A Topological Perspective of Cosmology.

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Abstract

Methods of continuous topological evolution, expressed in terms of Cartan's theory of exterior differential forms, are used to construct a cosmological model of the present universe. The methods invoke topological, non-statistical, thermodynamic principles without geometric constraints of metric, connection or gauge. Stars and galaxies appear as self organizing topological defects, or condensates, of Pfaff topological dimension 3, embedded in a turbulent dissipative, but very dilute, non-equilibrium medium of Pfaff topological dimension 4. Such defects form long lived states of Pfaff dimension 3, and as such are states far from equilibrium which are of Pfaff topological dimension of 2 or less. The Jacobian matrix of the 1-form of Action used to describe the physical system leads to a universal phase function as a characteristic polynomial of fourth degree. The similarity invariants of the Jacobian matrix may be used to express the holomorphic polynomial in an intrinsic manner. The phase function can be interpreted as a family of implicit surfaces in an intrinsic 4D space, with a complex family, or order, parameter. The envelope of the family is homeomorphic to the Gibbs swallowtail surface of a van der Waals gas. The singular set of the family is homeomorphic to the equation of state of a van der Waals gas. The implication is that a domain of Pfaff topological dimension 4 can be deformed into a representation that mimics a van der Waals gas. If the gas is near its critical point, there are large fluctuations in density, which Landau has shown are correlated with a $1/\mathrm{r}^2$ attractive force. Hence, a cosmological model based on the assumption that the universe is a dilute, non-equilibrium turbulent gas near its critical point explains the the granularity of the night sky, the inverse square law of gravitation, and the expansion of the universe.

1. Introduction

The objective of this article is to examine topological aspects and defects of thermodynamic physical systems and their possible continuous topological evolution, creation, and destruction on a cosmological scale. The creation and evolution of stars and galaxies will be interpreted herein in terms of the creation of topological defects and evolutionary phase changes in a very dilute turbulent, non-equilibrium, thermodynamic system of maximal Pfaff topological dimension 4. The cosmology so constructed is opposite in viewpoint to those efforts which hope to describe the universe in terms of properties inherent in the quantum world of Bose-Einstein condensates, super conductors, and superfluids [1]. Both approaches utilize the ideas of topological defects, but thermodynamically the approaches are opposite in the sense that the quantum method involves, essentially, equilibrium systems, while the approach presented herein is based upon non-equilibrium systems. Based upon the single assumption that the universe is a non-equilibrium thermodynamic system of Pfaff topological dimension 4 leads to a cosmology where the universe, at present, can be approximated in terms of the non-equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects and coherent - but not equilibrium - structures of Pfaff topological dimension 3 in this non-equilibrium system of Pfaff topological dimension 4.

The motivation for this conjecture is based on the classical theory of correlations of fluctuations presented in the Landau-Lifshitz volume on statistical mechanics [3]. However, the methods used herein are not statistical, not quantum mechanical, and instead are based on Cartan's methods of exterior differential forms and their application to the topology of thermodynamic systems and their continuous topological evolution [4]. Landau and Lifshitz emphasize that real thermodynamic substances, near the thermodynamic critical point, exhibit extraordinary large fluctuations of density and entropy. In fact, these authors demonstrate that for an almost perfect gas near the critical point, the correlations of the fluctuations can be interpreted as a $1/\mathrm{r}$ potential giving a $1/\mathrm{r}^2$ force law of attraction. Hence, as a cosmological model, the almost perfect gas - such as a very dilute van der Waals gas - near the critical point yields a reason for both the granularity of the night sky and for the $1/\mathrm{r}^2$ force law ascribed to gravitational forces between for massive aggregates.

In this article, a topological (and non statistical) thermodynamic approach is used to demonstrate how a four dimensional variety can support a turbulent, non-equilibrium, physical system with universal properties that are homeomorphic (deformable) to a van der Waals gas. The method leads to the necessary conditions required for the existence, creation or destruction of topological defect structures in such a non-equilibrium system. For those physical systems that admit description in terms of an exterior differential 1-form of Action potentials of maximal rank, a Jacobian matrix can be generated in terms of the partial derivatives of the coefficient functions that define the 1-form of Action. When expressed in terms of intrinsic variables, known as the similarity invariants, the Cayley-Hamilton 4 dimensional characteristic polynomial of the Jacobian matrix generates a universal phase function. Certain topological defect structures can be put into correspondence with constraints placed upon those (curvature) similarity invariants generated by the Cayley-Hamilton 4 dimensional characteristic polynomial. These constraints define equivalence classes of topological properties.

The characteristic polynomial, or Phase function, can be viewed as representing a family of implicit hypersurfaces. The hypersurface has an envelope which, when constrained to a minimal hypersurface, is related to a swallowtail bifurcation set. The swallowtail defect structure is homeomorphic to the Gibbs surface of a van der Waals gas. Another possible defect structure corresponds to the implicit hypersurface surface defined by a zero determinant condition imposed upon the Jacobian matrix. On 4 dimensional variety (space-time) , this non-degenerate hypersurface constraint leads to a cubic polynomial that always can be put into correspondence with a set of non-equilibrium thermodynamic states whose kernel is a van der Waals gas. Hence this universal topological method for creating a low density turbulent non-equilibrium media leads to the setting examined statistically by Landau and Lifschitz in terms of classical fluctuations about the critical point.

The conjecture presented herein is that non-equilibrium topological defects in a non-equilibrium 4 dimensional medium represent the stars and galaxies, which are gravitationally attracted singularities (correlations of fluctuations of density fluctuations) of a real gas near its critical point. Note that the Cartan methods do not impose (*a priori)* a constraint of a metric, connection, or gauge, but do utilize the topological properties associated with constraints placed on the similarity invariants of the universal phase function.

1.1. Topological Thermodynamics

The topological thermodynamic methods used herein are based upon Cartan's theory of exterior differential forms. The topological methods offer an understanding of the cosmos which is considerably different from the geometric approach assumed by the metrical theory of general relativity. The thermodynamic view assumes that the physical systems to be studied can be encoded in terms of a 1-form of Action Potentials, A, on a 4 dimensional variety of ordered independent variables, $\{\xi^1, \xi^2, \xi^3, \xi^4\}$. The variety supports a volume element $\Omega_4 = d\xi^1\hat{}d\xi^2\hat{}d\xi^3\hat{}d\xi^4$. No metric, no connection, no constraint of gauge symmetry is imposed upon the 4 dimensional variety. Topological constraints will be imposed in terms of exterior differential systems [2]

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables will be of the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety. For instance, it is NOT assumed that the variety is euclidean. In that which follows another useful formalism of independent variables will be constructed in terms of the ordered set of similiarity invariant functions, which are given the symbols $\{X_M, Y_G, Z_A, T_K\}$. The similarity invariant functions are those deduced from the Jacobian matrix of the coefficients of that 1-form of Action, A, presumed to encode the properties of a physical system.

The 1-form of Action, A, will have components that form a covariant direction field, $A_k(x, y, z, t)$, to within a non-zero factor. Evolutionary processes will be determined in terms of 4 dimensional contravariant direction fields, $\mathbf{V}_4(x, y, z, t)$, to within a non-zero factor. Continuous topological evolution [4] will be defined in terms of Cartan's magic formula for the Lie differential, which, when acting on an exterior differential 1-form of Action, $A = A_k dx^k$, is equivalent *abstractly* to the first law of thermodynamics.

$$\begin{aligned}
\text{Cartan's Magic Formula} \quad L_{(\mathbf{V}_4)}A &= i(\mathbf{V}_4)dA + d(i(\mathbf{V}_4)A) && (1.1)\\
\text{First Law of Thermodynamics} \quad &: W + dU = Q, && (1.2)\\
\text{Inexact 1-form of Heat} \quad L_{(\mathbf{V}_4)}A &= Q && (1.3)\\
\text{Inexact 1-form of Work} \quad W &= i(\mathbf{V}_4)dA, && (1.4)\\
\text{Internal Energy} \quad U &= i(\mathbf{V}_4)A. && (1.5)
\end{aligned}$$

In effect, Cartan's methods establish a topological basis of thermodynamics in terms of a theory of cohomology. The methods can be used to formulate precise mathematical definitions for many thermodynamic concepts in terms of topological properties -without the use of statistics or metric constraints. Moreover, the method applies to non-equilibrium thermodynamical systems and irreversible processes, again without the use of statistics or metric constraints.

1.1.1. The Pfaff Topological Dimension

One of the most useful topological tools is that defined as the Pfaff topological dimension. Recall that it is possible to define many topologies on the same set of elements. For any given exterior differential 1-form of functions, say $A = A_k(x, y, z, t)dx^k$, it is possible to construct the Pfaff sequence of terms, $\{A, dA, A\hat{}dA, dA\hat{}dA\}$. These elements may be used to construct a Cartan Topology (for any 1-form [5]). In the Cartan topology, the exterior derivative acts as limit point generator. Hence the union of a form and its exterior derivative create the topological closure of the form.

