COHERENT STRUCTURES IN FLUIDS ARE TOPOLOGICAL TORSION DEFECTS

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Abstract: Cartan's theory of a global 1-form of Action on a projective variety permits the algebraic evaluation of certain useful geometric and topological objects which can be singular. The projective algebraic methods therefore lend themselves to the development of a theory of coherent structures and defects in which the concept of translational shear dislocations and rotational shear disclinations can be put on equal footing. The topological methods not only lead to a precise definition of coherent structures in fluids, but also produce a non-statistical test for thermodynamic irreversibility on a symplectic manifold of dimension 4, and therefore yield a necessary criteria for turbulence.

1. Introduction

The objective is to devise non-statistical theoretical methods that will describe the one key feature of a turbulent flow that everyone agrees upon, the feature of irreversibility, and then to show that topological torsion defects in such irreversible regimes have long lived observable consequences that permit the defects to be defined as coherent structures. The intuitive suggestion is that starting from arbitrary initial conditions on 4 dimensional variety $\{x, y, z, t\}$, an irreversible process will decay to one of its (non-unique) "stationary states", a long-lived self-organized or coherent state. The mathematical suggestion is that irreversible processes occur on symplectic manifolds of Pfaff dimension 4 (or topological class) [1], and conformally decay or are attracted into closed sets of measure zero and Pfaff dimension 3. On the 4 dimensional manifold, the anholonomic differential (non-statistical) fluctuations in the classic kinematic formulas, which lead to irreversibility, disappear on the sub-manifolds of measure zero (the long-lived coherent structure).

Most of these ideas are based primarily upon the calculus of variations as extended by Cartan's theory of differential forms [2], and secondly upon Cartan's concept of the Repere Mobile on a projective manifold [3]. A particular topological feature of the Cartan method that has been ignored by the hydrodynamic community is the concept of the Torsion Current [4], perhaps because the idea involves non-Riemannian manifolds and their implicit non-uniqueness of solutions. It will be demonstrated below that when the Torsion

Current has a non-zero space-time divergence, then the associated dynamical system is irreversible in a thermodynamic sense, and is of topological dimension 4. The associated dynamical system decays to ”coherent” states where the divergence of the Torsion Current vanishes, yielding a conservation law for the evolution of the resulting (coherent) structure [5].

It is this Cartan idea of spaces with torsion [6] that is the major theme of this article. It is known that in a space with an affine connection it is possible to have torsion defects produced by shears of translation [7]. In this article, it is emphasized that in a fluid the dominant torsion defect is not induced by translational shears, but instead is induced by rotational shears, and their attendant accelerations. Such rotational torsion defects (disclinations) do not occur in affinely connected manifolds, but are latent in projective manifolds. Affine translational shears preserve parallelism; rotational projective shears do not. In hydrodynamics, such topological torsion defects are representatives of deRham period integrals, and are generated by Harmonic vector fields. As Harmonic vector fields do not produce any contributions to the RHS of the Navier-Stokes equations, no matter how large the kinematic viscosity, they do not induce dissipation. They are topological limit sets which will produce the visible wakes or coherent structures often seen in experiments [8]. Often these wakes, as coherent structures or topological defects, will appear as tangential discontinuities that, like minimal surface soap films, are globally stabilized. These concepts have been reported elsewhere [9].

In part 2, some historical background and motivation is provided for the present stage of the theory. In part 3, Cartan’s Magic Formula (from the calculus of variations) will be used to describe topological evolution, and to develop a thermodynamic criteria for irreversibility. Due to space limitations, part 4, the constructive development of Cartan’s Repere Mobile on a projective domain, the demonstration of a new equation of differential geometric structure involving rotational torsion 2-forms, and its application to disclination defects, rotational shears and coherent structures in hydrodynamics, will be presented elsewhere.

