# Continuous Topological Evolution

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## Abstract

A non-statistical theory of continuous, but irreversible evolution can be constructed in terms of the Cartan calculus. In such an evolutionary theory, the topology of the initial state is usually different from the topology of the final state. Three fundamental theorems of uniformly continuous evolution are established, yielding a set of global conservation laws for reversible and irreversible processes. The equation for the uniformly continuous evolution of Topological Torsion (an entropic idea) is compared to the equation for the uniformly continuous evolution of Topological Action.

## 1. Introduction

The objective of this article is to examine the topological and irreversible evolution of those physical systems that can be modeled in terms of exterior differential systems, when such systems are subjected to processes that can be defined in terms of contravariant vector fields. An arbitrary evolutionary process acting on such physical systems is defined as a map from an initial variety of independent variables to a final variety of independent variables, say $\{x, y, z, t...\}$.    Of all possible processes, the only processes to be considered herein will be those where the tangents to the fibers of the map are represented, to within a factor of reparameterization, by a (contravariant) vector field, $\mathbf{V}$. The evolutionary maps considered must be differentiable, but they need not be reversible. In particular, the evolutionary  processes considered are not necessarily homeomorphisms, for

the topology of the initial state need not be the same as the topology of the final state.

Although tensor fields, relative to processes that are not homeomorphic, are not deterministically predictable in a functional sense, differential forms (by means of functional substitution and the pullback) are well behaved in a deterministic, non-statistical, but retrodictive, sense. [1] Hence each admissable physical system considered in this article will be defined in terms of a set of exterior differential forms, $\Sigma$, constructed on the final state of independent variables $\{x, y, z, t...\}$.

Typically, a process may not have a continuous inverse, and/or the the Jacobian matrix of the process may not be invertible. The topology of the initial state can be different from the topology of the final state. If the physical system were to be defined in terms of exterior differential forms on the initial state, it is impossible (in terms of the differentiable map from initial to final state) to predict the functional forms of the tensor fields that make up the exterior differential forms on the final state. However, if the evolutionary map is differentiable, the definition of the generic physical system in terms of $\Sigma$ on the final state allows the functional form of the generic physical system to be well-defined retrodictively on the initial state, by means of functional substitution and the pullback. This result is due to the fact that covariant tensors pull back via the Jacobian matrix transpose, while contravariant tensor densities pull back via the adjoint of the Jacobian matrix. Both the transpose and the adjoint of a matrix exist, even though the matrix does not have an inverse. These facts effectively define a logical arrow of time with respect to differentiable but not homeomorphic processes. It is this mathematical feature that motivates this non-statistical treatment of irreversibility based upon topological evolution.

Any generic set of forms, $\Sigma$, by exterior differentiation, can be prolonged into its closure, $\{\Sigma, d\Sigma\}$, and into the Pfaff sequence that consists of all possible exterior products of the elements that make of the closure, $\{\Sigma, d\Sigma, \Sigma\hat{} d\Sigma, d\Sigma\hat{} d\Sigma...\}$. The coefficient functions of the independent variables used to define these exterior differential forms will consist of either (antisymmetric) covariant tensor fields or contravariant tensor densities. The number of non-zero elements in the Pfaff sequence defines an equivalence class of physical systems. This number, defined as the Pfaff dimension, or class, is determined by the functional format of the coefficients of the generic differential forms, $\Sigma$.

The elements of the Pfaff sequence for a given physical system (including possible constraints) may be used to define a topological structure, such that continuous evolutionary processes acting on the physical system can be separated

from those that are not continuous. The topological constraints form an exterior differential system [2]. The vector fields representing evolutionary processes then can be put into equivalence classes depending upon what topological properties are preserved under the evolution.

For example, on the domain $\{\mathbf{p}, \mathbf{q}, t\}$ there are many physical systems that appear to have an adequate description in terms of a single generic 1-form of Action,

$$A = \mathbf{p} \circ d\mathbf{q} - H(\mathbf{p}, \mathbf{q}, t)dt. \tag{1.1}$$

In addition, these physical systems can admit a closed 2-form (of forces), $F$, to form a system of differential forms, $\Sigma = \{A, F\}$. The closure of the generic system is the system $\{A, dA, F, dF\}$. The classical topological constraint subsumed herein is given in the form of the exterior differential system, $F - dA = 0$. The implication of this topological constraint is evident from Stokes theorem: the domain of support for finite non-zero $F$ *can not be compact without boundary, unless the Euler Characteristic is zero. (Hence there are only two exceptions, the Torus and the Klein bottle.)*

The Pfaff sequence is given by the set $\{A, dA, A\hat{\,}dA, dA\hat{\,}dA, ...\}$. It will be demonstrated below that the closed integral of $F = dA$ is a deformation invariant with respect to any vector field representing an evolutionary process. (This is the basis for the Helmholtz theorems of hydrodynamics, the conservation of flux theorems in electromagnetism, and the Poincare integral invariants of classical mechanics.) A specific class of vector fields can be defined as those that also leave the closed integral of Action, $\int_{closed} A$, a deformation invariant. Processes that belong to this restricted class are equivalent to the extremal fields of the calculus of variations. Cartan proved [3] that these extremal vector fields satisfy the equations of a Hamiltonian vector field. Such Hamiltonian processes imposed upon physical systems, as shown below, are reversible in a thermodynamic sense.

A result of this study of continuous topological evolution of constrained systems is the creation of a thermodynamic, non-statistical theory that may be used to describe the irreversible evolution observed for dissipative non-conservative physical systems. The motivation for a topological basis of the subject is due to the intuitive idea that physical irreversibility somehow must be associated with a map that does not have an inverse. Topological change can be effected by two processes, the discontinuous process described as cutting apart (or separation), or the continuous process of pasting (or glueing) together. In this article the focus will be on those continuous processes that admit topological change.

It was appreciated early on that group theory and symmetry concepts - so

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useful for the study of conservative reversible systems - are not appropriate to the understanding of irreversibility and non-conservative systems. If an evolutionary process is described by a map, $\Phi$, between initial and final states, and if that map does not have a continuous inverse, then the observable topology of the final state must be different from the observable topology of the initial state. Based upon this assumption, it was first thought that the observation of topological change, with the production and destruction of defects and holes, lines of self-intersection and other obstructions, should be the taken to be the signature of continuous but irreversible evolutionary processes. These ideas turn out to be necessary conditions for thermodynamic irreversibility, but they need not be sufficient. In that which follows, this idea of topological evolution and thermodynamic irreversibility will be made more precise. It is important to realize that the measureables of such processes are not geometrical issues of size and shape, but instead are those topological features which are independent from metric or metric based connections.

As mentioned above, both continuous and discontinuous processess can lead to topological evolution. The creation of disconnected components (by cutting or tearing) will be taken as the signature of discontinuous processes, and experience leads one to presume that such processes are irreversible. However, such discontinuous processes are not the main subject of this article, even though they are to be associated with *creation* of irreversible phenomena such as turbulence. The creation of turbulence by means of discontinuous processes will not be studied. It has been demonstrated that the creation of turbulence is a discontinuous process, while the decay of turbulence can be treated in a continuous manner.[5] Frankly, techniques for treating the discontinuous processes are beyond the present skill of the author. However, in this article, the irreversibility associated for example with the *decay* of turbulence by means of continuous processes of topological evolution will be examined.

## 1.1. Topological Change

Given a topology on the final state and a map from an initial state to the final state it is always possible to define a topology on the initial state such that the given transformation, or even a given set of transformations, is continuous. However, the topologies of the initial and final states need not be the same; hence the map need not have a unique inverse. Recall that with respect to a discrete topology all maps from the initial to final state are continuous, while relative

to the concrete topology, only the constant functions are continuous [8]. A first problem of a theory of topological evolution is to devise a rule for constructing a topology that is physically useful and yet is neither too coarse nor too fine. Such a rule is necessary for the concept of continuity of an evolutionary transformation is defined relative to the topologies of the initial and final states.

It has been the experience of this author that many colleagues do not associate topological change with continuous processes. Intuitively, cutting into parts (gasification) is a discontinous process, while pasting together (condensation) is a continuous process. In order to sensitize the reader to continuous processes that admit topological change, a number of transformations are presented graphically in Figure 1; some transformations are continuous and some are not. In all cases, the topology of the system changes because the observable hole count in the initial state is not the same as the hole count in the final state. It is remarkable that the production or destruction of holes can be accomplished in either a continuous or a discontinuous manner. Physical exhibitions of continuous and discontinuous transformations can be achieved through the deformations of a soap film attached to a wire frame. Figure 2 is intended to demonstrate the continuous deformation of a soap film which involves a change of the topological property of orientability. The soap film is transformed from a Mobius band into a cylindrical strip. In Figure 3 the continuous evolution of the number of handles (or holes) is also emulated by a continuous deformation of a wire frame supporting a soap film. In Figure 4, the evolution of the number of components is studied in terms of the dynamics of a soap film stretched between two rings. As the rings move further apart, the soap film stretches, still forming a catenoid of revolution. However, when the separation to radius ratio exceeds a critical value, the topological system becomes unstable, and without further perturbation from the environment, the soap film distorts until the original hyperbola of one sheet contracts to form a conical surface (with a critical point at the conical apex) and then separates to form a hyperbola of two sheets which ultimately contracts to form two isolated discs. The originally connected minimal surface undergoes a topological change to where it becomes two disconnected (minimal) surfaces. An example of this topological transition in the surface of null helicity density has been described in conjunction with the parametric saddle node Hopf bifurcation of a Navier-Stokes flow [9].

## 1.2. Continuity

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The classic definition [10] of a continuous transformation between a set $X$ with topology $T1$ to a set $Y$ with a topology $T2$ states that the transformation is continuous if and only if the inverse image of open sets of $T2$ are open sets of $T1$. In Figure 5 an example is given on two sets of four points in which the map from state $X$ to state $Y$ is continuous, but for which the inverse image is not continuous relative to the topologies $T1$ and $T2$. Hence such a transformation induces irreversible topological change, but in a continuous manner. The objective of this article is to exploit such results for evolutionary processes that are described by vector fields of flow.

There exists another more useful method of defining continuity which does not depend explicitly on being able to define open sets and their inverse images. This second method of defining continuity is based on the concept of closure. The closure of a set can be defined in (at least) two ways:

1. The closure of a set is the union of the interior and the boundary of a subset.

2. The closure of a set is the union of the set and its limit points.

The first definition of closure is perhaps the most common, and is often exploited in geometric situations, where a metric has been defined and a boundary can be computed easily. The second definition of closure is independent from metric and is the method of choice in this article, both for defining continuity and establishing a topological structure. In terms of the concept of closure, a transformation is continuous if and only if for every subset, the image of the closure of the initial subset is included in the closure of the image of that subset [5]. Another way of stating this idea is:

A map is continuous iff the limit points of every subset in the domain permute into the closure of the subsets in the range.

