^2 \tanh(z)\,(1-\tanh(z)^2)$$

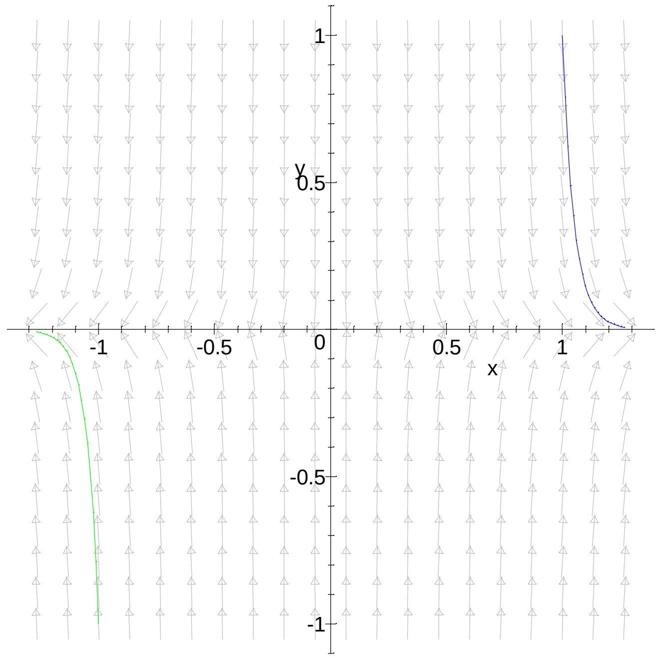
$$Poincare1 := -\frac{1}{4} B^2 (-2 m^2 \cosh(z)^2 \cosh(t)\sinh(t) - 4 m^2 y^2 \cosh(t)\sinh(t)\cosh(z)^2 + 4 m^2 y^2 \cosh(t)^2 \cosh(z)^2$$
$$+ 2 n^2 \cosh(z)^2 \cosh(t)\sinh(t) - 2 n^2 \cosh(z)^4 \cosh(t)\sinh(t) - 6 n^2 x^2 \cosh(t)\sinh(t) + 6 m^2 y^2 \cosh(t)\sinh(t)$$
$$+ 4 n^2 x^2 \cosh(t)^2 \cosh(z)^2 + 4 n^2 x^2 \cosh(t)\sinh(t)\cosh(z)^2 + 2 m^2 \cosh(z)^4 \cosh(t)\sinh(t)$$
$$- 2 m^2 \cosh(z)^4 \cosh(t)^2 + 2 m^2 \cosh(z)^2 \cosh(t)^2 + 2 n^2 \cosh(z)^2 \cosh(t)^2 - 2 n^2 \cosh(z)^4 \cosh(t)^2 + 2 n \cosh(z)^4 m$$
$$- 2 n \cosh(z)^2 m - 6 n^2 x^2 \cosh(t)^2 - 2 m^2 y^2 \cosh(z)^2 - 6 m^2 y^2 \cosh(t)^2 - 2 n^2 x^2 \cosh(z)^2 + 3 n^2 x^2 + 3 m^2 y^2$$
$$- m^2 \cosh(z)^2 + n^2 \cosh(z)^4 + m^2 \cosh(z)^4 - n^2 \cosh(z)^2) \Big/ (\cosh(t)^2 \cosh(z)^4)$$

```
>
```

THE Saddle case (Note signs)

```
>
>
> BXn:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-1.5,z=0.0,(B1)))):
> BYn:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-1.5,z=0.0,(B2)))):
> BZn:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-1.5,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BXn,diff(y(t),t)=BYn], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  2 B field at t = -1.5, m=n`,colour=grey,linecolor = [blue,green]);
```