For any given 1-form, the Pfaff sequence will contain M successive non-zero terms equal to or less than N, the number of geometric dimensions of the base independent variables. The number M is defined as the "Pfaff topological dimension" or class of the given 1-form. The three important 1-forms of thermodynamics, A, W, and Q, can have different Pfaff dimensions. Suppose the 1-form of work is defined in terms of two functions as $W = PdV$. The Pfaff sequence consists of the terms $\{W, dW, 0, 0\}$; hence in this example, the Pfaff dimension of W is 2. From the first law, under the assumption that $W = PdV$,

$$\begin{aligned} Q &= W + dU = PdV + dU, && (1.6)\\ dQ &= dW = dP\hat{}dV, && (1.7)\\ Q\hat{}dQ &= W\hat{}dW + dU\hat{}dW = 0 + dU\hat{}dP\hat{}dV && (1.8)\\ dQ\hat{}dQ &= 0. && (1.9) \end{aligned}$$

Hence, the Pfaff dimension of 2 for the work 1-form can be associated with a Pfaff dimension of 3 for the Heat 1-form, unless the Pressure is a function of the internal energy and the volume. In this latter case, the Pfaff dimension of Q and W are both 2.

In this article, attention will be focused on dissipative Turbulent systems with thermodynamic irreversible processes such that the Pfaff topological dimensions of A, W, and Q will be maximal and equal to 4. (The techniques can be extended to higher dimensional spaces.) These Turbulent systems of Pfaff dimension 4 are not topologically equivalent to Equilibrium systems (for which the topological dimension is 2, at most). Topological defects in the Turbulent state will be associated with sets of space time where the Pfaff topological dimensions are not maximal. It is remarkable that such topological defect sets can form attractors causing self organization and long lived states of Pfaff dimension 3, which are far from equilibrium.

1.1.2. Physical Systems

Isolated, Closed and Open Systems Physical systems and processes are elements of topological categories determined by the Pfaff topological dimension (or class) of the 1-forms of Action, A, Work, W, and Heat, Q. For example, the Pfaff topological dimension of the exterior differential 1-form of Action, A, determines the various species of thermodynamic systems in terms of distinct topological categories:

$$Systems : \text{defined by the Pfaff dimension of } A = \rho A^{(0)} \tag{1.10}$$
$$A\hat{}dA = 0 \quad \text{Isolated - Pfaff dimension 2} \tag{1.11}$$
$$d(A\hat{}dA) = 0 \quad \text{Closed - Pfaff dimension 3} \tag{1.12}$$
$$dA\hat{}dA \neq 0. \quad \text{Open - Pfaff dimension 4.} \tag{1.13}$$

In elementary thermodynamics it is often stated that isolated systems do not permit transport of energy or mass to the environment. Closed systems permit energy transport, but not mass transport to the environment. Open systems permit both energy and mass transport to the environment. Note that these topological specifications as given above are determined entirely from the functional properties of the physical system encoded as a 1-form of Action, A. The system topological categories do not involve a process, which is encoded by some vector direction field, $\mathbf{V}_4$. The cosmological model presented herein is based on an open, Pfaff dimension 4, non-equilibrium, Turbulent physical system, with internal defect structures of lesser Pfaff topological dimension acting as stars and galactic mass aggregates.

Equilibrium vs. Non-Equilibrium Systems The intuitive idea for an equilibrium system comes from the experimental recognition that the intensive variables of pressure and temperature become domain constants in an equilibrium state: $dP \Rightarrow 0$, $dT \Rightarrow 0$. A definition made herein is that the Pfaff dimension of a physical system in the equilibrium state is at most 2 [6]. The Cartan topology generated by the elements of the Pfaff sequence for A is then a connected topology of one component, $\{A \neq 0, dA \neq 0, A\hat{}dA = 0\}$. Although the Pfaff dimension of A is at most 2 in the equilibrium state, processes in the equilibrium state are such that the Work 1-form and the Heat 1-form must be of Pfaff dimension 1. For suppose $W = PdV$, then $dW = dP\hat{}dV \Rightarrow 0$ if the pressure is a domain constant. Similarly, suppose $Q = TdS$, the $dQ = dT\hat{}dS \Rightarrow 0$ if the temperature is a domain constant. Hence both W and Q are of Pfaff dimension 1 for this example.

A more stringent sufficient condition for equilibrium can be constructed in terms of the structure of the system, valid for any choice of process. For if the Pfaff dimension of the 1-form of Action is 1, then $dA \Rightarrow 0$. It follows that $W \Rightarrow 0$, hence the Pressure must vanish, and Heat 1-form is a perfect differential, $Q = d(U)$.

The cosmological model proposed herein is of Pfaff dimension 4, with defect structures of Pfaff dimension 3. Neither the dissipative system nor the defect structure is an "equilibrium" thermodynamic system, as the Pfaff dimension of such sets does not satisfy the criteria of equilibrium (where the Pfaff dimension is 2 or less). Although the defects in the Turbulent non-equilibrium regime are not necessarily equilibrium structures, once formed and self organized as coherent topological structures, they can evolve along extremal trajectories that are not dissipative, and may even have a Hamiltonian representation. These "stationary", if not long lived (excited) states, indeed are states far from equilibrium.

The descriptive words of self-organized states far from equilibrium are abstracted from the intuition and conjectures of I. Prigogine [7]. However, the topological theory presented herein presents for the first time a solid formal justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non-equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Thermodynamic irreversibility and the arrow of time are well defined in a topological sense [8], a technique that goes beyond (and without) statistical analysis.

Multiple Components One of the most remarkable properties of the Cartan topology generated by a Pfaff sequence is due to the fact that when $A\hat{}dA = 0$, (Pfaff dimension 2 or less) the physical system is reducible to a single connected topological component. On the other hand when $A\hat{}dA \neq 0$, (Pfaff dimension 3 or more) the physical system admits more than one topological component. The bottom line is that when the Pfaff dimension is 3 or greater (such that conditions of the Frobenius unique integrability theorem are not satisfied), solution uniqueness to the Pfaffian differential equation, $A = 0$, is lost. If there exist solutions, there is more than one. Such concepts lead to propagating discontinuities (signals), envelope solutions (Huygen wavelets), an edge of regression and lack of time reversal invariance, and the existence of irreducible affine torsion in the theory of connections. It is the opinion of this author that a dogmatic insistence on uniqueness historically has hindered the understanding of irreversibility and non-equilibrium systems.

1.1.3. Processes

Reversible and Irreversible Processes The Pfaff topological dimension of the exterior differential 1-form of Heat, Q, determines important topological categories of processes. From classical thermodynamics "The quantity of heat in a reversible process always has an integrating factor" [9] [10] . Hence, from the Frobenius unique integrability theorem, all reversible processes are such that the Pfaff dimension of Q is less than or equal to 2. Irreversible processes are such that the Pfaff dimension of Q is greater than 2. A dissipative irreversible topologically *turbulent* process is defined when the Pfaff dimension of Q is 4.

$$\begin{aligned} Processes &= \text{defined by the Pfaff dimension } Q && (1.14)\\ Q\hat{\ }dQ &= 0 \qquad \text{Reversible - Pfaff dimension 2} && (1.15)\\ d(Q\hat{\ }dQ) &\neq 0. \quad \text{Turbulent - Pfaff dimension 4.} && (1.16) \end{aligned}$$

Note that the Pfaff dimension of Q depends on both the choice of a process, $\mathbf{V}_4$, and the system, A, upon which it acts. As reversible thermodynamic processes are such that $Q\hat{\ }dQ = 0$, and irreversible thermodynamic processes are such that $Q\hat{\ }dQ \neq 0$, Cartan's formula of continuous topological evolution can be used to determine if a given process, $\mathbf{V}_4$, acting on a physical system, A, is thermodynamically reversible or not:

$$\begin{bmatrix} \text{Reversible Processes } \mathbf{V}_4: \ L_{(\mathbf{V}_4)}A\hat{\ }L_{(\mathbf{V}_4)}dA = 0, \\ \text{Irreversible Processes } \mathbf{V}_4: L_{(\mathbf{V}_4)}A\hat{\ }L_{(\mathbf{V}_4)}dA \neq 0. \end{bmatrix} \qquad (1.17)$$

In this article it is assumed that the cosmological background for space-time belongs to the dissipative irreversible Turbulent non-equilibrium category, where the Pfaff topological dimension (or class) is maximal and equal to 4, almost everywhere, for each of the 1-forms of Action, A, Work, W, and Heat, Q. Of particular interest will be those subsets of space and time where the Turbulent non-equilibrium category admits, or evolves into, topological defects such that the Pfaff topological dimension for all three 1-forms is no longer maximal and equal to 4. Remarkably, Cartan's magic formula can be used to describe the continuous dynamic possibilities of both reversible and irreversible processes, in equilibrium or non-equilibrium systems, even when the evolution induces topological change, transitions between excited states, and changes of phase, such as condensations.

It is important to note that the velocity field need not be topologically constrained such that it is singularly parameterized. That is, the evolutionary processes described by Cartan's magic formula are not necessarily restricted to vector fields that satisfy the topological constraints of kinematic perfection, $dx^k - V^k dt = 0$. A discussion of topological fluctuations and an example fluctuation process is described in the last section.