(See http://www.uh.edu/~rkiehn)

2. Some Historical Motivation

It is important to understand the motivation behind this article. It started in 1974, when, using Cartan’s techniques of exterior calculus [10], it was suggested (on intuitive grounds) to examine evolutionary systems that satisfied the equation:

$$i(\mathbf{V})d\mathrm{A} = \Gamma\mathrm{A} + d\Theta, \tag{1}$$

rather than the classic Cartan-Hamilton (extremal) equation:

$$i(\mathbf{V})dA = 0. \qquad (2)$$

These ideas were extended and compared to the projective features of the conformal group. Later, more detailed applications to hydrodynamics were made that led to a derivation of the Navier-Stokes equations on a 4-D space-time setting [11]. In 1975, Cartan's methods of differential topology were applied to the theory of period integrals, with the introduction of a novel 3-dimensional period integral, the integral of quantized spin [12]. The 3-dimensional spin integral is distinct from, but related to, the 3-dimensional period integral of Topological Torsion, which forms the basis of the current work.

Then in 1977 it was determined that irreversibility could be associated with continuous topological evolution, which, although not deterministically predictive, was deterministically retrodictive on the space of exterior forms (covariant anti-symmetric tensor fields) [13]. A natural logical arrow of time is built into the set of differentiable, but not homeomorphic, maps. It was also suggested about that time that the transition to turbulence must involve the failure of the Frobenius integrability theorem, but the details were not clear. It was argued that the streamline state of a fluid implied that the Frobenius condition, $A \hat{} dA = 0$, was satisfied; as the turbulent state was the antithesis of the streamline state, the Frobenius condition must fail in the turbulent regime. The key idea, however, was that the failure of the Frobenius theorem implied the necessity of including the topic of torsion into the analysis. In the current mathematics literature these ideas have migrated into what are called the Chern-Simons forms.

For a vector field that fails the Frobenius condition, the associated dynamical system can not be planar; the space curve has (Frenet) torsion and a helical signature. In 1979 it was determined that parity and time reversal symmetry breaking could occur in macroscopic electromagnetic systems, but the Pfaff dimension had to be 3 or greater [14] (a necessary condition for the failure of the Frobenius theorem). Moreover, such electromagnetic systems can support a new form of propagating discontinuities defined as a Torsion wave. In fact, it was determined that helical or torsional electromagnetic waves propagate with different speeds in different directions (a result verified by experiments in dual polarized ring laser systems)! The Torsional waves could not be represented as functions of a single variable (scalar longitudinal waves), or even an ordered pair of variables (complex transverse polarizable waves), but were irreducibly 4 dimensional [15]. The basis of the Torsional waves was the division algebra of quaternions. Such waves can also appear in fluids, but they have been little studied.

Then in 1986, while in Rio de Janeiro, the author became aware of what are now known as Falaco Solitons [16]. They are easily produced - easily observed - long lived topological defects, obviously involving rotational shears, in a dynamical fluid system. Observations of these long lived topological defects gave credence to the theory of topological defects in hydrodynamic systems. These defects are not to be associated with affine translational shears, as the Falaco effect is dominated by rotational shears. The two 2D surface defects whose Snell projections produce the black spots on the floor of the swimming pool are connected by a 1D string defect that is not visible in the photograph unless dye is injected into the water. The 1D string connects the vertices of the two dimensional surface dimples, and globally stabilizes the coherent structure. Helical torsion waves will propagate along the guiding center furnished by the invisible string connecting the surface defects, much in the fashion of whistlers along the earth's magnetic field lines. These topological defects will last for more than 15 minutes in a still pool of water. For more details and pictures, see [17]

In 1990, using the ideas of Pfaff reduction, some exact solutions to the Navier Stokes equations were obtained in a rotating frame of reference. The extraordinary feature of such solutions is that they replicated certain features of the Falaco Solitons, and exhibited topological phase changes as certain flow coefficients were varied. In one example, bifurcation into a Torsion bubble was produced as the mean flow speed parameter was increased beyond a critical value; the bifurcation took place at constant vorticity! These results are not too widely known, but should be of interest to those at this conference who study coherent structures in rotating systems. The results offer another alternative to the problem that goes by the name of "Vortex bursting" in the hydrodynamics literature. It is suggested herein that this phenomena has nothing to do with vorticity per se, but is an exhibition of one coherent structure of topological torsion transforming into another [18]. In thermodynamics, this event would be called a phase transition.