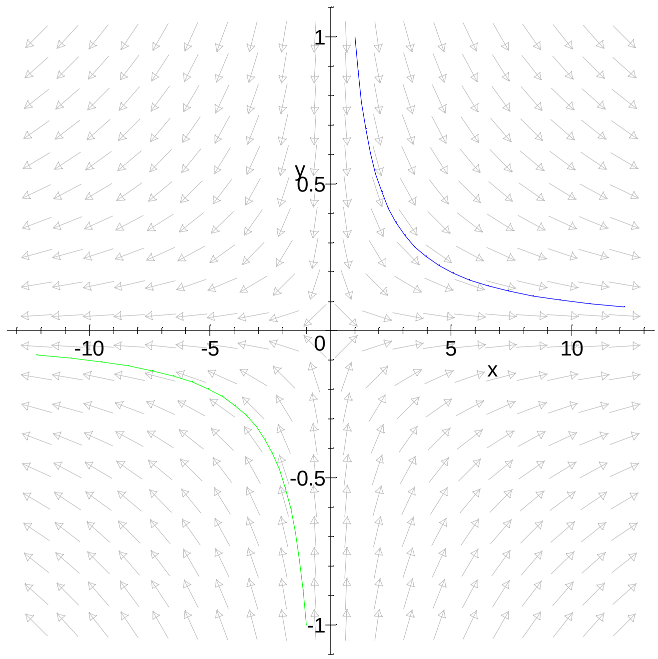
Cat.2 B field at t = -1.5, m=n



Category 2, early saddle t = -1.5 m=n

```
> BX:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-0,z=0.0,(B1)))):
> BY:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-0,z=0.0,(B2)))):
> BZ:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=-0,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BX,diff(y(t),t)=BY], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  2 B field at t = 0, m=n`,colour=grey,linecolor = [blue,green]);
>
```

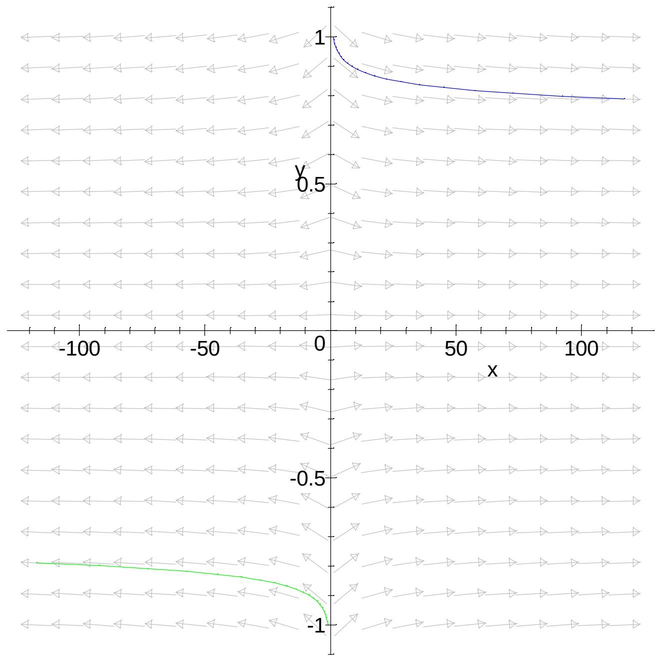
Cat. 2 B field at t = 0, m=n



Category 2: m=n, t=0 . The saddle.

```
>
> BXn:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=1.5,z=0.0,(B1)))):
> BYn:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=1.5,z=0.0,(B2)))):
> BZn:=simplify(evalm(subs(n=1.0,m=1.0,B=1.0,t=1.5,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BXn,diff(y(t),t)=BYn], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  2 B field at t = 1.5, m=n`,colour=grey,linecolor = [blue,green]);
```