If a method for constructing a closure operator ( a Kuratowski closure operator K of a subset relative to a topology) can be defined, then a strong version of continuity would imply that the Kuratowski closure operator commutes with those transformations which are continuous. The test for continuity would be to construct the closure of an arbitrary subset on the initial state, and then to propagate the elements of the closure to the final state by means of a transformation. If this result is the same as the result obtained by first propagating the

subset to the final state by means of the transformation, and then constructing its closure on the final state, then the map is continuous. Note that such a procedure has defined a topological structure which will be exploited in this article, for the subsets of interest will be defined as a Cartan system of exterior differential forms, $\Sigma$, on $X$. The topological base defined by this class of sets is too course to be of interest. Hence the Cartan exterior derivative will be used to generate additional sets of forms, $d\Sigma$, which when adjoined to the initial system of forms defines the Kuratowski closure of the Cartan system as the system of forms, $K(\Sigma) = \{\Sigma \cup d\Sigma\}$. The Cartan exterior product may be used as a convenient intersection operator between sets of differential forms. Starting from the system, $\{\Sigma\}$, the Cartan topology is then determined by the construction of the Cartan-Pfaff sequence, which consists of all possible intersections that may be constructed from the subsets of the closure of the differential system:

$$Pfaff\ Sequence:\ \{\Sigma, d\Sigma, \Sigma\hat{}d\Sigma, d\Sigma\hat{}d\Sigma, ...\}. \qquad (1.2)$$

The subsets of the Cartan topological space consist of all possible unions of the subsets that make up the Pfaff sequence. The Cartan topology will be constructed from a topological basis which consists of the odd elements of the Pfaff sequence, and their closures:

$$the\ Cartan\ topological\ base:\ \{\Sigma, K(\Sigma), \Sigma\hat{}d\Sigma, K(\Sigma\hat{}d\Sigma), ...\}. \qquad (1.3)$$

With respect to this topological base it is shown in Appendix A that the Cartan exterior derivative may be viewed as a closure or limit point operator. Given any subset of the Cartan topological space, the exterior derivative of that subset generates its limit points, if any. This is a remarkable result, for as will be demonstrated below, all C2 vector fields acting through the concept of the Lie derivative on a set of differential forms, with C2 coefficients, generate continuous transformations with respect to the Cartan topology. Moreover, the Cartan topology is disconnected if $\Sigma\hat{}d\Sigma \neq 0$.

### 1.3. The evolutionary process

An arbitrary evolutionary process, $X \Rightarrow Y$, is defined by a map $\Phi$. The map, $\Phi$, may be viewed as a propagator that takes the initial state, $X$, into the final state, $Y$. In this article the evolutionary processes to be studied are asserted to be generated by vector fields of flow, $\mathbf{V}$. The local trajectories defined by the vector

fields may be viewed as propagators that carry domains into ranges in the manner of a convective fluid flow. The evolutionary propagator of interest to this article is the Lie derivative with respect to a vector field ,$\mathbf{V}$, acting on differential forms, $\Sigma$ [6]. The Lie derivative has a number of interesting and useful properties.

1. The Lie derivative does not depend upon a metric or a connection.

2. The Lie derivative has a simple action on differential forms producing a resultant form that is decomposed into a transversal and an exact part:

$$L_{(\mathbf{V})}\Sigma = i(\mathbf{V})d\Sigma + di(\mathbf{V})\Sigma. \tag{1.4}$$

Marsden calls this Cartan's Magic Formula (see below).

3. The Lie derivative may be used to describe deformations and topological evolution.

4. With respect to vector fields and forms constructed over C2 functions, the Lie derivative commutes with the Kuratowski closure operator. Hence, the Lie derivative generates transformations on differential forms which are continuous with respect to the Cartan topology.

   In contrast to the covariant derivative, which is often defined in such a way as to consider only those evolutionary processes that preserve lengths, and therefore is useful only to the study of isometries (rigid body motions, and bending without shear), the Lie derivative may be used to describe deformations and shear processes associated with convective fluid flow. For example, the action of the Lie derivative on a 0-form (scalar function) is the same as the directional derivative of ordinary calculus,

$$L_{(\mathbf{V})}\varphi = i(\mathbf{V})d\varphi + 0 \Rightarrow \mathbf{V} \cdot grad\varphi. \tag{1.5}$$

In the examples given below, it will be demonstrated that the action of the Lie derivative on a 1-form typically will generate hydrodynamic equations of motion. The Lie derivative is not the same as the classic metric dependent covariant derivative (based upon Christoffel symbols), or generalizations of the metric connection used in certain gauge or fiber bundle theories. The reason is that the Lie derivative satisfies the equations

$$L_{(f\mathbf{V})}\Sigma = f \cdot L_{(\mathbf{V})}\Sigma + df \,\hat{}\, i(\mathbf{V})\Sigma. \tag{1.6}$$

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The covariant derivative, $\mathcal{D}$, and its generalizations are constrained [12] such that the second term on the right vanishes:

$$\mathcal{D}_{(f\mathbf{V})}\Sigma = f \cdot \mathcal{D}_{(\mathbf{V})}\Sigma. \tag{1.7}$$

This equation is often interpreted by saying that $f$ represents the action of some "group", and the covariant derivative is defined such that it commutes with the action of the group. The Lie derivative is not limited to the constraint of a specified group. However, there exists a special class of vector fields relative to the differential form, $\Sigma$, that permit the Lie derivative to be identified with a covariant derivative. This special class of vectors fields are called associated vectors, and are defined by the equation,

$$\text{Class of associated vectors :} \quad i(\mathbf{V})\Sigma = 0. \tag{1.8}$$

The first three properties of the Lie derivative appear in the literature, but the extraordinary property that all C2 vector fields that propagate C2 differential forms in the manner of a convective flow (Lie derivative) are continuous relative to the Cartan topology requires proof: Given $\Sigma$, first construct the closure, $\Sigma \cup d\Sigma$. Next propagate $\Sigma$ and $d\Sigma$ by means of the Lie derivative to produce the decremental or residue forms, say $Q$ and $Z$,

$$L_{(\mathbf{V})}\Sigma = Q \quad and \quad L_{(\mathbf{V})}d\Sigma = Z. \tag{1.9}$$

Now compute the contributions to the closure of the final state as given by $Q \cup dQ$. If $Z = dQ$, then the closure of the initial state is propagated into the closure of the final state, and the evolutionary process defined by $\mathbf{V}$ is continuous. However,

$$dQ = dL_{(\mathbf{V})}\Sigma = di(\mathbf{V})d\Sigma + dd(i(\mathbf{V})\Sigma) \tag{1.10}$$

and

$$Z = L_{(\mathbf{V})}d\Sigma = (i(\mathbf{V})dd\Sigma) + di(\mathbf{V})d\Sigma. \tag{1.11}$$

The difference becomes

$$Z - dQ = (i(\mathbf{V})dd\Sigma) - dd(i(\mathbf{V})\Sigma). \tag{1.12}$$

The concept of continuity requires that $Z - dQ \Rightarrow 0$, forming an exterior differential system. For vector fields and differential forms that are twice differentiable,

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the continuity condition is always satisfied relative to the Cartan topology (the Poincare lemma). The set $\{\Sigma, d\Sigma\}$ forms a differential ideal which is permuted into the differential ideal $\{Q, dQ\}$ by the action of the Lie derivative with respect to $\mathbf{V}$.

The Lie derivative also can be used to make some sense out of certain classes of discontinuous evolutionary processes (which are not C2). Note that the integral of the exterior differential system, $Z - dQ$, is exact for any process, if the coefficients that make up the exterior differential forms are C2. For example, consider a vector field $\mathbf{V} = \rho\mathbf{v}$ where the support function, $\rho$, is not C2. Then the action of the Lie derivative produces the discontinuity or excess function,

$$Z - dQ = -dd(i(\rho\mathbf{v})\Sigma) = d(d\rho\,\hat{}\,(i(\mathbf{v})\Sigma) + d\rho\,\hat{}\,d(i(\mathbf{v})\Sigma). \qquad (1.13)$$

This equation is of use in the study of certain discontinuous processes, for the concept of Cartan continuity is preserved when $Z - dQ = 0$, even though the support function $\rho$ is discontinuous. Such a special situation arises when $(i(\mathbf{v})\Sigma) = 0$. Such special vector fields were defined above to be *associated* vector fields, and have the properties that the Lie derivate is the same as the covariant derivative. Those vector fields that satisfy $(i(\mathbf{v})d\Sigma) = 0$ are defined, for reasons that will become apparent, as *extremal* vector fields relative to $\Sigma$. Vector fields that are both extremal and associated are defined as characteristic vector fields. Characteristic vector fields admit propagating discontinuities, which form the precise definition of a signal in electromagnetism [Luneberg]

## 2. Cartan's Magic Formula

### 2.1. Deformation Invariants

The evolutionary processes considered in this article are limited to those maps from the initial state to the final state whose fibers have tangents that can be described by a contravariant vector field, $\mathbf{V}$ (often singly parameterized). For such cases, the evolutionary process will be studied in terms of Cartan's magic formula [6] which describes the action of the Lie derivative (relative to $\mathbf{V}$) acting on the elements, $\Sigma$, that make up the exterior differential system:

$$L_{(\mathbf{V})}\Sigma = i(\mathbf{V})d\Sigma + di(\mathbf{V})\Sigma \Rightarrow Q \qquad (2.1)$$

Suppose $\Sigma$ is one of the (perhaps several) exterior differential forms required to define a physical category. Then a useful interpretation of Cartan's formula is

that $Q$ represents the change in that component $\Sigma$ of a physical category as it is is propagated along the trajectories (fibers) generated by $\mathbf{V}$. If $Q$ vanishes, the form(s) is said to be an invariant of the evolutionary process. If the trajectories of the evolution are reparameterized (or rescaled) by $\beta(x, y, z, t)$,, then the Cartan formula yields

$$L_{(\beta\mathbf{V})}\Sigma = i(\beta\mathbf{V})d\Sigma + di(\beta\mathbf{V})\Sigma \Rightarrow \beta Q + d\beta \,\hat{}\,(i(\mathbf{V})\Sigma) \tag{2.2}$$

If the parametrization is chosen to be $\beta = -1$, then the evolution is in the reverse direction. If the resultant, $Q$, of the action of the Lie derivative on the form $\Sigma$ is zero, independent of the parametrization (sign included) then the exterior form is not only a local invariant of the evolution, that evolutionary process for that form is reversible.

Integral properties of the exterior differential forms lead to the concept of deformation invariants. Consider the action of the Lie derivative acting on the closed integral:

$$L_{(\beta\mathbf{V})}\int_z \Sigma = \int_z i(\beta\mathbf{V})d\Sigma + \int_z di(\beta\mathbf{V})\Sigma \Rightarrow \int_z Q. \tag{2.3}$$

The second integral, $\int_z di(\beta\mathbf{V})\Sigma$, always vanishes for the closed integral of an exact form is zero, independent of the reparameterization function, $\beta$. For arbitrary choices of $\beta$, the first integral vanishes for that class (the extremal class) of vector fields that satisfy the equation

$$\text{Class of Extremal vectors:} \quad i(\mathbf{V})d\Sigma = 0. \tag{2.4}$$

With respect to the extremal class of vector fields (processes), the closed integral $\int_z \Sigma$ is a deformation invariant (and therefore a primitive topological property). The points on the trajectories that make up a closed integration chain enclose a tube of trajectories (fibers). If the closed integration chain and the integrand for a given value of the parameterization is re-evaluated at a different value of the parameter, (as if the points that make up the integration chain and the integrand flow in a hydrodynamic manner down the evolutionary fibers) then the same value is obtained for the integral (showing evolutionary invariance). Of even more interest is the fact that if the tube of trajectories is deformed in any manner by choosing another functional form for $\beta(x, y, z, t)$, thereby distorting the points that make up any individual trajectory and any closed integration chain connected to those points, the integral still remains the same, and therefore defines a deformation invariant; i.e, a topological property.