Adiabatic Processes - Reversible and Irreversible The topological formulation permits a precise definition to be made for both reversible and an irreversible adiabatic processes in terms of the topological properties of Q. On a geometrical space of N dimensions, a 1-form will admit N-1 vector fields such that $i(V_A)Q = 0$. Such processes V_A are defined as adiabatic processes [6]. Note that adiabatic processes are defined by vector direction fields, to within an arbitrary factor, $\beta(x, y, z, t)$. That is, if $i(V_A)Q = 0$, then it is also true that $i(\beta V_A)Q = 0$. The differences between the inexact 1-forms of Work and Heat become obvious in terms of the topological format. Both 1-forms depend on the process and on the physical system. However, Work is always transversal to the process, as $i(\mathbf{V}_4)W = i(\mathbf{V}_4)i(\mathbf{V}_4)dA = 0$, but Heat is not, as $i(\mathbf{V}_4)Q = i(\mathbf{V}_4)dU \Rightarrow 0$, only for adiabatic processes.

It is not obvious that the adiabatic direction fields are such that the Pfaff dimension of Q is 2. That is, it is not obvious that Q can be written in the form, $Q = TdS$, as is possible on the manifold of equilibrium states. From the Cartan formulation it is apparent that if Q is not zero, then

$$\begin{aligned} i(\mathbf{V}_A)L_{(\mathbf{V}_A)}A &= i(\mathbf{V}_A)i(\mathbf{V}_A)dA + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) \\ &= 0 + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) = i(\mathbf{V}_A)Q \end{aligned} \tag{1.18}$$

Hence, for an Adiabatic process:

$$\text{Adiabatic process } 0 + i(\mathbf{V}_A)d(i(\mathbf{V}_A)A) = i(\mathbf{V}_A)Q \Rightarrow 0, \quad Q \neq 0. \tag{1.19}$$

A reversible process is defined such that Q is less than Pfaff dimension 3, or $Q\hat{}dQ = 0$ Hence $i(\mathbf{V}_A)(Q\hat{}dQ) = 0$. But

$$i(\mathbf{V}_A)(Q\hat{}dQ) = (i(\mathbf{V}_A)Q)\hat{}dQ - Q\hat{}i(\mathbf{V}_A)dQ \tag{1.20}$$

which permits reversible and irreversible adiabatic processes to be well defined[1] when $Q \neq 0$:

$$\begin{aligned} \text{Reversible Adiabatic Process} &= -Q\hat{}i(\mathbf{V}_A)dQ \Rightarrow 0, \; i(\mathbf{V}_A)Q \Rightarrow 0, && (1.21) \\ \text{Irreversible Adiabatic Process} &= -Q\hat{}i(\mathbf{V}_A)dQ \neq 0, \quad i(\mathbf{V}_A)Q \Rightarrow 0 && (1.22) \end{aligned}$$

It is certainly true that if $L_{(\mathbf{V})}A = Q = 0$, *identically*, then all such processes are adiabatic, and reversible. In such cases, the Cartan formalism implies that $W + dU = 0$. Such systems are elements of the Hamiltonian class of processes, where $W = d\Theta$. Recall that *all* Hamiltonian processes are thermodynamically reversible. Hamiltonian processes are adiabatic when the internal energy $U = (i(V)A)$ is an evolutionary invariant.

$$\begin{aligned} \text{Hamiltonian Adiabatic Process} &= L_{(\mathbf{V})}\{i(V)A\} = i(V)Q = 0, && (1.23) \\ W &= i(V)dA = d\Theta, && (1.24) \\ i(V)W &= 0, \qquad i(V)A = U. && (1.25) \end{aligned}$$

Note that for a given 1-form of heat, Q, it is possible to construct a matrix of N-1 null vectors, and then to compute the adjoint matrix of cofactors transposed to create the unique direction field (to within a factor), $\mathbf{V}_{NullAdjoint}$. Evolution in the direction of $\mathbf{V}_{NullAdjoint}$ does not represent an adiabatic process path, as $i(\mathbf{V}_{NullAdjoint})Q \neq 0$. For a given Q, the N-1 null vectors need not span a smooth hypersurface whose surface normal is proportional to a gradient field. The components of the 1-form may be viewed as the normal vector to an implicit hypersurface, but the implicit hypersurface is not necessarily defined as the zero set of some function.

[1] It is apparent that i(V)Q= 0 defines an adiabatic process, but not necessarily a reversible adiabatic process. This topological point clears up certain misconceptions that appear in the literature.

1.2. Topological Torsion

For maximal, non-equilibrium, turbulent systems in space-time the maximal element in the Pfaff sequence generated by A, W, or Q, is a 4-form. On the geometric space of 4 independent variables, every 4-form is globally closed, in the sense that its exterior derivative vanishes everywhere. It follows that every 4-form is exact and can be generated by the exterior derivative of a 3-form. The exterior derivative of the 3-form is related to the concept of a divergence of a contravariant vector field. Most of the development in this article will be devoted to the study of such 3-forms, and their kernels. It is a remarkable fact that all 3-forms admit integrating denominators, such that their exterior derivative of a rescaled 3-form is zero almost everywhere. Space time points upon which the denominator has a zero value form defect topological structures.

When the Action for a physical system is of Pfaff dimension 4, there exists a unique direction field, $\mathbf{T}_4$, defined as the topological torsion 4-vector, that can be evaluated *entirely* in terms of those component functions of the 1-form of Action which define the physical system. To within a factor, this direction field[2] has the four components of the 3-form $A\hat{}dA$, with the properties such that

$$\begin{aligned}
i(\mathbf{T}_4)\Omega_4 &= A\hat{}dA && (1.26)\\
W &= i(\mathbf{T}_4)dA = \sigma\ A, && (1.27)\\
U &= i(\mathbf{T}_4)A = 0, && (1.28)\\
Q\hat{}dQ &= L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA && (1.29)\\
dA\hat{}dA &= 2\ \sigma\ \Omega_4. && (1.30)
\end{aligned}$$

Hence, evolution in the direction of $\mathbf{T}_4$ is thermodynamically irreversible, when $\sigma \neq 0$ and A is of Pfaff dimension 4. The kernel of this vector field is defined as the zero set under the mapping induced by exterior differentiation. In engineering language, the kernel of this vector field are those point sets upon which the divergence of the vector field vanishes. The Pfaff dimension of the Action 1-form is 3 in the defect regions defined by the kernel of $\mathbf{T}_4$.

For purposes of more rapid comprehension, consider a 1-form of Action, A, with an exterior differential, dA, and a notation that admits an electromagnetic

[2] A direction field is defined by the components of a vector field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

interpretation ($\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$, and $\mathbf{B} = \nabla\times\mathbf{A}$)[3]. The explicit format of $\mathbf{T}_4$ becomes:

$$
\begin{aligned}
\mathbf{T}_4 &= -[\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi,\ \mathbf{A}\circ\mathbf{B}] \text{ Topological Torsion 4 vector,} && (1.31)\\
A\hat{\ }dA &= i(\mathbf{T}_4)\Omega_4 && (1.32)\\
&= T_4^x dy\hat{\ }dz\hat{\ }dt - T_4^y dx\hat{\ }dz\hat{\ }dt + T_4^z dx\hat{\ }dy\hat{\ }dt - T_4^t dx\hat{\ }dy\hat{\ }dz, && (1.33)\\
dA\hat{\ }dA &= 2(\mathbf{E}\circ\mathbf{B})\ \Omega_4 && (1.34)\\
&= \{\partial T_4^x/\partial x + \partial T_4^y/\partial y + \partial T_4^z/\partial z + \partial T_4^t/\partial t\}\ \Omega_4. && (1.35)
\end{aligned}
$$

When the divergence of the topological torsion vector is not zero, $\sigma = (\mathbf{E}\circ\mathbf{B}) \neq 0$, and A is of Pfaff dimension 4, W is of Pfaff dimension 4, and Q is of Pfaff dimension 4. The process generated by $\mathbf{T}_4$ is thermodynamically irreversible. The evolution of the volume element relative to the irreversible process $\mathbf{T}_4$ is given by the expression,

$$
\begin{aligned}
L(\mathbf{T}_4)\Omega_4 &= i(\mathbf{T}_4)d\Omega_4 + d(i(\mathbf{T}_4)\Omega_4) && (1.36)\\
&= 0 + d(A\hat{\ }dA) = 2(\mathbf{E}\circ\mathbf{B})\ \Omega_4. && (1.37)
\end{aligned}
$$

Hence, the differential volume element (and therefore the turbulent cosmological universe) is expanding or contracting depending on the sign and magnitude of $\mathbf{E}\circ\mathbf{B}$.

If A is (or becomes) of Pfaff dimension 3, then $dA\hat{\ }dA \Rightarrow 0$ which implies that $\sigma^2 \Rightarrow 0$, but $A\hat{\ }dA \neq 0$. The differential volume element Ω_4 is subsequently an evolutionary invariant, and evolution in the direction of the topological torsion vector is thermodynamically reversible. The physical system is not in equilibrium, but the divergence free $\mathbf{T}_4$ evolutionary process forces the Pfaff dimension of W to be zero, and the Pfaff dimension of Q to be at most 1. Indeed, a divergence free $\mathbf{T}_4$ evolutionary process has a Hamiltonian representation. In the domain of Pfaff dimension 3 for the Action, A, the subsequent continuous evolution of the system, A, relative to the process $\mathbf{T}_4$, proceeds in an energy conserving manner, representing a "stationary" or "excited" state far from equilibrium. These excited states can be interpreted as the evolutionary topological defects in the Turbulent dissipative system of Pfaff dimension 4.