The concept that a 1-form of Action for a fluid system, when constrained with anholonomic differential fluctuations, would lead to a derivation of the Navier-Stokes equations was presented at the 1992 SECTAM conference in Tennessee [4]. The idea was to define a hydrodynamic action as the 1-form constructed from a classical Lagrange Action, but with possibly non-holonomic differential fluctuations $(d\mathbf{r} - \mathbf{v}dt) \neq 0$ included as constraints on the kinematics. In the following equation, the coefficients, $\mathbf{p}$, are to be considered as Lagrange multipliers.

$$\mathrm{A} = L(\mathbf{r}, \mathbf{v}, t)dt + \mathbf{p} \circ (d\mathbf{r} - \mathbf{v}dt) \tag{3}$$

If all of the variables are independent, the domain of definition is 10 dimensional, $\{\mathbf{r}, t, \mathbf{v}, \mathbf{p}\}$. For the 10 dimensional velocity vector $\mathbf{V} = \{\mathbf{v}, 1, \mathbf{a}, \mathbf{f}\}$, the virtual work 1-form becomes

$$\begin{aligned} W &= i(\mathbf{V})d\mathrm{A} \\ &= (\mathbf{f} - \partial L/\partial \mathbf{r}) \circ (d\mathbf{r} - \mathbf{v}dt) - (\mathbf{p} - \partial L/\partial \mathbf{v}) \circ (d\mathbf{v} - \mathbf{a}dt) \neq 0 \end{aligned} \tag{4}$$

The fundamental result is that if the system under consideration is without differential fluctuations $((d\mathbf{r} - \mathbf{v}dt) \Rightarrow 0,\ (d\mathbf{v} - \mathbf{a}dt) \Rightarrow 0)$, then the virtual work must vanish. But this can happen only on a manifold of odd Pfaff dimension! In contrast, if the system is a symplectic system of even Pfaff dimension, then the virtual work 1-form can never vanish. The key feature is that if the Pfaff dimension is even, then differential fluctuations are to be expected, and these lead to dissipation. The result implies that the evolution is described only imperfectly by a single parameter group of a dynamical system on the symplectic space. In the SECTAM reference explicit expressions were given for a Navier-Stokes system, for which the criteria of irreversibility required that

$$\nu \, curl\mathbf{v} \circ curl\, curl\mathbf{v} \neq 0. \tag{5}$$

Now it is known that a Lagrange system constrained by non-holonomic differential kinematic fluctuations leads to a non-compact symplectic manifold of dimension 2n+2. This (thermodynamic) manifold will not admit unique extremal vector fields that will leave the action integral stationary as a relative integral invariant (the virtual work must vanish for extremal fields, which is impossible on the symplectic manifold). There do exist non-extremal vector fields on the symplectic manifold that leave the Action integral invariant, but they are non unique and are dependent upon initial conditions that may require closed additions to be imposed on the Action 1-form. In modern language, the vector fields that produce stationary states (Bernoulli-Casimir functions) in a symplectic system are not gauge invariant. However, it has been observed that there does exist a unique, gauge independent, vector field on the symplectic manifold that would leave the Action integral a conformal, but not a stationary, invariant; this unique vector field, the Torsion vector field, will satisfy the thermodynamic criteria of irreversibility defined below.

3. **Differential Topology - Pfaff Dimension**

The basic tool for studying topological evolution is Cartan's magic formula [19], in which it is presumed that a physical (hydrodynamic) system can be described adequately by a 1-form of Action, A, and that a physical process can be represented by a contravariant vector field, **V**, which can be used to represent a dynamical system or a flow:

$$\begin{aligned} \mathrm{L}_{(\mathbf{V})} \int A &= \int \mathrm{L}_{(\mathbf{V})} A = \int \{ i(\mathbf{V}) dA + d(i(\mathbf{V})A) \} \\ &= \int \{ W + d(U) \} \;\; = \int Q. \end{aligned} \tag{6}$$

The base manifold will be the 4-dimensional variety $\{x, y, z, t\}$ of engineering practice, but no metrical features are presumed a priori. In fact, the defect analysis is based upon a projective space in which concept of length has been abrogated away. If the RHS of equation 6 is zero, the $\int A$ is said to be an integral invariant of the evolution generated by **V**.