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## 2.2. The first law of thermodynamics.

Cartan's invention of a deformation invariant can be used to show that those processes which leave the closed integral of Action, $A = \int_{z1} pdq - H(p,q,t)dt$, an invariant, independent of the parametrization $\beta$, define a Hamiltonian flow. On a contact manifold the Hamiltonian flow is a unique process that conserves the Hamiltonian function (energy).[3] Cartan thereby established the topological foundations of conservative Hamiltonian processes by requiring that such processes be the subsets of singly parameterized vector fields that leave the closed integral of the 1-form of Action a deformation invariant.

A primary thrust of this article will be to exploit certain topological concepts based on Cartan's theory of exterior differential systems, the Lie derivative, and the evolutionary equation described by Cartan's Magical Formula,

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q. \qquad (2.5)$$

and its closure:

$$dL_{(\mathbf{V})}A = L_{(\mathbf{V})}dA = di(\mathbf{V})dA + dd(i(\mathbf{V})A) = dW + ddU = dQ. \qquad (2.6)$$

If the Action $A$ is an invariant of the evolutionary process $\mathbf{V}$, then the differential residue 1-form $Q$ is zero. It is apparent that Cartan's Magic formula is equivalent to the cohomological expression of the first law of thermodynamics. In this article, the 1-form $W = i(\mathbf{V})dA$ is defined as the (non-exact) 1-form of virtual work, the 0-form $U = i(\mathbf{V})A$ is defined as the internal energy, and the 1-form $Q$ is defined as the (non-exact) 1-form of heat. These definitions are not whimisical, for it will be demonstrated below that they have utilization in many of the familiar formulas of classical physics.

In thermodynamics, an irreversible process is defined as a process for which the 1-form of heat, $Q$, does not admit an integrating factor (of reciprocal temperature). [7]. This definition may be made precise in terms of Cartan's magic formula and the Frobenius theorem, for if the 1-form of heat, $Q$, does not admit an integrating factor then the three form, $Q\,\hat{}\,dQ$, does not vanish. Hence, for a given physical system defined in terms of a 1-form of Action, $A$, and its Pfaff sequence, those processes, $\mathbf{V}$, that satisfy the equation $L_{(\mathbf{V})}A \,\hat{}\, L_{(\mathbf{V})}dA = 0$ are irreversible.

Definition of an irreversible process, $\mathbf{V}$ : $\qquad L_{(\mathbf{V})}A\,\hat{}\,L_{(\mathbf{V})}dA = Q\,\hat{}\,dQ \neq 0 \quad (2.7)$

This precise definition of topological irreversibility as stated above will be subsumed, and the cohomological equivalent of the first law of thermodynamics will be studied relative to the single constraint of continuous but irreversible topological evolution. Many intuitive thermodynamic concepts can be stated precisely in terms of the theory of continuous topological evolution based on the Cartan topology. For example, those processes for which $L_{(\mathbf{V})}A = Q = 0$ are adiabatic.

$$\text{Definition of a local adiabatic process, } \mathbf{V}: \qquad L_{(\mathbf{V})}A = Q = 0. \qquad (2.8)$$

A precise difference between the 1-form of (virtual) work and the 1-form of heat can be established: the 1-form of work is necessarily transversal to the process, while the 1 form of heat is not:

$$i(\mathbf{V})W = i(\mathbf{V})i(\mathbf{V})dA = 0 \quad but \quad i(\mathbf{V})Q = -dU \neq 0. \qquad (2.9)$$

Continuous processes on isolated systems satisfy the (extremal) equations

$$W = i(\mathbf{V})dA = 0\,. \qquad (2.10)$$

Continuous processes on closed but not isolated systems satisfy the (Helmholtz or symplectic) equations

$$dW = di(\mathbf{V})dA = 0\,. \qquad (2.11)$$

Continuous processes on open systems satisfy the equations

$$W\,\hat{}\,dW \neq 0. \qquad (2.12)$$

Continuous irreversible processes satisfy the equations

$$Q\,\hat{}\,dQ \neq 0. \qquad (2.13)$$

### 2.3. 5. Topological Evolution

If the flow field generated by $\mathbf{V}$ acting on a Cartan system of forms satisfies the equations

$$L_{(\mathbf{V})}\Sigma = 0 \quad and \quad L_{(\mathbf{V})}d\Sigma = 0. \qquad (2.14)$$

Then with respect to such evolutionary processes, the forms of the closure are said to be absolute invariants. It follows that each element that makes up the Cartan topological base is also invariant, such that the whole Cartan topology is invariant. As $\mathbf{V}$ is continuous, and the topology is preserved, those vector fields, $\mathbf{V}$, that satisfy the equations above must be homeomorphisms, and are reversible. In other words, $Q = 0$ and $dQ = 0$ are sufficient conditions that $\mathbf{V}$ be reversible.

However, for continuous transformations on the elements of the C2 Cartan topology the general equations of topological evolution become,

$$L_{(\mathbf{V})}\Sigma = Q$$

and

$$L_{(\mathbf{V})}d\Sigma = dQ,$$

from which it follows that

$$L_{(\mathbf{V})}\Sigma\hat{}d\Sigma = Q\hat{}d\Sigma + \Sigma\hat{}dQ$$

and

$$L_{(\mathbf{V})}d\Sigma\hat{}d\Sigma = 2dQ\hat{}d\Sigma.$$

As these equations of continuous topological evolution imply that the elements of the topological base may not be constant, then specific tests must be made to determine what features of the topology are changing, if any. For if it can be determined that the topology is indeed modified by the evolutionary process, then the process generated by this class of vector fields, $\mathbf{V}$, is continuous, but need not be reversible. When $dQ \neq 0$, the limit points are not invariants, and it would be natural to expect that the topology is not constant. However, even if $Q$ is closed, such that $dQ = 0$, it may be true that $Q$ contains harmonic components, such that DeRham cohomological classes of $\Sigma$ are not evolutionary invariants. Even though the topology of the initial state is not the same as the topology of the final state (for the "hole" count of the initial state is not the same as the hole count of the final state) it is not necessarily true that such continuous processes are thermodynamically irreversible.

## 3. Simple Systems

### 3.1. The Action 1-form and its Pfaff sequence.

For purposes of expose, the Cartan system, $\Sigma$, will be limited to a single 1-form of action, $A$, and perhaps a single pseudoscalar field, or N form density, $\rho$. The exterior derivative of $A$ produces a 2-form of closure points, $F = dA$, whose components are given by the expression, $F_{\mu\nu}dx^\mu {}^\wedge dx^\nu$. The combined set $\{A, F\}$ forms the closure of the set $\{A\}$. All possible intersections of the closure, $\{A, F, A^\wedge F, F^\wedge F...\}$, form the Pfaff sequence for the domain $\{x, y, z, t\}$. In this article these elements are defined as

$$Topo\log ical \;\; ACTION : A = A_\mu dx^\mu$$

$$Topo\log ical \;\; VORTICITY : F = dA = F_{\mu\nu}dx^\mu {}^\wedge dx^\nu$$

$$Topo\log ical \;\; TORSION : H = A^\wedge dA = H_{\mu\nu\sigma}dx^\mu {}^\wedge dx^\nu {}^\wedge dx^\sigma$$

$$Topo\log ical \;\; PARITY : K = dA^\wedge dA = K_{\mu\nu\sigma\tau}dx^\mu {}^\wedge dx^\nu {}^\wedge dx^\sigma {}^\wedge dx^\tau.$$

The union of all elements of the Pfaff sequence and their closures forms the elements of the Cartan topological base: $\{A, A \cup F, H, H \cup K...\}$.

In order to take into account projective (and certain discontinuous) features, the vector fields of interest often will be scaled by a support function, $\rho$, such that $\mathbf{J} = \rho\mathbf{V}$. The fundamental equations of continuous evolution become

$$L_{(\rho\mathbf{V})}A = Q \tag{3.1}$$

$$L_{(\rho\mathbf{V})}F = dQ \tag{3.2}$$

$$L_{(\rho\mathbf{V})}H = Q^\wedge F + A^\wedge Q$$

$$L_{(\rho\mathbf{V})}K = 2(dQ^\wedge F) = 2d(Q^\wedge F) \tag{3.3}$$

Note that for the even dimensional elements of the Pfaff sequence, the action of the Lie derivative is to produce an exact form, $dQ$ for 3.2 and $2d(Q^\wedge F)$ for 3.3. As integrals of exact forms over closed cycles or boundaries of support vanish, then it is possible to formulate the first theorem.

**Theorem 3.1.** *All even dimensional Pfaff classes of p-forms, $dA = F, dA^\wedge dA = K$ ... are relative integral deformation invariants of continuous evolutionary processes relative to the Cartan topology.*

The closed integrals of $F, K, \ldots$ are invariants of a continuous process as each integrand is exact, and the integral of an exact form over a closed domain vanishes. Hence if the functions are twice differentiable,

$$L_{(\rho\mathbf{V})} \int_{z2} F = \int_{z2} \{i(\rho\mathbf{V})dF + di(\rho\mathbf{V})F\} \Rightarrow 0. \qquad (3.4)$$

This theorem is an extension of Poincare's theorem for even dimensional p-forms which are absolute integral invariants (the integration domain is not necessarily closed) with respect to Hamiltonian processes. It is important to realize that the theorem expresses the existence of (relative) integral deformation invariants (topologocal properties) with respect to processess that may be thermodynamically reversible or irreversible. It should be noted that the domains of support of the even dimensional Pfaff classes can not be compact without boundary.

## 3.2. The Cartan-Hilbert integral example

The 1-form of Action, $A$ can be written in several equivalent formats but in this section the format of the Cartan-Hilbert invariant integral is subsumed:

$$A = A_\mu dx^\mu = \mathcal{L}(\mathbf{x}, \mathbf{v}, t)dt + \mathbf{p} \cdot (d\mathbf{x} - \mathbf{v}dt) = \mathbf{p} \cdot d\mathbf{x} - \mathcal{H}(\mathbf{x}, \mathbf{v}, \mathbf{p}, t)dt \qquad (3.5)$$

This representation indicates that the Action may be viewed abstractly in terms of the kinematic fluctuations in position, $\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt)$, on a fluctuation space of 10 dimensions $(\mathbf{x}, \mathbf{v}, \mathbf{p}, t)$. Recently it has been determined that the maximum Pfaff dimension of the sequence $\{A, dA, A\hat{\ }dA, dA\hat{\ }dA...)$ is of dimension 8 and not dimension 10. The 1-form of Action generates a non-compact symplectic manifold of dimension 8. If the Lagrange multipliers $\mathbf{p}$ of the kinematic fluctuations $(d\mathbf{x} - \mathbf{v}dt)$ are restricted to be the canonical momenta, such that $\mathbf{p} = \partial\mathcal{L}/\partial\mathbf{v}$, the maximum Pfaff dimension is 7, forming a contact manifold historically defined as state space. If the Lagrange function $\mathcal{L}(\mathbf{x}, \mathbf{v}, t)$ is homogeneous of degree 1 in $\mathbf{v}$, then the maximal Pfaff dimension is 6, forming a symplectic Finsler manifold of dimension 6, the phase space of classical mechanics. This manifold cannot be compact without boundary.