[3]The bold letter **A** represents the first 3 components of the 4 vector of potentials, with the order in agreement with the ordering of the independent variables. The letter A represents the 1-form of Action.

On a geometric domain of 4 dimensions, assume that the evolutionary process generated by $\mathbf{T}_4$ starts from an initial condition (or state) where the Pfaff topological dimension of A is also 4. Depending on the sign of the divergence of $\mathbf{T}_4$, the process follows an irreversible path for which the divergence represents an expansion or a contraction. If the irreversible evolutionary path is attracted to a region (or state) where the Pfaff topological dimension of the 1-form of Action is 3, then $\mathbf{E} \circ \mathbf{B}$ becomes (or has decayed to) zero. The zero set of the function $\mathbf{E} \circ \mathbf{B}$ defines a hypersurface in the 4 dimensional space. If the process remains trapped on this hypersurface of Pfaff dimension 3, $\mathbf{E} \circ \mathbf{B}$ remains zero, and the $\mathbf{T}_4$ process becomes an extremal field. Such extremal fields are such that the virtual work 1-form vanishes, $W = i(\mathbf{T}_4)dA = 0$, and the now reversible $\mathbf{T}_4$ process has a Hamiltonian representation. The system is conservative in a Hamiltonian sense, but it is in a "excited" state on the hypersurface that is far from equilibrium, as the Pfaff dimension of the 1-form of Action is 3, and not 2. (If the path is attracted to a region where the function $\mathbf{E} \circ \mathbf{B}$ is oscillatory, the system evolutionary path defines a limit cycle.)

The fundamental claim made in this article is that it is these topological defects that self organize from the dissipative irreversible evolution of the Turbulent state into "stationary" states far from equilibrium that form the stars and the galaxies of the cosmos. They are the long lived remnants or wakes generated from irreversible processes in the dissipative non-equilibrium turbulent medium.

2. Thermodynamic Cosmology

2.1. The Jacobian Matrix of the Action 1-form.

The idea is to express the Jacobian matrix of the coefficient functions that define the 1-form if Action, A, in terms of "universal" coordinates. These universal coordinates will be the similarity invariants of the Jacobian matrix. For a 1-form of Action of Pfaff topological dimension 4, the Cayley-Hamilton theorem produces a Universal Phase function as a polynomial of 4th degree. What is remarkable about this Universal Phase function is that it has properties that are homeomorphically deformable into the format of a classic van der Waals gas. It is this universality that gives credence to the idea that the universe could be a non-equilibrium van der Waals gas near its critical point.

2.1.1. The Universal Characteristic Phase Function

The 1-form of Action, used to encode a physical system, contains other useful topological information, as well as geometric information. Consider the Turbulent thermodynamic state generated by a 1-form of Action, A, of Pfaff topological dimension 4. The component functions of the Action 1-form can be used to construct a 4x4 Jacobian matrix of partial derivatives, $[\mathbb{J}_{jk}] = [\partial(A)_j/\partial x^k]$. In general, this Jacobian matrix will be a 4 x 4 matrix that satisfies a 4th order Cayley-Hamilton characteristic polynomial $\Theta(x, y, z, t;\ \Psi)$ with 4 perhaps complex roots representing the perhaps complex eigenvalues, ρ_k, of the Jacobian matrix.

$$\Theta(x, y, z, t;\ \Psi) = \Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi^1 + T_K \Rightarrow 0. \tag{2.1}$$

The functions $X_M(x, y, z, t), Y_G(x, y, z, t), Z_A(x, y, z, t), T_K(x, y, z, t)$ are the similarity invariants of the Jacobian matrix. If the eigenvalues are distinct, then the similarity invariants are given by the expressions:

$$X_M = \rho_1 + \rho_2 + \rho_3 + \rho_4, \tag{2.2}$$

$$Y_G = \rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1 + \rho_4\rho_1 + \rho_4\rho_2 + \rho_4\rho_3, \tag{2.3}$$

$$Z_A = \rho_1\rho_2\rho_3 + \rho_4\rho_1\rho_2 + \rho_4\rho_2\rho_3 + \rho_4\rho_3\rho_1, \tag{2.4}$$

$$T_K = \rho_1\rho_2\rho_3\rho_4. \tag{2.5}$$

The similarity invariants may be considered as a coordinate map from the original variety of independent variables, $\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\}$. When the similarity invariants are treated as generalized coordinates, then the characteristic polynomial becomes a Universal Phase function, and will be used to encode universal thermodynamic properties.

2.1.2. Minimal surfaces

The Universal Phase function, Θ, may be considered as a family of hypersurfaces in the 4 dimensional space, $\{X_M, Y_G, Z_A, T_K\}$ with a complex family (order) parameter, Ψ. Moreover, it should be realized that the Universal Phase Function is a holomorphic function, $\Theta = \phi + i\chi$ in the complex variable $\Psi = u + iv$. That is

$$\Theta(X_M, Y_G, Z_A, T_K;\ \Psi) \Rightarrow \phi + i\chi, \tag{2.6}$$

where

$$\phi = u^4 - 6u^2v^2 + v^4 - X_M(u^3 - 3uv^2) + Y_G(u^2 - v^2) - Z_A u + T_K \tag{2.7}$$

$$\chi = 4u^3v - 4uv^3 - X_M(3u^2v - v^3) + 2Y_G uv - Z_A v. \tag{2.8}$$

As such, in the 4D space of two complex variables, $\{\phi + i\chi, u + iv\}$, according to the theorem of Sophus Lie, any such holomorphic function produces a pair of conjugate *minimal* surfaces in the 4 dimensional space $\{\phi, \chi, u, v\}$. It follows that there exist a sequence of maps,

$$\{x, y, z, t\} \Rightarrow \{X_M, Y_G, Z_A, T_K\} \Rightarrow \{\phi, \chi, u, v\} \tag{2.9}$$

such that the family of hypersurfaces can be decomposed into a pair of conjugate minimal surface components. The criteria for a minimal surface is equivalent to the idea that $X_M = 0$. By suitable renormalization, the similarity invariant X_M is equivalent to the Mean Curvature of the hypersurface.

2.1.3. Envelopes

The theory of implicit hypersurfaces focuses attention upon the possibility that the Universal Phase function has an envelope. The existence of an envelope depends upon the possibility of finding a simultaneous solution to the two implicit surface equations of the family:

$$\Theta(x, y, z, t;\ \Psi) = \Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi + T_K \Rightarrow 0. \tag{2.10}$$

$$\partial\Theta/\partial\Psi = \Theta_\Psi = 4\Psi^3 - 3X_M\Psi^2 + 2Y_G\Psi - Z_A \Rightarrow 0. \tag{2.11}$$

For the envelope to be smooth, it must be true that $\partial^2\Theta/\partial\Psi^2 = \Theta_{\Psi\Psi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{}d\Theta_\Psi \neq 0$ subject to the constraint that the family parameter is a constant: $d\Psi = 0$. The envelope as a smooth hypersurface does not exist unless both conditions are satisfied. Recall that the envelope, if it exists, is a hypersurface in the space of similarity coordinates, $\{X_M, Y_G, Z_A, T_K\}$.

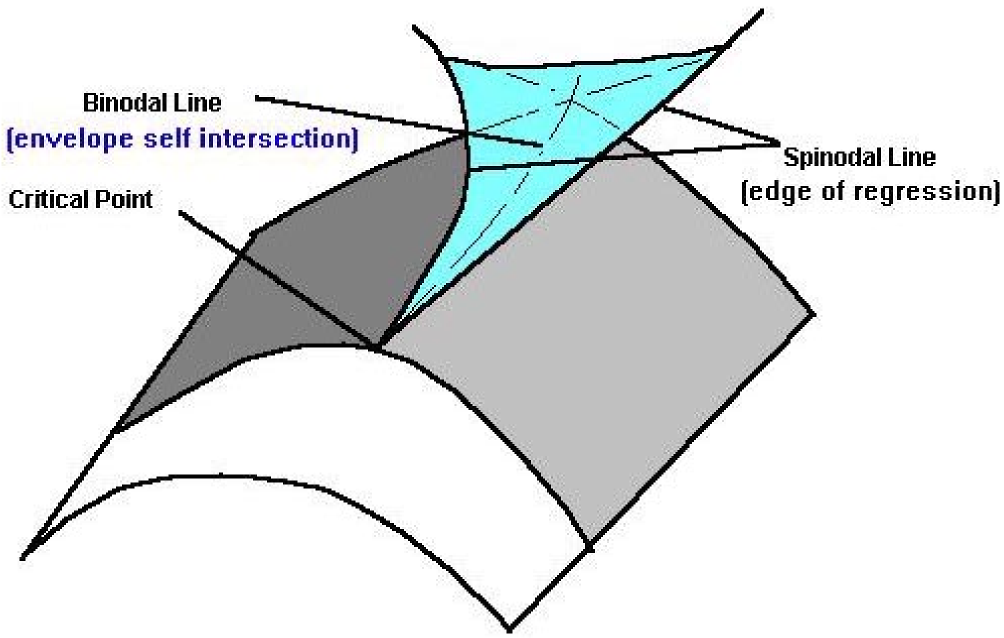
The envelope is determined by the discriminant of the Phase Function polynomial, which as a zero set is equal to a universal hypersurface in the 4 dimensional space of similarity variables $\{X_M, Y_G, Z_A, T_K\}$. This function can be written in terms of the similarity "coordinates" (suppressing the subscripts) :

$$\begin{aligned}
& \text{Discriminant of the Universal Phase Function} && (2.12)\\
= {} & 18X^3ZYT - 27Z^4 + Y^2X^2Z^2 - 4Y^3X^2T + 144YX^2T^2 && (2.13)\\
& +18XZ^3Y - 192XZT^2 - 6X^2Z^2T + 144TZ^2Y - 4X^3Z^3 && (2.14)\\
& -27X^4T^2 - 4Y^3Z^2 + 16Y^4T - 128Y^2T^2 + 256T^3 - 80XZY^2T. && (2.15)
\end{aligned}$$