From the point of view of differential topology, the key idea is that the Pfaff dimension, or class [20], of the 1-form of Action specifies topological properties of the system. Given the Action 1-form, A, the Pfaff sequence, $\{\mathrm{A}, d\mathrm{A}, \mathrm{A}\hat{}d\mathrm{A}, d\mathrm{A}\hat{}d\mathrm{A}, ...\}$ will terminate at an integer number of terms ≤ the number of dimensions of the domain of definition. On a 2n+2=4 dimensional domain, the top Pfaffian, $d\mathrm{A}\hat{}d\mathrm{A}$, will define a volume element with a density function whose singular zero set (if it exists) reduces the symplectic domain to a contact manifold of dimension 2n+1=3. This (defect) contact manifold supports a unique extremal field that leaves the Action integral ”stationary”, and leads to the Hamiltonian conservative representation for the Euler flow in hydrodynamics.

The irreversible regime will be on an irreducible symplectic manifold of Pfaff dimension 4, where $d\mathrm{A}\hat{}d\mathrm{A} \neq 0$. Topological defects (or coherent structures) appear as singularities of lesser Pfaff (topological) dimension, $d\mathrm{A}\hat{}d\mathrm{A} = 0$. On the non-singular symplectic domain, there do not exist unique extremal stationary states, but there can exist evolutionary invariant Bernoulli-Casimir functions, Θ, that generate non-extremal, ”stationary” states. Processes can be represente by the nested categories of vector fields, **V**. Recall that in order to be Extremal, the process, **V**, must satisfy the equation

$$Extremal\ --(unique\ Hamiltonian):\qquad i(\mathbf{V})d\mathrm{A} = 0; \tag{7}$$

in order to be Hamiltonian the process must satisfy the equation

$$Bernouilli--Casimir--Hamiltonian:\qquad i(\mathbf{V})d\mathrm{A} = d\Theta; \tag{8}$$

in order to be Symplectic, the process must satisfy the equation

$$Helmholtz--Symplectic:\qquad di(\mathbf{V})d\mathrm{A} = 0. \tag{9}$$

Extremal processes cannot exist on the symplectic domain, for an anti-symmetric matrix of maximal rank on space of even dimensions does not have null eigenvectors. The Bernoulli processes can correspond to energy dissipative symplectic processes, but they, as well as all symplectic processes are reversible in the thermodynamic sense described below. The mechanical energy need not be constant, but the Bernoulli-Casimir function(s), Θ, are evolutionary invariant(s), and may be used to describe non-unique stationary state(s). All of the nested processes above are reversible in a thermodynamic sense.

A crucial idea is the recognition that irreversible processes must support Topological Torsion, $A\hat{}dA \neq 0$, with its attendant properties of non-uniqueness, envelopes, regressions, and projectivized tangent bundles. The existence of Topological Torsion leads to the realization that the classical constraints of kinematic perfection,

$$\Delta\mathbf{x} = (d\mathbf{r} - \mathbf{v}dt) \Rightarrow 0,\ and\ \Delta\mathbf{v} = (d\mathbf{v} - \mathbf{a}dt) \Rightarrow 0, \tag{10}$$

put severe restrictions on the topology of the evolutionary process, restrictions that need not be realized in nature. Indeed, it appears that such constraints of null anholonomic differential fluctuations, such as $\Delta\mathbf{x} = 0, \Delta\mathbf{v} = 0,$ are not realized during the irreversible phase of a process, and such differential fluctuations can cause the lifetime of a ”stationary state” to be finite. Anholonomic differential fluctuations may be viewed as multiple parameter topological replacements for Langevin noise.

Although there does not exist a unique gauge independent stationary state on the symplectic manifold, remarkably there does exist a unique vector field on the symplectic domain, with components that are generated by the 3-form A^dA. This unique (to within a factor) vector field is defined as the Torsion Current, **T**, and satisfies (on the 2n+2=4 dimensional manifold) the equation,

$$i(\mathbf{T})dx\hat{}dy\hat{}dz\hat{}dt = \mathrm{A}\hat{}d\mathrm{A} \qquad (11)$$

This (four component) vector field, **T**, has a non-zero divergence almost everywhere, for if the divergence is zero, then the 4-form dA^dA vanishes, and the domain is no longer a symplectic manifold! The Torsion vector, **T**, can be used to generate a dynamical system that will decay to the stationary states $(div_4(\mathbf{T}) \Rightarrow 0)$ starting from arbitrary initial conditions. These processes are irreversible in the thermodynamic sense. It is remarkable that this unique evolutionary vector field is completely determined (to within a factor) by the physical system itself; e.g., the components of the 1-form, A, determine the components of the Torsion vector.