If the contact manifold of dimension 7 is constrained by the equations of kinematic closure, $d(\Delta\mathbf{x}) = d(d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0$, then the space of interest becomes

the configuration space of 4 dimensions, a submanifold of the original symplectic structure of 8 dimensions. The constraints of kinematic closure imply that the velocity field is expressible as functions of a single variable, $t$; $\mathbf{v} \Rightarrow \mathbf{v}(t)$. Note that the more severe constraint of kinematic perfection, $\Delta\mathbf{x} = (d\mathbf{x} - \mathbf{v}dt) \Rightarrow 0$, implies that the maximal Pfaff dimension is 2, as in this case $A \hat{} dA = 0$. The Action defines a completely integrable 2 dimensional submanifold that in this circumstance is not compact without boundary. These concepts will be exploited in other examples given below.

## 4. Cohomology and the Evolution of Energy

Consider the first equation 3.1 of the set, and write

$$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V})A) = W + dU = Q. \tag{4.1}$$

In this equation, the function $U$ is defined as $i(\mathbf{V})A$, and 1-form $W$ is defined as $W = i(\mathbf{V})dA$. Then equation 3.1 becomes equivalent to the statement of cohomology; that is, the difference between the inexact 1-form $Q$ and the inexact 1-form $W$ is equal to a perfect differential for continuous evolution: $W + dU = Q$. Of course this last result is precisely equivalent to the first law of thermodynamics, when the variables $Q$ and $W$ are interpreted as the 1-forms of heat and (virtual) work respectively, and $U$ is interpreted as the internal energy. Fundamentally, 3.1 is a topological law describing the evolution of energy. It is remarkable that the first law follows, without axiomatization, from the single and simple constraint of continuity on a 1-form of action. It is also intuitively pleasing to see that the inexact 1-forms, $Q$ and $W$, are defined in terms of a both a process, $\mathbf{V}$, and a physical system, $A$.

Elementary discussions of heat and work often emphasize the energy content of the first law, rather than the engineering idea that heat and work are related to processes. Further examples given below will demonstrate that equation 3.1 is indeed related to the evolution of energy, and Newton's laws of motion. Other authors have emphasized the topological foundations of thermodynamics [11], and from the time of Caratheodory have noted the connection to Pfaff systems [12]. However, these authors did not have access to, or did not utilize, the Cartan topology and DeRham cohomology. A remark by Tisza, "... the main content of thermostatic phase theory is to derive the topological properties of the sets of singular points in Gibbs phase space" [13], greatly stimulated the early developments of the theory presented in this article.

As must be the case in thermodynamics, there is a fundamental difference between the 1-form $W$ and the 1-form $Q$: the 1-form $W$ is necessarily transversal to the flow, while the 1-form $Q$ is not. From the definition of the virtual work 1-form, it follows that

$$i(\rho\mathbf{V})i(\rho\mathbf{V})dA = i(\rho\mathbf{V})W = 0 \quad (transversality). \quad (4.2)$$

This fact implies that the 1-form $W$ must be constructed from first integrals, $\varphi$, of the flow $\mathbf{V}$, or from transversal fluctuations in the kinematics:

$$W = d\varphi + \mathbf{f} \cdot (d\mathbf{x} - \mathbf{v}dt). \quad (4.3)$$

Although $W$ can be included in the concept of $Q$, there are parts of $Q$ that are not transformable into $W$: for $W$ is transversal and $Q$ is not. This issue is at the heart of the second law of thermodynamics. The argument is pleasing for it gives formal substance to the intuitive differences between the thermodynamic concepts of heat and work. All continuous processes may be put into equivalence classes as determined by the vector fields, $\mathbf{V}$, that generate the evolutionary process. For example, for any p-form, $\omega$, those vector fields that satisfy the transversal equation,

$$i(\rho\mathbf{V})\omega = 0 \quad (4.4)$$

are said to be elements of the associated class of vector fields relative to the form $\omega$. Those vectors that satisfy the equations,

$$i(\rho\mathbf{V})d\omega = 0 \quad (4.5)$$

are said to be elements of the extremal class of vector fields. Vectors which are both extremal and associated are said to be elements of the characteristic class of vector fields [13]. Note that characteristic flow lines generated by $\mathbf{V}$ preserve topology if they are continuous. Characteristics are often associated with wave phenomena. In the next section, the continuous vector fields that generate evolutionary processes will be put into distinct categories and subcategories, each of which has both physical and topological significance. The associated class of vector fields will have particular significance to the study of discontinuous processes. It is important to realize that $Q$ represents the inexact 1-form of heat, and its integral is the measurable quantity. If the evolution of $A$ is invariant, $L_{(\mathbf{V})}A = Q \Rightarrow 0$, and the topological evolution process is defined to be adiabatic. When

$L_{(\mathbf{V})}L_{(\mathbf{V})}A = L_{(\mathbf{V})}Q \Rightarrow 0$, the process is defined to be non-radiative, otherwise radiative.

In general a hierarchy of processes will be defined by the sequence of equivalence classes:

$$\{W = i(\mathbf{V})dA = 0, \quad dW = di(\mathbf{V})dA = 0, \quad i(\mathbf{V})dW = 0, \quad di(\mathbf{V})dW = 0, ...\}. \tag{4.6}$$

## 5. Continuous Processes

### 5.1. Closed Continuous Processes.

The continuous processes are naturally divided into two main categories: those for which $dQ = 0$ (closed processes) and those for which $dQ \neq 0$. Closed processes also will be defined as uniformly continuous processes, $L_{(\mathbf{V})}dA = 0$, (relative to the Cartan Topology) to distinguish them from non-uniformly continuous processes which are not closed: $L_{(\mathbf{V})}dA = \Gamma dA = dQ \neq 0$. Uniform continuity implies that the limit sets are invariant. Non-uniform continuity permits a permutation of the limit points, as in a folding and pleating process.

When $dQ = 0$, it is possible to formulate immediately the following theorem (Poincare):

**Theorem 5.1.** *All even dimensional Pfaff classes of p-forms, $dA = F, dAˆdA = K, ...$ are invariants of evolutionary processes that satisfy $L_{(\mathbf{V})}(dA) = dQ = 0$ relative to the Cartan topology. The forms $F, K, ...$ form a set of absolute integral invariants with respect to uniformly continuous processes.*

The proof of the theorem follows immediately from (9b) and by application of the Leibniz rule:

$$L_{(\mathbf{V})}(dAˆdAˆdA..ˆdA) = \text{integer} \times \{L_{(\mathbf{V})}(dA)\}ˆdAˆdA..ˆdA) = 0 \tag{5.1}$$

The integrands of the selected integrals are local invariants and so are their convected integrals.

The first application of theorem II gives,

$$L_{(\mathbf{V})}(dA) = L_{(\mathbf{V})}F = 0 \qquad (5.2)$$

which is the equivalent of Helmholtz' theorem [14] and the local conservation of angular momentum per unit moment of inertia, or the conservation of Topological Vorticity.

The second application of theorem II gives:

$$L_{(\mathbf{V})}(dA\hat{\ }dA) = L_{(\mathbf{V})}F\hat{\ }F = L_{(\mathbf{V})}K = 0 \qquad (5.3)$$

which leads to the local conservation of Topological Parity, with respect to uniformally continuous flows.

In general,

$$L_{(\mathbf{V})}(dA\hat{\ }dA\hat{\ }...dA) = 0 \qquad (5.4)$$

which expresses the invariance of a 2N dimensional area with respect to uniformally continuous flows.

## 5.2. Continuous Hydrodyamic Processes

Consider the domain of four independent variables of space time, $\{x, y, z, t\}$, and the three form of topological torsion

$$H = A\hat{\ }dA = A\hat{\ }F = i(\mathbf{T}_4)dx\hat{\ }dy\hat{\ }dz\hat{\ }dt. \qquad (5.5)$$

The continuous evolution of this 3-form is determined relative to an arbitrary process, $\mathbf{V}_4 = [\mathbf{V}, 1]$, by the equation:

$$L_{(\beta\mathbf{V}_4)}H = L_{(\beta\mathbf{V}_4)}(A\hat{\ }dA) = i(\beta\mathbf{V}_4)dH + di(\beta\mathbf{V}_4)H = Q\hat{\ }F + A\hat{\ }dQ \qquad (5.6)$$

For local invariance of the 3-form with respect to arbitrary parameterizations, the evolutionary vector $\beta\mathbf{V}_4$ must be collinear with the topological torsion vector $(\mathbf{T}_4)$ such that the term $i(\beta\mathbf{V}_4)H \Rightarrow 0$. This constraint implies that the three form $H$ then must be of the format:

$$H = A\hat{\ }F \approx \rho(x, y, z, t)(dx - V^x dt)\hat{\ }(dy - V^y dt)\hat{\ }(dz - V^z dt) = \rho\ i(\mathbf{V}_4)dx\hat{\ }dy\hat{\ }dz\hat{\ }dt \qquad (5.7)$$

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The invariance of the 3-form $H$ then requires that a function $\rho(x, y, z, t)$ exist such that $dH \Rightarrow 0$. But this constraint becomes the equivalent of the famous hydrodynamic equation of continuity:

$$dH = \{div_3\, \rho\mathbf{V} + \partial\rho/\partial t\}dx\hat{}\, dy\hat{}\, dz\hat{}\, dt \Rightarrow 0 \qquad (5.8)$$

which is interpreted physically as the conservation of mass. The implication is that those vector fields, $\beta\mathbf{V}_4$, that define a continuous hydrodynamic current, need not satisfy necessarily the formulas of topological kinematic constraint, $d\mathbf{x} - \mathbf{V}dt = 0$, but instead must be collinear with the topological torsion vector, $\mathbf{J}_4 = \lambda(x, y, z, t)\mathbf{T}_4$, if it exists. The important idea is that local deformable conservation of mass is to be associated with the conservation of the 3-form of Topological torsion as an absolute evolutionary invariant.

These results are to be compared with the even dimensional Poincare absolute integral invariants [12] for the more restrictive case of Hamiltonian (extremal) evolution of a Hamiltonian action,

$$A = A_\mu dx^\mu = \mathbf{p} \cdot d\mathbf{x} - H(\mathbf{x}, \mathbf{p}, t)dt \qquad (5.9)$$

on a 2N+1 dimensional state space. It is the result (8.4) which is interpreted in statistical mechanics as the invariant area of phase space with respect to extremal, or Hamiltonian, evolution. The fact of the matter is that uniform continuity alone (Equation 8.2) produces a set of absolute integral invariants for any action, in Hamiltonian format or not. Hamiltonian extremal flows satisfy equation (8.2) and are therefore uniformly continuous, but they are not the only flows that satisfy equation (8.2). The invariance of "phase space area" is a consequence of uniform continuity alone, and does not require the additional constraints of constant homogeneity that limit the set of continuous flows to that subset of continuous vector fields which are extremal, and Hamiltonian.