The discriminant has eliminated the family order parameter. Remarkably, choosing the constraint condition in terms of the dual condition that the Mean Curvature vanishes, $X_M \Rightarrow 0$, leads to a reduced discriminant, which defines a universal swallow tail surface homeomorphic (deformable) to the Gibbs surface of a van der Waals gas (subscripts suppressed):

$$\begin{aligned}&\text{Universal Gibbs Swallowtail Envelope } (X=0,Y,Z,T) \\ = \quad & -27Z^4+144TZ^2Y-4Y^3Z^2+16Y^4T-128Y^2T^2+256T^3 \Rightarrow 0.\end{aligned} \tag{2.16}$$

In other words, the Gibbs function for a van der Waals gas is a universal idea associated with minimal hypersurfaces, $X_K = 0$, of thermodynamic systems of Pfaff topological dimension 4. The similarity coordinate T_K plays the role of the Gibbs free energy, in terms of the Pressure ($\tilde{\ }Z_A$) and the Temperature ($\tilde{\ }Y_G$). The Spinodal line as a limit of phase stability, and the critical point are ideas that come from the study of a van der Waals gas, but herein it is apparent that these concepts are universal topological concepts that remain invariant with respect to deformations. The universal formulas for such constraints are presented in the next section. The result is that all thermodynamic systems of Pfaff topological dimension 4 are deformably equivalent to a van der Waals gas.



A Universal Thermodynamic Swallowtail
The Gibbs surface of a Van der Waals gas

It is important to recognize that the development of a universal non-equilibrium van der Waals gas has not utilized the concepts of metric, connection, statistics, relativity, gauge symmetries, or quantum mechanics.

2.1.4. The Edge of Regression and Self Intersections

The envelope is smooth as long as $\partial^2\Theta/\partial\Psi^2 = \Theta_{\Psi\Psi} \neq 0$, and that the exterior 2-form, $d\Theta\hat{\ }d\Theta_\Psi \neq 0$ subject to the constraint that the family parameter is a constant: $d\Psi = 0$. If $d\Theta\hat{\ }d\Theta_\Psi \neq 0$, but $\Theta_{\Psi\Psi} = 0$, then the envelope has a self intersection singularity. If $d\Theta\hat{\ }d\Theta_\Psi = 0$, but $\Theta_{\Psi\Psi} \neq 0$, there is no self intersection, and no envelope.

If the envelope exists, further singularities are determined by the higher order partial derivatives of the Universal Phase function with respect to Ψ.

$$\partial^2\Theta/\partial\Psi^2 = \Theta_{\Psi\Psi} = 12\Psi^2 - 6X_M\Psi + 2Y_G. \tag{2.17}$$

$$\partial^3\Theta/\partial\Psi^3 = \Theta_{\Psi\Psi\Psi} = 24\Psi - 6X_M \tag{2.18}$$

When $\partial^3\Theta/\partial\Psi^3 = \Theta_{\Psi\Psi\Psi} \neq 0$, and $d\Theta\hat{}d\Theta_\Psi\hat{}d\Theta_{\Psi\Psi} \neq 0$, the envelope terminates in a edge of regression. The edge of regression is determined by the simultaneous solution of $\Theta = 0, \Theta_\Psi = 0$ and $\Theta_{\Psi\Psi} = 0$. For the minimal surface representation of the Gibbs surface for a van der Waals gas, the edge of regression defines the Spinodal line of ultimate phase stability. The edge of regression is evident in the Swallowtail figure (Figure 2.1) describing the Gibbs function for a van der Waals gas.

If $\Theta_{\Psi\Psi\Psi} = 0$, then for $X_M = 0$, it follows that $Y_G = 0,\ Z_A = 0,\ T_K = 0$, which defines the critical point of the Gibbs function for the van der Waals gas. In other words, the critical point is the zero of the 4-dimensional space of similarity coordinates.

If $\Theta_{\Psi\Psi} = 0$, then for $X_M = 0$ the envelope has a self intersection. It follows from $\Theta_{\Psi\Psi} = 0$, that $\Psi^2 = -Y_G/6$, which when substituted into

$$\Theta_\Psi = 4\Psi^3 + 2Y_G\Psi - Z_A \Rightarrow 0, \tag{2.19}$$

yields the

$$\text{Universal Gibbs Edge of Regression: } Z_A^2 + Y_G^3(8/27) = 0, \tag{2.20}$$

which defines the Spinodal line, of the minimal surface representation for a universal non-equilibrium van der Waals gas, in terms of "similarity" coordinates.

Within the swallow tail region the "Gibbs" surface has 3 real roots and outside the swallow tail region there is a unique real root. The edge of regression furnished by the Cardano function defines the transition between real and imaginary root structures. The details of the universal non-equilibrium van der Waals gas in terms of envelopes and edges of regression with complex molal densities or order parameters will be presented elsewhere. These systems are not equilibrium systems for the Pfaff dimension is not 2. Of obvious importance is the idea that the a zero value for both Z_G and T_K are required to reduce the Pfaff dimension to 2, the necessary condition for an equilibrium system.

2.1.5. Ginsburg Landau Currents

The Universal Phase function can be solved for the determinant of the Jacobian matrix, which is equal to the similarity invariant T_K,

$$T_K = -\{\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi\}. \tag{2.21}$$

All determinants are in effect N - forms on the domain of independent variables. All N-forms can be related to the exterior derivative of some N-1 form or current, J. Hence

$$dJ = K\Omega_4 = div\mathbf{J} + \partial\rho/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi)\Omega_4. \tag{2.22}$$

For currents of the form

$$\mathbf{J} = grad\ \Psi, \tag{2.23}$$
$$\rho = \Psi, \tag{2.24}$$

the Universal Phase function generates the universal Ginsburg Landau equations

$$\nabla^2\Psi + \partial\Psi/\partial t = -(\Psi^4 - X_M\Psi^3 + Y_G\Psi^2 - Z_A\Psi). \tag{2.25}$$

2.2. Singularities as defects of Pfaff dimension 3

The family of hypersurfaces can be topologically constrained such that the topological dimension is reduced, and/or constraints can be imposed upon functions of the similarity variables forcing them to vanish. Such regions in the 4 dimensional topological domain indicate topological defects or thermodynamic changes of phase. It is remarkable that for a given 1-form of Action there are an infinite number rescaling functions, λ, such that the Jacobian matrix $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$ is singular (has a zero determinant). For if the coefficients of any 1-form of Action are rescaled by a divisor generated by the Holder norm,

$$\text{Holder Norm:}\ \ \lambda = \{a(A_1)^p + b(A_2)^p + c(A_3)^p + e(A_4)^p\}^{m/p}, \tag{2.26}$$

then the rescaled Jacobian matrix

$$\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k] \tag{2.27}$$

will have a zero determinant, for any index p, any set of isotropy or signature constants, a, b, c, e, if the homogeneity index is equal to unity: $m = 1$. This homogeneous constraint implies that the similarity invariants become projective invariants, not just equi-affine invariants. Such species of topological defects can have the image of a 3-dimensional implicit characteristic hypersurface in space-time:

$$\text{Singular hypersurface in 4D: } \det[\partial(A/\lambda)_j/\partial x^k] \Rightarrow 0 \tag{2.28}$$

The singular fourth order Cayley-Hamilton polynomial of $[\mathbb{J}_{jk}]$ then will have a cubic polynomial factor with one zero eigenvalue.

For example, consider the simple case where the determinant of the Jacobian vanishes: $T_K \Rightarrow 0$. Then the Phase function becomes

$$\begin{aligned} \text{Universal Equation of State} &: \quad \Theta(\{X_M, Y_G, Z_A, T_K = 0\};\ \Psi) & (2.29) \\ &= \quad \Psi(\Psi^3 - X_M\Psi^2 + Y_G\Psi - Z_A) \Rightarrow 0. & (2.30) \end{aligned}$$

The space has been topologically reduced to 3 dimensions (one eigen value is zero), and the zero set of the resulting singular Universal Phase function becomes a universal cubic equation that is homeomorphic to the cubic equation of state for a van der Waals gas.

When the rescaling factor λ is chosen such that $p = 2, a = b = c = 1, m = 1$, then the Jacobian matrix, $[\mathbb{J}_{jk}]$, is equivalent to the "Shape" matrix for an implicit hypersurface in the theory of differential geometry. (See appendix 1.) Recall that the homogeneous similarity invariants can be put into correspondence with the linear Mean curvature, $X_M \Rightarrow C_M$, the quadratic Gauss curvature, $Y_G \Rightarrow C_G$, and the cubic Adjoint curvature, $Z_A \Rightarrow C_A$, of the hypersurface. The characteristic cubic polynomial can be put into correspondence with a nonlinear extension of an ideal gas *not necessarily* in an equilibrium state.