To understand what is meant by thermodynamic irreversibility, realize that Cartan’s magic formula of topological evolution is equivalent to the first law of thermodynamics.

$$\mathrm{L}_{(\mathbf{v})}\mathrm{A} = i(\mathbf{V})d\mathrm{A} + d(i(\mathbf{V})\mathrm{A}) = W + dU = Q. \qquad (12)$$

A is the ”Action” 1-form that describes the hydrodynamic system. **V** is the vector field that defines the evolutionary process. W is the 1-form of (virtual) work. Q is the 1-form of heat. From classical thermodynamics, a process is irreversible when the heat 1-form Q does not admit an integrating factor. From the Frobenius theorem, the lack of an integrating factor implies that $Q\hat{}dQ \neq 0$. Hence a simple test may be made for any process, **V**, relative to a physical system described by an Action 1-form, A:

$$\textit{If}\ \ \mathrm{L}_{(\mathbf{v})}\mathrm{A}\hat{}\mathrm{L}_{(\mathbf{v})}d\mathrm{A} \neq 0 \ \textit{then the process is irreversible}.$$

This definition implies that symplectic (and therefor Hamiltonian) processes, **S**, are reversible (as $\mathrm{L}_{(\mathbf{S})}d\mathrm{A} = dQ = 0$), but vectors in the direction of the Torsion, **T**, vector are irreversible. For the Torsion vector, the fundamental equations are given by the constraints of conformal invariance,

$$\mathrm{L}_{(\mathbf{T})}\mathrm{A} = \sigma\mathrm{A} \quad and \quad i(\mathbf{T})\mathrm{A} = 0, \qquad (13)$$

such that

$$L_{(\mathbf{T})}A\wedge L_{(\mathbf{T})}dA = Q\wedge dQ = \sigma^2 A\wedge dA \neq 0. \qquad (14)$$

Turbulent flows must have a component along the Torsion vector to be irreversible ($\sigma \neq 0$). A coherent structure is the end result of an irreversible decay process that forms a set of measure zero, $dA\wedge dA = 0$, on space time, but such that the integral over a closed 3-dimensional hypersurface, $\iiint_{closed} A\wedge dA \neq 0$, is a relative integral invariant for the remainder of the evolution. In such domains,

$$L_{(\mathbf{V})}\iiint_z A\wedge dA = \iiint_z \{i(\mathbf{V})(dA\wedge dA) + d(i(\mathbf{V})(A\wedge dA))\} = 0 + 0, \qquad (15)$$

if $dA\wedge dA \Rightarrow 0$, hence the closed integral is an evolutionary, although deformable, invariant.

For a hydrodynamic system, consider the Action 1-form defined by the equation

$$A = \mathbf{v}\circ d\mathbf{x} - (\mathbf{v}\circ\mathbf{v}/2 + \int dP/\rho + \lambda\, div\ \mathbf{v})dt, \qquad (16)$$

with a topological (non-Hamiltonian) constraint involving non-holonomic fluctuations in the kinematic velocity field:

$$i(\mathbf{V})dA = \upsilon\, curl\ curl\mathbf{v}\circ(d\mathbf{r} - \mathbf{v}dt). \qquad (17)$$

Substitution of the Action 1-form, A, into the constraint yields the Navier-Stokes equations as the equations of constrained topological evolution [4]. By direct evaluation of equation 11, the Torsion vector has 4 space time components $\{with\ \ h = \mathbf{v}\circ curl\,\mathbf{v}\}$:

$$\mathbf{T} = \{h\mathbf{v} - (\mathbf{v}\circ\mathbf{v}/2)\, curl\,\mathbf{v} - \nu\, curl\, curl\,\mathbf{v};\ \ h\}, \qquad (18)$$

and a 4 divergence given by the expression:

$$div_4\mathbf{T} = -2\nu\ curl\,\mathbf{v}\circ curl\, curl\,\mathbf{v} = -2\sigma. \qquad (19)$$

When the 4 divergence, -2σ, does not vanish, it follows from equation 14 that the flow, $\mathbf{v}$, is thermodynamically irreversible. Such irreversible solutions to the viscous Navier-Stokes equations must generate lines of vorticity that have non-zero helicity, and can exist only on domains where the Action 1-form is of Pfaff dimension 4.