### 5.3. DeRham categories of Closed Vector Fields

DeRham's cohomology theory [13] may be used to classify p-forms, and such ideas may be applied to the 1-form $W$ defined by equation (8b). Correspondingly, the vector fields that are used to construct the 1-forms $W$ of virtual work perit processes to be put into the following categories, depending on whether the virtual work, $W$, is null, exact, closed, or not closed with respect to exterior differentiation. These categories are defined as:

| Closed Flows<br>$Categories\ \ for\ Q - W = dU$ | $W = i(\rho\mathbf{V})F$ | $Q$ | $dW$ | $dQ$ |
|---|---|---|---|---|
| $Hamiltonian - extremal$ | $0$ | $dU$ | $0$ | $0$ |
| $Bernoulli - Eulerian$ | $d\Theta$ | $d(U + \Theta)$ | $0$ | $0$ |
| $Helmholtz - Symplectic$ | $d\Theta + \gamma$ | $d(U + \Theta) + \gamma$ | $0$ | $0$ |
| Open Flows<br>$Navier - Stokes - Torsion$ | $arbitrary$ | $arbitrary$ | $dW \neq 0$ | $dQ \neq 0$ |

$$(5.10)$$

The Bernoulli-Casimir functions, $\Theta$, must be first integrals as in general,

$$i(\mathbf{V})W = i(\mathbf{V})d\Theta = 0. \tag{5.11}$$

For closed flows the first law (8c) insures that the 1-form $W$ is closed, $dW = dQ = 0$, but $W$ need not be exact and may contain harmonic components. That is, the 1-form $W$ is not necessarily representable over the variety $x, y, z, t$ in terms of the gradient of a single scalar function. The classic example of a non-exact 1-form is given by the expression,

$$\Gamma = \sigma_z(ydx - xdy)/(x^2 + y^2) \tag{5.12}$$

for which $d\Gamma = 0, but \ \int_{z1}\Gamma = 2\pi\sigma_z$. The coefficient $\sigma_z$ is assumed to be a constant. Such forms, $\Gamma$, generate period integrals and the DeRham cohomology classes. The number of independent forms of the type given by equation (20) determine the Betti numbers of a variety for which the singular point (at the origin in the example) has been excised. The Betti numbers can be interpreted as a method for counting the number of holes or handles in the variety. It is these contributions to the general differential form that carry topological information about the domain of support. The duals to these forms are also closed, leading to the definition, harmonic forms.

From the first law (8c) the harmonic contributions to $W$ are equal to the harmonic contributions to $Q$. If the harmonic contributions to $Q$ are not zero, then the number of "holes and handles" in the Cartan topology of the final state is different from the number of holes and handles in the Cartan topology of the initial state, and the evolutionary process is continuous but not reversible.

In order to make (20) transversal, use the Cartan trick of substituting $dx^i - V^i dt$ for each $dx^i$. The transversal harmonic form becomes

$$\Gamma = \sigma_z \{ydx - xdy + (\mathbf{r} \times \mathbf{V})_z dt\}/(x^2 + y^2) \tag{5.13}$$

which demonstrates the close relationship to transversal harmonic forms and angular momentum. The format may be extended to a spin vector of components

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3] = [\sigma_x/(y^2 + z^2), \sigma_y/(z^2 + x^2), \sigma_z/(x^2 + y^2)] \tag{5.14}$$

such that the harmonic form becomes

$$\Gamma = \boldsymbol{\sigma}_1(zdy - ydz) + \boldsymbol{\sigma}_2(xdz - zdx) + \boldsymbol{\sigma}_3(ydx - xdy) + (\boldsymbol{\sigma} \circ \mathbf{r} \times \mathbf{V})dt. \tag{5.15}$$

The last term is recognized as a "spin orbit" coupling term. The idea of harmonic contributions to a 1-form is closely related to the concept of a complex number or ordered pair representation; i.e., the form cannot be represented by a map to a space of 1 dimension. Other formats for harmonic 1-forms are given by the expressions:

$$\Gamma = \{\phi d\chi - \chi d\phi\}/(a\phi^p + b\chi^p)^{2/p}, \tag{5.16}$$

where $\phi$ and $\chi$ are arbitrary functions on the base space, and for the complex function, $\psi$,

$$\Gamma = \{\psi d\psi^* - \psi^* d\psi\}/(\psi^* \psi). \tag{5.17}$$

The last representation of a harmonic form is in the format of the "probability current" of quantum mechanics, and gives a clue as how to adapt the formalism of this article to quantum systems. Such a development is deferred to a later article.

For closed flows on space time, the fundamental equations of evolution are given by the expressions for the odd 1-form and the odd 3-form. The even forms are invariant. The two fundamental equations of uniformly continuous evolution are:

$$L_{(\rho\mathbf{V})}A = Q \quad and \tag{5.18}$$
$$L_{(\rho\mathbf{V})}H = Q\,\hat{}\,F \tag{5.19}$$

It should be remarked that if the 1-form of Action, $A$, is completely integrable in the sense of Frobenius, then the 3-form $H$ is evanescent, and the evolutionary equation for $H$ has no applicability. Such evolutionary processes $(H = 0)$ are

the equivalent to laminar flows in fluid dynamics and completely integrable, non-chaotic, Hamiltonian systems. It is known that if a Lagrangian system is not chaotic, then the action, $A$, is reducible to two variables (or less), and the 3-form $H$ is necessarily zero. However when there exists a sense of helicity in the evolutionary process, or chaos is present, then the formula for $H$ describes the appropriate topological evolution.

The first expression (5.18) may be put into correspondence with the evolution of energy, while the second fundamental equation (5.19) may be described as the evolution of complexity, or perhaps better as the evolution of defects, links, knots, or in abstract terms, the evolution of an entropic concept. If the heat 1-form $Q$ is zero, then the evolutionary process is adiabatic, and topology is preserved. However, as the Cartan topology is not connected when $H \neq 0$, then continuous evolution of $H$ can be accomplished only between connected subsets. The transition from a connected topology with $H = 0$ to a disconnected topology with $H \neq 0$ can only take place via a discontinuous transformation. The idea is that the continuous rate of change of $H$ is definite (and arbitrarily taken to be positive). This feature is one of the key properties of entropy. Entropy can never change its sign. The creation of topological torsion, $H$, is a discontinuous process from a state of zero topological torsion, but once created, the growth (or decay) of $H$ can be described by a continuous process (relative to the Cartan topology). These entropic features of the topological torsion 3-form will be useful in the description of the transition to turbulence.

## 5.4. The Hamiltonian Sub-Category

It should be remarked, that Cartan has proved, on a domain of dimension 2n+1, that if

$$i(\mathbf{V})F = W = 0, \qquad Q = dU \qquad (5.20)$$

for any reparametrization, $\rho$, then $\mathbf{V}$ generates a Hamiltonian system, and visa versa [14]. This remarkable result indicates that Hamiltonian flows are not only continuous, but preserve many topological properties. The 1-form $Q$ must be exact for Hamiltonian flows. Hence the observable holes and handles are topological invariants of Hamiltonian flows, as the $\rho$ terms vanish. However, the fact that $Q$ is exact for Hamiltonian flows does not completely establish a proof that Hamiltonian systems preserve all topological properties of the Cartan topology.

In the calculus of variations, vector fields that satisfy (21) are defined as extremal vector fields. Characteristic vector fields are a subclass of extremal fields that satisfy the equations

$$L_{(\mathbf{V})}A = 0 \quad and \qquad (5.21)$$
$$L_{(\mathbf{V})}F = 0. \qquad (5.22)$$

In other words, continuous characteristics preserve the Cartan topology ($Q = 0$ and $dQ = 0$). Characteristic Hamiltonian vector fields generate waves in systems that can be endowed with the additional structure of a metric.

## 5.5. The Bernoulli-Euler subcategory

The Bernoulli-Euler category is not quite Hamiltonian. $W$ is not zero, but must be a perfect differential, $W = d\Theta$. However, this perfect differential must be a first integral in order to satisfy the transversality condition, $i(\rho\mathbf{V})W = 0$. The 1 form $Q$ is not necessarily so constrained. The abstract flows of this category are to be compared with the equations of motion of a compressible Eulerian fluid in which there may be stratification. If the pressure, $P$, is a function of the density, $\rho$, alone, then the Eulerian flow can be reduced to a Hamiltonian system [15]. If the exists some anisotropy due to stratification, then the Hamiltonian reduction is not perfect. Note that the first integral, $\Theta$, acts as a Bernoulli constant along a given streamline, but the constant can vary from streamline to streamline because the function is transversal.

## 5.6. The Stokes subcategory

The Stokes category admits topological evolution in the sense that the harmonic contributions to $W$ are not null, and therefore the "hole and handle" count of the Cartan topology is changing in an evolutionary manner. Such closed flows are not reversible. Note that all closed flows preserve topological vorticity and topological parity, and so if the flow is without vorticity in the initial state, then the flow is without vorticity in the final state. The Pfaff dimension [16] remains less than 2. However, if the initial state has vorticity, that vorticity will be preserved, but the Topological Torsion 3-form can change. In fact the Topological Torsion 3-form could be non-zero in the initial state, and zero in the final state, for the decay rate of topological torsion is proportional to $Q\hat{\ }F$ (See Figure 6). Both the 1-form of

action and its hole count, and the 3-form of Topological Torsion, and its twisted handle count, are not necessarily invariants of a Stokes flow.

A method of distinguishing between "holes and twisted handles" is of some interest. Note that physically a handle can be constructed by deforming the rims of two holes in a surface into tubes and pasting the tubular ends together. If the rims are twisted by half integer or integer multiples of pi before the ends are glued together, then the handles have torsion (see Figure 7). Note that a handle cannot be constructed in the plane, so it is an intrinsically 3-dimensional thing. If the 3-form $H$ vanishes, then there are no handles in the initial state, and as the Hamiltonian evolution produces no more new holes, there can be no more new handles in a Hamiltonian flow. However, existing handles may become twisted or knotted, because $Q\char94 F \neq 0$, even for Hamiltonian flows. These facts correspond to the physical result that Hamiltonian systems are not dissipative and preserve energy, but that does not mean that entropy must be conserved.

It should be noted that for all closed flows, $dW = 0$. It follows that for closed flows, the transversality condition $i(\rho\mathbf{V})W = 0$ implies that the 1-form of virtual work $W$ is an absolute invariant of the flow :

$$Closed\ Flows: \quad L_{(\rho\mathbf{V})}W = 0. \tag{5.23}$$

## 5.7. The Navier-Stokes category of open flows

It should be noted that the 1-form $Q$ may be use to construct the Pfaff sequence, $\{Q, dQ, Q\char94 dQ, dQ\char94 dQ\}$, and generates another Pfaff dimension depending upon the rank or class of the elements of the Pfaff sequence for $Q$. For closed flows, $dQ = 0$ and the Pfaff dimension generated by $Q$ is 1. The bulk of this article is devoted to closed flows. For open flows, $dQ \neq 0$, but the Pfaff sequence demonstrates that the topological features of open flows can have various levels of complexity. For example, the criteria that the Pfaff dimension of $Q$ be 2 or less is equivalent to the Frobenious integrability constraint, $Q\char94 dQ = 0$. This is precisely the Caratheodory condition that there exist "inaccessible paths" [17], and that (on a simply connected neighborhood) the 1-form of heat be representable as, $Q = TdS$. The topological evolution theory presented herein permits an analysis to be made for non-equilibrium processes, where the heat 1-form is not of the equilibrium monomial format, $Q \neq TdS$.