2.2.1. The Universal van der Waals gas

More that 100 years ago van der Waals introduced into the science of thermodynamics the equation of state now called the van der Waals gas:

$$P = \rho RT/(1 - b\rho) + a\rho^2 \tag{2.31}$$

The van der Waals equation may be considered as a cubic constraint on the space of variables $\{n; P, V, T\}$ where $\rho = n/V$ is defined as the molar density.

$$\rho^3 - (1/b)\rho^2 + \{-(RT + bP)/ab\}\rho + P/ab = 0. \tag{2.32}$$

This cubic equation is to be compared with the characteristic polynomial written in terms of the similarity invariants, M, G, and A. Note that the roots of the characteristic polynomial are not necessarily real. This observation leads to a well defined procedure for treating non-equilibrium thermodynamics systems as complex deviations from the real, or equilibrium, systems. The reality condition is determined by the Cardano function that describes an edge of regression discontinuity.

For a transformation such that

$$(8T + P)/3 = Y_G/(M/3)^2, \tag{2.33}$$

$$\begin{aligned} P &= Z_K/(M/3)^3, \quad (2.34) \\ \lambda &= -\rho/(M/3), \quad (2.35) \end{aligned}$$

the characteristic polynomial becomes an equation in terms of dimensionless parameters,

$$U(\lambda, T, P) = (\lambda)^3 - 3(\lambda)^2 + [(8T + P)/3](\lambda) - P = 0. \tag{2.36}$$

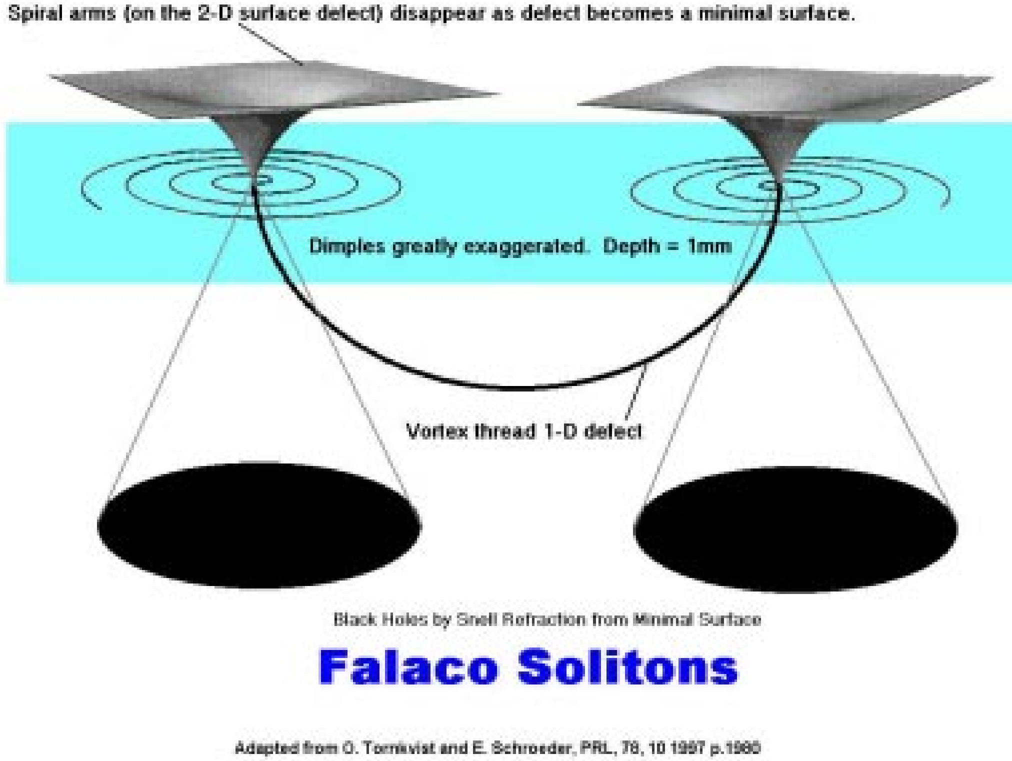
The last format given above is to be recognized as the Equation of State of a van der Waals Gas (compare to 2.29), in terms of dimensionless Pressure, Temperature relative to their values at the critical point.

2.3. The Falaco Cosmological Soliton

Although of importance to the cosmological concept of a universe expressible as a low density (non-equilibrium) van der Waals gas near its critical point, the factorization of the Jacobian characteristic polynomial into a cubic is not the only cosmological possibility. Of particular interest is the factorization that leads to a Hopf bifurcation. In this case the characteristic determinant vanishes, the Adjoint cubic curvature vanishes, the mean curvature vanishes (indicating a minimal surface), but the Gauss curvature is positive, and the two remaining eigenvalues of the characteristic polynomial are pure imaginary conjugates. Such results indicate rotations or oscillations (as in the chemical Brusselator reactions) and the possibility of spiral concentration or density waves on such minimal surfaces. Such structures at a cosmological level would appear to explain the origin of spiral arm galaxies. The Hopf type minimal surfaces of positive Gauss curvature do not represent thermodynamic equilibrium systems, for their curvatures, although two in number, are pure imaginary. The molal density distributions (or order parameters) are complex.

Evidence of such topological defects (at the macroscopic level) can be demonstrated by the creation of Falaco Solitons in a swimming pool [14]. These experiments demonstrate that such topological defects are available at all scales. The

Falaco Solitons consist of spiral "vortex defect" structures (analogous to CGL theory) on a two dimensional minimal surface, one at each end of a 1-dimensional "vortex line" or thread (analogous to GPG theory). Remarkably the topological defect surface structure is locally unstable, as the surface is of negative Gauss curvature. Yet the pair of locally unstable 2-D surfaces is *globally* stabilized by the 1-D line defect attached to the "vertex" points of the minimal surfaces.



For some specific physical systems it can be demonstrated that period (circulation) integrals of the 1-form of Action potentials, A, lead to the concept of "vortex defect lines". The idea is extendable to "twisted vortex defect lines" in three dimensions. The "twisted vortex defects" become the spiral vortices of a Complex Ginsburg Landau (CGL) theory , while the "untwisted vortex lines" become the defects of Ginzburg-Pitaevskii-Gross (GPG) theory [13].

In the macroscopic domain, the experiments visually indicate "almost flat" spiral arm structures during the formative stages of the Falaco solitons. In the cosmological domain, it is suggested that these universal topological defects represent the ubiquitous "almost flat" spiral arm galaxies. Based on the experimental creation of Falaco Solitons in a swimming pool, it has been conjectured that M31 and the Milky Way galaxies could be connected by a topological defect thread [14]. Only recently has photographic evidence appeared suggesting that galaxies may be connected by strings.

At the other extreme, the rotational minimal surfaces of negative Gauss curvature which form the two endcaps of the Falaco soliton, like quarks, apparently are confined by the string. If the string (whose "tension" induces global stability of the unstable endcaps) is severed, the endcaps (like unconfined quarks in the elementary particle domain) disappear (in a non-diffusive manner). In the microscopic electromagnetic domain, the Falaco soliton structure offers an alternate, topological, pairing mechanism on a Fermi surface, that could serve as an alternate to the Cooper pairing in superconductors.

2.4. The Adjoint Current and Topological Spin

From the singular Jacobian matrix, $\left[\mathbb{J}_{jk}^{scaled}\right] = [\partial(A/\lambda)_j/\partial x^k]$, it is always possible to construct the Adjoint matrix as the matrix of cofactors transposed:

$$\text{Adjoint Matrix}: \left[\widehat{\mathbb{J}}^{kj}\right] = adjoint \left[\mathbb{J}_{jk}^{scaled}\right] \tag{2.37}$$

When this matrix is multiplied times the rescaled covector components, the result is the production of an adjoint current,

$$\text{Adjoint current}: \left|\widehat{\mathbf{J}}^k\right\rangle = \left[\widehat{\mathbb{J}}^{kj}\right] \circ \left|\mathbf{A}_j/\lambda\right\rangle \tag{2.38}$$

It is remarkable that the construction is such that the Adjoint current 3-form, if not zero, has zero divergence globally:

$$\widehat{J} = i(\widehat{\mathbf{J}}^k)\Omega_4 \tag{2.39}$$

$$d\widehat{J} = 0. \tag{2.40}$$

From the realization that the Adjoint matrix may admit a non-zero globally conserved 3-form density, or current, $\widehat{J}$, it follows abstractly that there exists a 2-form density of "excitations", $\widehat{G}$, such that

$$\text{Adjoint current}: \widehat{J} = d\widehat{G}. \tag{2.41}$$

$\widehat{G}$ is not uniquely defined in terms of the adjoint current, for $\widehat{G}$ could have closed components (gauge additions $\widehat{G}_c$, such that $d\widehat{G}_c = 0$), which do not contribute to the current, $\widehat{J}$.