Consider sets of measure zero (topological Torsion defects) on the space of 4 dimensions such that $div_4\mathbf{T} = 0$. Such domains are at most of Pfaff dimension 3 (relative to the given 1-form of Action) and define a coherent structure. Note that these defect domains (in a Navier-Stokes fluid) do not require that the viscosity coefficient vanish, $\nu \neq 0$, and yet they support thermodynamically reversible processes. Such domains usually evolve in a deformable manner that preserves both the topological property of Pfaff dimension 3 and the topological Torsion integral defined in equation (15). A common feature of such coherent structures is that the vorticity field satisfies the integrability criteria of Frobenius, e.g., as a three vector field, the vorticity vector must be proportional to a gradient. It follows that the velocity field may have helicity, but the vorticity field does not.

It would appear that the concept of two dimensional turbulence is paradoxical, for it requires four dimensions to support an irreversible flow according to the definitions above. It should be remarked that the definition of irreversibility, $Q\hat{}dQ \neq 0$, implies that there are two topological classes of irreversibility. Either $dQ\hat{}dQ = 0$, implying that the "heat current" does not stop or start in the interior, or $dQ\hat{}dQ \neq 0$, implying internal sources of heat current (pinch points).

Similar results will hold for coherent structures created in plasmas. From the electromagnetic 1-form of Action, defined in terms of the vector and scalar potentials as,

$$\mathrm{A} = \Sigma_{k=1}^{3} A_k(x,y,z,t)dx^k - \phi(x,y,z,t)dt, \tag{20}$$

the topological torsion 3-form, $\mathrm{A}\hat{}d\mathrm{A}$, induces the torsion current

$$\mathbf{T} = \{(\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi); \mathbf{A}\circ\mathbf{B}\} \equiv \{\mathbf{S}, h\}. \tag{21}$$

If $div_4\mathbf{T} = -2\,\mathbf{E}\circ\mathbf{B} \neq 0$, the electromagnetic 1-form defines a domain of Pfaff dimension 4. Such domains cannot support transverse waves. Evolutionary processes (currents) that are proportional to the Torsion current are thermodynamically irreversible, if $\mathbf{E}\circ\mathbf{B} \neq 0$. Electromagnetic coherent structures are evolutionary deformable domains of Pfaff dimension 3, where $\mathbf{E}\circ\mathbf{B} = 0$. The conformal dissipation function, $\mathbf{E}\circ\mathbf{B}$, is the electromagnetic analogue of the Navier-Stokes function, $\nu\, curl\,\mathbf{v}\circ curl\,curl\,\mathbf{v}$.

Epilogue

It is a rare thing to attend a conference where on one day a new theoretical prediction is made, and then on the following day of the conference experimental evidence is presented to support the abstract theory. During the presentation of the material described above on May 27 of the SIMFLO conference, it was stated that in an irreversible turbulent flow there should exist a 4 dimensional defect of topological torsion. For a Navier-Stokes fluid, the signature of such a defect would be a curve of vorticity in the form of a twisted helix, and the basic requirement for the existence of the 4 dimensional symplectic manifold is given by the condition, $curl\mathbf{v}\circ curl\,curl\mathbf{v} \neq 0$. The following day Kuibin and Okulov presented experimental observations with a detailed analysis of a dynamical helical curve of vorticity in a swirling fluid. On the following day, they determined that their independent analysis supported the idea that $curl\mathbf{v}\circ curl\,curl\mathbf{v} \neq 0$, thereby giving credence to the abstract theory of Topological Torsion defects presented above.

References

The entire article in expanded form with hot linked references in the form of pdf files can be found at

http://www.uh.edu/~rkiehn/pd2/pd2home.pdf

1. Zhitomirskii, M. (1992) *Typical Singularities of Differential* 1*-forms and Pfaffian Equations*. Translations of Mathematical Monographs, AMS Providence . See also, Kiehn, R.M. (1990) Topological Torsion, Pfaff Dimension and Coherent Structures, in H. K. Moffatt and A. Tsinober, editors, *Topological Fluid Mechanics*, Cambridge University Press, p. 225.