For the Navier-Stokes flow, the key feature is that $dQ = dW \neq 0$, but it still must be true that $W$ is transversal. Therefore the 1-form $W$ must be constructed

from fluctuations, in the format,

$$W = f(dx - \mathbf{V}dt) + \text{closed additions transversal to} \mathbf{V}. \qquad (5.24)$$

For open flows W is no longer a flow invariant. In the examples below, a particular choice is made for f which will generate the Navier-Stokes equations, which may have equilibrium or non-equilibrium solutions.

## 5.8. The Kinematic Topological Base

For continuous evolution in space-time, the key idea is that the exterior differential system consists of a Pfaff sequence constructed from a single 1-form of Action A, plus (perhaps) some additional constraints defining a domain of support and its boundary. The work of Arnold (and others) [18] has established that the singular points (zero's) of a global 1-form carry topological information. This idea is to be extended to the singular points of all elements of the Pfaff sequence, or topological base. In Appendix A, the idea of how a global 1-form of Action, A, existing on a space of dimension N+1 can be put into correspondence with a line bundle on a variety of dimension N is worked out in detail. The key features are that the Jacobian matrix of the projectivized 1-form of Action carries most of the information about the subspace. The trace and determinant of the Jacobian matrix determine the mean and Gaussian curvature of the subspace. The anti-symmetric components of the Jacobian are the functions that make up the 2-form, $F = dA$. The polynomial powers of $F$ form the Chern classes for the line bundle.

For continuous transformations on a variety of $\{x, y, z, t\}$, the Cartan Action, $A$, can be defined kinematically as:

$$A = \sum_{1}^{3} \mathbf{v}_k dx^k - \mathcal{H}dt, \qquad (5.25)$$

where the "Hamiltonian" function, $\mathcal{H}$, is defined as,

$$\mathcal{H} = \mathbf{v} \bullet \mathbf{v}/2 + \int dP/\rho \qquad (5.26)$$

Substitute this 1-form into the constraint equation given by **??**. Carry out the indicated operations of exterior differentiation and exterior multiplication to yield a system of necessary partial differential equations yields of the form,

$$\partial \mathbf{v}/\partial t + grad(\mathbf{v} \bullet \mathbf{v}/2) - \mathbf{v} \times curl \mathbf{v} = -grad P/\rho. \qquad (5.27)$$

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These equations are exactly the Euler partial differential equations for the evolution of a perfect fluid.

By direct computation, the 2-form $F = dA$ has components,

$$F = dA = \omega_z dx\char`^dy + \omega_x dy\char`^dz + \omega_y dz\char`^dx + a_x dx\char`^dt + a_y dy\char`^dt + a_z dz\char`^dt, \quad (5.28)$$

where by definition

$$\boldsymbol{\omega} = curl\ \mathbf{v}, \quad \mathbf{a} = -\partial\mathbf{v}/\partial t - grad\mathcal{H} \quad (5.29)$$

These vector fields always satisfy the Poincare-Faraday induction equations, $dF = ddA = 0$ for C2 functions, or,

$$curl\ \mathbf{a} - \partial\boldsymbol{\omega}/\partial t = 0, \quad div\boldsymbol{\omega} = 0. \quad (5.30)$$

The 3-form of Helicity or Topological Torsion, $H$, is constructed from the exterior product of $A$ and $dA$ as,

$$\begin{aligned} H &= A\char`^dA = H_{ijk}dxi\char`^dxj\char`^dxk & (5.31) \\ &= -\mathbf{T}_x dy\char`^dz\char`^dt - \mathbf{T}_y dz\char`^dx\char`^dt - \mathbf{T}_z dx\char`^dy\char`^dt + hdx\char`^dy\char`^dz, & (5.32) \end{aligned}$$

where $\mathbf{T}$ is the fluidic Torsion axial vector current, and h is the torsion (helicity) density:

$$\mathbf{T} = \mathbf{a} \times \mathbf{v} + \mathcal{H}\boldsymbol{\omega}, \quad h = \mathbf{v} \bullet \boldsymbol{\omega} \quad (5.33)$$

The Torsion current, $\mathbf{T}$, consists of two parts. The first term represents the shear of translational accelerations, and the second part represents the shear of rotational accelerations. The topological torsion tensor, $H_{ijk}$ , is a third rank completely anti-symmetric covariant tensor field, with four components on the variety $\{x, y, z, t\}$.

The Topological Parity becomes

$$K = dH = dA\char`^dA = -2(\mathbf{a} \bullet \boldsymbol{\omega})dx\char`^dy\char`^dz\char`^dt. \quad (5.34)$$

This equation is in the form of a divergence when expressed on $\{x, y, z, t\}$,

$$div\mathbf{T} + \partial h/\partial t = -2(\mathbf{a} \bullet \boldsymbol{\omega}), \quad (5.35)$$

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and yields the helicity-torsion current conservation law if the anomaly, $-2(\mathbf{a} \bullet \boldsymbol{\omega})$, on the RHS vanishes. It is to be observed that when $K = 0$, the integral of $K$ vanishes, which implies that the Euler index, $\chi$, is zero. It follows that the integral of $H$ over a boundary of support vanishes by Stokes theorem. This idea is the generalization of the conservation of the integral of helicity density in an Eulerian flow. Note the result is independent from viscosity, subject to the constraint of zero Euler index, $\chi = 0$.

The torsion vector, $\mathbf{T}$, consists of two parts. The first term represents the shear of translational accelerations, and the second part represents the shear of rotational accelerations. The pseudo scalar function, $K$, acts as the source for the divergence of the torsion vector, $T$, and the torsion or helicity density, $h$. When $K = 0$, the evolutionary "lines" associated with the torsion tensor never cross, implying that the system is free of defects in space time. If $K$ is positive or negative, the defects in the system are either growing or decaying. Equation (5.35) is the fundamental new law of topological physics that governs the specific realizations of controlled processes that minimize or maximize defect evolution.

Recall that if $H = A\hat{\ }dA = 0$, the 1-form of action satisfies the complete integrability condition of Frobenius. Similar to the Caratheodory equilibrium result for $Q$, the flow can be described then in terms of two variables; i.e., the flow is laminar. Turbulent flow is not laminar, and the transition from the laminar to the turbulent state must involve the topological evolution of $H$. It was the evolution of the 3-form of topological torsion as displayed in Figure 6 that galvanized the author's interest in topological evolution. The 3-form, $H$, and its evolution is intuitively related to the thermodynamic property of entropy. The fact that the Cartan topology is disconnected if the topological torsion, $H$, is not zero implies that the turbulent state cannot be created from the laminar state by means of a continuous transformation. Turbulence must be created by a discontinuous process. However, the decay of turbulence can be described by means of continuous process.

## 6. Global Conservation Laws

### 6.1. First Variation

Extremal (or Hamiltonian) flows and Eulerian flows induce a set of global conservation laws in the sense that the closed integrals of all odd dimensional Pfaff classes of the fundamental forms are relative integral invariants of uniformly con-

tinuous evolution. The result follows from the fact that the evolutionary rates, $Q$ and $Q\hat{\ }F$ respect to such flows are zero. Integrals of exact forms evaluated over closed cycles, whether the cycle ($Z1\ or\ Z3$) is a boundary or not, vanish. Hence all closed integrals of odd dimensional sets, $\int_{z1} A$ and $\int_{z3} H$, are evolutionary invariants of Hamiltonian and Eulerian flows.

For the closed flows of the Stokes category, the evolutionary rates of all odd Pfaff classes are closed, but not necessarily exact. That is,

$$dQ = 0, \ \ and \ \ d(Q\hat{\ }F) = 0, \tag{6.1}$$

implying closure, but $Q$ and $Q\hat{\ }F$ are not exact. The DeRham classes are not empty and are not flow invariants. Topology changes during such evolutionary processes.

Hence a global set of conservation laws in terms of closed integrals of $A$ and $H$ can be devised only for those closed chains that satisfy Stokes theorem, and those chains must be boundaries (of support). Arbitrary closed integrals are not evolutionary invariants. This lack of relative integral invariance [19] for $\int_{z3} H$ corresponds to the production or destruction of 3 dimensional defects, and these new defects are indications of changing topology and changing inhomogeneity. Formally, a closed integral over a closed form is a period integral whose value, by Brouwer's theorem [20], is an integer multiple of some smallest value. A variation of a period integral signals a change in a Betti number and hence a change in topology. Such flows can produce three dimensional defects.

These results point out the limitations of Moffatt's and Gaffet's claims [21] that the volume integral of helicity density, $\mathbf{v} \bullet curl\mathbf{v}$, is an evolutionary invariant. Helicity is NOT necessarily an invariant of a continuous flow. Moreover, open or closed integrals of Helicity are not necessarily integral invariants of continuous evolution. In particular, the closed volume integral of helicity density, the fourth component of the Helicity four current, is not an invariant of continuous flows for which there is a torsion current .

A theorem depending on only the first variation can be stated for the continuous evolution of flows restricted to Hamiltonian or Eulerian flows:

**Theorem 6.1.  *III***: *The (uniformly) continuous evolution of all odd dimensional Pfaff classes of the Cartan base with respect to Hamiltonian or Eulerian flows ($dQ = 0$, $Q$ exact) are exact. Hence, the closed integrals of $A$ and $H = A\hat{\ }dA$ over closed cycles or boundaries are relative integral invariants with respect to Hamiltonian or Eulerian flows.*

The proof of the theorem is as follows:

$L_{(\mathbf{V})}A = i(\mathbf{V})dA + d(i(\mathbf{V}))A) = d[P + i(\mathbf{V}))A] = Q$ and is exact.

Therefore $L_{(\mathbf{V})} \int_{z1} A = \int_{z1} Q = \int_{z1} d[P + i(\mathbf{V}))A] = 0 \supset$ invariance of $\int_{z1} A$.

Similarly,

$L_{(\mathbf{V})}H = L_{(\mathbf{V})}(A\hat{\ }F) = (L_{(\mathbf{V})}A)\hat{\ }F = Q\hat{\ }F = d[P + i(\mathbf{V}))A]\hat{\ }F$ is exact such that

$L_{(\mathbf{V})} \int_{z3} H = \int_{z3} d[P + i(\mathbf{V}))A]\hat{\ }F \supset$ invariance of $\int_{z3} H$. Q.E.D.