Cat. 2 B field at t = 1.5, m=n



Category 2 m=n late times

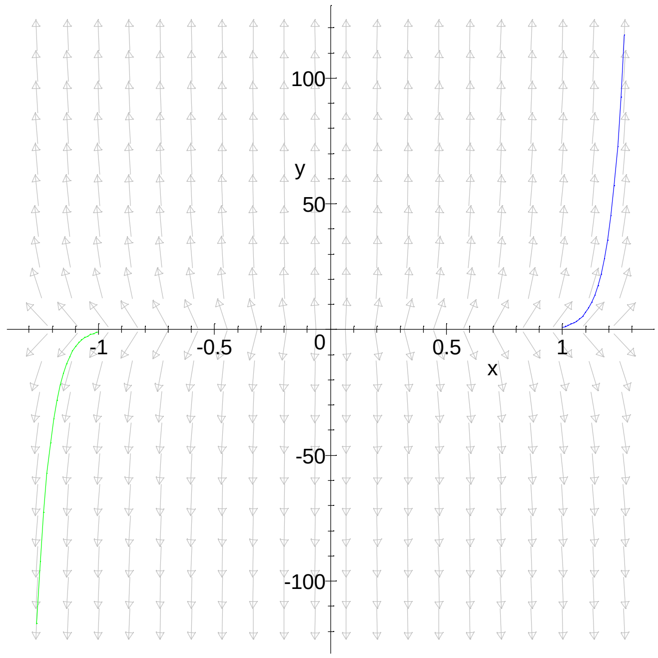
>

The SOURCE and SINK case

m= - n

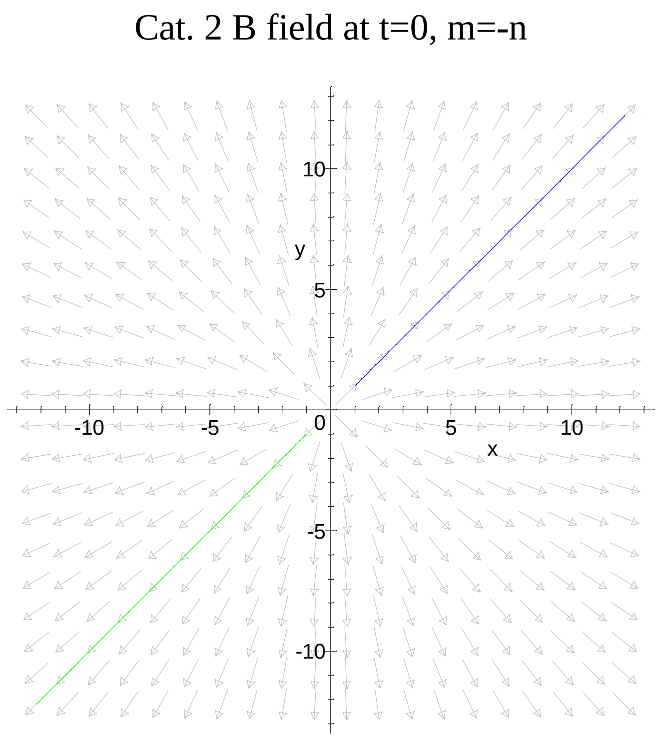
```
> BXn:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-1.5,z=0.0,(B1)))):
> BYn:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-1.5,z=0.0,(B2)))):
> BZn:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-1.5,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BXn,diff(y(t),t)=BYn], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat
  2 B field at t = -1.5, m=-n`,colour=grey,linecolor = [blue,green]);
```

Cat 2 B field at t = -1.5, m=-n



Category 2 m=-n t=0 Early Source

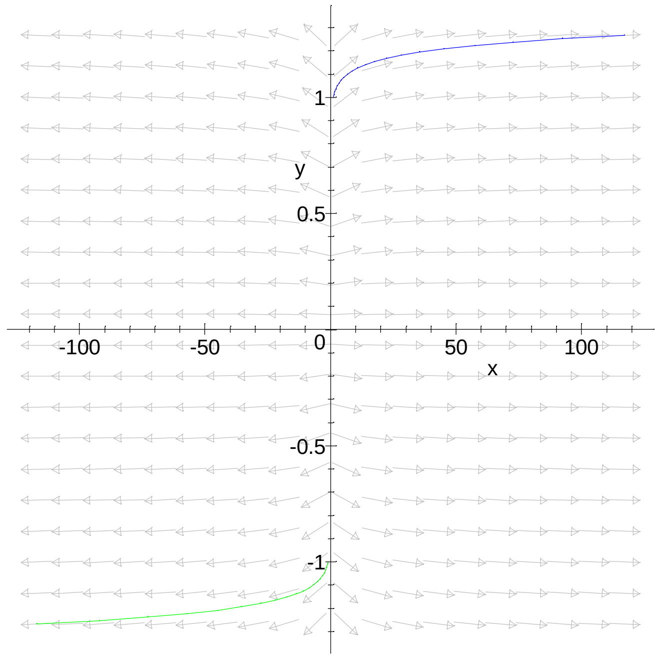
```
> BX:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-0,z=0.0,(B1)))):
> BY:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-0,z=0.0,(B2)))):
> BZ:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=-0,z=0.0,(B3)))):
> with(DEtools):
  DEplot([diff(x(t),t)=BX,diff(y(t),t)=BY], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat.
  2 B field at t=0, m=-n`,colour=grey,linecolor = [blue,green]);
>
```



Category 2 B field, at t = 0. A source singularity in the z = 0 plane

```
>
> BXn:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=1.5,z=0.0,(B1)))):
> BYn:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=1.5,z=0.0,(B2)))):
> BZn:=simplify(evalm(subs(n=1.0,m=-1.0,B=1.0,t=1.5,z=0.0,(B3))))>
> with(DEtools):
  DEplot([diff(x(t),t)=BXn,diff(y(t),t)=BYn], \
  [x(t),y(t)],t=0..5,[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=-1]],arrows=medium,title=`Cat
  2 B field at t = 1.6 m=-n`,colour=grey,linecolor = [blue,green]);
```

Cat 2 B field at t = 1.6 m=-n



Category 2 B field at late times. m = -n a late source.