2. Giaquinta, M. and Hildebrandt, S. (1995) *Calculus of Variations*, Springer Verlag, Vol 1 p.398

3. Cartan, E. (1937) *La Theorie des Spaces a Connexion Projective*, Hermann, Paris.

4. Kiehn, R. M. (1992) Topological Defects, Coherent Structures and Turbulence in Terms of Cartan's Theory of Differential Topology, in B. N. Antar, R. Engels, A.A. Prinaris and T. H. Moulden, Editors, *Developments in Theoretical and Applied Mathematics*, Proceedings of the SECTAM XVI conference, The University of Tennessee Space Institute, Tullahoma, TN 37388 USA, p. III.IV.2

5. Kiehn, R. M. (1997) When does a dynamical system represent an Irreversible Process, SIAM Snowbird May 1997 poster. Also see http://www.uh.edu/~rkiehn/pdf/siam.pdf

6. Brillouin, L. (1964) *Tensors in Mechanics and Elasticity*, Academic Press, N.Y., p. 93. In 1938, Brillouin wrote: "If one does not admit the symmetry of the (connection) coefficients, Γ_{ijk}, one obtains the twisted spaces of Cartan, spaces which scarcely have been used in physics to the present, but which seem to be called to an important role."

7. Kondo, K. (1962) Unifying Study of Basic Problems in Engineering and Physical Sciences by means of Geometry in RAAG memoirs Vol 3 Tokyo. See also, F.R.N. Nabarro (1987) *The Theory of Crystal Dislocations*, Dover, p. 562.

8. Kiehn, R. M. (1993) Instability Patterns, Wakes and Topological Limit Sets, in J.P.Bonnet and M.N. Glauser, (eds) *Eddy Structure Identification in Free Turbulent Shear Flows*, Kluwer Academic Publishers, p. 363

9. Kiehn, R. M. (1995) Hydrodynamic Wakes and Minimal Surfaces with Fractal Boundaries, in J.M. Redondo, O. Metais, (eds) *Mixing in Geophysical Flows*, CIMNE, Barcelona p.52

10. Kiehn, R. M. (1974) Extensions of Hamilton's Principle to include Dissipative Systems, J. Math Phys. **5**, 9

11. Kiehn, R. M. (1975) Intrinsic Hydrodynamics with Applications to Space-Time Fluids, Int. J. of Engng Sci. Vol 13, pp 941-949

12. Kiehn, R. M. (1977) Periods on manfolds, quantization and gauge, J. of Math Phys **18**, no. 4, p. 614

13. Kiehn, R. M. (1976) Retrodictive Determinism, Int. J. of Eng. Sci. **14**, p. 749

14. Schultz, A., Kiehn, R. M., Post, E. J., and Roberds, R. B., (1979) Lifting of the four fold EM degeneracy and PT asymmetry, Phys Lett 74A, p. 384.

15. Kiehn, R. M., Kiehn, G. P., and Roberds, R. B. (1991) Parity and Time-reversal Symmetry Breaking, Singular Solutions, Phys Rev A, **43**, p. 5665

16. The Falaco Effect as a topological defect was first noticed by the present author in the swimming pool of an old MIT friend, during a visit in Rio de Janeiro, at the time of Halley's comet, March 1986. The concept was

presented at the Austin Meeting of Dynamic Days in Austin, January 1987, and caused some interest among the resident topologists. The easily reproduced experiment added to the credence of topological defects in fluids. It is now perceived that this topological phenomena is universal, and will appear at all levels from the microscopic to the galactic.

17. See references [18], [4] and http://www.uh.edu/~rkiehn/pd2/pd2homep.htm

18. Kiehn, R. M., (1991) Compact Dissipative Flow Structures with Topological Coherence Embedded in Eulerian Environments, in R.Z. Sagdeev, U. Frisch, F. Hussain, S. S. Moiseev and N. S. Erokhin, (eds) *Non-linear Dynamics of Structures*, World Scientific Press, Singapore p.139-164.

19. Marsden, J.E. and Riatu, T. S. (1994) *Introduction to Mechanics and Symmetry*, Springer-Verlag, p.122

20. Libermann, P. and Charles-Michel, M., (1986) *Symplectic Geometry and Analytical Mechanics*, Riedel -Kluwer, p. 284