In the hydrodynamic case of a compressible Eulerian fluid, this theorem is the generalization of the "invariance of Helicity theorem" often stated for a barotropic domain or isentropic constraints. Closed flows therefore exhibit global conservation laws based on relative integral invariants of $A$ and $H$, as well as absolute integral invariants of $F$ and $K$. As will be demonstrated below, the integral of the 3-form of topological torsion, not the helicity density, over a boundary is an invariant of all flows that satisfy the Navier-Stokes equations and for which the vorticity vector field satisfies the Frobenius complete integrability conditions. This result is independent from the magnitude of the viscosity coefficient. On the other hand, the continuous destruction of 3-dimensional defects can be associated with closed flows of the Stokes category. Helicity is NOT necessarily a relative integral invariant of Stokes flows. Remarkably, such flows also admit a set of relative integral invariants, but these are determined only in terms of a second variational process.

## 6.2. Second Variation

It should be noted that the second Lie derivative of the odd dimensional Pfaff classes (represented by $A$ and $H$) does produce a set of global conservation laws for uniformly continuous processes. The result follows from the fact that the second Lie derivative of the Action with respect to closed flows is exact, where the first Lie derivative is closed!

The fundamental theorem is then:

**Theorem 6.2.** *$IV$: The (uniformly) continuous evolution of all odd dimensional Pfaff classes of the Cartan base with respect to closed flows ($dQ = 0$) are closed , but not necessarily exact. The second Lie derivative is always exact so that $\int_{z1} Q$ and $\int_{z3} Q\hat{\ }F$ are relative integral invariants of (uniformly) continuous ($dQ = 0$) evolution.*

The proof of the fundamental theorem is as follows:

$L_{(\rho\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V}))A) = Q$
$L_{(\rho\mathbf{V})}L_{(\rho\mathbf{V})}A = L_{(\rho\mathbf{V})}Q = R$
$= i(\rho\mathbf{V})d(i(\rho\mathbf{V})dA) + di(\rho\mathbf{V})di(\rho\mathbf{V})A = i(\rho\mathbf{V})d(Q) + di(\rho\mathbf{V})di(\rho\mathbf{V})A = 0 + d(\Lambda)$

which is exact.

Similarly,

$L_{(\rho\mathbf{V})}L_{(\rho\mathbf{V})}A\hat{\ }dA = L_{(\rho\mathbf{V})}Q\hat{\ }F = d(\Delta\ F)$

which is exact. It follows that

$L_{(\rho\mathbf{V})}\int_{z3}Q\hat{\ }F = \int_{z3}d(\Delta\ F) = 0$

such that $\int_{z3}Q\hat{\ }F$ is a relative integral invariant. Q.E.D.

Uniform continuity requires that $d(L_{(\rho\mathbf{V})}A) = L_{(\rho\mathbf{V})}dA = dQ = 0$, which insures that $Q$ and $Q\hat{\ }F$ are closed. Hence closed integrals of the odd dimensional p-forms of $Q$ and $Q\hat{\ }F$ (and not necessarily $A$ and $H$) are relative integral invariants of uniformly continuous evolution. The integrals *$Q$ and * $Q\hat{\ }F$ generate global conservation laws for uniformly continuous processes in which $dQ = 0$. In elementary terms, on a space time variety, the fundamental theorem of uniformly continuous evolution states that the Lorentz force has zero curl, and the torsion defect production rate has zero divergence ($K = 0$), whether the system is dissipative or not.

The successive Lie derivation with respect to a uniformly continuous vector field J = *V produces an exact sequence, starting from the concept of action-angular momentum, A, evolving to a closed set, $Q$, which under continued Lie derivation evolves to an exact kernel of radiation-power, R [20]. See Figure 8 which represents the exact sequence mod an exact kernel pictorially. A similar exact sequence can be constructed for all odd dimensional Pfaff classes, $A, A\hat{\ }dA, A\hat{\ }dA\hat{\ }dA, ....$

## 6.3. Continuity and the Integers

A most remarkable feature of the fundamental theorem of uniformly continuous evolution is that the integral of any radiation 1-form, $R$, through a container which is a maximal cycle is in relation to the integers. This concept is another application of the Brouwer degree of a map theorem, that says that all period integrals are integer multiples of some smallest value. The maximal cycle is a closed set that is not a boundary but can contain a system with internal defects, hence the name, the "container". As a simple example consider a disc with several internal holes; the maximal cycle is the cycle which would be the boundary if the

disc had no holes. The global conservation laws stated above imply that radiation through the maximal cycle must be compensated by a change in the cohomology class, or the production of a defect of inhomogeneity in the interior. Radiation defects ("holes and torsion handles") are quantized, for it is impossible to create half a hole.

It would appear from the above argument that Planck's hypothesis of quantized radiation oscillators may be considered a consequence of theorem IV and Uniformly CONTINUOUS evolution as defined by equation (17).

## 6.4. The Navier-Stokes Fluid

Although the bulk of this article is limited to the study of uniformly continuous evolution $(dQ = 0)$, some remarks should be made about continuous evolution of the Navier-Stokes category $(dQ \neq 0)$. The evolution equations are then constrained by the format given by equations (8-11) rather than by (17,18) and (24,25).

The kinematic topology is often too course for direct application to a typical physical system. Additional topological constraints must be applied. For a Navier-Stokes fluid, the additional topological constraints on the admissible flow fields, $V = \{\mathbf{v}, 1\}$ implies a specific format is required for the dissipative force, $f$, given in (28); let $f$ take the form $\mathbf{v} \circ curl\mathbf{v}$ such that upon dividing through by *, (28) becomes:

$$i(V)dA = -\{(*curl*)_i(dx^i - V^i dt)\}. \qquad (6.2)$$

Evaluating both sides explicity and comparing coefficients of the terms dxi yields the Navier-Stokes partial differential equations,

$$*v/*t + grad(v*v/2) - vxcurlv = -gradP/* + *curlcurlv \qquad (6.3)$$

This process is typical of the Cartan method, where by the coefficients of a system of differential forms are equivalent to a system of partial differential equations. For the kinematic Action, equation (39) is the differential form equivalent to the Navier-Stokes equations; the constraint limits the class of all V to those V that are solutions to the Navier-Stokes partial differential equations.

The constraint given by (39) may be used evaluate the behavior of the topological base with respect to the evolution described by V. For example, the evolution of the Action is given by the expression,

$$L(V)A = i(V)dA + d\{i(V)A\} = -\{(*curl*)i(dxi - Vidt)\} + d\{(v*v/2) + H\}(40)$$

$$(6.4)$$

The evolution of the limit sets is given by

$$L(V)dA = -d\{(*curl*)i(dxi - Vidt)\}.(41) \tag{6.5}$$

If the flow V is uniformly continuous, then the RHS of (41) must vanish, making F = dA a flow invariant. The Navier-Stokes equations have C2 solutions that belong to the Stokes category of closed flows. This result is an extension of the Helmholtz theorem on the conservation of vorticity. It would follow that the 4-form, K = dAˆdA is also a flow invariant, for uniformly continuous flows. A remarkable result is that even for dissipative Navier Stokes flows where * curl * * 0, it is still possible that the RHS of (41) may vanish, and the flow is uniformly continuous. Examples of such harmonic solutions to the Navier Stokes equations were presented by this author at the Permb conference on Large Scale Structures [3]. One such harmonic closed form solution was shown to develop a tertiary Hopf bifurcation in terms of the parameter of mean flow. The surface of null helicity density, h = v * * = 0 went through a topological phase change as the bifurcation took place similar to that presented by the soap films in Figure 4.

According to theorem II, the even dimensional topological properties {F , K} are invariants of a uniformly continuous flow. If topology is to change in a uniformly continuous manner, the only possible candidates for topological evolution must be the 1-dimensional circulation, A, and the 3-dimensional torsion, H. For incompressible flows (div v = 0) circulation defects must be associated with boundaries; however, if K * 0, then according to (37) torsion defects can occur within the bulk media. It is the author's perception that the production of torsion defects is the key to the understanding of large scale structures in continuous media, and the transition to turbulence.

In general, as has been stated above, if the flow is continuous, then the limit sets d* must remain within the closure of *. Abstractly this idea can be written as,

$$L(V)d* = d * + *ˆ*(42) \tag{6.6}$$

Uniform continuity is the stronger constraint,

$$L(V)d* = 0.(43) \tag{6.7}$$

For the Navier-Stokes flows, where the evolution is not necessarily uniformly continuous, the Navier-Stokes constraint (39) may be used to express the acceleration term, a, dynamically; i.e.,

$$a = -gradH - *v/*t = -vxcurlv + *\{curlcurlv)(44)$$  (6.8)

By substituting this expression for a into equation (34) a simple engineering representation is obtained for the torsion vector current, T, of a Navier-Stokes fluid:

$$T = \{hv - Lcurlv\} - *\{vx(curlcurlv)\}(45)$$  (6.9)

Note that the torsion axial vector current persists even for Euler flows, where * = 0. When h = 0, the torsion axial vector is proportional to the vorticity of the flow. It is the opinion of this author that many of the visual phenomena of fluid dynamics which have been associated with "vortices" are actually representations of torsion defects. In fact, a closed form solution to the Navier-Stokes equations was presented at the Perm conference [3] which indicates that the experimental phenomena of "vortex" bursting can be emulated by the streamlines of a flow for which there is no parametric evolutionary change of vorticity, but for which there is a parametric evolution and topological phase change of the 3-form of topological torsion. As the critical value of flow is achieved, a re-entrant compact torsion bubble is produced in what was originally a unidirectional flow. The measurement of the components of the Torsion vector have been completely ignored by experimentalists (and theorists) in hydrodynamics (and other dynamical systems).

By a similar substitution using the value of a given by the constraint (39), the topological parity pseudo-scalar becomes expressible in terms of engineering quantities as,

The measurement of the components of the Torsion vector have been completely ignored by experimentalists in hydrodynamics.

The Topological Parity 4-form can be evaluated by exterior differentiation as

$$K = dH = dA\hat{}dA = -2 * (* * curl*)dx\hat{}dy\hat{}dz\hat{}dt.$$  (6.10)

From this expression it is apparent that if the vorticity field is integrable in the sense of Frobenius, then viscosity does NOT contribute to the creation of torsion defects. As described below, the integral of K over {x,y,z,t} gives the Euler index induced by the flow on the space time variety. If K = 0, the flow lines never intersect.

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# 7. Pfaff's Problem, Characteristics, and the Torsion Current.

Closely related to the concept of topological torsion is the Pfaff problem that asks about the solubility of the system of differential equations defined by setting each element of the Cartan closure to zero. The problem is equivalent to finding characteristic vector fields which, if continuous, generate an evolutionary flow that preserves the Cartan topology. The key idea of Pfaff's problem is to find maps from spaces of q dimensions into the variety, X, such that when these maps and their differentials are substituted into the system of forms that make up the Cartan closure, then the new forms are equal to zero. In this sense, the pullback of the forms of the Cartan closure to the spaces of dimension q are zero. In the case of usual interest to physics, the maps are of a single parameter which almost always is associated with the concept of time. However, they may exist higher dimensional solutions of say two parameters or more.

The question arises as to the largest dimension of such a "solution" and is determined in terms of the "characters" and "genus" of the Pfaff system [22]. It is the objective of this section to demonstrate that the genus of the Pfaff system built from a single 1-form of action is 3 if the Torsion current, $\mathbf{T}$, vanishes, and can be 2 only if $\mathbf{T} \neq 0$. The genus is an arithmetic invariant and a topological property. A change of genus implies topological evolution. However for the special Pfaff system described, the characters are such that only 1-parameter solutions are possible, when $\mathbf{T} = 0$, and a unique 2 parameter solution is admissible only when $\mathbf{T} \neq 0$. In other words the Pfaff problem admits a "string" solution (a two parameter solution) only when the Torsion current is not zero.