From the topological theory of electromagnetism [11] [12] there exists a fundamental 3-form, $A\hat{}G$, defined as the "topological Spin" 3-form,

$$\text{Topological Spin 3-form} : A\hat{}G. \tag{2.42}$$

The exterior derivative of this 3-form produces a 4-form, with a coefficient energy density function that is composed of two parts:

$$d(A\hat{}G) = F\hat{}G - A\hat{}\widehat{J}. \tag{2.43}$$

The first term is twice the difference between the "magnetic" and the "electric" energy density, and is a factor of 2 times the Lagrangian usually chosen for the electromagnetic field in classic field theory:

$$\text{Lagrangian Field energy density}: F\hat{}G = \mathbf{B}\circ\mathbf{H} - \mathbf{D}\circ\mathbf{E} \tag{2.44}$$

The second term is defined as the "interaction energy density"

$$\text{Interaction energy density}: A\hat{}\widehat{J} = \mathbf{A}\circ\widehat{\mathbf{J}} - \rho\phi. \tag{2.45}$$

For the special (Gauss) choice of integrating denominator, λ with $(p = 2, a = b = c = 1, m = 1)$ it can be demonstrated that the Jacobian similarity invariants are equal to the classic curvatures:

$$\{X_M, Y_G, Z_A, T_K\} \Rightarrow \{C_{M(mean\_linear)}, C_{G(gauss\_quadratic)}, C_{A(adjoint\_cubic)}, 0\}. \tag{2.46}$$

It can be demonstrated that the interaction density is exactly equal to the Adjoint curvature energy density:

$$\text{Interaction energy } A\hat{}\widehat{J} = C_A\ \Omega_4 \quad \text{(The Adjoint Cubic Curvature).} \tag{2.47}$$

The conclusion reached is that a non-zero interaction energy density implies the thermodynamic system is not in an equilibrium state.

However, it is always possible to construct the 3-form, $\widehat{S}$:

$$\text{Topological Spin 3-form : } \widehat{S} = A\hat{}\widehat{G} \tag{2.48}$$

The exterior derivative of this 3-form leads to a co-homological structural equation similar the first law of thermodynamics, but useful for non-equilibrium systems. This result, now recognized as a statement applicable to non-equilibrium thermodynamic processes, was defined as the "Intrinsic Transport Theorem" in 1969 [15] :

$$\text{Intrinsic Transport Theorem (Spin)} \quad : \quad d\widehat{S} = F\hat{}\widehat{G} - A\hat{}\widehat{J}, \tag{2.49}$$

$$\text{First Law of Thermodynamics (Energy)} \quad : \quad dU = Q - W \tag{2.50}$$

If one considers a collapsing system, then the geometric curvatures increase with smaller scales. If Gauss quadratic curvature, C_G, is to be related to gravitational collapse of matter, then at some level of smaller scales a term cubic in curvatures, C_H, would dominate. It is conjectured that the cubic curvature produced by the interaction energy effect described above could prevent the collapse to a black hole. Cosmologists and relativists apparently have ignored such cubic curvature effects.

2.5. Topological Fluctuations

Topological fluctuations are admitted when the evolutionary vector direction fields are not singly parametrized:

$$\begin{aligned} \text{Fluctuations in position (pressure)} &: \quad d\mathbf{x} - \mathbf{v}dt = \Delta\mathbf{x} \neq 0 \quad (2.51) \\ \text{Fluctuations in velocity (temperature)} &: \quad d\mathbf{v} - \mathbf{a}dt = \Delta\mathbf{v} \neq 0 \quad (2.52) \\ \text{Fluctuations in momenta (viscosity)} &: \quad d\mathbf{p} - \mathbf{f}dt = \Delta\mathbf{p} \neq 0. \quad (2.53) \end{aligned}$$

These failures of kinematic perfection undo the topological refinements imposed by a "kinematic particle" point of view, and place emphasis on the continuum methods inherent in fluids and plasmas. For example, consider the Cartan-Hilbert 1-form of Action on a space of 3n+1 independent variables (the p_μ are presumed to be independent Lagrange multipliers):

$$A = L(\mathbf{x}, \mathbf{v}, t)dt + p_\mu(dx^\mu - v^\mu dt) = L(x, v, t)dt + p_\mu \Delta x^\mu) \qquad (2.54)$$

The Top Pfaffian in the Pfaff sequence is

$$(dA)^{n+1} = (n+1)!\{\Sigma_{\mu=1}^{n}(\partial L/\partial v^\mu - p_\mu) \bullet d\mathbf{v}^\mu\}\hat{}dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt, \qquad (2.55)$$

and yields a Pfaff dimension of 2n+2 for the 1-form of Action, defined on the geometric space of 3n+1 variables $\{x^\mu, p_\mu, v^\mu, t\}$. This even dimensional space defines a symplectic manifold.

For the maximal non-canonical symplectic physical system of Pfaff dimension 2n+2, consider evolutionary processes to be representable by vector fields of the form $\gamma V_{3n+1} = \gamma\{\mathbf{v}, \mathbf{a}, \mathbf{f}, 1\}$, relative to the independent variables $\{\mathbf{x}, \mathbf{v}, \mathbf{p}, t\}$. Define the "virtual work" 1-form, W, as $W = i(\mathbf{W})dA$, a 1-form which must vanish for the extremal case, and be non-zero, but closed, for the symplectic case. For any n, it may be shown by direct computation that the virtual work 1-form consists of two distinct terms, each involving a different fluctuation:

$$W = \{\mathbf{p} - \partial L/\partial\mathbf{v}\} \bullet \Delta\mathbf{v} + \{\mathbf{f} - \partial L/\partial\mathbf{x}\} \bullet \Delta\mathbf{x} \qquad (2.56)$$

When the fluctuations in velocity are zero (temperature) and the fluctuations in position are zero (pressure), then the work 1-form will vanish, and the process and physical system admits a Hamiltonian representation. On the other hand if the fluctuations in velocity are not zero and the fluctuations in position are not zero, then the Work 1-form vanishes only if the momenta (the Lagrange multipliers, $\mathbf{p}$, are canonically defined ($\{\mathbf{p} - \partial L/\partial\mathbf{v}\} \Rightarrow 0$) and the Newtonian force is a gradient,

$\{\mathbf{f} - \partial L/\partial \mathbf{x}\} \Rightarrow 0$. These topological constraints are ubiquitously assumed in classical mechanics.

When $\Delta x^k \Rightarrow 0$, such that all topological fluctuations vanish, then the Pfaff dimension of the physical system defined in terms of the Cartan-Hilbert 1-form of Action, A, is 2 (the equilibrium requirement).

3. Examples

In order to demonstrate content to the thermodynamic topological theory, two algebraically simple examples are presented below. The first corresponds to a Jacobian characteristic equation that has a cubic polynomial factor, and hence can be identified with a van der Waals gas. The second example exhibits the features associated with a Hopf bifurcation, where the characteristic equation has a quadratic factor with two pure imaginary roots, and two null roots. The third example demonstrates how a bowling ball, given initial angular momentum and energy, skids and/or slips changing its angular momentum and kinetic energy irreversibly via friction effects, until the dynamics is such that the ball rolls with out slipping. Once that "excited" state is reached, and topological fluctuations are ignored, the motion continues without dissipation. The system is in an excited state far from equilibrium.

3.0.1. Example 1: van der Waals properties from rotation and contraction

In this example,the Action 1-form is presumed to be of the form

$$A_0 = a(ydx - xdy) + b(tdz + zdt). \tag{3.1}$$

The 1-form of Potentials depends on the coefficients a and b. The results of the topological theory are (for $r^2 = x^2 + y^2 + z^2 + t^2$):

$$\begin{aligned}
\text{Mean curvature} &: \quad C_M = -2btz/(r^2)^{3/2} && (3.2)\\
\text{Gauss curvature} &: \quad C_G = -\{b^2(x^2+y^2) - a^2(z^2+t^2)\}/(r^2)^2 && (3.3)\\
\text{Adjoint curvature} &: \quad C_A = A\hat{}J_s = -2a^2btz/(r^2)^{5/2} && (3.4)\\
Top_Torsion &= \quad 2ab \cdot [0, 0, z, -t]/(r^2) && (3.5)\\
\text{Adjoint Current} &: \quad J_s = (a^2b^2 \cdot [x, y, z, t])\ /(r^2)^2 && (3.6)\\
\text{Pfaff Dimension 4} &: \quad dA\hat{}dA = 2ba(t^2 - z^2)/(r^2)^2\ \Omega_4 && (3.7)
\end{aligned}$$

The Jacobian matrix has 1 zero eigen value and three non-zero eigenvalues. Hence, the cubic polynomial will yield an interpretation as a van der Waals gas. The Adjoint current represents a contraction in space-time, while the flow associated with the 1-form has a rotational component about the z axis.