Consider an electromagnetic format. For the electromagnetic case, the Cartan 1-form may be defined in terms of the vector and scalar potentials,

$$A = \mathbf{A} \bullet d\mathbf{r} - \varphi dt. \tag{7.1}$$

Using the classical notation of Sommerfeld, define the $\mathbf{E}$ and $\mathbf{B}$ field intensities as

$$\mathbf{B} = curl\mathbf{A}, \quad \mathbf{E} = -\partial\mathbf{A}/\partial t - grad\varphi. \tag{7.2}$$

Then the components of the Darboux-Cartan-Maxwell field, $F_{\mu\nu}$, may be written as an anti-symmetric matrix ( or as a Sommerfeld six-vector) of components :

$$F_{12} = B_z, \ F_{13} = -B_y, \ F_{23} = B_x, \ F_{14} = E_x, \ F_{24} = E_y, \ F_{34} = E_z \tag{7.3}$$

such that the components of $dA = F = F_{\mu\nu}dx^\mu \char94 dx^\nu$

The Topological torsion, $H$, becomes

$$H = A\char94 dA = -i\{\mathbf{E} \times \mathbf{A} + \varphi\mathbf{B}, \mathbf{A} \bullet \mathbf{B}\}dx\char94 dy\char94 dz\char94 dt. \qquad (7.4)$$

with the torsion current defined as,

$$\mathbf{T} = \mathbf{E} \times \mathbf{A} + \varphi\mathbf{B} \qquad (7.5)$$

and the helicity density,

$$h = \mathbf{A} \bullet \mathbf{B}. \qquad (7.6)$$

The Topological Parity 4-form becomes the global top Pfaffian on the 4 dimensional space-time variety, and is equal to

$$K = dA\char94 dA = -2\mathbf{E} \bullet \mathbf{B}dx\char94 dy\char94 dz\char94 dt. \qquad (7.7)$$

Note that

$$div\mathbf{T} + \partial h/\partial t = -2\mathbf{E} \bullet \mathbf{B} \qquad (7.8)$$

. The 3-form of axial current, $H$, is NOT conserved when $K \neq 0$. This result has been observed by Berger [23]. Following Chern, the Euler index on a compact manifold would be the integral

$$\chi = \int_{z4} 2\mathbf{E} \bullet \mathbf{B}dx\char94 dy\char94 dz\char94 dt. \qquad (7.9)$$

Now the Pfaff problem is determined by the equations

$$A = 0, \quad F = 0. \qquad (7.10)$$

Following Slebodzinsky, as there is only one 1-form in the Pfaff system, the first character, $s0$, of the Pfaff system is equal to 1. Multiply $F$ by $\varphi$, and use $A = 0$ to eliminate $\varphi dt$ in the equation $F = 0$. The result is given by the equation,

$$\{\mathbf{E} \times \mathbf{A} + \varphi\mathbf{B}\}_{\mu\nu}dx^\mu \char94 dx^\nu = \{\mathbf{T}\}_{\mu\nu}dx^\mu \char94 dx^\nu = 0, \qquad (7.11)$$

which is an expression that does not contain $dt$. The polar system of these resultant equations determines the genus of the Pfaff system. In particular, if $\mathbf{T}$, the torsion current vanishes, then (7.11) vanishes, the second character, $s1$ is zero and the genus of the Pfaff system is 3. All higher characters vanish, so the Pfaff

system is special. Only 1-parameter homeomorphic evolutionary solutions are possible for the Pfaff system in 4 dimensions, when $\mathbf{T} = 0$.

On the other hand, for any arbitrary vector field, $\mathbf{V}$, such that the two 1-forms $\{\mathbf{T} \times \mathbf{V}\}_\mu dx^\mu$ and $A$, are linearly independent, then the second character, $s1$, equals 1, and the genus is 2. There then exists a two parameter characteristic evolutionary system (a string). In other words, the presence of the torsion current is necessary for the existence of a two parameter solution to the Pfaff problem. There are no 3 parameter solutions to this Pfaff problem in 4-dimensions. This extraordinary connection between the concept of the Torsion current and the solubility of Pfaff's problem serves to further emphasize the content of the often neglected quantity of topological torsion.

### 7.1. The Euler index

The coefficients of the Action 1-form globally define a covariant vector field on the variety. This vector field need not be a section without singularities. As mentioned in section 13 Arnold has shown how the singular points (zeros) of the Action 1-form, A, can be used to define the Euler index of the topology induced on the variety. Another method for evaluating this key topological property has been devised by Chern [24]. Following Chern, the Euler index becomes the integral

$$\chi = \int_{z4} K = \int_{z4} 2\mathbf{E} \bullet \mathbf{B} \, dx\hat{\ }dy\hat{\ }dz\hat{\ }dt. \qquad (7.12)$$

In Lagrangian field theories, a non-zero value for K implies that the second Chern class is not empty and signals the demise of time reversal and parity symmetry [25] (hence, the name Topological Parity 4-form). It should be remarked that K is the exterior derivative of the 3-form of topological torsion, H, and that this 3-form can be put into correspondence with the Chern-Simons 3-form of differential geometry. In effect the evolutionary law for the 3-form of Topological Torsion given by (10) is a Lagrangian field theory built on a Chern-Simons action. In this article, no constraint of self dualism is imposed, as is usually the case in current string theories.

When the electric field is orthogonal to the magnetic field, then the Euler index is zero. The idea that this Poincare invariant might have deeper meaning led Eddington [26] to state: "It is somewhat curious that the scalar-product of the electric and magnetic forces is of so little importance in classical theory, for ..(eq (53)) .. would seem to be the most fundamental invariant of the field. Apart from the fact that it vanishes for electromagnetic waves propagated in the absence of

any bound electric field (i.e., remote from electrons), this invariant seems to have no significant properties. Perhaps it may turn out to have greater importance when the study of electron-structure is more advanced."

A non-zero value of the Topological Parity 4-form, $K$, implies that the divergence of $\mathbf{T}$ is not zero. Therefore, torsion lines can stop or start within the variety even though the evolution is C2 continuous. The torsion current is not necessarily conserved and 3-dimensional defects can be produced internally. String theorists describe this effect as an anomaly of the axial (Torsion) current. In the same sense that the closed but not exact 1-form leads to a complex representation involving ordered pair of variables, a closed but not exact 3-form leads to a quaternionic representation.

The concept of a domain of non-null Euler index ($K \neq 0$) now appears to be useful to the theory of magnetic reconnection in the electromagnetic case [27] and to vortex reconnection [28] in the hydrodynamic case. The correspondence between the bridging and rib structures produced in numerical simulations of turbulent fluid flows and the 4-string interaction of superstring theory is remarkable [29]. The concept ($K \neq 0$) appears to be applicable to the understanding of the stretching of lines and surfaces in turbulent flows where time-reversal symmetry is violated [30]. The appearance of large scale structures in certain flows has been associated with the lack of parity invariance [31]. The concepts of macroscopic violations of P and T symmetries appear to have application to the theory of the quantum Hall effect [32].

With regards to hydrodynamic systems, the evolution of a flow from a laminar flow to a turbulent flow involves topological evolution. For the Navier-Stokes system, the Euler index depends upon the viscosity and the lack of Frobenius integrability of the vorticity field (see equation 36). Such a term yields a local source for the creation of Torsion currents. The lack of reversibility of such flows, and the irreducible time dependent, 3 dimensional features of such flows, implies that $K$ can not be zero for the turbulent state. It is conjectured that the Euler index of the flow (the integral of $K$ over the domain) is not zero during the transition to turbulence. That is, $K$ is not a last multiplier of the spatial volume element, dx^dy^dz for the flow describing the continuous (relative to the Cartan C2 topology) transition to turbulence. If $dQ\hat{\ }F = 0$ then the function $K$ defines an integrating actor in the sense of a mass density such that

$$div(K\mathbf{V}) + \partial K/\partial t = 0. \qquad (7.13)$$

If $K$ were a mass density, this equation is often called the "equation of conti-

nuity", but it is more accurately described as the "conservation of mass". Relative to the Cartan topology all C2 vector fields are continuous. The transition to the turbulent state, however, must be discontinuous, for the Cartan topology in the turbulent state is disconnected.

# 8. SUMMARY

To review, a topology has been constructed on a variety in terms of the elements of closure of a Cartan system of C2 differential forms and their intersections. The associated topological structure indicates that all processes generated by the Lie convective derivative (relative to a C2 vector field, $\mathbf{V}$) are continuous relative to the Cartan topology. However, the processes so generated are not necessarily homeomorphisms for they need not be reversible; i.e., the topology of the initial state can evolve continuously into a different topology on the final state. The method for constructing the Cartan topology is the same on both the initial and the final state, but, for example, the "hole and handle" count on the initial state can be different from the "hole and handle" count in the final state.

In terms of a single 1-form of Action, $A$, a Cartan topological base was constructed in terms of a set of distinct elements, defined as a Pfaff sequence, and their closures. The fundamental laws of evolution of each of the elements of the topological base was formulated relative to an arbitrary vector field. It was determined that there are two categories of continuous flows, those which are "closed" and those which are "open". A special sub-category of closed flows describe a Hamiltonian evolution, an evolutionary process which preserves the number of "holes and handles".

Relative to the closed category of continuous processes, all even dimension elements of the Cartan topological base are evolutionary invariants. For closed flows, topological evolution takes place only in terms of the odd elements of the topological base. The first odd element of the topological base is the Action, and its law of evolution is equivalent to the evolution of energy. The next odd element (and the only other odd element on space-time) of the Cartan topological base is formulated as the novel 3-form of Topological Torsion. The evolution of this 3-form is studied, for although it does not necessarily satisfy a local conservation law, the anomalous source term, defined as topological parity, can be computed. It is a source of system evolutionary defects. However, it is still possible to establish a set of global conservation laws for the category of closed, (uniformly) continuous but irreversible evolutionary flows. Although the evolution of topoologigcal torsion

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may be described by a continuous process, the creation of topological torsion from a state without topological torsion is not described by a continuous process. As the Cartan topology is not connected, the creation of topological torsion must involve discontinuous processes or shocks.

The fundamental equation of topological evolution, $L_{(\rho\mathbf{V})}A = Q$, is equivalent to cohomological format of the first law of thermodynamics, $W + dU = Q$. The heat 1-form $Q$ may be used to form a Pfaff sequence whose Pfaff dimension may be used to further classify evolutionary flows. For example, if the Pfaff dimension of $Q$ is 2 or less, then $Q$ can be written in the equilibrium format, $Q = TdS$. An example of an open system of flows (defined as $dQ \neq 0$) was presented in terms of the Navier-Stokes equations, for which the anomalous source term, can be computed. In effect it was demonstrated that C2 irreversible flows are among the solution set to the Navier-Stokes system. An abstract example was also given for an electromagnetic Action, in which the concept of time reversal and parity symmetry breaking was associated with a non-null Euler characteristic of the Cartan topology.

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