3.0.2. Example 2: A Hopf 1-form

In this example,the Hopf 1-form is presumed to be of the form

$$A_0 = a(ydx - xdy) + b(tdz - zdt). \tag{3.8}$$

The 1-form of Potentials depends on the coefficients a and b. There are two cases corresponding to left and right handed "polarizations": $a = b$ or $a = -b$. The results of the topological theory are (for $r^2 = x^2 + y^2 + z^2 + t^2$):

$$\begin{aligned}
\text{Mean curvature} &: C_M = 0, & (3.9)\\
\text{Gauss curvature} &: C_G = \{b^2(x^2 + y^2) + a^2(z^2 + t^2)\}/(r^2)^2 & (3.10)\\
\text{Adjoint Cubic curvature} &: C_A = A\hat{}J_s = 0 & (3.11)\\
Top_Torsion &= 2ab \cdot [x, y, z, t]/(r^2) & (3.12)\\
\text{Adjoint Current} &: J_s = (ab/2) \cdot Top_Torsion & (3.13)\\
\text{Pfaff Dimension 4} &: dA\hat{}dA = 4ab/(r^2)\ \Omega_4 & (3.14)
\end{aligned}$$

What is remarkable for this Action 1-form is that both the mean curvature and the Adjoint curvature of the implicit hypersurface in 4D vanish, for any choice of a or b. The Gauss curvature is non-zero, positive real and is equal to the inverse square of the radius of a 4D euclidean sphere, when $a^2 = b^2$. The Adjoint cubic interaction energy density is zero. The two non-zero curvatures are pure imaginary conjugates equal to

$$\rho = \pm\sqrt{-b^2(x^2 + y^2) - a^2(z^2 + t^2)}/(r^2). \tag{3.15}$$

The Hopf surface is a 2D imaginary *minimal* two dimensional hyper surface in 4D and has two non-zero imaginary curvatures! Strangely enough the charge-current density is not zero, but it is proportional to the Topological Torsion vector that generates the 3 form $A\hat{}F$. The topological Parity 4 form is not zero, and depends on the sign of the coefficients a and b. In other words the 'handedness' of the different 1-forms determines the orientation of the normal field with respect to the implicit surface. It is known that a process described by a vector proportional to the topological torsion vector in a domain where the topological parity is non-zero $4ba/(x^2 + y^2 + z^2 + t^2) \neq 0$ is thermodynamically irreversible.

3.0.3. The sliding bowling ball

Assume that the physical system of a bowling ball may be represented by a 1-form of Action constructed from a primitive Lagrange function with constraints. The Lagrange function is defined as

$$L = L(x, \theta, v, \omega, t) = \{\beta m(\lambda\omega)^2/2 - mv^2/2\} \tag{3.16}$$

Let the topological constraints be defined anholonomically by the Pfaffian system:

$$\{dx - vdt\} \Rightarrow 0, \qquad \{d\theta - \omega dt\} \Rightarrow 0, \quad \{dx - \lambda d\theta\} \Rightarrow 0 \tag{3.17}$$

The constrained 1-form of Action becomes:

$$A = L(x, \theta, v, \omega, t)dt + p\{dx - vdt\} + l\{d\theta - \omega dt\} + s\{\lambda d\theta - dx\} \tag{3.18}$$

where $\{p, l, s\}$ are Lagrange multipliers. For simplicity, assume initially that two of the Lagrange multipliers (momenta) are defined canonically; e.g.,

$$p = \partial L/\partial v \Rightarrow -mv, \qquad l = \partial L/\partial \omega \Rightarrow \beta m\lambda^2\omega \tag{3.19}$$

which implies that

$$A = (-mv - s)dx + (\beta m\lambda^2\omega + \lambda s)d\theta - \{-mv^2/2 + \beta m(\lambda\omega)^2/2\}dt. \tag{3.20}$$

The Pfaff dimension of this action 1-form is 6. The volume element of the associated symplectic manifold is given by the expression

$$6Vol = 6m^2\beta\lambda^2\{v - \lambda\omega\}dx\hat{}d\theta\hat{}dv\hat{}d\omega\hat{}ds\hat{}dt = dA\hat{}dA\hat{}dA \tag{3.21}$$

The symplectic manifold has a singular subset upon which the Pfaff dimension of the Action 1-form is 2n+1 = 5. The constraint for such a contact manifold is precisely the no-slip condition:

$$\{v - \lambda\omega\} \Rightarrow 0. \tag{3.22}$$

On the 5 dimensional contact manifold there exists a unique extremal (Hamiltonian) field which (to within a projective factor) defines the conservative reversible part of the evolutionary process.

However, on the 6 dimensional symplectic manifold, there does not exist a unique extremal field, nor a unique stationary field, that can be used to define the dynamical system. There does exist a unique Topological Torsion direction field (or current) defined (to within a projective factor, σ) by the 6 components of the 5 form,

$$Torsion = A\hat{}dA\hat{}dA \tag{3.23}$$

This unique vector, T, has the properties that

$$L_{(\mathbf{T})}A = \Gamma \cdot A \quad and \quad i(\mathbf{T})A = 0. \tag{3.24}$$

This "Torsion" vector direction field satisfies the equation

$$L_{(\mathbf{T})}A\hat{}L_{(\mathbf{T})}dA = Q\hat{}dQ \neq 0. \tag{3.25}$$

Hence a process having a component constructed from this unique Torsion vector field becomes a candidate to describe the initial irreversible decay of angular momentum and kinetic energy.

Solving for the components of the Torsion vector for the bowling ball problem leads to the (unique) decaying dynamical system:

$$dv/dt = -\sigma/2\{\beta\lambda^2\omega^2 - 2\lambda v\omega + v^2\}/(v - \lambda\omega) \tag{3.26}$$

$$d\omega/dt = -\sigma/2\{-\beta\lambda^2\omega^2 + 2\beta\lambda v\omega - v^2\}/\lambda\beta(v - \lambda\omega) \tag{3.27}$$

$$ds/dt - \sigma s = \sigma/2\{-m\beta\lambda^2\omega^2 + mv^2\}/(v - \lambda\omega) \tag{3.28}$$

$$dx/dt = v \tag{3.29}$$

$$d\theta/dt = \omega \tag{3.30}$$

It is to be noted that the non-canonical "symplectic momentum" variables, defined by inspection from the constrained 1-form of Action as

$$P_x \doteq -(mv + s), \qquad P_\theta \doteq (m\beta\lambda^2\omega + s\lambda), \tag{3.31}$$

both decay irreversibly at the same (unit σ) rate on the manifold of Pfaff topological dimension 6, until the NoSlip condition is satisfied and the topology has become of Pfaff topological dimension 5.

Once the NoSlip condition is reached (irreversibly), the evolution proceeds without further topological change, in a conservative Hamiltonian manner.

4. Conclusions

Based upon the single assumption that the universe is a non-equilibrium thermodynamic system of Pfaff topological dimension 4 leads to a cosmology where the universe, at present, can be approximated in terms of the non-equilibrium states of a very dilute van der Waals gas near its critical point. The stars and the galaxies are the topological defects and coherent (but not equilibrium) self-organizing structures of Pfaff topological dimension 3 formed by irreversible topological evolution in this non-equilibrium system of Pfaff topological dimension 4.

The turbulent non-equilibrium thermodynamic cosmology of a real gas near its critical point yields an explanation for:

1. The granularity of the night sky as exhibited by stars and galaxies.
2. The Newtonian law of gravitational attraction proportional to $1/\mathrm{r}^2$.
3. The expansion of the universe.

5. Appendix 1. The shape matrix for implicit hypersurfaces.

Hypersurfaces can be defined in several ways. The most common is a differentiable parametric map from M dimensions into a variety of M+1 dimensions. The surface is uniquely defined, with 1 component, and can be orientable or not orientable. A second way is to define the surface implicitly, in terms of the zero set of a function. Such surfaces are orientable, but can consist of multiple components. Some surfaces have both a parametric formulation and an implicit representation.

If in the parametric process the mapping is to a space constrained by a metric (not necessarily euclidean), then the M+1 x M+1 covariant metric on space M+1 has a well defined pullback, or preimage on the hypersurface, M. The induced M x M metric defines the first fundamental form of the surface. The more interesting properties are determined by the second fundamental form, or the shape matrix, which is related to properties of the normal field to the surface in M+1 space. Classical texts on differential geometry cover the parametric methods quite well.

However, the same cannot be said for implicit hypersurfaces, especially for the development of the equivalent of the shape matrix in terms of an implicit function representation. A somewhat obscure presentation is given by Harley Flanders. A more detailed presentation for implicit surface functions is given in O'Neill [16]. For surfaces that have both a parametric and an implicit representation, the present author discovered that there is an equivalence between the shape matrix, the Repere Mobile, and the Jacobian matrix of the surface normal field (the gradient of the implicit function) - if the normal field is scaled (divided) by the euclidean norm (the Gauss map). These results led to a more important formulation, where the previous results are extended to more general implicit surfaces, where the normal field is given, yet the normal field cannot be defined as a gradient of a single function. The normal field when expressed as a 1-form, need not be exact as it is in the classic theory of implicity surfaces. Indeed, the 1-form whose coefficients are form the normal field at a point can be of any Pfaff topological dimension in the general case, where it is always zero in the classic case.

The scaling of the normal field can also be done using the generalization of the euclidean distance given by the Holder norm 2.26. The key idea is that any form of the Holder norm which is homogeneous of degree 1 in the components of the normal field always leads to a Jacobian matrix with zero determinant. The constraint of a zero determinant forms a projective invariant system. Application to 4 dimensional spaces indicates a change of topological dimension as evolutionary properties evolve or are attracted to topological defect regions of zero determinant. The result is a domain of Pfaff dimension 3 occurring in regions of Pfaff topological dimension 4 [17]. For the Gauss map, the origin becomes a singular point, and is "removed" as a topological defect. For other versions of the Holder norm, the singular "points" have a variety of shapes. The theory of Homogeneous normal fields and the applications of the Holder norm will appear elsewhere.

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